THE ZERO-REMOVING PROPERTY AND LAGRANGE-TYPE INTERPOLATION SERIES

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The classical Kramer sampling theorem, which provides a method for obtaining orthogonal sampling formulas, can be formulated in a more general nonorthogonal setting. In this setting, a challenging problem is to characterize the situations when the obtained nonorthogonal sampling formulas can be expressed as Lagrange-type interpolation series. In this article a necessary and sufficient condition is given in terms of the zero removing property. Roughly speaking, this property concerns the stability of the sampled functions on removing a finite number of their zeros.

Keywords Analytic Kramer kernels; Lagrange-type interpolation series; Zero-removing property.

AMS Subject Classification 46E22; 42C15; 94A20.

1. STATEMENT OF THE PROBLEM

The classical Kramer sampling theorem provides a method for obtaining orthogonal sampling theorems [5, 13, 15, 21]. The statement of this general result is as follows. Let \( K \) be a complex function defined on \( D \times I \), where \( I \subset \mathbb{R} \) is an interval and \( D \) is an open subset of \( \mathbb{R} \), and such that for every \( t \in D \) the sections \( K(\cdot, t) \) are in \( \mathcal{L}^2(I) \). Assume that there exists a sequence of distinct real numbers \( \{ t_n \} \subset D \), indexed by a subset of \( \mathbb{Z} \), such that \( \{ K(x, t_n) \} \) is a complete orthogonal sequence of functions for \( \mathcal{L}^2(I) \). Then for any \( f \) of the form

\[
f(t) = \int_I F(x)K(x, t) \, dx \quad t \in D,
\]

Received 15 February 2011; Revised 28 April 2011; Accepted 4 May 2011.
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where $F \in L^2(I)$, we have
\[
f(t) = \sum_{n} f(t_n) S_n(t), \quad t \in D,
\]
with
\[
S_n(t) := \frac{\int_I K(x, t) \overline{K(x, t_n)} \, dx}{\int_I |K(x, t_n)|^2 \, dx}.
\]
The series in (2) converges absolutely and uniformly on subsets of $D$ where
$\|K(\cdot, t)\|_{L^2(I)}$ is bounded.

For instance, taking $I = [-\pi, \pi]$, $K(x, t) = e^{itx}$ and $\{t_n = n\}_{n \in \mathbb{Z}}$, we get
the well-known Whittaker–Shannon–Kotel’nikov sampling formula
\[
f(t) = \sum_{n = -\infty}^{\infty} f(n) \frac{\sin \pi(t - n)}{\pi(t - n)}, \quad t \in \mathbb{R},
\]
for functions in $L^2(\mathbb{R})$ whose Fourier transform has support in $[-\pi, \pi]$.

Now, if we take $I = [0, 1]$, $K(x, t) = \sqrt{xt} J_v(xt)$ and $\{t_n\}$, the sequence of
the positive zeros of the Bessel function $J_v$ of $v$th order with $v > -1$, then
\[
f(t) = \sum_{n} f(t_n) \frac{2 \sqrt{t_n} J_v(t)}{J_v(t_n)(t^2 - t_n^2)}, \quad t \in \mathbb{R},
\]
for every $f$ of the form $f(t) = \int_0^1 F(x) \sqrt{xt} J_v(xt) \, dx$, where $F \in L^2(0, 1)$
(see [13, p. 83]).

The Kramer sampling theorem has played a very significant role in
sampling theory, interpolation theory, signal analysis and, generally, in
mathematics (see, e.g., the survey articles [3, 4]).

In [6], an extension of the Kramer sampling theorem has been
obtained to the case when the kernel is analytic in the sampling parameter
$t \in D \subseteq \mathbb{C}$. Namely, assume that the Kramer kernel $K$ is an entire function
for any fixed $x \in I$, and that the function $h(t) = \int_I |K(x, t)|^2 \, dx$ is locally
bounded on $D \subseteq \mathbb{C}$. Then any function $f$ defined by (1) is an entire
function, as are all the sampling functions (3).

A straightforward discrete version of Kramer’s theorem can be
obtained. Namely, let $K(n, z)$ be a kernel such that, as function of $n$,
the sequence $\{K(n, z)\} \in \ell^2(\mathbb{I})$ for any $z \in D \subseteq \mathbb{C}$, where $\mathbb{I}$ is a countable
index set. Assume that, for a suitable sequence $\{z_n\} \subset D$, the sequence
$\{K(\cdot, z_n)\}$ is an orthogonal basis for $\ell^2(\mathbb{I})$. Then, any function of the
form $f(z) = \sum_{n \in \mathbb{I}} c_n K(n, z)$, where $\{c_n\} \in \ell^2(\mathbb{I})$, can be expanded by means
of a sampling series like (2) (see [8]). As examples of discrete kernels
for which a sampling formula works we can consider discrete kernels
\( K(n,z) := P_n(z), \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( z \in \mathbb{C} \), where \( \{P_n(z)\}_{n \in \mathbb{N}_0} \) denotes a sequence of orthonormal polynomials associated with an indeterminate Hamburger or Stieltjes moment problem (see [8, 9] for the details).

The Kramer sampling theorem has been the cornerstone for a significant mathematical literature of sampling theory associated with differential or difference problems. See, among others, [1, 5, 8, 9, 13, 21] and the references therein.

Thus an abstract analytic formulation of the Kramer sampling theorem raises in a natural way: Let \( \mathcal{H} \) be a complex, separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \), and let \( \{x_n\}_{n=1}^{\infty} \) be a Riesz basis for \( \mathcal{H} \). Suppose \( K \) is a \( \mathcal{H} \)-valued function defined on \( \mathbb{C} \). For each \( x \in \mathcal{H} \), define the function \( f_x(z) = \langle K(z), x \rangle_{\mathcal{H}} \) on \( \mathbb{C} \), and let \( \mathcal{H}_K \) denote the collection of all such functions \( f_x \). Furthermore, each element in \( \mathcal{H}_K \) is an entire function if and only if \( K \) is analytic on \( \mathbb{C} \). In this setting, an abstract version of the analytic Kramer theorem is obtained assuming the existence of two sequences, \( \{z_n\}_{n=1}^{\infty} \) in \( \mathbb{C} \) and \( \{a_n\}_{n=1}^{\infty} \) in \( \mathbb{C} \setminus \{0\} \), such that \( K(z_n) = a_n x_n \) for each \( n \in \mathbb{N} \). Namely, for any \( f_x \in \mathcal{H}_K \) we have

\[
 f_x(z) = \sum_{n=1}^{\infty} f_x(z_n) \frac{S_n(z)}{a_n}, \quad z \in \mathbb{C},
\]

where \( S_n(z) = \langle K(z), y_n \rangle, \ n \in \mathbb{N} \), being \( \{y_n\}_{n=1}^{\infty} \) the dual Riesz basis of \( \{x_n\}_{n=1}^{\infty} \) (see sections 2 and 4 infra for all the details).

