Unsteady Effects in Droplet Evaporation and Combustion

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1. INTRODUCTION

The evaporation or simultaneous evaporation and combustion of a fuel droplet is frequently treated by using quasisteady theories (see the review papers by Williams (1973) and Rosner (1972)). This assumption is based on the facts that the droplet is burning or vaporizing in a steady environment and that the liquid density is much higher than the gas density (Williams, 1959). The purpose of this work is to make a perturbation analysis in which, by considering the ratio of the densities of gas and liquid as a small parameter, the asymptotic nature of the quasisteady solution is shown. Corrections to the "quasisteady" (QS) theory are found to be of the order of the square root of the ratio of the densities. It is also found that the solution is "non-steady" (NS) at distances from the droplet of the order of the droplet radius divided by the square root of the ratio of the densities.

A situation in which NS effects are important is when the pressure and temperature at the droplet surface approach the critical pressure and temperature of the fuel (Rosner and Chang, 1973). This may happen in liquid propellant rocket motors and in diesel engines (Wieber, 1963; Rosner, 1967; Manrique and Borman, 1969). For high temperature and pressure the non-ideal gas effects and the dissolution of the ambient fluid in the liquid phase have to be taken into account (Manrique and Borman, 1969; Matlosz, Leipziger and Torda, 1972). Manrique and Borman used the QS approximation in their works; this may be justified when the ambient density is much smaller than the critical density of the fuel; in this case the QS results can be improved with an analysis similar to the one

presented here. In our analysis, however, we will not take into account the realistic gas mixture properties and it will be assumed that the ambient gas is not dissolved in the droplet, and that the droplet temperature is the boiling temperature at ambient pressure. This last assumption is based on the fact that the heat of vaporization divided by the specific heat at constant pressure is large compared to the boiling temperature, and implies the existence of a heat-up period that precedes the vaporization period, during which there is no significant vaporization. This heat up period is not considered here. In this work we will mainly concentrate in calculating the perturbations introduced to the results of the QS analysis by NS effects during the vaporization period.

A comparison will be made of our perturbation theory results and the numerical results of Duda and Vrentas (1971) for the problem of sphere dissolution, recently shown by Rosner and Chang (1973) to be applicable to the prediction of fuel droplet lifetimes when QS -approximations fail.

The problem of droplet evaporation with combustion is reduced to the pure evaporation case, for Lewis number equal to one, by using the thin flame assumption. It is found that when the stoichiometric ratio is not large, the flame position is correctly predicted by the QS theory, and corrections to this result are of the order of the square root of the ratio of the densities. However, in many practical cases the stoichiometric ratio is large and the flame has to go far from the droplet so that fuel and oxidizer will reach it in stoichiometric ratio. When the flame is in the NS region, the QS theory will not give correctly its position even in first

approximation. The analysis presented here explains the fact (experimentally confirmed) that in many cases the *QS* analysis gives valid results regarding burning rates but not regarding flame position (Nuruzzaman and Beer, 1971; Krier and Wronkiewicz, 1972; Rush and Krier, 1973; Isoda and Kumagai, 1958; Kumagai, Sakai, Okajima, 1971). An earlier attempt to solve this contradiction has been made by Chervinsky (1969), however, the mathematical approximations he uses do not appear to be justified, and his theory does not lead to useful results for the *NS*-droplet lifetime. Kotake and Okazaki (1969) obtained numerical solutions of the *NS* equations of droplet combustion that show a qualitative agreement with the above-mentioned experimental results.

The results obtained here show that the time elapsed until the flame radius vanishes (extinction time) is longer than the vaporization time. When the stoichiometric ratio is not large, these two times are approximately the same, however, for sufficiently large values of that ratio the extinction time may even become much larger than the vaporization time. In this last limiting case our results coincide with those obtained by Spalding (1959) who replaces the droplet by an instantaneous source of vapor.

The problem of supercritical droplet combustion has been shown by Tarifa, Crespo, and Fraga (1972), using a gas pocket model, to be similar in many aspects to the subcritical case if the initial pocket temperature is much smaller than the ambient temperature. However, for supercritical droplet combustion the *NS* effects are more important than in the case analyzed here, because the ratio of the two temperatures mentioned earlier is usually not so small as the ratio of the gas and liquid densities. These *NS* effects have been considered by Tarifa and Crespo (1972) in an analysis similar to the one presented here.

