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A RECURSIVE ALGORITHM TO COMPUTE A BASIS OF A SIMILARITY
Luis Garmendia¹, Jordi Recasens², Adela Salvador³

Complutense University of Madrid, lgarmend@fdi.ucm.es
Technical University of Catalonia, j.recasens@upc.edu
Technical University of Madrid, ma09@caminos.upm.es

Abstract
This paper studies some theory and methods to build a representation theorem basis of a similarity from the basis of its subsimilarities, providing an alternative recursive method to compute the basis of a similarity.

Keywords: T-indistinguishabilities representation theorem, similarity, basis of a similarity, dimension of a similarity, similarity representation theorem.

1 INTRODUCTION
Similarity fuzzy relations were introduced by Zadeh [1971] [10] to represent a degree of equality or closeness between the elements of a universe. They generalise the equivalence relations; in fact, the similarities are the only T-indistinguishabilities that satisfy that all their alpha cuts are crisp equivalence relations. Similarities are not only a powerful tool to represent equality information, but they are also useful to classify the universe into clusters with uncertainty.

The Valverde’s representation theorem of T-indistinguishabilities is one of the strongest theorems in fuzzy logic theory. It opened an interesting area of investigation on T-indistinguishabilities, the computation of its basis, specially for both the minimum T-norm and Archimedean T-norms, and the study of special T-indistinguishabilities such as the one dimensional ones.

On the other hand, the computational investigation on the computation of T-transitive closures, or the search of the structure of a similarity, specially its decomposition into subsimilarities, were not related in previous studies with the computation of the basis of a similarity.

Joan Jacas [4] studied two algorithms to compute basis of a similarity in 1990, but did not consider the decomposition of similarities to do it. The aim of this paper is to introduce a new approach to compute them, providing useful theory and methods for both better understanding the concept of structure of a similarity and the Valverde’s representation theorem for similarities and the computation of their basis using a new decomposition approach.

2 PRELIMINARIES
Let X = {x₁, ..., xₙ} be a finite universe.
Let T be a t-norm. A T-indistinguishability operator E on X is a fuzzy relation E: X×X → [0, 1], satisfying for all x, y, z in X:
1. E(x, x) = 1 (Reflexivity)
2. E(x, y) = E(y, x) (Symmetry)
3. T(E(x, y), E(y, z)) ≤ E(x, z) (T-transitivity)

Definition 2.2. A similarity [Zadeh, 1971] [10] is a reflexive, symmetric and min-transitive fuzzy relation, it is, a similarity is a Min-indistinguishability operator.

Notation 2.1.
We can denote xᵢⱼ = E(xᵢ, xⱼ).

Lemma 2.1. Let π be a permutation on X. If E is a similarity on X, then the fuzzy relation P_π(E) is also a fuzzy similarity.

Proof. It is obvious. P_π(E) is reflexive and symmetric. If xᵢⱼ \geq \min(xᵢₖ, xⱼₖ) for all i, j, k then xᵢₖ = xᵢ(ₖ) xⱼₖ \geq \min(xᵢₖ, xⱼₖ(ₖ)) = T(xᵢₖ, xⱼₖ) for all 1 \leq r, s, k \leq n.

2.1. CONSTRUCTION OF A FUZZY SIMILARITY FROM SUBSIMILARITIES
Let C and D be two similarities on two disjoint sets X₁ and X₂; A similarity relation E(F; C, D) on X₁ ∪ X₂ can be built with the following shape:

E(F; C, D) = \begin{bmatrix} C & F' \\ F & D \end{bmatrix}

A method for giving the bridging values fᵢ₀ in F, (when j ≤ \text{card}(X₁) < i) is the assignation of a unique value fᵢ in all the \text{card}(X₁) \times \text{card}(X₂) values in F. This value must be chosen in an interval [0, a] where a = \min(\min(C), \min(D)) \{fᵢₖ\}. The values in F' are the symmetric values fᵢ of the computed F.
So the computed values in F are equal and satisfy that f ≤ \min(\min(C), \min(D)).

