# Computing the Hessenberg matrix associated with a self-similar measure

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**Abstract**

We introduce in this paper a method to calculate the Hessenberg matrix of a sum of measures from the Hessenberg matrices of the component measures. Our method extends the spectral techniques used by G. Mantica to calculate the Jacobi matrix associated with a sum of measures from the Jacobi matrices of each of the measures.

We apply this method to approximate the Hessenberg matrix associated with a self-similar measure and compare it with the result obtained by a former method for self-similar measures which uses a fixed point theorem for moment matrices. Results are given for a series of classical examples of self-similar measures.

Finally, we also apply the method introduced in this paper to some examples of sums of (not self-similar) measures obtaining the exact value of the sections of the Hessenberg matrix.


*Keywords:* Self-similar measures; Orthogonal polynomials; Moment matrix; Hessenberg matrix

## 1. Introduction

In a recent work [8] we have obtained a method to approximate the moment matrix of a self-similar measure using a fixed point theorem for moment matrices. The Cholesky factorization of this moment matrix allows us to obtain an approximation of the Hessenberg matrix of the measure.

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In this paper we introduce a new method to calculate exactly the Hessenberg matrix of a sum of measures from the Hessenberg matrices of the component measures. This method extends the spectral techniques used by Mantica [17] to calculate the Jacobi matrix associated with a sum of measures from the Jacobi matrices of each of the measures (see also [5,11]).

Moreover, for the particular case of a self-similar measure $\mu$, by iteratively applying the above method to a suitable system of measures approximating $\mu$, we obtain a method to approximate the Hessenberg matrix associated with $\mu$.

The study of the Hessenberg matrix associated with a self-similar measure might help to understand the structure of this measure. In [8,14], it was shown how geometric transformations of an iterated function system can be translated to transformations of moment matrices. Our method leads to similar transformations for the associated Hessenberg matrices. Our work is also related to the problem of Bernoulli convolutions [6,13,19].

In the first section of the paper we recall the concepts of self-similar measure and iterated function system (IFS) and some results about moment matrices and Hessenberg matrices that we will need in the paper.

The new methods to calculate Hessenberg matrices introduced in this paper will be presented in Sections 2 and 3. In Section 4 we will illustrate our methods with some numerical experiments.

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### *1.1. Moments and Hessenberg matrices*

Let $\mu(z)$ be a positive measure with compact support $\Omega$ in the complex plane. Let $\mathcal{P}$ be the space of polynomials. Then, there exists a unique orthonormal polynomials sequence (ONPS) $\{P_n(z)\}_{n=0}^{\infty}$ associated with the measure $\mu$ (see [3,10] or [20]). Given two polynomials $Q(z), R(z) \in \mathcal{P}$, the expression

$$\langle Q(z), R(z)\rangle_{\mu} = \int_{\text{Supp}(\mu)} Q(z)\overline{R(z)}\mathrm{d}\mu(z)$$

defines an inner product. Recall that we can define a hermitian moment matrix $M = (c_{jk})_{j,k=0}^{\infty}$, where $c_{jk} = \int_{\Omega} z^j \overline{z}^k \mathrm{d}\mu,\ j,k \in \mathbb{Z}_+ . M$ is the matrix of the inner product in the canonical basis. We denote by $M_n = (c_{j,k})_{j,k=0}^{n-1}$ the $n$th-section of the matrix $M$.

In the space $\mathcal{P}^2(\mu)$, closure of the polynomials space $\mathcal{P}$, we consider the multiplication by $z$ operator. Let $D = (d_{jk})_{j,k=0}^{\infty}$ be the infinite upper Hessenberg matrix of this operator in the basis of ONPS $\{P_n(z)\}_{n=0}^{\infty}$, hence

$$zP_n(z) = \sum_{k=0}^{n+1} d_{k,n} P_k(z), \quad n \geq 0, \tag{1}$$

with $P_0(z) = 1$ when $c_{00} = 1$.

This Hessenberg matrix $D$ is the natural generalization of the tridiagonal Jacobi matrix to the complex plane. The matrices $M$ and $D$ are related by the formula $D = T^H S_R T^{-H}$, where $T$ is the infinite matrix whose $n$th-section is the lower triangular matrix, with real diagonal, obtained from the Cholesky factorization of the $n$th-section $M_n = T_n T_n^H$ of the moment matrix $M$, the superscript $H$ applied to a matrix denotes its conjugate transpose matrix, and $S_R$ is the shift-right matrix which is null everywhere with the exception of a subdiagonal of ones.

For more information on this subject see the books [3,20] by Chihara and Szegö, respectively.