2. MODEL AND ASSUMPTIONS

The configuration to be used is shown in Figure 1. An isolated fuel droplet is enveloped by a very thin flame where fuel and oxidizer react irreversibly at an infinitely fast rate. The fuel concentration is zero outside the flame, and the oxidizer concentration is zero inside the flame. Both concentrations vanish at the flame. For the pure vaporization case the fuel concentration at infinity may be different from zero.

It will be assumed that the fluid properties

FIG. 1. Schematic showing temperature and concentration distributions, for droplet evaporation with thin flame combustion.

initially are uniform and at infinity are uniform and constant, and that there are no forced convection effects. These assumptions imply that there is spherical symmetry. Tarifa and Crespo (1972) show that forced convection effects are negligible compared to *NS* effects when the Reynolds number corresponding to the external velocity is small compared to the square root of the ratio of gas and liquid density.

We consider conditions at the vapor/liquid interface to be well below the critical conditions of the fuel, so that we can assume that there is no dissolution of ambient gas in the droplet. Since, under these conditions, the heat of vaporization is large, the droplet temperature is very close to the boiling temperature of the fuel, except for a heat-up period which is not analyzed in this work; the droplet is here assumed to be always at the boiling temperature.

The gas is assumed to be a perfect gas mixture with a constant value of the mean molecular mass and of the specific heat, c_p . Since we will neglect the effects of the small spatial variations of pressure in the equation of state, the product of the density times the temperature will be constant

$$\rho T = \rho_\infty T_\infty$$

Only a diffusion coefficient has been considered, which, for simplicity in the presentation, has been assumed to be inversely proportional to the density

$$\rho D = \rho_\infty D_\infty$$

and equal to the thermal diffusivity of the mixture

$$\rho c_p D / \lambda = 1$$

In the following equations non-dimensional variables will be used. The temperature and the

density have been made non-dimensional with their corresponding values at infinity:

$$\theta = T/T_\infty \quad \bar{\rho} = \rho/\rho_\infty$$

The radial distance and the instantaneous droplet radius are made non-dimensional with the initial droplet radius:

$$x = r/r_{s0} \quad x_s = r_s/r_{s0}$$

The non-dimensional time is:

$$\tau = \varepsilon D_\infty t / r_{s0}$$

where ε is the ratio of the ambient gas and liquid densities

$$\varepsilon = \rho_\infty / \rho_l$$

This non-dimensional form of the time has been chosen so as to make the non-dimensional vaporization time of order unity.

The velocity is made non-dimensional with a characteristic gas velocity obtained by dividing the initial droplet radius by the diffusion time

$$\bar{V} = V(r_{s0}^2 / D_\infty) / r_{s0}$$

In the following the subscript s also means gas conditions at the droplet surface.

3. DROPLET VAPORIZATION

Although the theory presented here could be applied to the case of vaporization with combustion, the method is better illustrated with this more simple case. Nevertheless, the main results of the combustion case will be exposed at the end.

The equations to be considered are, in non-dimensional form, as follows:

Mass conservation

$$\varepsilon \frac{\partial \bar{\rho}}{\partial \tau} + \frac{1}{x^2} \frac{\partial}{\partial x} (x^2 \bar{V} \bar{\rho}) = 0 \quad (1)$$

Species conservation for the mass fraction, Y_1 , of the fuel vapor

$$\mathcal{L}(Y_1) = 0 \quad (2)$$

Energy conservation

$$\mathcal{L}(\theta) = 0 \quad (3)$$

where

$$\mathcal{L} = \varepsilon \bar{\rho} \frac{\partial}{\partial \tau} + \bar{\rho} \bar{V} \frac{\partial}{\partial x} - \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial}{\partial x} \right) \quad (4)$$

Equation of state:

$$\bar{\rho} \theta = 1 \quad (5)$$

The boundary conditions are:

Mass conservation at the droplet surface

$$\frac{dx_s}{d\tau} = -\frac{1}{\theta_s} \left(\bar{V}_s - \varepsilon \frac{dx_s}{d\tau} \right) \quad (6)$$

Mass conservation for evaporation fuel

$$(1 - Y_{1s}) \left(\bar{V}_s - \varepsilon \frac{dx_s}{d\tau} \right) + \theta_s \frac{\partial Y_1}{\partial x} \Big|_s = 0 \quad (7)$$

Energy conservation

$$\frac{\partial \theta}{\partial x} \Big|_s = \frac{1}{\theta_s} \left(\bar{V}_s - \varepsilon \frac{dx_s}{d\tau} \right) \frac{L}{c_p T_\infty} + \frac{x_s c_{pl}}{3 c_p} \frac{d\theta_s}{d\tau} \quad (8)$$

Clapeyron equation, for constant heat of vaporization, L , per unit mass of fuel,

$$Y_{1s} = \exp \left[-\frac{L}{R_1 T_B} \left(\frac{\theta_B - \theta_s}{\theta_s} \right) \right] \quad (9)$$

where R_1 is the gas constant for the fuel vapor. If $L/R_1 T_B$ is large, Y_{1s} will be very small until θ_s becomes almost equal to θ_B . Thus, there will be a heat-up period during which the droplet temperature rises to the boiling temperature, followed by the vaporization period which we shall analyze here with the assumption

$$\theta_s = \theta_B = T_B / T_\infty \quad (9')$$

The conditions at infinity are:

$$x \rightarrow \infty: \theta = 1 \quad Y_1 = Y_{1\infty} \quad (10)$$

$$\tau = 0 \quad x > 1: \theta = 1 \quad Y_1 = Y_{1\infty} \quad (11)$$

In these equations and boundary conditions ε is a small parameter; if it is made equal to zero, the classical results of the QS theory of droplet vaporization are obtained.

3.1 QS Solution

The mass conservation equation (1) gives

$$x^2 \bar{V} / \theta = m(\tau) x_s \quad (12)$$

where m is an integration constant that, from equation (6), is given by

$$m = x_s \bar{V}_s / \theta_s = -\frac{1}{2} dx_s^2 / d\tau \quad (13)$$

By integrating equation (3) with boundary conditions (8) and (9')

$$\theta = \theta_B - \frac{L}{c_p T_\infty} + \frac{L}{c_p T_\infty} \exp [m(1 - x_s/x)] \quad (14)$$

This expression for large values of x behaves like

$$\theta = \theta_B - \frac{L}{c_p T_\infty} + \frac{L}{c_p T_\infty} (1 - mx_s/x) \exp m \quad (15)$$

showing that, far from the droplet, the temperature distribution is that corresponding to a heat source of strength $-mx_s \exp(m)L/c_p T_\infty$, without radial convective effects, in an atmosphere of temperature $\theta_{\infty a}$

$$\theta_{\infty a} = \theta_B + \frac{L}{c_p T_\infty} (\exp m - 1) \quad (16)$$

If the boundary condition (10) is taken into account, $\theta_{\infty a}$ is equal to one, obtaining

$$m = m_0 = \ln \{1 + (1 - \theta_B)c_p T_\infty/L\} \quad (17)$$

and from equation (13) and the initial condition $x_s = 1$ at $\tau = 0$

$$x_s^2 = x_{s0}^2 = 1 - 2m_0\tau \quad (18)$$

By integrating equation (3) with boundary conditions (10) and (7), the concentration field is obtained

$$Y_1 = 1 - (1 - Y_{1\infty}) \exp(-mx_s/x) \quad (19)$$

The temperature and concentration fields are given by equations (15) and (19) with m and x_s given by equations (17) and (18). Notice that, far from the droplet, the concentration distribution corresponds to a mass source of strength $(1 - Y_{1\infty})m_0 x_{s0}$. When trying to improve this QS solution by using a regular expansion procedure, the next terms, of order ε , are found to diverge at infinity (Williams, 1959).