If C and D are fuzzy similarities, then E(f; C, D) is also a fuzzy similarity, ∀f ∈ {0, \min(\min(C), \min(D))}.
2.2. THE REPRESENTATION THEOREM OF T-INDISTINGUISHABILITY OPERATORS

The representation theorem allows us to generate a $T$-indistinguishability operator on a set $X$ from a family of subsets on $X$, and reciprocally states that every $T$-indistinguishability can be obtained in this form.

Definition 2.2.1 The residuation $\overline{T}$ or quasi inverse of a $t$-norm $T$ is the map $\overline{T} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined for all $x, y \in [0, 1]$ by

$$\overline{T}(x, y) = \sup \{ \alpha \in [0, 1] | T(x, \alpha) \leq y \}.$$

The residuation $\overline{T}$ of a $t$-norm $T$ is a $T$-preorder (reflexive and $T$-transitive) on $[0, 1]$, and then it is a useful operator to generate implication relations from fuzzy sets on $X$.

Note that $\overline{\text{Min}}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$

Definition 2.2.2 The bisimulation $\overline{E}$ (or also $E_T$) of a $t$-norm $T$ is the map $\overline{E} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined for all $x, y \in [0, 1]$ by

$$\overline{E}(x, y) = \overline{T}(x, y) = \overline{\text{Min}}(\overline{T}(x, y), \overline{T}(y, x)).$$

Note that $\overline{\text{Min}}(x, y) = \overline{\text{Min}}(\text{Max}(x, y), \text{Min}(x, y)) = \begin{cases} 1 & \text{if } x = y \\ \text{Min}(x, y) & \text{if } x \neq y \end{cases}$

The bisimulation $\overline{E}$ of a $t$-norm $T$ is a $T$-indistinguishability operator on $[0, 1]$, and then it is a useful operator to generate $T$-indistinguishability relations on $X$ from two fuzzy sets on $X$.

Lemma 2.2.1.

Let $\mu$ be a fuzzy set on $X$, and $T$ a continuous $t$-norm. The fuzzy relation $E_\mu$ on $X$ defined for all $x, y \in X$ by $E_\mu(x, y) = E_T(\mu(x), \mu(y))$ is a $T$-indistinguishability operator.

Note that $E_\mu$ is a one-dimensional $T$-indistinguishability generated by a basis of one fuzzy set $\mu$.

Lemma 2.2.2.

Let $\{E_i\}_{i \in I}$ be a family of $T$-indistinguishability operators on a set $X$. The relation $E$ on $X$ defined for all $x, y \in X$ by $E(x, y) = \inf_{i \in I} E_i(x, y)$ is a $T$-indistinguishability operator.

The next theorem is crucial to understand the structure of a $T$-indistinguishability operator. It allows us to generate $T$-indistinguishabilities from a family of fuzzy sets, and reciprocally that any $T$-indistinguishability can be generated from a family of fuzzy sets.

Representation Theorem 2.2.1. [9]

Let $R$ be a fuzzy relation on $X$ and $T$ a continuous $t$-norm. $R$ is a $T$-indistinguishability operator if and only if there exists a family $\{\mu_i\}_{i \in I}$ of fuzzy sets on $X$ such that for all $x, y \in X$

$$R(x, y) = \inf_{i \in I} E_i(x, y).$$

Definition 2.2.1. Dimension and basis of a $T$-indistinguishability operator

The dimension of a $T$-indistinguishability operator $E$ is the minimal of the cardinals $d$ of the generating families of $E$, in the sense of the representation theorem. That minimal basis of generating fuzzy sets is called a basis of the $T$-indistinguishability operator.

3 DECOMPOSITION OF SIMILARITIES AND REPRESENTATION THEOREM

This chapter gives some theoretical results toward a method to compute a basis of a similarity from the bases of its subsimilarities.