A challenging problem is to give a necessary and sufficient condition to ensure that the above sampling formula can be written as a Lagrange-type interpolation series, that is

\[
 f_x(z) = \sum_{n=1}^{\infty} f_x(z_n) \frac{P(z)}{(z - z_n)P(z_n)}, \quad z \in \mathbb{C},
\]

where \( P \) denotes an entire function having only simple zeros at all the points of the sequence \( \{z_n\}_{n=1}^{\infty} \). Roughly speaking, the aforesaid necessary and sufficient condition concerns the stability of the functions belonging to the space \( \mathcal{H}_K \) on removing a finite number of their zeros; this is an ubiquitous algebraic property in the mathematical literature (see section 3 infra) and it will be called the zero-removing property along the article.

Let us consider the following toy example: Given a basis \( \{e_1, e_2\} \) in \( \mathbb{C}^2 \), for the kernel \( K(z) := z^2(e_2 - e_1) + e_1 \) consider the corresponding space \( \mathcal{H}_K \), which coincides with \( \{az^2 + b \mid a, b \in \mathbb{C}\} \). Obviously, this space has not the zero-removing property: if we remove a zero from an element in \( \mathcal{H}_K \), the resulting polynomial does not belong to \( \mathcal{H}_K \). Besides, the sampling formula \( f(z) = f(0)(1 - z^2) + f(1)z^2 \), which holds in \( \mathcal{H}_K \) cannot be written as a Lagrange interpolation formula. The study of all these topics will be carried out throughout the remaining sections.
2. SOME PRELIMINARIES ON THE SPACE $\mathcal{H}_K$

Suppose we are given a separable complex Hilbert space $X$ and an abstract kernel $K$ which is nothing but a $\mathcal{H}$-valued function on $\mathbb{C}$. Set $f_x(z) := \langle K(z), x \rangle_\mathcal{H}$ and denote by $\mathcal{H}_K$ the collection of all such functions $f_x, x \in \mathcal{H}$. It is a reproducing kernel Hilbert space (RKHS) coming from the transforms $K(z), z \in \mathbb{C}$, and corresponding to the reproducing kernel $(z, w) \mapsto \langle K(z), K(w) \rangle_\mathcal{H}$. Notice that the mapping $\mathcal{T}$ given by

$$\mathcal{H} \ni x \mapsto f_x \in \mathcal{H}_K$$

is an antilinear mapping from $\mathcal{H}$ onto $\mathcal{H}_K$ (henceforth we omit the subscript $x$ for denoting the elements in $\mathcal{H}_K$). The mapping $\mathcal{T}$ is injective if and only if the set $\{K(z)\}_{z \in \mathbb{C}}$ is a complete set in $\mathcal{H}$. In particular, if there exists a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ such that $\{K(z_n)\}_{n=1}^\infty$ is a Riesz basis for $\mathcal{H}$, then $\mathcal{T}$ is an antilinear isometry from $\mathcal{H}$ onto $\mathcal{H}_K$. Recall that a Riesz basis in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a boundedly invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m \rangle_\mathcal{H} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^\infty \langle x, y_n \rangle_\mathcal{H} x_n = \sum_{n=1}^\infty \langle x, x_n \rangle_\mathcal{H} y_n$$

hold for every $x \in \mathcal{H}$ (see [20] for more details and proofs).

The convergence in the norm $\| \cdot \|_\mathcal{H}$ implies pointwise convergence which is uniform on those subsets of $\mathbb{C}$ where the function $z \mapsto \|K(z)\|_\mathcal{H}$ is bounded.

Like in the classical case the following result holds: The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel $K$ is analytic in $\mathbb{C}$ [19, p. 266]. Another characterization of the analyticity of the functions in $\mathcal{H}_K$ is given in terms of Riesz bases. Suppose that a Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$ is given and let $\{y_n\}_{n=1}^\infty$ be its dual Riesz basis; expanding $K(z)$, for each fixed $z \in \mathbb{C}$, with respect to the basis $\{x_n\}_{n=1}^\infty$ we obtain

$$K(z) = \sum_{n=1}^\infty \langle K(z), y_n \rangle_\mathcal{H} x_n,$$

where the coefficients $\langle K(z), y_n \rangle_\mathcal{H}$ as functions in $z$ are in $\mathcal{H}_K$. The following result holds: The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if all the functions

$$S_n(z) := \langle K(z), y_n \rangle_\mathcal{H}, \quad z \in \mathbb{C}$$

are entire and $\|K(\cdot)\|_\mathcal{H}$ is bounded on compact sets of $\mathbb{C}$ (see [11]).
3. THE ZERO-REMOVING PROPERTY

In this section, we introduce the zero-removing property for classes of entire functions.

**Definition 1** (Zero-Removing Property). A set $\mathcal{S}$ of entire functions has the zero-removing property (ZR property hereafter) if for any $g \in \mathcal{S}$ and any zero $w$ of $g$ the function $g(z)/(z-w)$ belongs to $\mathcal{S}$.

The ZR property is ubiquitous in mathematics; for instance, the set $\mathcal{P}_N(\mathbb{C})$ of polynomials with complex coefficients of degree less or equal $N$ has the ZR property. Another more involved examples sharing this property are:

- The entire functions in the Pólya class have the ZR property [2, p. 15]. Recall that an entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if $|E(x-iy)| \leq |E(x+iy)|$ for $y > 0$, and if $|E(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed $x$.
- The entire functions in the Paley-Wiener class $\mathcal{P}_\pi$ of bandlimited functions to $[-\pi, \pi]$, that is, $\mathcal{P}_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \text{supp} \hat{f} \subseteq [-\pi, \pi]\}$, where $\hat{f}$ stands for the Fourier transform of $f$, satisfy the ZR property; it follows from the classical Paley-Wiener theorem [20, p. 101], which says that this space can be written as $\mathcal{P}_\pi = \{f \text{ entire function : } |f(z)| \leq A e^{\pi|z|}, f|_\mathbb{R} \in L^2(\mathbb{R})\}$. From this characterization the ZR property immediately comes out.
- In general, de Branges spaces $\mathcal{H}(E)$ with strict de Branges function $E$ have the ZR property [2, p. 52]. Let $E$ be an entire function verifying $|E(x-iy)| < |E(x+iy)|$ for all $y > 0$. The de Branges space $\mathcal{H}(E)$ is the set of all entire functions $F$ such that
  \[
  \|F\|_E^2 := \int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^2 \, dt < \infty,
  \]
and such that both ratios $F/E$ and $F^*/E$, where $F^*(z) := \overline{F(\overline{z})}$, are of bounded type and of non-positive mean type in the upper half-plane. The structure function or de Branges function $E$ has no zeros in the upper half plane. A de Branges function $E$ is said to be strict if it has no zeros on the real axis. We require that $F/E$ and $F^*/E$ be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function is of bounded type if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$ and it is of nonpositive mean type if it grows no faster than $e^{\epsilon y}$ for each $\epsilon > 0$ as $y \to \infty$ on the positive imaginary axis $\{iy : y > 0\}$. Note that the Paley-Wiener space $\mathcal{P}_\pi$ is a de Branges space for the structure function $E_\pi(z) = \exp(-\pi \ell^2)$. 