### 1.2. *Self-similar measures*

Given a family $\{\varphi_i\}_{i=1}^m$ of contractive maps defined on a complete metric space, there exists a unique compactum $K$ satisfying $K = \bigcup_{i=1}^m \varphi_i(K)$. This compactum is obtained as a limit in the metric space of compacta with the Hausdorff metric, iterating the maps, taking as initial set any compactum of the space. We call this family $\{\varphi_i\}_{i=1}^m$ an Iterated Functions System (IFS) [2]. In all this work, the maps $\varphi_i$ $(i = 1, \ldots, m)$ are contractive similarities ($\varphi$ is a contractive similarity when $|\varphi(x) - \varphi(y)| = r|x - y|, 0 \le r < 1,$ for all $x, y$) and we will call it an Iterated Functions System of Similarities (IFSS).

If we assign a probability $p_i > 0$ to every $\varphi_i$, with $\sum_{i=1}^m p_i = 1$, there exists a unique probability measure $\mu$ invariant for the Markov operator $T$, defined over the set of Borel regular probability measures as $T\nu = \sum_{i=1}^m p_i \nu\varphi_i^{-1}$. This measure is called the self-similar measure $\mu$ associated with the IFSS with probabilities $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_m; p_1, p_2, \ldots, p_m\}$. If we denote by $K = \mathrm{Supp}(\mu)$ the support of $\mu$, we have that

$$\mu = \sum_{i=1}^m p_i \mu\varphi_i^{-1}, \qquad \int_K f \mathrm{d}\mu = \sum_{i=1}^m p_i \int_K f \circ \varphi_i \mathrm{d}\mu,$$

for any continuous function on $K$. Moreover, if the $\varphi_i(K)$ are disjoints sets, then the measure $\mu$ restricted to each subset $\varphi_i(K)$ is, up to similarity, the same measure [12,15].

For more information on this subject see the books [9,18] by Falconer and Mattila, respectively.

## 2. Hessenberg matrix associated with a sum of measures

Throughout this work, we will consider a family of measures $\{\mu_i\}_{i=1}^m$ with compact support $\Omega_i \subset \mathbb{C}$ and $\mu_i(\Omega_i) = 1$. Let $\mu$ be the sum measure, i.e.,

$$\mathrm{d}\mu = \sum_{i=1}^m p_i \mathrm{d}\mu_i,$$

where $\sum_{i=1}^m p_i = 1$ and $p_i \ge 0$ for all $i = 1, 2, \ldots, m$.

Let $\{D^{(i)}\}_{i=1}^m$ be the associated Hessenberg matrices, and let $D = (d_{jk})_{j,k=0}^\infty$ be the Hessenberg matrix associated with $\mu$. We will give a technique to calculate $D$ in terms of $\{D^{(i)}\}_{i=1}^m$.

First, note that the matrices $D^{(i)}$ are bounded as operators on $\ell^2 = \ell^2(\{0, 1, 2, \ldots\})$ because the support of every $\mu_i$ is compact. Second, remark that every matrix defines a subnormal operator in $\ell^2$ [1,21,23]. These two properties allow us to extend Mantica's spectral techniques [17] to the complex plane.

We will need the following result from spectral theory for subnormal operators on $l^2$, to establish our main result.

**Proposition 1.** *Let the polynomials $\{P_n(z)\}_{n=0}^\infty$ satisfy the recurrence formula* (1). *Let $S$ be a bounded and subnormal operator on $H = \ell^2$. Then $S$ satisfies the identity:*

$$d_{n+1,n} P_{n+1}(S) = (S - d_{nn} I) P_n(S) - \sum_{k=0}^{n-1} d_{k,n} P_k(S), \quad n \ge 0. \tag{2}$$

**Proof.** We can express the recurrence formula (1) as follows

$$d_{n+1,n}P_{n+1}(z) = (z - d_{nn})P_n(z) - \sum_{k=0}^{n-1} d_{k,n}P_k(z), \quad n \geq 0,$$

where, for $n = 0$, the sum $\sum_{k=0}^{-1} d_{k,0}P_k(z)$ is considered to be equal to 0.

Applying functional calculus for a normal operator $N$ we have

$$d_{n+1,n}P_{n+1}(N) = (N - d_{nn})P_n(N) - \sum_{k=0}^{n-1} d_{k,n}P_k(N), \quad n \geq 0. \tag{3}$$

The main idea of the proof is to use the minimal normal extension of $S$, $N = \text{mne}(S)$ [4] (given $S$, there exists a Hilbert space $K \supset H$ and there exists a normal extension $N : K \to K$ such that $N|_H = S$). We can decompose $K = H \bigoplus H^\perp$, and the operator $N$ can be expressed as a $2 \times 2$-block matrix in the following way

$$N = \begin{pmatrix} S & X \\ 0 & Y \end{pmatrix},$$

where the block 0 is due to $N(H) \subset H$ and $N|_H = S$.