Lemma 3.1. Let $S$ be the following similarity on $X_1 \cup X_2$, $S(F; C, D) = \begin{pmatrix} C & F \\ \bar{C} & \bar{D} \end{pmatrix}$.

Then $\dim(S) \geq \dim(C)$.

Lemma 3.2.

Let $\{\mu_i\}_{i \in I}$ be a representation theorem basis of a similarity $C$ of dimension $r$ on a finite set $X_1$. Let $\{\gamma_i\}_{i \in J}$ be a representation theorem basis of a similarity $D$ of dimension $s$ on a finite set $X_2$.

Let $S$ be the following similarity on $X_1 \cup X_2$, $S(F; C, D) = \begin{pmatrix} C & F \\ \bar{C} & \bar{D} \end{pmatrix}$ where $F = \begin{pmatrix} f_{11} & \ldots & f_{1s} \\ \vdots & \ddots & \vdots \\ f_{r1} & \ldots & f_{rs} \end{pmatrix}$ is a $\text{card}(X_1) \times \text{card}(X_2)$ matrix.

Suppose that $r > s$ and $f < \min(C)$ and $f \leq \min(D)$ then $\dim(S) \leq \dim(C)$ and a generator set of $S$ is $\{ (\mu_{i_1}), \ldots, (\mu_{i_s}), (\mu_{i_{s+1}}), \ldots, (\mu_{i_r}) \}$ where $F^* = \begin{pmatrix} f_{11} & \ldots & f_{1s} \\ \vdots & \ddots & \vdots \\ f_{r1} & \ldots & f_{rs} \end{pmatrix}$ is a $\text{card}(X_2) \times 1$ matrix.

Proposition 3.1.
Let \((\mu_i)_{i \in I}\) be a representation theorem basis of a similarity \(C\) of dimension \(r\) on a finite set \(X_1\). Let \((\gamma_i)_{i \in I}\) be a representation theorem basis of a similarity \(D\) of dimension \(s\) on a finite set \(X_2\).

Let \(S\) be the following similarity on \(X_1 \cup X_2\), \(S(F; C, D) = \begin{pmatrix} C & F^T \\ F & D \end{pmatrix}\) where \(F = \begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}\) is a \(\text{card}(X_1) \times \text{card}(X_2)\) matrix.

Suppose that \(r > s\) and \(f < \min(C)\) and \(f \leq \min(D)\) then

1) \(\text{dim}(S) = \text{dim}(C) = r\)

2) a basis of \(S\) is

\[
\left\{ \left( \begin{array}{c} \mu_1 \\ \vdots \\ \mu_s \\ \mu_{s+1} \\ \vdots \\ \mu_{r+s} \\ \vdots \\ \mu_{r+s} \\ F' \\ \end{array} \right) \right\}
\]

where \(F' = \begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}\) is a \(\text{card}(X_2) \times 1\) matrix.

**Proof:**

By Lemma 3.1 \(\text{dim}(S) \geq \text{dim}(C)\).

By Lemma 3.2 \(\text{dim}(S) \leq \text{dim}(C)\). Also \(\text{dim}(S) = \text{dim}(C) = r\) and

\[
\left\{ \left( \begin{array}{c} \mu_1 \\ \vdots \\ \mu_s \\ \mu_{s+1} \\ \vdots \\ \mu_{r+s} \\ \vdots \\ \mu_{r+s} \\ F' \\ \end{array} \right) \right\}
\]

is a basis of \(S(F; C, D)\). □

**Corollary 3.1.**

Let \((\mu_i)_{i \in I}\) be a representation theorem basis of a similarity \(C\) on a finite set \(X = \{x_1, \ldots, x_n\}\).

Let \(S\) be the similarity on \(X \cup \{x_{n+1}\}\) such that \(S(F; C, 1) = \begin{pmatrix} C & F^T \\ F^T & 1 \end{pmatrix}\) where \(F' = (f \quad \cdots \quad f)\) is a \(1 \times \text{card}(X)\) matrix.