Assume that the space $\mathcal{H}_K$ in section 2 comes from a polynomial kernel $K$ with coefficients in $C$; concerning the ZR property in $\mathcal{H}_K$, the following result holds:

**Theorem 1.** The space $\mathcal{H}_K$ associated with a polynomial kernel $K(z) := \sum_{n=0}^{N} p_n z^n$, where $p_n \in C$ and $p_N \neq 0$, has the ZR property if and only if the set $\{p_0, p_1, \ldots, p_N\}$ is linearly independent in $\mathcal{H}$.

**Proof.** Consider $f(z) = a_N z^N + \cdots + a_1 z + a_0 \in \mathcal{H}_K$ with $a_N \neq 0$; there exists $x \in \mathcal{H}$ such that $f(z) = \langle K(z), x \rangle$ and, consequently, $a_j = \langle p_j, x \rangle$ for $j = 0, 1, \ldots, N$. If the space $\mathcal{H}_K$ has the ZR property and $a_0, a_1, \ldots, a_N$ are the roots of the polynomial $f$ then the constant $a_N$ and the polynomials $a_N(z - a_N), a_N(z - a_N)(z - a_{N-1}), \ldots, a_N(z - a_N)(z - a_{N-1})\cdots(z - a_1)$ belong to $\mathcal{H}_K$. Let $b_0, b_1, \ldots, b_N \in C$ such that

$$b_N p_N + b_{N-1} p_{N-1} + \cdots + b_0 p_0 = 0. \quad (6)$$

The vector $(b_N, \ldots, b_0)$ is orthogonal in $C^{N+1}$ to any vector $(c_N, \ldots, c_0) \in C^{N+1}$ with $c_N z^N + \cdots + c_0 \in \mathcal{H}_K$. As a consequence, since $a_N \in \mathcal{H}_K$, $b_0 a_N = 0$, which implies that $b_0 = 0$. Analogously, since $a_N(z - a_N)$ belongs to $\mathcal{H}_K$ we have that $a_N b_1 - (a_N a_N) b_0 = 0$ and consequently $b_1 = 0$. Proceeding iteratively it is straightforward to obtain that $b_2 = \cdots = b_{N-1} = 0$; finally, from (6) we conclude that $b_N = 0$.

Now suppose that the set $\{p_0, p_1, \ldots, p_N\}$ is linearly independent in $\mathcal{H}$. In this case, the mapping $\Phi : \mathcal{H} \to C^{N+1}$ given by $\Phi(x) = (\langle p_0, x \rangle, \langle p_1, x \rangle, \ldots, \langle p_N, x \rangle)$ is surjective. As a consequence, any complex polynomial of degree less than or equal to $N$ belongs to $\mathcal{H}_K$. Let $f(z) = a_N z^N + \cdots + a_1 z + a_0 \in \mathcal{H}_K$ and let $w \in C$ be a root of $f$. Hence, $f(z)/(z - w) = c_0 + c_1 z + \cdots + c_{N-1} z^{N-1}$ is a polynomial of degree less than or equal to $N - 1$. Since $\Phi$ is onto there exists $x \in \mathcal{H}$ such that $\Phi(x) = (c_0, c_1, \ldots, c_{N-1}, 0)$. From the definition of $\Phi$, we conclude that $f(z)/(z - w) = \langle K(z), x \rangle$, that is, the function $f(z)/(z - w) \in \mathcal{H}_K$.

Giving a necessary and sufficient for a general analytic kernel $K$ remains as an open problem. It is worth to mention that a straightforward application of Cauchy–Schwarz inequality shows that entire functions in $\mathcal{H}_K$ inherit the finite order and the type of the vector-valued entire function $K$ provided it has finite order.

As examples of spaces $\mathcal{H}_K$ where the ZR property does not hold let us mention the following:

- Consider the spaces $\mathcal{H}_K, i = 1, 2$, associated with the analytic kernels $K_i : C \to L^2[0, \pi]$ defined by $K_1(z)[x] := \sin zx$ and $K_2(z)[x] := \cos zx$. The space $\mathcal{H}_{K_i}$ corresponds to the space of odd bandlimited functions in $PW_\pi$.
while $\mathcal{H}_{K_2}$ corresponds to the space of even bandlimited functions in $PW_g$. It is clear that the ZR property does not hold in these spaces.

- Let $K : \mathbb{C} \to \mathcal{H}$ be an analytic kernel such that $K(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Then all the functions in the associated space $\mathcal{H}_K$ have a zero at $z_0$ and the ZR property does not hold in $\mathcal{H}_K$. Indeed, let $f$ be a nonzero entire function in $\mathcal{H}_K$ and let $r$ denote the order of its zero $z_0$. The function $f(z)/(z-z_0)^r$ is not in $\mathcal{H}_K$ since it does not vanish at $z_0$.

- A little more sophisticated example is the following: For $m \geq 2$ let $K_m : \mathbb{C} \to L^2[-\pi, \pi]$ be defined as $K_m(z) = \frac{1}{\sqrt{2\pi}} e^{izm} \in L^2[-\pi, \pi]$. It is straightforward to show that $K_m$ is an analytic kernel; the corresponding space $\mathcal{H}_{K_m}$ does not have the ZR property. Indeed, expanding $K_m(z)$ as power series around the origin we obtain

$$[K_m(z)](x) = \sum_{k=0}^{\infty} \frac{(ix)^k z^{mk}}{k!} = 1 + ixz - \frac{x^2 z^{2m}}{2!} - \frac{x^3 z^{3m}}{3!} + \cdots.$$  

Thus, for any function $f(z) = \langle K_m(z), F \rangle$ with $F \in L^2[-\pi, \pi]$ we have

$$f(z) = \sum_{k=0}^{\infty} c_k z^{mk},$$

where $c_k = \langle (ix)^k/k!, F \rangle$, $k = 0, 1, \ldots$. Let $G \in L^2[-\pi, \pi]\setminus\{0\}$ be such that $G$ is orthogonal to $K(0)$ and let $g(z) = \langle K_m(z), G \rangle$. Since $\langle K(0), G \rangle = 0$ we have $g(0) = 0$. Hence, the Taylor expansion of $g(z)/z$ around the origin has the form

$$\frac{g(z)}{z} = d_1 z^{m-1} + d_2 z^{2m-1} + \cdots$$

where $d_k = \langle (ix)^k/k!, G \rangle$, $k = 1, 2, \ldots$. Since $G$ is not the zero function the function $g(z)/z$ does not belong to $\mathcal{H}_{K_m}$.