Therefore, we have

$$N^j = \begin{pmatrix} S^j & \square \\ 0 & Y^j \end{pmatrix}, \tag{4}$$

where the symbol $\square$ indicates some quantities that are not relevant for our goal. From identity (3), we have

$$d_{n+1,n}P_{n+1}\left(\begin{pmatrix} S & X \\ 0 & Y \end{pmatrix}\right) = \left[\begin{pmatrix} S & X \\ 0 & Y \end{pmatrix} - d_{nn}\begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix}\right] P_n\left(\begin{pmatrix} S & X \\ 0 & Y \end{pmatrix}\right)$$
$$- \sum_{k=0}^{n-1} d_{k,n}P_k\left(\begin{pmatrix} S & X \\ 0 & Y \end{pmatrix}\right),$$

where $I' : H^\perp \to H^\perp$ is the identity operator on the orthogonal complement space of $H$ in $K$. Taking into account (4) we obtain

$$\begin{pmatrix} d_{n+1,n}P_{n+1}(S) & \square \\ 0 & \square \end{pmatrix} = \left[\begin{pmatrix} S - d_{nn}I & \square \\ 0 & \square \end{pmatrix}\right]\begin{pmatrix} P_n(S) & \square \\ 0 & \square \end{pmatrix} - \sum_{k=0}^{n-1}\begin{pmatrix} d_{k,n}P_k(S) & \square \\ 0 & \square \end{pmatrix}.$$

By taking the (1, 1) block entry of this equation, we obtain the desired result. $\square$

**Proposition 2.** *Let $\{\mu_i\}_{i=1}^m$ be a family of measures with compact support on $\mathbb{C}$ and let $\mu$ be the sum measure. Let $\{P_n\}_{n=0}^\infty$ be the associated orthonormal polynomials sequence (ONPS) and let $D = (d_{jk})_{j,k=0}^\infty$ and $\{D^{(i)}\}_{i=1}^m$ be the Hessenberg matrices as above. Then*

$$d_{n+1,n}v_{n+1}^{(i)} = [D^{(i)} - d_{nn}I]v_n^{(i)} - \sum_{k=0}^{n-1} d_{kn}v_k^{(i)}, \quad n \geq 0,\ i = 1, \ldots, m \tag{5}$$

$$d_{kn} = \sum_{i=1}^{m} p_i \langle D^{(i)}v_n^{(i)}, v_k^{(i)}\rangle, \quad k = 0, 1, 2, \ldots, n, \tag{6}$$

*where*

$$v_n^{(i)} = P_n(D^{(i)})e_0, \quad n \geq 0,\ i = 1, \ldots, m \text{ and } e_0 = (1, 0, 0, \ldots)^T$$

*are families of vectors in* $\ell^2$.

**Proof.** Note that the matrices $D^{(i)}$ are bounded in $\ell^2$ because the support of every $\mu_i$ is compact and moreover every matrix defines a subnormal operator on $\ell^2$. As a consequence, every matrix satisfies identity (2). Applying this identity to the vector $e_0$ and taking into account the definition of $v_n^{(i)}$ we obtain (5).

Identity (6) is obtained from the long recurrence formula (1) as follows. If we multiply by $P_k(z)$ in both sides of (1), using the inner product induced by $\mu$, since the polynomials $\{P_n(z)\}_{n=0}^{\infty}$ are orthonormal with respect to this measure, we obtain that

$$d_{kn} = \langle z P_n(z), P_k(z)\rangle_\mu = \int_\Omega \overline{P_k(z)} z P_n(z) \mathrm{d}\mu, \tag{7}$$

where $\Omega = \mathrm{Supp}(\mu)$.

On the other hand, we take the function $\overline{P_k(z)} z P_n(z)$ and we apply the spectral theorem for the minimal normal extension $N^{(i)}$ of $D^{(i)}$. This yields

$$\left[P_k(N^{(i)})\right]^H \left[N^{(i)} P_n(N^{(i)})\right] = \int_{\sigma(N^{(i)})} \overline{P_k(z)} z P_n(z) \mathrm{d}E_{\mu_i},$$

where $\mathrm{d}E_{\mu_i}$ is the spectral measure. Using the matrix expression (4) we obtain

$$\begin{pmatrix} \left[P_k(D^{(i)})\right]^H \left[D^{(i)} P_n(D^{(i)})\right] & \square \\ \square & \square \end{pmatrix} = \int_{\sigma(N^{(i)})} \overline{P_k(z)} z P_n(z) \mathrm{d}E_{\mu_i}.$$