This difficulty is mathematically similar to that appearing when trying to find a second approximation to the slow viscous motion around a sphere (Whitehead's paradox), and the method used by Kaplun and Lagerstrom (1957) to solve this problem will be applied here. There is an "Outer Region" located at distances from the droplet of the order of

$$x \sim 1/\sqrt{\varepsilon}$$

where the NS and diffusion terms are of the same order in equations (2) and (3), and where the temperature and concentration differ from their values at infinity (10) by small terms, of order $\sqrt{\varepsilon}$, which will be obtained below by linearization of equations (2) and (3). In an "Inner Region", located at distances from the droplet of the order of the droplet radius, the QS differential equations describe the temperature and concentration distributions, with

errors of order ε , if the appropriate matching conditions with the outer expansion are used. Since the first approximation to the outer solution is given by conditions at infinity (10), the classical QS results presented above, with the condition $\theta_{\infty a}$ equal to one, correspond to the first approximation to the solution with errors of order $\sqrt{\varepsilon}$.

3.2 Outer Solution

In this region the appropriate variables are τ , and

$$X = x\sqrt{\varepsilon} \quad (20)$$

and we use the expansion

$$\theta = 1 + \sqrt{\varepsilon} \theta^* + 0(\varepsilon) \quad (21)$$

$$Y_1 = Y_{1\infty} + \sqrt{\varepsilon} Y_1^* + 0(\varepsilon) \quad (22)$$

where θ^* and Y_1^* are functions of X and τ .

The continuity equation (1) will only be used to show that $\bar{V}$ is of order ε in this region, and consequently the convective term is of order $\sqrt{\varepsilon}$ relative to the NS term in equations (2) and (3). By taking equations (20) and (21) into equation (3), the equation for the temperature perturbation becomes

$$\frac{\partial \theta^*}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial \theta^*}{\partial x} \right) \quad (23)$$

A similar equation is obtained for the concentration. In order to match with the inner solution (15), the solution of (23), satisfying the initial condition $\theta^* = 0$ (equation (11)), is written in the form

$$\theta^* = \frac{1}{2\sqrt{\pi}} \int_0^\tau g(\xi) \frac{e^{-X^2/4(\tau-\xi)}}{(\tau-\xi)^{3/2}} d\xi \quad (24)$$

that expresses that the droplet, when viewed from a large distance, behaves like a point heat source of varying intensity, $g(\tau)$. The same solution (24) is valid for the concentration, Y_1^* , with a different source intensity, $h(\tau)$.

At small values of X , and large values of x , given by $\tilde{x} \sim 1$ in an intermediate variable

$$\tilde{x} = \eta x \quad (25)$$

where

$$\sqrt{\varepsilon} \ll \eta \ll 1$$

the first two terms of the outer expansion (21) lead to

$$\theta = 1 + \frac{g(\tau_1)\eta}{\tilde{x}} - \frac{\sqrt{\varepsilon}}{\sqrt{\pi}} \times \left[\frac{g(0)}{\sqrt{\tau_1}} + \int_0^{\tau_1} \frac{g'(\tau_1 - \xi)}{\sqrt{\xi}} d\xi \right] + o(\sqrt{\varepsilon}) \quad (26)$$

and a similar equation for Y_1 with g replaced by h and the first term in the right hand side replaced by $Y_{1\infty}$. The matching conditions with the first approximation to the inner solution, or classical QS solution, give

$$g(\tau) = -\frac{L}{c_p T_\infty} m_0 \exp(m_0)(1 - 2m_0\tau)^{1/2} \quad (27)$$

$$h(\tau) = (1 - Y_{1\infty})m_0(1 - 2m_0\tau)^{1/2} \quad (28)$$

3.3 Inner Solution

As it was indicated before, the inner solution is given in first approximation by equations (14) and (19), with m and x_s given by equations (17) and (18) which result from the matching condition with the first approximation of the outer solution

$$\theta = 1 \quad Y_1 = Y_{1\infty} \quad \text{at } x \rightarrow \infty \quad (29)$$

To calculate corrections of order $\sqrt{\varepsilon}$ to this QS solution, the QS differential equations and boundary conditions at the droplet surface can still be used, however, the outer boundary condition (29) should be substituted by the matching conditions with the two-term outer expansion. Then,

$$\begin{aligned} \theta &= 1 + \sqrt{\varepsilon} \theta_a^*(\tau) \\ Y_1 &= Y_{1\infty} + \sqrt{\varepsilon} Y_{1a}^*(\tau) \quad \text{at } x \rightarrow \infty \end{aligned} \quad (30)$$

where $\sqrt{\varepsilon} \theta_a^*$ is the third term in the right hand side of equation (26), and $\sqrt{\varepsilon} Y_{1a}^*$ is the corresponding term for Y_1 .