### 4. Lagrange-Type Interpolation Series

In this section, we introduce the analytic Kramer kernels $K$ for which a nonorthogonal sampling theorem in $\mathcal{H}_K$ holds. We prove a converse result: From a sampling formula in $\mathcal{H}_K$ we deduce when $K$ is an analytic Kramer kernel. Finally, we prove the main result: a necessary and sufficient condition ensuring that the Kramer sampling result can be expressed as a Lagrange-type interpolation series.

#### 4.1. The Abstract Kramer Sampling Result

Consider the data

$$\{z_n\}_{n=1}^{\infty} \in \mathbb{C} \quad \text{and} \quad \{a_n\}_{n=1}^{\infty} \in \mathbb{C}\setminus\{0\}. \quad (7)$$
Definition 2 (Analytic Kramer Kernel). An analytic kernel $K : \mathbb{C} \rightarrow \mathcal{H}$ is said to be an analytic Kramer kernel (with respect to the data (7)) if it satisfies $K(z_n) = a_n x_n$, $n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^{\infty}$ of $\mathcal{H}$.

A sequence $\{S_n\}_{n=1}^{\infty}$ of functions in the space $\mathcal{H}_K$ is said to have the interpolation property (with respect to the data (7)) if

$$S_n(z_n) = a_n \delta_{{n,m}}.$$  

Thus, an analytic kernel $K$ is an analytic Kramer one if and only if the sequence of functions $\{S_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$ given by (5), where $\{y_n\}_{n=1}^{\infty}$ is the dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$, has the interpolation property with respect to the same data (7).

Concerning the existence of analytic Kramer kernels, it has been proved in [11] that, associated with any arbitrary sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ such that $\lim_{n \to \infty} |z_n| = +\infty$, there exists an analytic Kramer kernel $K$.

Under the notation introduced so far an abstract version of the classical Kramer sampling theorem sampling [15] holds in $\mathcal{H}_K$; this is a slight modification of a sampling result in [14]. For notational purposes we include its proof.

Theorem 2 (Kramer Sampling Theorem). Let $K : \mathbb{C} \rightarrow \mathcal{H}$ be an analytic Kramer kernel, and assume that the interpolation property (8) holds for some sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{0\}$. Let $\mathcal{H}_K$ be the corresponding RKHS of entire functions. Then any $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^{\infty}$ by means of the sampling series

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{S_n(z)}{a_n}, \quad z \in \mathbb{C},$$  

where the reconstruction functions $S_n$ are given in (5). The series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Proof. First, notice that $\lim_{n \to \infty} |z_n| = +\infty$; otherwise the sequence $\{z_n\}_{n=1}^{\infty}$ contains a bounded subsequence and, hence, the entire function $S_n \equiv 0$ for all $n \in \mathbb{N}$, which contradicts (8). The anti-linear mapping $\mathcal{F}$ given by (4) is a bijective isometry between $\mathcal{H}$ and $\mathcal{H}_K$. As a consequence, the functions $\{S_n = \mathcal{F}(y_n)\}_{n=1}^{\infty}$ form a Riesz basis for $\mathcal{H}_K$; let $\{T_n\}_{n=1}^{\infty}$ be its dual Riesz basis. Expanding any $f \in \mathcal{H}_K$ in this basis we obtain

$$f(z) = \sum_{n=1}^{\infty} \langle f, T_n \rangle_{\mathcal{H}_K} S_n(z).$$
Moreover,
\[
\langle f, T_n \rangle_{L^2} = \langle x, x_n \rangle_{L^2} = \left( \frac{K(z_n)}{a_n}, x \right)_{L^2} = \frac{f(z_n)}{a_n}.
\] (10)

Since a Riesz basis is an unconditional basis, the sampling series will be pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory since \( z \mapsto \|K(z)\|_{L^2} \) is bounded on compact subsets of \( \mathbb{C} \).

Riesz bases theory (see, e.g., [20]) assures the existence of two positive constants \( 0 < A \leq B \) such that
\[
A\|f\|_{L^2}^2 \leq \sum_{n=1}^{\infty} |f(z_n)/a_n|^2 \leq B\|f\|_{L^2}^2 \quad \text{for all } f \in \mathcal{H}_K, \tag{11}
\]
that is, \( \|f\| := \left( \sum_{n=1}^{\infty} |f(z_n)/a_n|^2 \right)^{1/2} \) defines an equivalent norm in \( \mathcal{H}_K \). Following [12], we can say that the data (7) is a sampling set for \( \mathcal{H}_K \); here the sequence of samples belongs to a weighted \( \ell^2 \) space. In [12], the authors characterize the reproducing kernel Hilbert spaces having a fixed sampling set.

The Whittaker–Shannon–Kotel’nikov sampling formula in \( PW_\pi \) becomes a particular case of formula (9) in Theorem 2. Indeed, any \( f \in PW_\pi \) can be written as
\[
f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(w) e^{iw} dw = \left( \frac{e^{iw}}{\sqrt{2\pi}}, f \right)_{L^2([-\pi, \pi])}, \quad z \in \mathbb{C}.
\]
The Fourier kernel \( K(z) := \frac{e^{iz}}{\sqrt{2\pi}} \in L^2[-\pi, \pi] \) is an analytic Kramer kernel for the data \( \{z_n = n\}_{n \in \mathbb{Z}} \) and \( \{a_n = 1\}_{n \in \mathbb{Z}} \). In this case, as \( \{e^{imn}/\sqrt{2\pi}\}_{n \in \mathbb{Z}} \) is an orthonormal basis for \( L^2[-\pi, \pi] \) we get
\[
S_n(z) = \frac{1}{2\pi} \langle e^{iz}, e^{im} \rangle_{L^2[-\pi, \pi]} = \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}.
\]
As a consequence, we obtain the WSK sampling formula in \( PW_\pi \):
\[
f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}. \tag{12}
\]
The series converges absolutely and uniformly on horizontal strips of the complex plane.

It is worth to remark that a kernel \( K \) can be an analytic Kramer kernel with respect to different data (7). For instance, the Fourier kernel is also
an analytic Kramer kernel with respect to the data \( \{z_n = n + x\}_{n \in \mathbb{Z}} \) where \( x \in \mathbb{R} \) and \( \{a_n = 1\}_{n \in \mathbb{Z}} \). More generally, it is an analytic Kramer kernel with respect to any data \( \{t_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) and \( \{a_n = 1\}_{n \in \mathbb{Z}} \), where the points \( t_n \) satisfy Kadec’s condition \( \sup_n |t_n - n| < 1/4 \) since the sequence \( \{e^{i_n w}/\sqrt{2\pi}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(-\pi, \pi) \) [20, p. 42].

