Differential invariants of second-order ordinary differential equations

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Second-order ordinary differential equations

Let M be a manifold, $\dim M = n$.
Let $p\colon \mathbb{R} \times M \to \mathbb{R}$ be the natural projection. Let $p^k\colon J^k(\mathbb{R}, M) \to \mathbb{R}$ be the k-jet bundle of p, $J^0(\mathbb{R}, M) = \mathbb{R} \times M$, with natural projections

$$p^k_h\colon J^k(\mathbb{R}, M) \to J^h(\mathbb{R}, M), \quad k \geq h.$$

Coordinates induced by (x^i) on $J^2(\mathbb{R}, M)$:

$$(t, x^i; \dot{x}^i, \ddot{x}^i),\ 1 \leq i \leq n.$$

A second-order ordinary differential equation (SODE)

$$\ddot{x}^i = F^i\left(t, x^i, \dot{x}^i\right), \quad F^i \in C^\infty\left(J^1(\mathbb{R}, M)\right),\ 1 \leq i \leq n,$$

can be viewed as a section σ of the natural projection $p^2_1\colon J^2(\mathbb{R}, M) \to J^1(\mathbb{R}, M)$, by setting

$$\ddot{x}^i \circ \sigma = F^i.$$

Vertical automorphisms

Every p-vertical automorphism Φ of p, $\Phi(t,x) = (t, \phi(t,x))$, induces, for $r \geq 1$, a diffeomorphism

$$\begin{array}{ccc} J^r(\mathbb{R}, M) & \xrightarrow{\Phi^{(r)}} & J^r(\mathbb{R}, M), \\ \downarrow & & \downarrow \\ \mathbb{R} \times M & \xrightarrow{\Phi} & \mathbb{R} \times M, \end{array} \qquad \begin{array}{c} \Phi^{(r)}\left(j^r_t \gamma\right) = j^r_t\left(\Phi \circ j^0 \gamma\right), \\ \forall \gamma \in C^\infty(\mathbb{R}, M) \end{array}$$

$\left(p^2_1\right)^r \colon J^2(p^2_1) \to J^1(\mathbb{R}, M)$, r-jet bundle of $p^2_1 \colon J^2(\mathbb{R}, M) \to J^1(\mathbb{R}, M)$.

Every $\Phi \in \mathrm{Aut}^v(p)$ induces a transformation $(\Phi^{(2)})^{(r)}$ of $J^r(p^2_1)$ given by

$$(\Phi^{(2)})^{(r)}(j^r_\xi \sigma) = j^r_{\Phi^{(1)}(\xi)}(\Phi \cdot \sigma),$$

where

$$\Phi \cdot \sigma = \Phi^{(2)} \circ \sigma \circ \left(\Phi^{(1)}\right)^{-1}.$$

The definition of a differential invariant

A smooth function $\mathcal{I}\colon \mathcal{U} \subseteq J^r(p_1^2) \to \mathbb{R}$ is a differential invariant of order r with respect to the group $\mathrm{Aut}^v(p)$ if

$$\mathcal{I} \circ \left(\Phi^{(2)}\right)^{(r)} = \mathcal{I}, \quad \forall \Phi \in \mathrm{Aut}^v(p).$$

For a given SODE σ on M, we set

$$I(\sigma, \xi) = \mathcal{I}(j^r_\xi \sigma), \quad \xi \in J^1(\mathbb{R}, M).$$

The invariance condition reads as:

$$I\left(\Phi \cdot \sigma, \Phi^{(1)}(\xi)\right) = I(\sigma, \xi),$$

$\forall \xi \in J^1(\mathbb{R}, M)$ and $\forall \Phi \in \mathrm{Aut}^v(p)$.

Infinitesimal differential invariant

If $\Phi_t \in \mathrm{Aut}^v(p)$ is the flow of a p-vertical $X \in \mathfrak{X}(\mathbb{R} \times M)$, then

- $\Phi_t^{(2)}$ flow of a p^2-vertical $X^{(2)} \in \mathfrak{X}(J^2(\mathbb{R}, M))$,
- $(\Phi_t^{(2)})^{(r)}$ flow of a vector field $(X^{(2)})^{(r)} \in \mathfrak{X}(J^r(p_1^2))$.