Then, the first two terms of the inner expansion can be easily obtained from equation (14) with m given by the linearized form of equation (16) with

$$\theta_{\infty a} = 1 + \sqrt{\varepsilon} \theta_a$$

and x_s^2 given by (13). Thus, we obtain

$$(m - m_0)/\sqrt{\varepsilon} = m_1 = (1 - \theta_B + L/c_p T_\infty)^{-1} \theta_a^* \quad (31)$$

and using equations (26) and (30)

$$m_1 = \frac{m_0^{3/2}}{\sqrt{\pi}} \left[\frac{1}{\sqrt{\tau m_0}} - \frac{1}{\sqrt{2}} \ln \frac{1 + \sqrt{2m_0\tau}}{1 - \sqrt{2m_0\tau}} \right] \quad (32)$$

From equation (13)

$$\begin{aligned} X_s^2 &= 1 - 2m_0\tau - (2\varepsilon m_0/\pi)^{1/2} \\ &\times \left[\sqrt{2m_0\tau} + \frac{1 - 2m_0\tau}{2} \ln \frac{1 + \sqrt{2m_0\tau}}{1 - \sqrt{2m_0\tau}} \right] \end{aligned} \quad (33)$$

The proportionality of the heat and mass sources, equations (27) and (28), allows us to write

$$Y_a^* = -(1 - Y_{1\infty}) \left[\exp(m_0) \frac{L}{c_p T_\infty} \right]^{-1} \theta_a^* \quad (34)$$

and then to calculate corrections to the concentration field.

Equation (32) shows that the vaporization rate coefficient, that is constant in the classical QS analysis, is corrected due to unsteady effects by a term of order $\sqrt{\varepsilon}$ which varies with time. The square of the droplet radius varies no longer linearly with time, equation (33) if the NS effects are retained.

The vaporization time, τ_v , is obtained from equation (33) by making x_s equal to zero. We thus obtain an explicit relation between εm_0 and $2m_0\tau_v$. For small values of εm_0 , the error of the QS analysis that gives:

$$\tau_v = 1/2m_0 \quad (35)$$

is found to be small. If terms of order $\varepsilon \ln \varepsilon$ are neglected, we obtain from equation (33)

$$2m_0\tau_v = 1 - \sqrt{2\varepsilon m_0/\pi} \quad (36)$$

The fact that the droplet life-time, and thus the boundary conditions for the equations, change with the order of approximation indicates that one should use a stretched time variable,

$$\tau' = \tau(1 + b_1\sqrt{\varepsilon} + \dots),$$

instead of the non-dimensional time τ that has been used; the coefficients b_i should be chosen so that the vaporization time τ'_v does not change with higher order approximations. If the analysis presented above is remade in this way, no changes are found in the previous results if they are written in terms of τ' , with the exception of equation (33) which is replaced by

$$\begin{aligned} x_s^2 &= 1 - 2m_0\tau'(1 - b_1\sqrt{\varepsilon}) - \sqrt{2\varepsilon m_0/\pi} \\ &\times \left[\sqrt{2m_0\tau'} + \frac{1 - 2m_0\tau'}{2} \ln \frac{1 + \sqrt{2m_0\tau'}}{1 - \sqrt{2m_0\tau'}} \right] \end{aligned} \quad (33')$$

So that $b_1 = \sqrt{2m_0/\pi}$ if we want the droplet lifetime to be given by $2m_0\tau'_v = 1$ also in the second approximation. In terms of the variable τ , to this approximation, we obtain for τ_v

$$2m_0\tau_v = \{1 + \sqrt{2\varepsilon m_0/\pi}\}^{-1} \quad (36)$$

FIG. 2. Vaporization time as a function of ϵm_0 . Comparison with the numerical sphere dissolution lifetime results of Duda and Vrentas (1971), shown by Rosner and Chang (1973) to be applicable to fuel droplet evaporation-combustion under *NS*-conditions.