If \(f < \min(C)\), then a basis of \(S\) is

\[
\left\{ \left( \begin{array}{c} \mu_1(x_1) \\ \vdots \\ \mu_s(x_s) \\ \mu_{s+1}(x_{s+1}) \\ \vdots \\ \mu_{r+s}(x_{r+s}) \\ \vdots \\ \mu_{r+s}(x_{r+s}) \\ f \\ \end{array} \right) \right\}
\]

**Example 3.1.**

Let \(S\) be the similarity \(\begin{pmatrix} 1 & a & b & c \\ a & 1 & b & c \\ b & b & 1 & c \\ c & c & c & 1 \end{pmatrix}\), already ordered with \(c < b < a\).

\[
S = \begin{pmatrix} 1 & a & b & c \\ a & 1 & b & c \\ b & b & 1 & c \\ c & c & c & 1 \end{pmatrix}
\]

and a base of \(\begin{pmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{pmatrix}\) is \(\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}\), then by the theorem 3.1 a basis of \(S\) is

\[
\begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \times \{c\} = \begin{pmatrix} 1 \\ a \\ c \end{pmatrix}
\]

Note that \(S\) is one dimensional.

**Example 3.2.**

Let \(S\) be the similarity \(\begin{pmatrix} 1 & a & a & b \\ a & 1 & a & b \\ a & a & 1 & b \\ b & b & b & 1 \end{pmatrix}\), already ordered with \(b < a\).

\[
S = \begin{pmatrix} 1 & a & a & b \\ a & 1 & b & b \\ a & a & 1 & 1 \\ b & b & b & 1 \end{pmatrix}
\]

A basis of \(\begin{pmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & 1 \end{pmatrix}\) is \(\begin{pmatrix} 1 \\ a \\ a \\ a \end{pmatrix}\).

Then, as \(b < \min(a, 1, a, a)\), by theorem 3.1, a basis of \(S\) is

\[
\left\{ \begin{pmatrix} 1 \\ a \\ a \\ a \end{pmatrix} \times \{b\} = \begin{pmatrix} 1 \\ a \\ b \\ b \end{pmatrix}
\]

**Lemma 3.2.**

Let \((\mu_i)_{i \in I}\) be a representation theorem basis of a similarity \(C\) of dimension \(r\) on a finite set \(X_1\). Let \((\gamma_i)_{i \in I}\) be a representation theorem basis of a similarity \(D\) of dimension \(s\) on a finite set \(X_2\).

Let \(S\) be the following similarity on \(X_1 \cup X_2\), \(S(F; C, D) = \begin{pmatrix} C & F^T \\ F & D \end{pmatrix}\), where \(F = \begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}\) is a \(\text{card}(X_1) \times \text{card}(X_2)\) matrix.

Suppose that \(r > s, f = \min(C)\) and \(f \leq \min(D)\), then:

1) \(\text{dim}(S) > \text{dim}(C) = r\).

**Lemma 3.3.**

Let \((\mu_i)_{i \in I}\) be a representation theorem basis of a similarity \(C\) of dimension \(r\) on a finite set \(X_1\). Let \((\gamma_i)_{i \in I}\) be a representation theorem basis of a similarity \(D\) of dimension \(s\) on a finite set \(X_2\).
Let $S$ be the following similarity on $X_1 \cup X_2$, $S(F; C, D) = \begin{bmatrix} C & F^T \\ F & D \end{bmatrix}$, where $F = \begin{bmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{bmatrix}$ is a $\text{card}(X_1) \times \text{card}(X_2)$ matrix.

Suppose that $r > s, f = \min(C)$ and $f \leq \min(D)$, then:

1) $\dim(S) = \dim(C) + 1 = r + 1$

2) A generator set of $S$ is a basis of $S$ is

\[
\begin{align*}
\{(\mu_1, (y_1)), \ldots, (\mu_s, (y_s)), (\mu_{s+1}, F'), \ldots, (\mu_r, F')\},
\end{align*}
\]

where $(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a $\text{card}(X_1) \times 1$ matrix and $F' = \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$ is a $\text{card}(X_2) \times 1$ matrix.