### 4.2. A Converse Result

An interesting converse problem is to decide whether a sampling formula as (9), pointwise convergent in \( \mathcal{H}_K \), implies the Kramer kernel condition in definition 2 for \( K \). From formula (9) in Theorem 2 we derive that:

- From (5), for each \( z \in \mathbb{C} \), the sequence \( \{S_n(z)\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \).
- The sequence \( \{f(z_n)/a_n\}_{n=1}^{\infty} \) belongs to \( \ell^2(\mathbb{N}) \) for any \( f \in \mathcal{H}_K \), and
- \( \sum_{n=1}^{\infty} a_n S_n(z) = 0 \) for all \( z \in \mathbb{C} \) and \( \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \) implies \( x_n = 0 \) for all \( n \in \mathbb{N} \), due to the uniqueness of a Riesz basis expansion in the RKHS \( \mathcal{H}_K \).

It is worth to point out that these conditions are also sufficient to prove that \( K \) is an analytic Kramer kernel.

**Theorem 3.** Let \( \mathcal{H}_K \) be the range of a mapping \( \mathcal{T} \) as in (4) considered as a RKHS with reproducing kernel \( k(z, w) = \langle K(z), K(w) \rangle_\mathcal{F} \). Let \( \{S_n\}_{n=1}^{\infty} \) be a sequence in \( \mathcal{H}_K \) such that \( \{S_n(z)\}_{n=1}^{\infty} \) belongs to \( \ell^2(\mathbb{N}) \) for each \( z \in \mathbb{C} \). Suppose that the following conditions are fulfilled:

- (i) \( \sum_{n=1}^{\infty} a_n S_n(z) = 0 \) for all \( z \in \mathbb{C} \) and \( \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \) implies \( x_n = 0 \) for all \( n \).
- (ii) There exist sequences \( \{z_n\}_{n=1}^{\infty} \) in \( \mathbb{C} \) and \( \{a_n\}_{n=1}^{\infty} \) in \( \mathbb{C}\setminus\{0\} \) such that

\[
\left\{ \frac{f(z_n)}{a_n} \right\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{S_n(z)}{a_n}, \quad \text{for any} \ f \in \mathcal{H}_K,
\]

where the sampling series is pointwise convergent in \( \mathbb{C} \).

Then, the sequence \( \{S_n\}_{n=1}^{\infty} \) is a Riesz basis for \( \mathcal{H}_K \) and the kernel \( K \) of the mapping \( \mathcal{T} \) evaluated at \( z \in \mathbb{C} \) can be expressed as \( K(z) = \sum_{n=1}^{\infty} S_n(z) y_n \), where \( \{y_n\}_{n=1}^{\infty} \) is the dual Riesz basis of the Riesz basis \( \{x_n = \mathcal{T}^{-1}(S_n)\}_{n=1}^{\infty} \) in \( \mathcal{H} \). In particular, \( K(z_n) = a_n y_n \) for any \( n \in \mathbb{N} \).

**Proof.** By defining \( \tilde{k}(z, w) := \sum_{n=1}^{\infty} S_n(z) \overline{S_n(w)} \), we obtain a positive definite function which defines a RKHS \( \tilde{\mathcal{H}} \), such that \( \tilde{\mathcal{H}} \subseteq \mathcal{H}_K \). Condition (i) implies that the sequence \( \{S_n\}_{n=1}^{\infty} \) is an orthonormal basis for \( \tilde{\mathcal{H}} \) (see [17]).
Now we prove that \( X = X_K \) and that the identity mapping \( X \rightarrow X_K \) is continuous. Take \( f \in X_K \), by condition ii), the sequence \( \{f(z_n)a_n^{-1}\}_{n=1}^{\infty} \) is in \( \ell^2(\mathbb{N}) \). As a consequence, the series \( \sum_{n=1}^{\infty} f(z_n)a_n^{-1}S_n \) converges in the norm of \( X_K \). By the reproducing kernel property, we have that the series \( \sum_{n=1}^{\infty} f(z_n)a_n^{-1}S_n(z) \) is pointwise convergent. Comparing this with what we get from the sampling formula for \( f \) we deduce that \( f = \sum_{n=1}^{\infty} f(z_n)a_n^{-1}S_n \), where the convergence is in \( X_K \) and, consequently, \( f \in X_K \).

Next we show the continuity of the identity mapping by applying the closed graph theorem. Indeed, let \( \{f_n\}_{n=1}^{\infty} \) be a sequence such that \( f_n \rightarrow f \) in \( X_K \) and \( f_n \rightarrow g \) in \( X_K \) as \( n \rightarrow \infty \). Using the reproducing property in both \( X_K \) and \( X \), for \( z \in \mathbb{C} \) we have

\[
|f_n(z) - f(z)| \leq \|f_n - f\| \sqrt{k(z,z)};
\]

\[
|f_n(z) - g(z)| \leq \|f_n - g\| \sqrt{k(z,z)}.
\]

Therefore, \( \lim_{n \rightarrow \infty} f_n(z) = f(z) = g(z) \) for each \( z \in \mathbb{C} \), and hence \( f = g \).

Since it is also surjective, we infer that the norms \( \| \cdot \|_{X_K} \) and \( \| \cdot \|_{X_K} \) are equivalent from the open mapping theorem. As a consequence, the orthonormal basis \( \{S_n\}_{n=1}^{\infty} \) in \( X_K \) is a Riesz basis for \( X_K \).

Assuming that the mapping \( F \) is one-to-one, the sequence \( \{x_n = F^{-1}(S_n)\}_{n=1}^{\infty} \) is a Riesz basis for \( X \); denote by \( \{y_n\}_{n=1}^{\infty} \) its dual Riesz basis. Expanding \( K(z) \) with respect to \( \{y_n\}_{n=1}^{\infty} \), for each fixed \( z \in \mathbb{C} \) we obtain

\[
K(z) = \sum_{n=1}^{\infty} \langle K(z), x_n \rangle y_n = \sum_{n=1}^{\infty} S_n(z) y_n,
\]

that is, the required expansion for \( K(z) \).

Notice that the interpolatory condition \( S_n(z_n) = a_n \delta_{n,m} \) comes out of a direct application of condition (ii) to \( S_n \), followed by condition (i).

As to the case when, a priori, \( F \) is not known to be one-to-one, let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( X \) with \( P(x_n) \neq 0 \) for all \( n \), where \( P \) denotes the orthogonal projection onto the closed subspace \( (\text{Ker} F)^\perp \). Consider \( S_n = F(x_n) \in X_K \), and suppose that these functions satisfy the hypotheses in Theorem 3. In this case, \( \{S_n\}_{n=1}^{\infty} \) is a Riesz basis for \( X_K \). Consequently, since \( S_n = F[P(x_n)] \) and \( F|_{P(\text{Ker} F)} = 0 \), we obtain that \( \{P(x_n)\}_{n=1}^{\infty} \) is a Riesz basis for \( P(\hat{X}) = (\text{Ker} F)^\perp \). The result comes out taking into account the orthogonal sum \( \hat{X} = (\text{Ker} F)^\perp \oplus (\text{Ker} F) \).