Every differential invariant $\mathcal{I}$ of order r satisfies

$$(X^{(2)})^{(r)}(\mathcal{I}) = 0, \quad \text{for all } p\text{-vertical } X \in \mathfrak{X}(\mathbb{R} \times M).$$

Let $\mathcal{D}^{(r)}$ be the distribution on $J^r(p_1^2)$ spanned by all the r-jet prolongations $(X^{(2)})^{(r)}$ of p-vertical vector fields $X \in \mathfrak{X}(\mathbb{R} \times M)$.

Theorem

$$\operatorname{rank} \mathcal{D}^{(2)} = \begin{cases} \frac{1}{2} n \left(3n^2 + 11n + 10\right), & n \geq 2 \\ 11, & n = 1 \end{cases}$$

Goal of the talk

- **GOAL:** To determine second-order invariants of a SODE.
  The only zero- and first-order invariants are $f(t)$, $f \in C^\infty(\mathbb{R})$.
- For $n \leq 2$, it is possible to determine a explicit basis for second-order differential invariants of a SODE.

D. D. Kosambi, *Systems of differential equations of the second order*, Quart. J. Math. Oxford **6** (1935), 1–12.

- For $n \geq 3$, the main result states that invariant functions factor through the curvature mapping attached to each SODE, which almost coincides with the torsion tensor of the Chern connection.
- We also remark the similarity between this result and the geometric version of the Utiyama theorem in gauge theories.

D. Bleecker, *Gauge Theory and Variational Principles*, Addison-Wesley Publishing Company, Inc., Reading, MA, 1981.

R. Utiyama, *Invariant Theoretical Interpretation of Interaction*, Phys. Rev. **101** (1956), 1597–1607.

The splitting induced by a SODE (1)

As is known, $p_0^1 \colon J^1(\mathbb{R}, M) \to \mathbb{R} \times M$ is an affine bundle modelled over $p'^* TM$ where

$$p' \colon \mathbb{R} \times M \to M, \quad p'(t,x) = x.$$

The following exact sequence holds:

$$0 \to \left(p' \circ p_0^1\right)^* TM \xrightarrow{\varepsilon} V(p_0^1) \to T\left(J^1(\mathbb{R}, M)\right) \xrightarrow{(p_0^1)_*} (p_0^1)^* T(\mathbb{R} \times M) \to 0$$

where ε is defined by the directional derivative and $V(p_0^1)$ denotes the vector subbundle of p_0^1-vertical vectors.

We look for a section of $(p_0^1)_*$ in such a way that the exact sequence splits.

The splitting induced by a SODE (2)

Dynamical flow associated to a SODE

Every SODE σ defines a vector field $X^\sigma \in \mathfrak{X}(J^1(\mathbb{R}, M))$, called the dynamical flow associated to σ, as follows:

$$(X^\sigma)_\xi = (j^1\gamma)_* \left(\frac{d}{dt}\right)_{t_0}, \quad \forall \xi \in (p^1)^{-1}(t_0),$$

where by $\gamma^i = x^i \circ \gamma$, $1 \leq i \leq n$, is the only solution to $\ddot{x}^i = F^i$ satisfying

$$\gamma^i(t_0) = x^i(\xi), \quad \frac{d\gamma^i}{dt} = \dot{x}^i(\xi).$$

In coordinates, we have

$$X^\sigma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial \dot{x}^i}.$$

The splitting induced by a SODE (3)

The Lie derivative of the fundamental tensor field

$$J = \omega^i \otimes \frac{\partial}{\partial \dot{x}^i}, \quad \omega^i = dx^i - \dot{x}^i dt,$$

on $J^1(\mathbb{R}, M)$ along X^σ is

$$L_{X^\sigma} J = -\omega^i \otimes X_i^\sigma + \varpi^j \otimes \frac{\partial}{\partial \dot{x}^j}, \quad \varpi^j = d\dot{x}^j - F^j dt - \tfrac{1}{2}\tfrac{\partial F^j}{\partial \dot{x}^i}\omega^i.$$