Multiplying by $\widetilde{e_0} = (e_0, \overline{0})$, where $\overline{0}$ is the vector zero of $(\ell^2)^\perp$, and taking into account that $\mathrm{d}\mu_i = \langle \mathrm{d}E_{\mu_i} \widetilde{e_0}, \widetilde{e_0}\rangle = \langle \mathrm{d}E_{\mu_i} e_0, e_0\rangle$, we have

$$\begin{aligned}
\left\langle \left[P_k(D^{(i)})\right]^H \left[D^{(i)} P_n(D^{(i)})\right] e_0, e_0 \right\rangle &= \langle D^{(i)} P_n(D^{(i)}) e_0, P_k(D^{(i)}) e_0\rangle \\
&= \int_{\sigma(N^{(i)})} \overline{P_k(z)} z P_n(z) \langle \mathrm{d}E_{\mu_i} e_0, e_0\rangle \\
&= \int_{\sigma(N^{(i)}) = \Omega_i} \overline{P_k(z)} z P_n(z) \mathrm{d}\mu_i
\end{aligned}$$

and, since for all $i = 1, 2, \ldots, m$,

$$\left\langle D^{(i)} v_n^{(i)}, v_k^{(i)} \right\rangle = \left\langle \left[P_k(D^{(i)})\right]^H \left[D^{(i)} P_n(D^{(i)})\right] e_0, e_0 \right\rangle,$$

we have

$$\begin{aligned}
\sum_{i=1}^m p_i \langle D^{(i)} v_n^{(i)}, v_k^{(i)}\rangle &= \sum_{i=1}^m p_i \int_{\Omega_i} \overline{P_k(z)} z P_n(z) \mathrm{d}\mu_i \\
&= \int_\Omega \overline{P_k(z)} z P_n(z) \left[\sum_{i=1}^m p_i \mathrm{d}\mu_i\right] = d_{kn}. \quad \square
\end{aligned}$$

**Proposition 3.** *In the conditions of Proposition* 2*, the following holds,*

$$d_{n+1,n}^2 = \sum_{i=1}^{m} p_i \ \langle w_{n+1}^{(i)}, w_{n+1}^{(i)} \rangle,$$

*where*

$$w_{n+1}^{(i)} = d_{n+1,n} v_{n+1}^{(i)}, \quad i = 1, 2, \ldots, m,$$

*is a family of vectors in* $\ell^2$.

**Proof.** Since $\langle w_{n+1}^{(i)}, w_{n+1}^{(i)} \rangle = d_{n+1,n}^2 \langle v_{n+1}^{(i)}, v_{n+1}^{(i)} \rangle$, using the same spectral techniques, we have

$$\begin{aligned}\sum_{i=1}^{m} p_i \langle w_{n+1}^{(i)}, w_{n+1}^{(i)} \rangle &= \sum_{i=1}^{m} p_i d_{n+1,n}^2 \langle v_{n+1}^{(i)}, v_{n+1}^{(i)} \rangle \\ &= d_{n+1,n}^2 \sum_{i=1}^{m} p_i \langle P_{n+1}(D^{(i)}) e_0, P_{n+1}(D^{(i)}) e_0 \rangle \\ &= d_{n+1,n}^2 \int_{\Omega} |P_{n+1}(z)|^2 \mathrm{d}\mu = d_{n+1,n}^2. \quad \square\end{aligned}$$

**Theorem 4.** *Let the sum measure* $\mu$*, the ONPS* $\{P_n\}_{n=0}^{\infty}$ *and the Hessenberg matrices* $D$ *and* $\{D^{(i)}\}_{i=1}^{m}$ *be as above. Define the semi-infinite vector* $v_0^{(i)} = (1, 0, 0, \ldots)^T$ *for every* $i = 1, \ldots, m$*. Then the elements of the matrix* $D = (d_{jk})_{j,k=0}^{\infty}$ *associated with* $\mu$ *can be calculated recursively from the matrices* $\{D^{(i)}\}_{i=1}^{m}$ *using the following formulas for* $n = 0, 1, 2, \ldots$.

$$d_{k,n} = \sum_{i=1}^{m} p_i \langle D^{(i)} v_n^{(i)}, v_k^{(i)} \rangle, \quad k = 0, 1, \ldots, n \tag{8}$$

$$w_{n+1}^{(i)} = \left[ D^{(i)} - d_{nn} I \right] v_n^{(i)} - \sum_{k=0}^{n-1} d_{k,n} v_k^{(i)}, \quad i = 1, \ldots, m \tag{9}$$

$$d_{n+1,n} = \sqrt{\sum_{i=1}^{m} p_i \langle w_{n+1}^{(i)}, w_{n+1}^{(i)} \rangle}, \tag{10}$$

$$v_{n+1}^{(i)} = \frac{w_{n+1}^{(i)}}{d_{n+1,n}}, \quad i = 1, \ldots, m. \tag{11}$$