4.3. Lagrange-Type Interpolation Series

A more difficult question concerns whether the sampling expansion (9) can be written, in general, as a Lagrange-type interpolation series.
For instance, for \( f \in PW_n \) the WSK formula (12) can be written as the Lagrange-type interpolation series

\[
f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{P(z)}{(z-n)P'(n)}, \quad z \in \mathbb{C},
\]

by taking \( P(z) = (\sin \pi z)/\pi \), an entire function having only simple zeros at \( \mathbb{Z} \).

The case where the sequence \( \{x_n\}_{n=1}^{\infty} \) in Definition 2 is an orthonormal basis for \( \mathcal{H} \) was studied in [7]: A necessary and sufficient condition involves the ZR property. Next, we prove that the same necessary and sufficient condition holds in the general case of analytic Kramer kernels \( K \) involving Riesz bases.

**Theorem 4.** Let \( \mathcal{H}_K \) be a RKHS of entire functions obtained from an analytic Kramer kernel \( K \) with respect to the data \( \{z_n\}_{n=1}^{\infty} \subset \mathbb{C} \) and \( \{a_n\}_{n=1}^{\infty} \in \mathbb{C}\setminus\{0\} \), that is, \( K(z_n) = a_n x_n, \quad n \in \mathbb{N} \), for some Riesz basis \( \{x_n\}_{n=1}^{\infty} \) for \( \mathcal{H} \). Then, the sampling formula (9) for \( \mathcal{H}_K \) can be written as a Lagrange-type interpolation series

\[
f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}, \quad z \in \mathbb{C}, \quad \tag{13}
\]

where \( P \) denotes an entire function having only simple zeros at \( \{z_n\}_{n=1}^{\infty} \) if and only if the space \( \mathcal{H}_K \) satisfies the ZR property.

**Proof.** For the sufficient condition we have to prove that sampling formula (9) can be written as a Lagrange-type interpolation series (13) for some entire function \( P \). First, we prove that the only zeros of the sampling function \( S_n \) are given by \( \{z_r\}_{r \neq n} \). Suppose that \( S_n(w) = 0 \), then by hypothesis the function \( S_n(z)/(z-w) \) is in \( \mathcal{H}_K \). Hence, the function

\[
\frac{z-z_n}{z-w} S_n(z) = S_n(z) + \frac{w-z_n}{z-w} S_n(z)
\]

also belongs to \( \mathcal{H}_K \). If \( w \notin \{z_r\}_{r \neq n} \), the function \( \frac{z-z_n}{z-w} S_n(z) \) in \( \mathcal{H}_K \) vanishes at the sequence \( \{z_r\}_{r=1}^{\infty} \) which implies that \( S_n \equiv 0 \), to give a contradiction. In addition, the zeros of \( S_n \) are simple; indeed, suppose that \( z_m \) is a multiple zero of \( S_n \). Proceeding as above, the function \( \frac{z-z_m}{z-w} S_n(z) \) belongs to \( \mathcal{H}_K \) and vanishes at \( \{z_r\}_{r=1}^{\infty} \) which again implies that \( S_n \equiv 0 \).

Consequently, choosing an entire function \( Q \) having only simple zeros at \( \{z_n\}_{n=1}^{\infty} \), for each \( n \in \mathbb{N} \) there exists an entire function \( A_n \) without zeros such that \( (z-z_n)S_n(z) = Q(z)A_n(z), \quad z \in \mathbb{C} \). Next, we prove that there exists an entire function \( A \) without zeros and a sequence \( \{a_n\}_{n=1}^{\infty} \) in \( \mathbb{C}\setminus\{0\} \) such
that \( A_n(z) = \sigma_n A(z) \) for all \( z \in \mathbb{C} \). For \( m \neq n \) the function \( \frac{z-z_n}{z-z_m} S_n(z) \) in \( \mathcal{H}_K \) has its zeros at \( \{z_r : r \neq m\} \). Thus, the sampling formula (9) gives

\[
\frac{z-z_n}{z-z_m} S_n(z) = \left[ (z_m-z_n) S_n'(z_m) \right] \frac{S_m(z)}{a_m}, \quad z \in \mathbb{C}.
\]

Fixing \( m = 1 \), we conclude that \( A_n(z) = \sigma_n A(z) \) where \( A = A_1 \) and \( \sigma_n = (z_1-z_n) S_n'(z_1) \neq 0 \) for \( n \in \mathbb{N} \setminus \{1\} \) and \( \sigma_1 = 1 \). Hence, \( S_n(z) = \frac{\sigma_n Q(z) A(z)}{z-z_n} \) for \( z \neq z_n \) and \( S_n(z_n) = a_n = \sigma_n Q'(z_n) A(z_n) \). Substituting in (9) we obtain the Lagrange-type interpolation series (13) where \( P(z) = A(z) Q(z) \).

For the necessary condition, assume that the sampling formula in \( \mathcal{H}_K \) takes the form of a Lagrange-type interpolation series (13). Given \( g \in \mathcal{H}_K \), there exists \( x \in \mathcal{H} \) such that \( g(z) = \langle K(z), x \rangle, \ z \in \mathbb{C} \). Assuming that \( g(w) = 0 \), we have to prove that the function \( g(z)/(z-w) \) belongs to \( \mathcal{H}_K \). The sampling expansion for \( g \) at \( w \) gives

\[
\sum_{n=1}^{\infty} g(z_n) \frac{P(w)}{(w-z_n)P'(z_n)} = 0. \tag{14}
\]

We distinguish two cases:

(i) \( w \in \mathbb{C} \setminus \{z_n \}_{n=1}^{\infty} \). As \( P(w) \neq 0 \), from (14) we obtain

\[
\sum_{n=1}^{\infty} g(z_n) \frac{1}{(w-z_n)P'(z_n)} = 0.
\]

Thus,

\[
g(z) = \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{(z-z_n)P'(z_n)} - \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{(w-z_n)P'(z_n)}
\]

\[
= (z-w) \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{P'(z_n)} \frac{1}{(z-z_n)(z-w)}.
\]

Therefore, the entire function \( G(z) := g(z)/(z-w) \) can be recovered from its samples at \( \{z_n \}_{n=1}^{\infty} \) through the formula

\[
G(z) = \sum_{n=1}^{\infty} G(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}, \quad z \in \mathbb{C}. \tag{15}
\]

Moreover, the function \( G \) is in \( \mathcal{H}_K \) because \( G(z) = \langle K(z), y \rangle \), where \( y \in \mathcal{H} \) has the expansion \( y = \sum_{n=1}^{\infty} \langle y, x_n \rangle y_n \) with respect to the dual Riesz basis.
\( \{ y_n \}_{n=1}^{\infty} \) of \( \{ x_n \}_{n=1}^{\infty} \), where the coefficients are given by

\[
\left\{ \langle y, x_n \rangle := \frac{1}{z_n - w} \langle x, x_n \rangle \right\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}).
\]

Indeed, sampling formula (13) for \( S_n(z) = a_n \frac{P(z)}{(z-z_n)P'(z_n)} \). Hence, by using the biorthogonality \( \langle x_n, y_n \rangle = \delta_{n,m} \), we obtain

\[
\langle K(z), y \rangle = \sum_{n=1}^{\infty} \frac{S_n(z) \langle x, x_n \rangle}{w - z_n} = G(z), \quad z \in \mathbb{C},
\]

where we have used (15), and the result that \( \langle x, x_n \rangle = g(z_n)/a_n, \ n \in \mathbb{N} \).