which has been plotted in Figure 2 and is found to correlate well the numerical results of Duda and Vrentas (1971). It should be remarked that the quantity $2m_0\tau_v$ is identical to the dimensionless droplet lifetime τ_{lte} introduced by Rosner and Chang (1973). It is simply the ratio of the actual droplet lifetime to that formally predicted by the *QS*-approximation.

Figure 3 gives the deviation from the linear *QS* result in the square of the droplet radius as a function of $2m_0\tau' = \tau/\tau_v$, given by equation (33').

3.4 Temperature and Concentration Distributions for Times Larger Than the Vaporization Time.

During the droplet vaporization period, $\tau < \tau_v$, the inner region is scaled with x_s that vanishes at $\tau = \tau_v$. At times $\tau > \tau_v$ the temperature and concentration distributions are given by the outer expansion (21), which is now valid everywhere, if the upper limit of integration in equation (24) is replaced by $1/2m_0$, because there are no sources of heat and mass at $X = 0$ at $\tau > \tau_v \simeq 1/2m_0$.

FIG. 3. Deviation from the linear law of variation of x_s^2 with time.

4. DROPLET VAPORIZATION WITH COMBUSTION

In this case the vapor leaving the droplet reacts exothermically with the oxidizer leading to an increase in the heating of the droplet that enhances its vaporization rate. This reaction is represented by production terms that appear in the right hand side of equations (2) and (3). However, the variables

$$\bar{Y} = Y_1 - Y_2/\nu \quad (37)$$

$$\bar{\theta} = \theta + q_r Y_2/\nu c_p T_\infty \quad (38)$$

satisfy the following equations without the chemical production terms,

$$\mathcal{L}(\bar{Y}) = \mathcal{L}(\bar{\theta}) = 0 \quad (39)$$

where ν and q_r are respectively the mass of oxidizer consumed and heat released per unit mass of fuel consumed. The operator $\mathcal{L}$ is defined in formula (4).

If the thin flame assumption is used, we find that inside the flame Y_2 is equal to zero and, thus, $\bar{Y} = Y_1$ and $\bar{\theta} = \theta$, so that $\bar{Y}$ and $\bar{\theta}$ satisfy not only the same equations, (2) and (3), as Y_1 and θ in the pure vaporization case, but also the same boundary conditions at the droplet surface, equations (5)–(8'). However, since $Y_1 = 0$ outside the thin flame, the initial and boundary conditions at infinity for $\bar{Y}$ and $\bar{\theta}$ are

$$\bar{Y} = -Y_{2\infty}/\nu \quad \bar{\theta} = 1 + q_r Y_{2\infty}/c_p T_\infty \nu \quad (40)$$

instead of the equations (9) and (10) used for Y_1 and θ in the pure vaporization case.

The results obtained in the previous section can now be used directly to generate the first and second approximation for $\bar{Y}$ and $\bar{\theta}$ in the inner and outer regions. In particular, the first approximation, m_0 , for the vaporization coefficient, m , is given by

$$m_0 = \ln \{ (1 + q_r Y_{2\infty}/\nu L) + (1 - \theta_B) c_p T_\infty / L \} \quad (41)$$

that can be used to calculate, in the outer region, the first perturbations of $\bar{Y}$ and $\bar{\theta}$ from their values at infinity; these are given by equation (24) together with equations (27) and (28). Notice that the factor $(1 - Y_{1\infty})$ in equation (28) must be replaced by $(1 + Y_{2\infty}/\nu)$. Then, as in the previous section, we can proceed to calculate the second approximation for $\bar{Y}$ and $\bar{\theta}$, as well as m and r_s that are given by equations (32) and (33') together with (41). The second approximation of the vaporization time is given by equations (36') and (41).