**Proof.** We see first how to obtain $(\{\{v_k^{(i)}\}_{i=1}^{m}\}_{k=0}^{n}, D_{n+1})$ by induction.

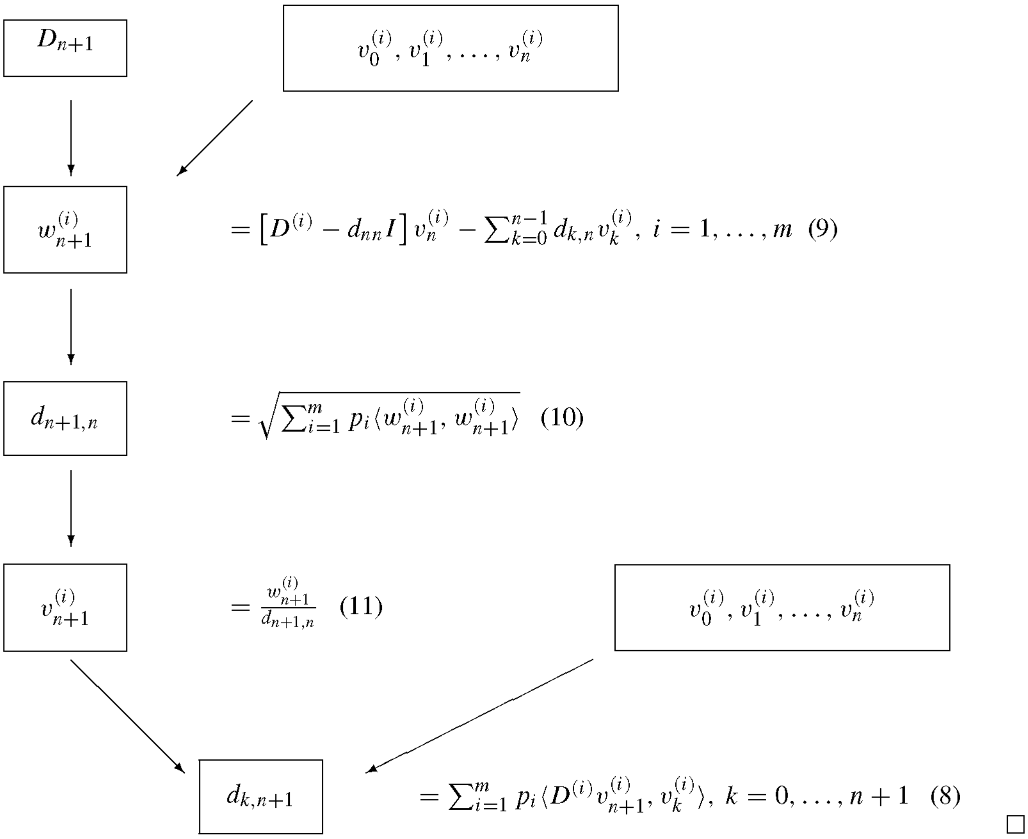
For $n = 0$, we know $(\{v_0^{(i)}\}_{i=1}^{m}, D_1)$:

- $v_0^{(i)} = e_0$ for every $i = 1, \ldots, m$, and
- $D_1 = (d_{00})$ where $d_{00} = \sum_{i=1}^{m} p_i \langle D^{(i)} e_0, e_0 \rangle = \sum_{i=1}^{m} p_i d_{00}^{(i)}$.

Suppose that we know the value of $\left(\left\{v_0^{(i)}, v_1^{(i)}, \dots, v_n^{(i)}\right\}_{i=1}^{m}, D_{n+1}\right)$, i.e., we have

$$\left\{v_0^{(i)}, v_1^{(i)}, \dots, v_n^{(i)}\right\}_{i=1}^{m}, \qquad D_{n+1} = \begin{pmatrix} d_{00} & d_{01} & \dots & d_{0n} \\ d_{10} & d_{11} & \dots & d_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix}.$$

We will show how to obtain $\left(\left\{v_0^{(i)}, v_1^{(i)}, \dots, v_{n+1}^{(i)}\right\}_{i=1}^{m}, D_{n+2}\right)$.




□

Note that the above formulas can be written in a matricial form.

**Corollary 5.** *Let $V^{(i)}$ denote the upper triangular matrix with the vectors $v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \dots,$ of $\ell^2$, as columns, i.e., $V^{(i)} = (v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \dots)$. Then,*

$$D = \sum_{i=1}^{m} p_i \left[V^{(i)}\right]^H D^{(i)} V^{(i)}.$$

## 3. Hessenberg matrix associated with a self-similar measure

To apply the above result to self-similar measures, we will use the following result by Torrano [22] to obtain the Hessenberg matrix of the measure $\mu \circ \varphi^{-1}$ obtained from the transformation of a measure $\mu$ by a similarity function $\varphi$.

**Lemma 6.** *Let D be the Hessenberg matrix associated with a measure. If $D^*$ is the Hessenberg matrix associated with the transformation of this measure by a similarity $\varphi(z) = \alpha z + \beta$, $\alpha, \beta \in \mathbb{C}$, then*

$$D^* = \alpha U^H DU + \beta I,$$

*where, if $\alpha = |\alpha| e^{\theta i}$, then $U = \left(\delta_{jk} e^{(k-1)\theta i}\right)_{j,k=1}^{\infty}$.*

Consider a self-similar measure $\mu$ associated with an iterated functions system of similarities (IFSS) with probabilities $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_m; p_1, p_2, \ldots, p_m\}$. This measure $\mu$ satisfies

$$\mu = \sum_{i=1}^{m} p_i \mu \varphi_i^{-1},$$

i.e., $\mu$ is the sum of the transformations of itself by the similarities $\varphi_i$. Then, applying Corollary 5, we obtain the following result.