Then $T\left(J^1(\mathbb{R}, M)\right) = T^0 \oplus T^- \oplus T^+$, where

$$\begin{aligned} T^0 &= \ker(L_{X^\sigma} J) = \langle X^\sigma \rangle, \\ T^- &= \ker(L_{X^\sigma} J + I) = \langle X_i^\sigma \rangle, \quad X_i^\sigma = \tfrac{\partial}{\partial x^i} + \tfrac{1}{2}\tfrac{\partial F^j}{\partial \dot{x}^i}\tfrac{\partial}{\partial \dot{x}^j}, \\ T^+ &= \ker(L_{X^\sigma} J - I) = V\left(p_0^1\right) = \left\langle \tfrac{\partial}{\partial \dot{x}^i} \right\rangle. \end{aligned}$$

The splitting induced by a SODE (4)

The inverse mapping of the isomorphism

$$(p_0^1)_*\big|_{T^0 \oplus T^-} : T^0 \oplus T^- \xrightarrow{\cong} (p_0^1)^* T\left(\mathbb{R} \times M\right),$$

determines a section H^σ of $(p_0^1)_* : T\left(J^1(\mathbb{R}, M)\right) \to (p_0^1)^* T\left(\mathbb{R} \times M\right)$ given by

$$H^\sigma = dt \otimes X^\sigma + \omega^i \otimes X_i^\sigma .$$

Every X in $T\left(J^1(\mathbb{R}, M)\right)$ can uniquely be written as $X = X^v + X^h$,

$$X^h = H^\sigma\left((p_0^1)_* X\right) \in T^0 \oplus T^-,$$
$$X^v = X - X^h \in V(p_0^1).$$

The curvature of the splitting

The curvature form $K^\sigma \in \wedge^2 T^*\left(J^1(\mathbb{R}, M)\right) \otimes V(p_0^1)$, of the splitting H^σ,

$$K^\sigma(X, Y) = \left[X^h, Y^h\right]^v, \quad \forall X, Y \in \mathfrak{X}(J^1(\mathbb{R}, M)).$$

- From the formula for K^σ in local coordinates we have: $K^\sigma \in (p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1)$.
- CURVATURE MAPPING

$$\mathcal{K}\colon J^2(p_1^2) \to (p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1),$$
$$\mathcal{K}(j_\xi^2 \sigma) = (K^\sigma)_\xi .$$

- Let ∇^σ be the Chern connection attached to σ and let T^σ be the torsion tensor field of ∇^σ. We have:

$$T^\sigma = K^\sigma + dt \wedge \varpi^i \otimes X_i^\sigma .$$

Functoriality of the Chern connection

The Chern connection ∇^σ is functorial with respect to $\mathrm{Aut}^v(p)$; i.e.,

$$\Phi \cdot \nabla^\sigma = \nabla^{\Phi \cdot \sigma}, \quad \forall \Phi \in \mathrm{Aut}^v(p),$$

where $\Phi \cdot \nabla^\sigma$ is the linear connection defined by,

$$(\Phi \cdot \nabla^\sigma)_X Y = \Phi^{(1)} \cdot \left((\nabla^\sigma)_{(\Phi^{(1)})^{-1} \cdot X} \left((\Phi^{(1)})^{-1} \cdot Y \right) \right),$$

$\forall X, Y \in \mathfrak{X}(J^1(\mathbb{R}, M))$ and $\Phi \cdot \sigma$ is the SODE given by,

$$\Phi \cdot \sigma = \Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1}.$$

Second-order invariants

$\left(p_1^2\right)^2 : J^2(p_1^2) \to J^1(\mathbb{R}, M)$, 2-jet bundle of $p_1^2 \colon J^2(\mathbb{R}, M) \to J^1(\mathbb{R}, M)$.