**Proof:**

Now

\[
\inf \left\{ \inf_{s \leq s' \leq \text{sir}} E((\mu_1, (y_1)), \ldots, (\mu_s, F'), (\mu_{s+1}, F'), \ldots, (\mu_r, F') \right\} = \begin{bmatrix} C & F^T \\ F & D \end{bmatrix} = S(F; C, D).
\]

Also

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sigma_{\text{sir}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

the infimum of all the similarities is $\begin{bmatrix} C & F^T \\ F & D \end{bmatrix} = S(F; C, D)$.

**Proposition 3.2.**

Let $(\mu_i)_{i \in I}$ be a representation theorem basis of a similarity $C$ of dimension $r$ on a finite set $X_1$. Let $(y_i)_{i \in I}$ be a representation theorem basis of a similarity $D$ of dimension $s$ on a finite set $X_2$.

Let $S$ be the following similarity on $X_1 \cup X_2$, $S(F; C, D) = \begin{bmatrix} C & F^T \\ F & D \end{bmatrix}$, where $F = \begin{bmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{bmatrix}$ is a $\text{card}(X_1) \times \text{card}(X_2)$ matrix.

Suppose that $r > s, f = \min(C)$ and $f \leq \min(D)$, then:

1) $\dim(S) = \dim(C) + 1 = r + 1$

2) A basis of $S$ is:

\[
\{((\mu_1, (y_1)), \ldots, (\mu_s, (y_s)), (\mu_{s+1}, F'), \ldots, (\mu_r, F')) \}
\]

where $(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a $\text{card}(X_1) \times 1$ matrix and $F' = \begin{bmatrix} \vdots \\ 1 \end{bmatrix}$ is a $\text{card}(X_2) \times 1$ matrix.

**Proof:**

By Lemma 3.2 $\dim(S) > \dim(C) = r$, and by Lemma 3.3 $\dim(S) \leq \dim(C) + 1 = r + 1$, then $\dim(S) = \dim(C) + 1 = r + 1$, and a basis of $S$ is

\[
\{((\mu_1, (y_1)), \ldots, (\mu_s, (y_s)), (\mu_{s+1}, F'), \ldots, (\mu_r, F')) \}
\]

**Example 3.5**

Let $S$ be the similarity $\begin{bmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & b \\ b & b & b & 1 \end{bmatrix}$, already ordered with $b < a$.

Let $S = \begin{bmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{bmatrix}$ and a basis of $\begin{bmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{bmatrix}$, is

\[
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

then by the propositions 3.1 and 3.2 a generator set of $S$ is

\[
\begin{bmatrix} a \\ b \\ b \\ b \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

**Proposition 3.3.**

Let $(\mu_i)_{i \in I}$ be a representation theorem basis of a similarity $C$ of dimension $r$ on a finite set $X_1$. Let $(y_i)_{i \in I}$ be a representation theorem basis of a similarity $D$ of dimension $s$ on a finite set $X_2$.

Let $S$ be the following similarity on $X_1 \cup X_2$, $S(F; C, D) = \begin{bmatrix} C & F^T \\ F & D \end{bmatrix}$, where $F = \begin{bmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{bmatrix}$ is a $\text{card}(X_1) \times \text{card}(X_2)$ matrix.

Suppose that $r = s$ (dim(C) = dim(D) and f < min(C, D), then a generator set of $S$ is:

\[
\{((\mu_1, (y_1)), \ldots, (\mu_r, (y_2)), (1)) \}
\]

and $\dim(S) \leq r + 1$.