**Corollary 7.** *Let $\Phi = \{\varphi_i(z) = \alpha_i z + \beta_i; p_i\}$ be an IFS of similarities with probabilities and let $\mu$ be the corresponding self-similar measure. Then, the Hessenberg matrix D associated with the self-similar measure $\mu$ satisfies the following equation*

$$D = \sum_{i=1}^{m} p_i \, [V^{(i)}]^H \left[\alpha_i [U^{(i)}]^H D U^{(i)} + \beta_i I\right] V^{(i)},$$

*where $U^{(i)}$ and $\alpha_i$ are as in the above lemma and $V^{(i)}$ is as in Corollary 5 for $\mu_i = \mu\varphi_i^{-1}$.*

**Definition 8.** Given $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_m; p_1, p_2, \ldots, p_m\}$ an IFSS with probabilities as above, we define the following transformation in the space of all Hessenberg matrices associated with a measure by

$$\mathcal{T}_\Phi(D_\nu) = \sum_{i=1}^{m} p_i \, [V^{(i)}]^H \left[\alpha_i [U^{(i)}]^H D_\nu U^{(i)} + \beta_i I\right] V^{(i)},$$

where $U^{(i)}$ and $\alpha_i$ are as in the above lemma and $V^{(i)}$ is as in Corollary 5 for $\mu_i = \mu\varphi_i^{-1}$.

**Remark 9.** Note that the transformation $\mathcal{T}_\Phi$ is well defined because for every Hessenberg matrix $D_\nu$ associated with a measure $\nu$, the transformation $\mathcal{T}_\Phi(D_\nu)$ is the Hessenberg matrix of the sum measure $\sum_{i=1}^{m} p_i \nu \varphi_i^{-1}$ as we proved before.

**Theorem 10.** *Let $\Phi = \{\varphi_i(z) = \alpha_i z + \beta_i; p_i\}$ be an IFSS with probabilities, let $\mu$ be the corresponding self-similar measure and let $\mathcal{T}_\Phi$ be as above. Then, for every Hessenberg matrix $D_\nu$ associated with a measure $\nu$, the sequence $\mathcal{T}_\Phi^n(D_\nu)$ converges element by element to the Hessenberg matrix $D_\mu$, where $\mathcal{T}_\Phi^n$ denotes the nth-composition of $\mathcal{T}_\Phi$.*

**Proof.** For every Hessenberg matrix $D_\nu$ associated with a measure $\nu$, the sequence $\mathcal{T}_\Phi^n(D_\nu)$ are the Hessenberg matrices corresponding to the moment matrices of the measures given by the iteration of Markov operator. We proved in an earlier work [8] that this sequence of moment matrices converges to the moment matrix of the self-similar measure $\mu$ (invariant for Markov operator). We can see the convergence in the following diagram

$$\begin{array}{ccccccccc} \nu & \longrightarrow & \mathcal{T}_\Phi(\nu) & \longrightarrow & \mathcal{T}_\Phi^2(\nu) & \cdots & \mathcal{T}_\Phi^n(\nu) & \longrightarrow & \mu \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ M_\nu & \longrightarrow & \mathcal{T}_\Phi(M_\nu) & \longrightarrow & \mathcal{T}_\Phi^2(M_\nu) & \cdots & \mathcal{T}_\Phi^n(M_\nu) & \longrightarrow & M_\mu \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ D_\nu & \longrightarrow & \mathcal{T}_\Phi(D_\nu) & \longrightarrow & \mathcal{T}_\Phi^2(D_\nu) & \cdots & \mathcal{T}_\Phi^n(D_\nu) & \longrightarrow & D_\mu \end{array} \qquad \square$$

The speed of convergence and the numerical stability of the algorithm in Theorem 10 would be addressed in a future work. Nevertheless, the speed of convergence should be at least linear, since the order of convergence of the algorithm in [8], depending on a contractive function, is at least linear. On the other hand, the stability of these computations could be deduced from the stability proved in [17] of Mantica's algorithm. These facts are observed in the experimental results showed in the examples in the next section.

**Remark 11.** The above theorem allows to obtain approximate values of the sections of the Hessenberg matrix of a self-similar measure. On the other hand, the recurrent formula for the moments of self-similar measures given in [16] (later generalized in [7] to measures with support in the complex plane) allow to obtain, in an exact way, the moments of self-similar measures. Then, using Cholesky factorization we can obtain the $n$th-section of the desired Hessenberg matrix.

Even though the latter method allows to obtain the exact value of the sections of the Hessenberg matrix, to obtain the exact value, it must work symbolically and therefore it has a high computational cost.