Theorem

Every second-order differential invariant $\mathcal{I} \colon J^2(p_1^2) \to \mathbb{R}$ with respect to $\mathrm{Aut}^v(p)$ factors uniquely through the curvature mapping: $\mathcal{I} = \bar{\mathcal{I}} \circ \mathcal{K}$,

$$\begin{array}{ccl} J^2(p_1^2) & \xrightarrow{\mathcal{K}} & (p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1) \\ & \bar{\mathcal{I}} & \\ \mathcal{I} \downarrow & \swarrow & \\ \mathbb{R} & & \end{array}$$

where $\bar{\mathcal{I}} \colon (p_0^1)^* \wedge^2 T^*(\mathbb{R} \times M) \otimes V(p_0^1) \to \mathbb{R}$ is an invariant smooth function under the natural action of $\mathrm{Aut}^v(p)$.

Concluding remarks

As $\mathcal{D}^{(2)}$ is involutive, the number of functionally independent second-order differential invariants is

$$\dim J^2(p_1^2) - \operatorname{rank}\mathcal{D}^{(2)} = \begin{cases} \frac{1}{2}n^2(n-1)+1, & n \geq 2 \\ 2, & n = 1 \end{cases}$$

An endomorphism $\tilde{K}^\sigma \colon (p' \circ p_0^1)^* TM \to (p' \circ p_0^1)^* TM$ is defined:

$$\tilde{K}^\sigma = \varepsilon^{-1} \circ i_{X^\sigma} K^\sigma|_{T^-} \circ (\iota_1)^{-1} \circ \iota_2,$$

where ι_1, ι_2 are the isomorphisms:

$$\iota_1 \colon T^- \xrightarrow{\cong} (p_0^1)^* T(\mathbb{R} \times M) / (p_0^1)_* T^0,$$
$$\iota_2 \colon (p' \circ p_0^1)^* TM \hookrightarrow (p_0^1)^* T(\mathbb{R} \times M) \to (p_0^1)^* T(\mathbb{R} \times M) / (p_0^1)_* T^0.$$

We have: $\tilde{K}^\sigma(\partial/\partial x^j) = -P_j^h \partial/\partial x^h$. The coefficients of the characteristic polynomial of $\tilde{K}^\sigma$ determine n second-order invariants.

References

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The curvature form:

$$K^\sigma = -(P^h_j dt \wedge \omega^j + \sum_{i<j} T^h_{ij} \omega^i \wedge \omega^j) \otimes \frac{\partial}{\partial \dot{x}^h},$$

$$T^h_{ij} = \tfrac{1}{2} \left(\frac{\partial^2 F^h}{\partial x^i \partial \dot{x}^j} - \frac{\partial^2 F^h}{\partial x^j \partial \dot{x}^i} \right) + \tfrac{1}{4} \left(\frac{\partial F^r}{\partial \dot{x}^i} \frac{\partial^2 F^h}{\partial \dot{x}^r \partial \dot{x}^j} - \frac{\partial F^r}{\partial \dot{x}^j} \frac{\partial^2 F^h}{\partial \dot{x}^r \partial \dot{x}^i} \right),$$

$$P^h_j = \tfrac{1}{2} X^\sigma \left(\frac{\partial F^h}{\partial \dot{x}^j} \right) - \frac{\partial F^h}{\partial x^j} - \tfrac{1}{4} \frac{\partial F^r}{\partial \dot{x}^j} \frac{\partial F^h}{\partial \dot{x}^r}.$$

Formulas in local coordinates
Chern connection

Given a SODE σ, the Chern connection ∇^σ as defined by [2], [5], is locally given by