(ii) \( w = z_m \) for some \( m \in \mathbb{N} \). As \( g(z_m) = 0 \), the sampling expansion for \( g \) reads

\[
g(z) = \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}, \quad z \in \mathbb{C}.
\]

Setting \( P(z) = (z-z_m)Q_m(z) \) we have \( P'(z) = Q_m(z) + (z-z_m)Q'_m(z) \) and, hence,

\[
P'(z_k) = \begin{cases} (z_k - z_m)Q'_m(z_k) & \text{if } k \neq m \\ Q_m(z_m) & \text{if } k = m \end{cases}
\]

Hence,

\[
g(z) = \sum_{n=1}^{\infty} g(z_n) \frac{Q_m(z)}{z_n - z_m (z-z_n)Q'_m(z_n)}, \quad z \in \mathbb{C}. \quad (16)
\]

Using the uniform convergence of the series in (16) we deduce that this series defines a continuous function. Hence, taking the limit as \( z \to z_m \) we obtain

\[
g'(z_m) = \sum_{n=1}^{\infty} g(z_n) \frac{Q_m(z_m)}{z_n - z_m (z-z_n)Q'_m(z_n)} \quad (17)
\]

Now we prove that

\[
\frac{g(z)}{z-z_m} = \sum_{n=1}^{\infty} g(z_n) \frac{P(z)}{z_n - z_m (z-z_n)P'(z_n)} + g'(z_m) \frac{P(z)}{(z-z_m)P'(z_m)}. \quad (18)
\]
Indeed, substituting (17) into (18) we obtain

\[
\sum_{n=1 \atop n \neq m}^{\infty} \left[ \frac{g(z_n)}{z_n - z_m} \frac{P(z)}{(z - z_n)P'(z_n)} + \frac{g(z_n)}{z_n - z_m} \frac{Q_m(z)}{(z_m - z_n)Q_m'(z_n)} \right]
\]

\[
= \sum_{n=1 \atop n \neq m}^{\infty} \frac{g(z_n)}{z_n - z_m} \frac{Q_m(z)}{Q_m'(z_n)} \left[ \frac{z - z_m}{(z_m - z_n)(z - z_n)} - \frac{1}{z_n - z_m} \right]
\]

\[
= \sum_{n=1 \atop n \neq m}^{\infty} \frac{g(z_n)}{z - z_m} \frac{Q_m(z)}{(z_n - z_m)(z - z_n)Q_m'(z_n)}
\]

\[
= \frac{g(z)}{z - z_m}.
\]

Thus, defining \( y \in \mathcal{H} \) by the expansion \( y = \sum_{n=1}^{\infty} \langle y, x_n \rangle y_n \) where the coefficients \( \{\langle y, x_n \rangle\}_{n=1}^{\infty} \) in \( \ell^2(\mathbb{N}) \) are given by

\[
\langle y, x_n \rangle := \begin{cases} \frac{\langle x, x_n \rangle}{z_n - z_m} & \text{if } n \neq m \\ \frac{g'(z_m)}{a_m} & \text{if } n = m \end{cases}
\]

and proceeding as in case (i), it may be shown that

\[
\frac{g(z)}{z - z_m} = \langle K(z), y \rangle, \quad z \in \mathbb{C},
\]

which proves that the function \( g(z)/(z - z_m) \) belongs to \( \mathcal{H}_K \). This concludes the proof of the theorem. \( \square \)

Some comments concerning Theorem 4 are in order:

1. In the proof of Theorem 4 we have found that the entire function \( P \) satisfies:

\[
(z - z_n)S_n(z) = \sigma_n P(z), \quad z \in \mathbb{C},
\]

for some sequence \( \{\sigma_n\}_{n=1}^{\infty} \in \mathbb{C} \setminus \{0\} \). In the case where \( P \) can be factorized as \( P(z) = A(z)Q(z) \), where \( Q \) denotes a canonical product having its simple zeros at \( \{z_n\}_{n=1}^{\infty} \) and \( A \) is an entire function.
without zeros, then the Lagrange-type interpolation series (13) can be expressed as

\[ f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{A(z)}{A(z_n)} \frac{Q(z)}{(z - z_n)Q'(z_n)}, \quad z \in \mathbb{C}. \]

2. In particular, as de Branges space satisfy the ZR property the orthogonal sampling formulas in these spaces, first proved in [16], can be expressed as Lagrange-type interpolation series (see [11] for some nontrivial examples).

3. It is worth to mention that if one particular sampling formula (9) can be written as a Lagrange-type interpolation formula, then the same occurs for all the sampling formulas (9) obtained from other compatible data (7). Besides, if the space \( \mathcal{H}_K \) does not satisfy the ZR property, we conclude that it does not exist any data (7) for which the kernel \( K \) is an analytic Kramer kernel and the associated sampling formula (9) can be written as a Lagrange-type interpolation series.

### 4.4. Some Illustrative Examples

Closing the article, we show some examples illustrating Theorems 2 and 4.

#### 4.4.1. Classical Polynomial Interpolation

Let \( \mathcal{P}_N(\mathbb{C}) \) be the set of polynomials with complex coefficients of degree less or equal \( N \). As we proved in Theorem 1, \( \mathcal{P}_N(\mathbb{C}) \) coincides with the corresponding \( \mathcal{H}_K \) space where \( K(z) := \sum_{n=0}^{N} p_n z^n \) being \( \{p_0, p_1, \ldots, p_N\} \) any basis for the euclidean space \( \mathcal{H} := \mathbb{C}^{N+1} \). Consider \( N + 1 \) different points \( \{z_n\}_{n=0}^{N} \) in \( \mathbb{C} \); it is easy to prove that \( K \) is an analytic Kramer kernel with respect the data \( \{z_n\}_{n=0}^{N} \) and \( \{a_n = 1\}_{n=0}^{N} \). Indeed, the set \( \{K(z_n) = \{q_n\}_{n=0}^{N}\} \) is linearly independent in \( \mathbb{C}^{N+1} \) by using Vandermonde determinants, that is, it forms a (Riesz) basis for \( \mathbb{C}^{N+1} \). Thus, Theorems 2 and 4 give, for any \( f \in \mathcal{P}_N(\mathbb{C}) \)

\[ f(z) = \sum_{n=0}^{N} f(z_n) S_n(z) = \sum_{n=0}^{N} f(z_n) \frac{P(z)}{(z - z_n)P'(z_n)}, \quad z \in \mathbb{C}, \]

where \( S_n(z) = (K(z), q_n^*) \), being \( \{q_n^*\}_{n=0}^{N} \) the dual basis of \( \{q_n\}_{n=0}^{N} \) in \( \mathbb{C}^{N+1} \), and \( P(z) = \prod_{n=0}^{N} (z - z_n) \).
4.4.2. The Paley–Wiener–Levinson Theorem Revisited