As an illustration, we show how to obtain the first polynomials of the Cantor measure $\mu_C$ on the Cantor set $C$ in the interval $[-1, 1]$. This measure is self-similar for the following IFSS $\Phi = \left\{\varphi_1(z) = \frac{1}{3}z - \frac{2}{3}, \varphi_2(z) = \frac{1}{3}z + \frac{2}{3};\ p_i = \frac{1}{2}\right\}$. Using the recurrent formula in [7],

$$c_{i,j} = \frac{1}{1 - \sum\limits_{s=1}^{k} p_s \alpha_s^i \overline{\alpha_s}^j} \sum_{s=1}^{k} p_s \sum_{m=0,l=0(m,l)\neq(i,j)}^{i,j} \binom{i}{m}\binom{j}{l} \beta_s^{i-m} \overline{\beta_s}^{j-l} \alpha_s^m \overline{\alpha_s}^l c_{m,l},$$

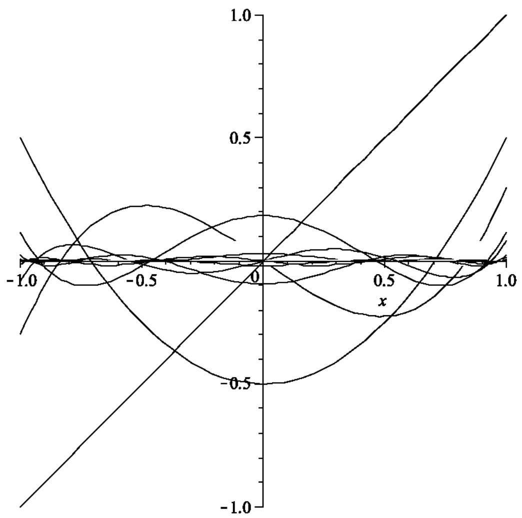
to compute the moments of an invariant measure for an IFSS with probabilities $\Phi = \{\varphi_i = \alpha_i z + \beta_i;\ p_i\}_{i=1}^k$, we may obtain any moment $s_{i+j} = c_{i,j}$ of the measure $\mu_C$:

$$s_0 = 1, s_1 = 0, s_2 = \frac{1}{2}, s_3 = 0, s_4 = \frac{7}{20}, s_5 = 0, s_6 = \frac{205}{728}, s_7 = 0, s_8 = \frac{10241}{42640}.$$

Using the expression for monic polynomials $\tilde{P}_n(z) = \det(M_n^{-1} M_n' - z I_n)$, we can obtain in an exact way the first monic orthogonal polynomials for Cantor measure

$$x,$$
$$x^2 - \frac{1}{2},$$
$$x^3 - \frac{7}{10}x,$$

$$x^4 - \frac{97}{91}x^2 + \frac{333}{1820},$$
$$x^5 - \frac{1785}{1517}x^3 + \frac{143833}{552188}x,$$
$$x^6 - \frac{189964505}{112825966}x^4 + \frac{7410073867}{9251729212}x^2 - \frac{156207248595}{1683814716584},$$
$$x^7 - \frac{4548711144551}{2534028699430}x^5 + \frac{6972489245973139}{7481466332197132}x^3 - \frac{4855955749246420947}{39876215550610713560}x.$$




## 4. Conclusions and examples

In Theorem 4 we have shown a method to obtain exactly finite sections of the Hessenberg or Jacobi matrices associated with a sum of measures with compact support in $\mathbb{C}$ or $\mathbb{R}$, respectively. We can apply this method to every measure given by the Markov operator to approximate Hessenberg or Jacobi matrices associated with self-similar measures (Theorem 10). We will call this method Algorithm II.

On the other hand, we will call Algorithm I the iterative process for moment matrices of self-similar measures described in [8] applying then Cholesky factorization to obtain an approximation of the Jacobi or Hessenberg matrix.

We will apply these two algorithms (with ten digits of precision) to four examples of self-similar measures. We will use different number of iterations in each case, obtaining different degrees of approximation: for instance, in the first two examples we compute 30 iterations while in the third example we only compute 7, due to the lack of symmetry, which increases the computational cost.