$$\begin{array}{lll}
\nabla^\sigma_{X^\sigma} X^\sigma = 0, & \nabla^\sigma_{X^\sigma} X^\sigma_i = -\frac{1}{2}\frac{\partial F^j}{\partial \dot{x}^i} X^\sigma_j, & \nabla^\sigma_{X^\sigma} \frac{\partial}{\partial \dot{x}^i} = -\frac{1}{2}\frac{\partial F^j}{\partial \dot{x}^i}\frac{\partial}{\partial \dot{x}^j}, \\
\nabla^\sigma_{X^\sigma_i} X^\sigma = 0, & \nabla^\sigma_{X^\sigma_j} X^\sigma_i = -\frac{1}{2}\frac{\partial^2 F^k}{\partial \dot{x}^i \partial \dot{x}^j} X^\sigma_k, & \nabla^\sigma_{X^\sigma_i} \frac{\partial}{\partial \dot{x}^j} = -\frac{1}{2}\frac{\partial^2 F^k}{\partial \dot{x}^i \partial \dot{x}^j}\frac{\partial}{\partial \dot{x}^k}, \\
\nabla^\sigma_{\frac{\partial}{\partial \dot{x}^i}} X^\sigma = 0, & \nabla^\sigma_{\frac{\partial}{\partial \dot{x}^i}} X^\sigma_j = 0, & \nabla^\sigma_{\frac{\partial}{\partial \dot{x}^i}} \frac{\partial}{\partial \dot{x}^j} = 0.
\end{array}$$

From the general formulas of jet prolongation of vector fields $X = u^i \frac{\partial}{\partial x^i}$, $u^i \in C^\infty(\mathbb{R} \times M)$, one obtains

$$X^{(2)} = u^i \frac{\partial}{\partial x^i} + \left(\frac{\partial u^i}{\partial t} + \frac{\partial u^i}{\partial x^h} \dot{x}^h \right) \frac{\partial}{\partial \dot{x}^i} + \left(\frac{\partial^2 u^i}{\partial t^2} + 2 \frac{\partial^2 u^i}{\partial t \partial x^h} \dot{x}^h + \frac{\partial^2 u^i}{\partial x^h \partial x^k} \dot{x}^h \dot{x}^k + \frac{\partial u^i}{\partial x^h} \ddot{x}^h \right) \frac{\partial}{\partial \ddot{x}^i}.$$

As the values of u^i and its derivatives can arbitrarily be taken at a given point $j^2_t \gamma \in J^2(\mathbb{R}, M)$, one has

$$\mathcal{D}^{(0)} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \dot{x}^i}, \dot{x}^h \frac{\partial}{\partial \dot{x}^i} + \ddot{x}^h \frac{\partial}{\partial \ddot{x}^i}, \frac{\partial}{\partial \ddot{x}^i}, 2\dot{x}^h \frac{\partial}{\partial \ddot{x}^i}, \dot{x}^h \dot{x}^k \frac{\partial}{\partial \ddot{x}^i} \right\rangle = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \dot{x}^i}, \frac{\partial}{\partial \ddot{x}^i} \right\rangle.$$

The only differential invariants of order 0 are the functions in $(p^2)^* C^\infty(\mathbb{R})$.

Formulas in local coordinates

First-order invariants

By collecting the derivatives of the functions u^i in $(X^{(2)})^{(1)}$ we have

$$
\begin{aligned}
(X^{(2)})^{(1)} &= u^r \tfrac{\partial}{\partial x^r} + \tfrac{\partial u^r}{\partial t}\chi^r_t + \tfrac{\partial u^r}{\partial x^a}\chi^r_a \\
&+ \tfrac{\partial^2 u^r}{\partial t^2}\chi^r_{tt} + \tfrac{\partial^2 u^r}{\partial t \partial x^a}\chi^r_{ta} + \sum_{a \leq b} \tfrac{\partial^2 u^r}{\partial x^a \partial x^b}\chi^r_{ab} \\
&+ \tfrac{\partial^3 u^r}{\partial t^3}\chi^r_{ttt} + \tfrac{\partial^3 u^r}{\partial t^2 \partial x^a}\chi^r_{tta} + \sum_{a \leq b} \tfrac{\partial^3 u^r}{\partial t \partial x^a \partial x^b}\chi^r_{t,ab} + \sum_{a \leq b \leq c} \tfrac{\partial^3 u^r}{\partial x^a \partial x^b \partial x^c}\chi^r_{abc}.
\end{aligned}
$$

for certain $\chi^r_t, \chi^r_a, \chi^r_{tt}, \chi^r_{ta}, \chi^r_{ab}, \chi^r_{ttt}, \chi^r_{tta}, \chi^r_{tab}, \chi^r_{abc} \in \mathfrak{X}(J^1(p^2_1))$. Is is proved that