Let \( \{z_n\}_{n \in \mathbb{Z}} \) be a sequence in \( \mathbb{C} \) for which \( \sup_n |\text{Re} z_n - n| < 1/4 \) and \( \sup_n |\text{Im} z_n| < \infty \). It is known that the system \( \{e^{i\pi n}/\sqrt{2\pi}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2[-\pi, \pi] \) (see [20, p. 196]). The Fourier kernel \( K(z) = e^{i\pi z}/\sqrt{2\pi} \in L^2[-\pi, \pi] \) is an analytic Kramer kernel for the data \( \{z_n\}_{n \in \mathbb{Z}} \) and \( \{a_n = 1\}_{n \in \mathbb{Z}} \). Thus, Theorems 2 and 4 give, for any \( f \in \text{PW}_H \)

\[
    f(z) = \sum_{n=-\infty}^{\infty} f(z_n) S_n(z) = \sum_{n=-\infty}^{\infty} f(z_n) \frac{P(z)}{(z - z_n)P'(z_n)}, \quad z \in \mathbb{C},
\]

where, for \( n \in \mathbb{Z} \), the sampling function \( S_n(z) = \langle K(z), h_n \rangle_{L^2[-\pi, \pi]} \), being \( \{h_n(w)\}_{n \in \mathbb{Z}} \) the dual Riesz basis of \( \{e^{i\pi n}/\sqrt{2\pi}\}_{n \in \mathbb{Z}} \) in \( L^2[-\pi, \pi] \), and \( P \) is the entire function having only simple zeros at \( \{z_n\}_{n \in \mathbb{Z}} \). Since a result from Titchmarsh [18] assures that the functions in \( \text{PW}_H \) are completely determined by their zeros, we derive that, up to a constant factor, the entire function \( P \) coincides with the infinite product

\[
    (z - z_0) \prod_{n=1}^{\infty} \left( \frac{1 - \frac{z}{z_n}}{1 - \frac{z}{z-n}} \right).\]

Indeed, the function \( S_0 \in \text{PW}_H \) has only simple zeros at \( \{z_m\}_{m \neq 0} \) \( (S_0(z_m) = \delta_{0,m}) \). Suppose on the contrary that \( s \notin \{z_n\}_{m \neq 0} \) is a zero of \( S_0 \). According to the classical Paley–Wiener theorem, the function \( S(z) := (z - z_0)S_0(z)/(z - s) \) belongs to \( \text{PW}_H \) and vanishes at every \( z_n \). If we take into account the completeness of the Riesz basis \( \{e^{i\pi n}/\sqrt{2\pi}\}_{n \in \mathbb{Z}} \), this implies that \( S \equiv 0 \), a contradiction. Therefore, by using the Titchmarsh's result, the function \( S_0 \) coincides, up to a constant factor, with the (convergent) product \( \prod_{n=1}^{\infty} \left( 1 - \frac{\frac{s}{z}}{1 - \frac{\frac{s}{z}}{z-n}} \right) \). Since Theorem 4 gives \( (z - z_n)S_n(z) = \sigma_n P(z) \) for all \( n \in \mathbb{Z} \), we obtain the desired result.

4.4.3. Finite Cosine Transform

It is known that any function \( f(z) = \langle \cos z x, F(x) \rangle_{L^2[0,\pi]} \), \( z \in \mathbb{C} \), can be expanded as the sampling formula [13, p. 5]

\[
    f(z) = f(0) \sin \frac{\pi z}{\pi} + \frac{2}{\pi} \sum_{n=0}^{\infty} f(n) \frac{(-1)^n \sin \frac{\pi z}{z^2 - n^2}}, \quad z \in \mathbb{C}.
\]

This sampling formula cannot be expressed as a Lagrange-type interpolation series since, as we noticed in section 3, the corresponding \( \mathcal{H}_K \) space does not satisfy the ZR property.
4.4.4. An Example Involving a Sobolev Space

Finally, we give an example taken from [10] of a RKHS $\mathcal{H}_K$, built from the Sobolev Hilbert space $\mathcal{H} := H^1(-\pi, \pi)$, where the ZR property fails. Namely, consider the Sobolev Hilbert space $H^1(-\pi, \pi)$ with its usual inner product

$$\langle f, g \rangle_1 = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx + \int_{-\pi}^{\pi} f'(x) \overline{g'(x)} \, dx, \quad f, g \in H^1(-\pi, \pi).$$

The sequence $\{e^{inx}\}_{n \in \mathbb{Z}} \cup \{\sinh x\}$ forms an orthogonal basis for $H^1(-\pi, \pi)$: it is straightforward to prove that the orthogonal complement of $\{e^{inx}\}_{n \in \mathbb{Z}}$ in $H^1(-\pi, \pi)$ is a one-dimensional space for which $\sinh x$ is a basis. For a fixed $a \in \mathbb{C}\setminus\mathbb{Z}$ we define a kernel

$$K_a : \mathbb{C} \rightarrow H^1(-\pi, \pi)$$

$$z \mapsto K_a(z),$$

by setting

$$[K_a(z)](x) = (z - a) e^{ix} + \sin \pi z \sinh x, \quad \text{for } x \in (-\pi, \pi).$$

Clearly, $K_a$ defines an analytic Kramer kernel. Expanding $K_a(z) \in H^1(-\pi, \pi)$ in the former orthogonal basis we obtain

$$K_a(z) = [1 - i(z - a)] \sin \pi z \sinh x + (z - a) \sum_{n=-\infty}^{\infty} \frac{1 + zn}{1 + n^2} \sin c(z - n) e^{inx}.$$

As a consequence, Theorem 2 gives the following sampling result in $\mathcal{H}_K$:

Any function $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(a)\} \cup \{f(n)\}_{n \in \mathbb{Z}}$ by means of the sampling formula

$$f(z) = [1 - i(z - a)] \frac{\sin \pi z}{\sin \pi a} f(a) + \sum_{n=-\infty}^{\infty} f(n) \frac{z - a}{n - a} \frac{1 + zn}{1 + n^2} \sin c(z - n).$$

The function $(z - a) \text{sinc } z$ belongs to $\mathcal{H}_K$ since $(z - a) \text{sinc } z = \langle K_a(z), 1/2\pi \rangle_1$ for all $z \in \mathbb{C}$. However, by using the sampling formula for $\mathcal{H}_K$ it is straightforward to check that the function $\text{sinc } z$ does not belong to $\mathcal{H}_K$; as a consequence, the above sampling formula cannot be expressed as a Lagrange-type interpolation series.

ACKNOWLEDGMENTS

This work has been supported by the grant MTM2009–08345 from the Spanish Ministerio de Ciencia e Innovación (MICINN).
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