The flame position can be obtained as a function of time by solving the equation $\bar{Y}(x_f, \tau) = 0$. The solution will be obtained, first, for the case in which the flame is in the inner region that correspond to $Y_{2\infty}/\nu \sim 1$, and, then, for the case in which the flame is in the outer region when $Y_{2\infty}/\nu \sim \sqrt{\varepsilon}$.

4.1 Flame in the Inner Region

The first two terms of the inner expansion for $\bar{Y}$ can be obtained from the solution of the QS forms of equation (39) and of the boundary condition (7). Then

$$\bar{Y} = 1 - (1 - \bar{Y}_{a\infty}) \exp(-mx_s/x) \quad (42)$$

The apparent concentration at infinity

$$\bar{Y}_{a\infty} = -Y_{2\infty}/\nu + \sqrt{\varepsilon} Y_a^* \quad (43)$$

includes a term of order $\sqrt{\varepsilon}$ obtained from the matching conditions with the outer expansion. According to equation (34),

$$Y_a^*/(1 + Y_{2\infty}/\nu) = -m_1 \quad (44)$$

The equation $\bar{Y} = 0$ can now be solved, and then the first two terms of an expansion of the ratio x_f/x_s are obtained.

$$\begin{aligned} \frac{x_f}{x_s} = & \frac{m_0}{\ln(1 + Y_{2\infty}/\nu)} \\ & \times \left[1 + \sqrt{\varepsilon} \frac{m_1}{m_0} \left(1 - \frac{m_0}{\ln(1 + Y_{2\infty}/\nu)} \right) \right] \end{aligned} \quad (45)$$

4.2 Flame in the Outer Region

The first and second terms of the expansion for (x_f/x_s) , given by equation (45), become of the same order, $1/\sqrt{\varepsilon}$, when $Y_{2\infty}/\nu m_0 \sim \sqrt{\varepsilon}$. This occurs when the oxidizer concentration at infinity is low or the stoichiometric ratio is high; then the flame has to go far from the droplet, to the outer region, so that fuel and oxidizer will arrive to the flame in the stoichiometric ratio.

The flame position in this case has to be found by solving the equation

$$\begin{aligned} \bar{Y} = & -Y_{2\infty}/\nu + \frac{\sqrt{\varepsilon} m_0 (1 + Y_{2\infty}/\nu)}{2\sqrt{\pi}} \\ & \times \int_0^\gamma \frac{\sqrt{1 - 2m_0\xi}}{(\tau - \xi)^{3/2}} e^{-X_f^2/4(\tau - \xi)} d\xi = 0 \end{aligned} \quad (46)$$

where $\gamma = \tau$ if $2m_0\tau < 1$, and $2m_0\gamma = 1$ if $2m_0\tau > 1$. In this case $Y_{2\infty}/\nu$ is small, so that the factor $(1 + Y_{2\infty}/\nu)$ will be left out of equation (46).

Notice that when the droplet disappears at $\tau = \tau_v$, the flame radius does not become zero and takes the value given by the equation

$$E_1(m_0 X_f^2/2) = (2\pi/m_0^3 \epsilon)^{1/2} Y_{2\infty}/\nu = \alpha \quad (47)$$

obtained from (46) by setting $\gamma = \tau = \tau_v = 1/2m_0$, where E_1 is the exponential integral function. For large values of the parameter α , X_f becomes very small; this means that in this limit, corresponding to the flame in the inner region, the flame and droplet radius become zero simultaneously.

The extinction time, τ_e , obtained by setting $X_f = 0$ in equation (46), is given by the relation

$$\alpha = -\frac{2}{\sqrt{\tau_e/\tau_v}} + \ln \frac{\sqrt{\tau_e/\tau_v} + 1}{\sqrt{\tau_e/\tau_v} - 1} \quad (48)$$

For small values of α , $\tau_e/\tau_v \gg 1$. Then, during most of the flame lifetime $\tau \gg \tau_v > \xi$ and the integral in equation (46) can be simplified

$$m_0 X_f^2/2 = (\tau/\tau_v) \ln \{ (2/3\alpha)(\tau/\tau_v)^{-3/2} \} \quad (49)$$

The same result was obtained by Spalding (1959) considering the droplet as an instantaneous source of vapor.