$$
\begin{aligned}
\mathcal{D}^{(1)} &= \langle \partial/\partial x^r, \chi^r_t, \chi^r_a, \chi^r_{tt}, \chi^r_{ta}, \chi^r_{ab}, \chi^r_{ttt}, \chi^r_{tta}, \chi^r_{tab}, \chi^r_{abc} \rangle \\
&= \langle \partial/\partial x^r, \partial/\partial \dot{x}^r, \partial/\partial \ddot{x}^r, \partial/\partial \ddot{x}^r_t, \partial/\partial \ddot{x}^r_a, \partial/\partial \ddot{x}^r_{\dot{a}} \rangle .
\end{aligned}
$$

The first-order differential invariants are $f(t)$, $f \in C^\infty(\mathbb{R})$.

Equations of the curvature mapping

$$
\begin{aligned}
t\circ\mathcal{K} &= t,\ x^i\circ\mathcal{K}=x^i,\ \dot{x}^i\circ\mathcal{K}=\dot{x}^i,\\
y_a^i\circ\mathcal{K} &= -\tfrac{1}{2}\left(\ddot{x}^i_{t\dot{a}}+\dot{x}^h\ddot{x}^i_{h\dot{a}}+\ddot{x}^h\ddot{x}^i_{\dot{h}\dot{a}}\right)+\ddot{x}^i_a+\tfrac{1}{4}\ddot{x}^k_{\dot{a}}\ddot{x}^i_{\dot{k}},\\
y_{ab}^k\circ\mathcal{K} &= -\tfrac{1}{2}\left(\ddot{x}^k_{a\dot{b}}-\ddot{x}^k_{b\dot{a}}+\tfrac{1}{2}\left(\ddot{x}^h_{\dot{a}}\ddot{x}^k_{h\dot{b}}-\ddot{x}^h_{\dot{b}}\ddot{x}^k_{h\dot{a}}\right)\right),\ a<b,
\end{aligned}
$$

$(t,\ x^i,\ \dot{x}^i,\ \ddot{x}^i,\ \ddot{x}^i_t,\ \ddot{x}^i_a,\ \ddot{x}^i_{\dot{a}},\ \ddot{x}^i_{tt},\ \ddot{x}^i_{ta},\ \ddot{x}^i_{t\dot{a}},\ \ddot{x}^i_{a\le b},\ \ddot{x}^i_{a\dot{b}},\ddot{x}^i_{\dot{a}\le\dot{b}})$ induced coordinate system on $J^2(p_1^2)$ and

$$
\eta=\left(y_i^j(\eta)\left(dt\wedge\omega^i\right)_{(t_0,x_0)}+\sum\nolimits_{h<i}y^j_{hi}(\eta)\left(\omega^h\wedge\omega^i\right)_{(t_0,x_0)}\right)\otimes\left(\frac{\partial}{\partial\dot{x}^j}\right)_{\xi},
$$

coordinates are introduced in $(p_0^1)^*\wedge^2T^*(\mathbb{R}\times M)\otimes V(p_0^1)$.

The curvature mapping

The curvature mapping is $\mathrm{Aut}^v(p)$-equivariant with respect to the natural actions, i.e.,

$$\Phi \cdot \mathcal{K}(j^2_\xi \sigma) = \mathcal{K}(\Phi \cdot j^2_\xi \sigma), \quad \forall \Phi \in \mathrm{Aut}^v(p),$$

where the action on the left-hand side is defined by

$$\Phi \cdot \eta = \left(\wedge^2((\Phi^{(1)})^{-1})^* \otimes (\Phi^{(1)})_* \right)(\eta),$$

$\forall \eta \in (p^1_0)^* \wedge^2 T^* (\mathbb{R} \times M) \otimes V(p^1_0)$, and the action on the right-hand side is defined as follows:

$$\Phi \cdot j^2_\xi \sigma = j^2_{\Phi^{(1)}(\xi)}(\Phi \cdot \sigma).$$