**Proof:**

Let $(\mu_1, \ldots, \mu_r)$ be a representation theorem basis of a similarity $C$, so

\[
C = \inf_{i \in I} E_{\mu_i} \text{ where } I = \{1, \ldots, r\}.
\]
In the same way, let \( \{ \gamma_1, ..., \gamma_r \} \) be a representation theorem basis of a similarity \( D \), so 
\[ D = \inf_{\gamma \in J} E_{\gamma} \] 
where \( J = \{ 1, ..., r \} \) 

Now, for all \( 1 \leq i \leq r \),
\[ (\mu_i) \sigma_{\gamma_i} (\gamma_i) = (E_{\mu_i}) (x_{nm}) (E_{\gamma_i}) \]

Also,
\[ (1) F' \sigma_{\gamma_i} (1) F' = (1) \]
\[ F' \]

So the infimum of all the similarities is
\[ \begin{pmatrix} C & F' \\ F & D \end{pmatrix} = S(F; C, D) \]

and then
\[ \{ (\mu_1), ..., (\mu_r), (1) \} \]

is a generator set of \( S \). \( \square \)

**Example 3.4.**

Let \( S \) be the similarity \[ \begin{pmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & a \\ b & a & 1 & 1 \end{pmatrix} \]

with \( b < a \).

\[ S = \begin{pmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & a \\ b & b & 1 & 1 \end{pmatrix} \]

A basis of \( \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \), is \[ \{ (1) \} \], then by the proposition 3.3 a generator set (and a basis in this example) of \( S \) is
\[ \{ (\mu_1), ..., (\mu_r), (1) \} \]

\[ = \begin{pmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & a \\ b & b & 1 & 1 \end{pmatrix} \]

Note that there exists other bases of \( S \), for example
\[ \{ (1) \}, \{ (b) \}, \{ (b) \}, \{ (b) \} \]

Let \( S \) be the similarity \[ \begin{pmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & a \\ b & b & 1 & 1 \end{pmatrix} \]

with \( b < a \).

\[ S = \begin{pmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & a \\ b & b & 1 & 1 \end{pmatrix} \]

A basis of \( \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \) is \[ \{ (1) \} \].

Then by proposition 3.3 a basis of \( S \) is
\[ \{ (1) \}, \{ (b) \}, \{ (b) \}, \{ (b) \} \]

**4 GENERATING A DECOMPOSITION OF A GIVEN SIMILARITY**

Input: A similarity \( S \) on \( X \).

Output:
- Two similarities \( S_1 \) and \( S_2 \) on \( X_1 \) and \( X_2 \) such that \( X = X_1 \cup X_2 \)
- A bridging value \( f \) such that \( f < \min(S_1) \) and \( f \leq \min(S_2) \)

**Algorithm 5.1:**

1. Sort the universe \( X \) having descending columns under the diagonal using algorithm 4.1
2. Compute the frequency \( k \) of the lowest value \( v \) in the first column
3. If \( k \geq \text{card}(X) - 1 \) then STOP, \( S_1 = S \).
4. Let \( X_1 \) be \( \{ x_1, ..., x_{k+1} \} \) and let \( X_2 \) be \( \{ x_{k+2}, ..., x_{\text{card}(X)} \} \)
5. Let \( S_1 = S \setminus X_1, S_2 = S \setminus X_2, f = v \)

**5 A RECURSIVE ALGORITHM TO COMPUTE A GENERATING SET OF A SIMILARITY**

Input: a similarity \( S \) on a universe \( X \).

Output: A generating set of \( S \), and so, an upper bound of the dimension of \( S \).