In Figure 4 the flame radius multiplied by $\sqrt{m_0/2}$, as function of the ratio τ/τ_v , is represented for different values of the parameter α , as obtained from equation (46). Figure 4 includes a comparison with the *QS* theory for a large value, 10, of α , and with Spalding's result for a small value, 0.05, of α .

FIG. 4. Flame radius as a function of time, and comparison with Spalding (1959) source solution for $\alpha = 0.05$, ---, and with the *QS* solution for $\alpha = 10$, — x — x —.

FIG. 5. Ratio of flame to droplet radius as a function of time. Comparison of the present results, equations (45) and (46), with the experimental results of Kumagai *et al.* (1971).

As it was pointed out in the Introduction, the experimental results on the combustion of single droplets show that the “ d^2 law” of burning is valid, but the ratio of flame to droplet diameter changes during combustion, whereas the *QS* theory will predict that it is constant. The results shown above suggest that this may be due to the fact that the flame is located far from the droplet. For example for *n*-heptane burning in air $Y_{2\infty}/\nu = 0.065$, and the value of ϵ , for a temperature at infinity of 1200 °K, is $\epsilon = 0.42 \times 10^{-3}$. Consequently, in a situation like this, $\sqrt{\epsilon}$ and $Y_{2\infty}/\nu$ are of the same order, and the flame could be located in the outer *NS* region. While the expected errors in using the classical *QS* theory for vaporization are of order $\sqrt{\epsilon}$, the relative errors will be of order unity in calculating the flame position.

In Figure 5 a comparison is given of the experimental results of Kumagai *et al.* (1971) and our theoretical calculations. The conditions under which the experiments of Kumagai *et al.* have been carried out are such that the flame is in the zone of common validity of the outer (46) and inner (45) solutions, and for this reason both of them have been drawn in Figure 5.

Regarding this point, it should be mentioned that if the flame would have been located clearly in the outer region, the theory here presented should not be applicable to this experiment, because the heat of reaction would be so large as to make the temperature of the flame considerably larger than the temperature at infinity, thus invalidating the assumption $\theta = 1 + \sqrt{\varepsilon} \theta^*$ that allows to linearize the equations in the outer region.

5. CONCLUDING REMARKS

Unsteady effects in droplet vaporization are due on one hand to the thermal inertia of the liquid, so that an initial period exists in which the liquid temperature rises from its initial value to a temperature close to the boiling temperature; for moderately large values of the nondimensional heat of vaporization $L/R_1 T_B$ no significant vaporization occurs during this heat up period and $T = T_B$ afterwards. The analysis presented here does not consider the first heat-up period, and it refers to the main period of vaporization, when the liquid temperature is constant, and the unsteady effects are represented by the time derivative terms in the gas phase equations. We have found that in an inner region, $r \sim r_s$, these terms become of order ε relative to the convective or diffusive terms after a short time period of adjustment of the temperature and concentration distributions to their quasi-steady forms. This period of time is of order ε relative to the droplet lifetime. We have not taken into account the existence of this initial stage because it leads to changes in the vaporization rate and droplet lifetime of order ε .

For the droplet vaporization time we have found the value

$$t_v = (r_{so}^2 \rho_l c_p / 2 \lambda m_0) \{1 + \sqrt{2 m_0 \rho_\infty / \rho_l \pi}\}^{-1} \quad (50)$$

where m_0 is given by equation (17) for the pure vaporization case, and by equation (41) for the case of vaporization enhanced by combustion. The factor in curly brackets accounts for the unsteady effects. They are dependent on the parameter εm_0 which should have been used instead of ε as the small parameter in the asymptotic analysis. Equation (50) is found to correlate very well, even for values of εm_0 of order unity, the results obtained by Duda and Vrentas (1971) by numerical integration of the unsteady differential equations for the problems of the dissolution of a sphere. Indeed, provided ρ_∞ / ρ_l is sufficiently small equation (50) can

now be safely exploited to extend useful crossplots of Duda and Vrentas' numerical results (of Rosner and Chang (1973)) to values of the driving parameter m_0 in the range of combustion interest ($m_0 > \ln 2$).

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