A recursive algorithm:

1) If \( \text{card}(X) \leq 2 \) then STOP. \( \dim(S) = 1 \) and a basis of \( S \) is the first column.
2) Decompose \( S \) into \( S_1 \) and \( S_2 \) using algorithm 5.1. Note that \( f < \min(S_1) \)
3) If \( 2 \leq \text{card}(X) \leq 7 \) then use corollary 3.1:
   a. Recursively compute a generating set of \( S_1 (\mu_i)_{i \in I} \)
   b. A generating set of \( S \) is \( \{ (\mu_i)_{i \in I} \} \times \{ f \} \)
   c. STOP
4) Recursively compute a generating set of \( S_1, (\mu_i)_{i \in I} \) and \( S_2, (\gamma_i)_{i \in I} \)
5) If \( \dim(S_1) > \dim(S_2) \) - vice versa - then use proposition 3.1
   a. A generating set of \( S \) is
   \[ \begin{pmatrix} (\mu_1) \cdots (\mu_s) \\ (\gamma_1) \cdots (\gamma_s) \end{pmatrix} \]
   b. STOP
6) If \( \dim(S_1) = \dim(S_2) \) then use proposition 3.3
   a. A generating set of \( S \) is
   \[ \begin{pmatrix} (\mu_1) \cdots (\mu_s) \\ (\gamma_1) \cdots (\gamma_s) \\ (1) \end{pmatrix} \]
   b. STOP
Example 6.1.

Let $S$ be the sorted similarity with descending columns under the diagonal given by the following matrix:

$$
S = \begin{pmatrix}
1 & 0.9 & 0.4 & 0.4 & 0.3 \\
0.9 & 1 & 0.4 & 0.4 & 0.3 \\
0.4 & 0.4 & 1 & 0.7 & 0.3 \\
0.4 & 0.4 & 0.7 & 1 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.3 & 1
\end{pmatrix}
$$

It will be computed the algorithm 6.1 step by step.

A decomposition of $S$ is given by algorithm 5.1 is

$$
S = \begin{pmatrix}
1 & 0.9 & 0.4 & 0.4 & 0.3 \\
0.9 & 1 & 0.4 & 0.4 & 0.3 \\
0.4 & 0.4 & 1 & 0.7 & 0.3 \\
0.4 & 0.4 & 0.7 & 1 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.3 & 1
\end{pmatrix},
S_1 = \begin{pmatrix}
1 & 0.9 & 0.4 & 0.4 \\
0.9 & 1 & 0.4 & 0.4 \\
0.4 & 0.4 & 1 & 0.7 \\
0.4 & 0.4 & 0.7 & 1
\end{pmatrix}
$$

Now that $S$ is ordered and decomposed, we can build its basis as follows:

A basis of $\begin{pmatrix} 1 \\ 0.9 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 0.9 \end{pmatrix}$. A basis of $\begin{pmatrix} 1 \\ 0.7 \\ 1 \end{pmatrix}$

is $\begin{pmatrix} 1 \\ 0.7 \end{pmatrix}$, and $\dim C = 2 = \dim D, f = 0.4 < \min(C, D)$.

Then by proposition 3.3, a generator set of $S_1 = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \\ 0.4 & 0.4 \\ 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix}$

is $\begin{pmatrix} \left( \mu_1 \right)_{\left( Y_1 \right)} \\ \ldots \end{pmatrix}, \begin{pmatrix} \left( \mu_r \right)_{\left( Y_r \right)} \end{pmatrix}, \begin{pmatrix} \left( F' \right) \end{pmatrix}$

And then, by proposition 3.1, a generator set of $S = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}$

is $\begin{pmatrix} \left( \mu_1 \right)_{\left( Y_1 \right)} \\ \ldots \end{pmatrix}, \begin{pmatrix} \left( \mu_s \right)_{\left( Y_s \right)} \end{pmatrix}, \begin{pmatrix} \left( \mu_{s+1} \right)_{\left( Y_{s+1} \right)} \end{pmatrix}, \ldots, \begin{pmatrix} \left( \mu_r \right)_{\left( Y_r \right)} \end{pmatrix}$

6 CONCLUDING REMARKS

This paper’s main contribution is a method to compute a representation theorem basis of a similarity from the bases of its subsimilarities.

These results can be used to propose an alternative algorithm to build a basis of similarities.

The bases of all structures of similarities with dimension four are computed in the examples using the new construction method.

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8 REFERENCES