**Example I.** Let $\mathcal{L}$ be the normalized Lebesgue measure in the interval $[-1, 1]$. This is a self-similar measure for the IFSS

$$\Phi = \left\{\varphi_1(x) = \frac{1}{2}x - \frac{1}{2}, \varphi_2(x) = \frac{1}{2}x + \frac{1}{2};\ p_1 = p_2 = \frac{1}{2}\right\}.$$

*Algorithm I.* If we iterate the transformation $\mathcal{T}_\Phi(M_\nu) = \sum_{i=1}^{2} \frac{1}{2} A_{\varphi_i}^H M_\nu A_{\varphi_i}$ [8], 30 times starting with the sixth order identity matrix, we obtain the following approximation of the 6th-section moment matrix for the Lebesgue measure

$$\begin{pmatrix} 1.0 & 0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 \\ 0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 \\ 0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 \\ 0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 \\ 0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 \\ 0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 & 0.09090909 \end{pmatrix}.$$

This matrix agrees (with ten digits of precision) with the 6th order moment matrix $M_{\mathcal{L}}$. Then, applying Cholesky factorization, we have the following approximation of the 5th-section of Jacobi matrix $J_{\mathcal{L},5}$

$$\begin{pmatrix} 0.0 & 0.5773502693 & 0.0 & 0.0 & 0.0 \\ 0.5773502691 & 0.0 & 0.5163977795 & 0.0 & -0.7577722133 \cdot 10^{-9} \\ 0.0 & 0.5163977796 & 0.0 & 0.5070925551 & 0.0 \\ 0.3023715782 \cdot 10^{-9} & 0.0 & 0.5070925521 & 0.0 & 0.5039526136 \\ 0.0 & -0.2639315569 \cdot 10^{-8} & 0.0 & 0.5039526419 & 0.0 \end{pmatrix}.$$

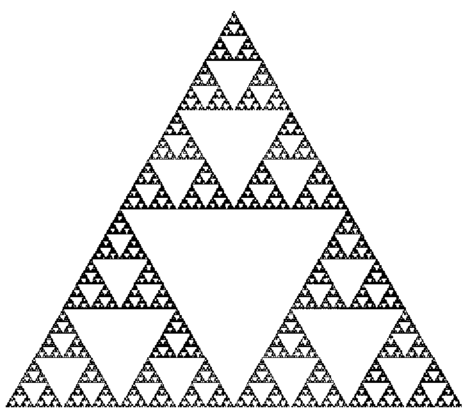
*Algorithm II.* Starting with the 5th-section of the shift right matrix, and making 30 iterations of the transformation $\mathcal{T}_\Phi(D) = \sum_{i=1}^{m} p_i \; [V^{(i)}]^H \left[\alpha_i [U^{(i)}]^H D U^{(i)} + \beta_i I\right] V^{(i)}$ in Theorem 10, we obtain the following matrix

$$\begin{pmatrix} 0.0 & 0.5773502692 & 0.0 & -0.2133333332 \cdot 10^{-9} & 0.0 \\ 0.5773502691 & 0.0 & 0.5163977796 & 0 & -0.1 \cdot 10^{-9} \\ 0.0 & 0.5163977796 & 0.0 & 0.5070925526 & 0.0 \\ 0.0 & 0.0 & 0.5070925529 & 0.0 & 0.5039526304 \\ 0.0 & 0.0 & 0.0 & 0.5039526307 & 0.0 \end{pmatrix}.$$

These two matrices agree with 6 and 9 digits of precision respectively, with the 5th order Jacobi matrix of this measure whose diagonal is null and the sub- and superdiagonal are given by $d_{n+1,n} = d_{n,n+1} = \frac{n+1}{\sqrt{4(n+1)^2-1}}$.

**Example II.** Let $T$ be the Sierpinski triangle with basis on the $[-1, 1]$ interval.

Consider the uniform measure $\mu$, i.e., the $\frac{\log 3}{\log 2}$-dimensional Hausdorff measure on $T$



This is a self-similar measure for the IFSS given by

$$\Phi = \left\{ \varphi_1(z) = \frac{z}{2} - \frac{1}{2}, \varphi_2(z) = \frac{z}{2} + \frac{1}{2}, \varphi_3(z) = \frac{z}{2} + \frac{1\sqrt{3}i}{2}; \; p_i = \frac{1}{3} \right\}.$$

*Algorithm I.* With 30 iterations, starting with the identity matrix, we obtain an approximation of the 4th-section of the Hessenberg matrix of the measure:

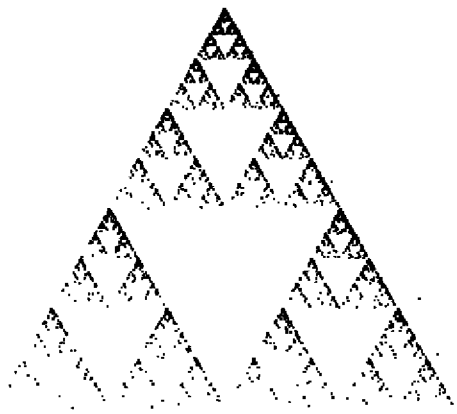
$$\begin{pmatrix} 0.5773502693i & 0.3 \cdot 10^{-9} & -0.4182428890i & -0.2457739408 \cdot 10^{-8} \\ 0.6666666673 & 0.5773502691i & 0.1267731382 \cdot 10^{-8} & -0.3487499915i \\ 0 & 0.7888106373 & 0.5773502706i & 0.1292460659 \cdot 10^{-8} \\ -0.406877 \cdot 10^{-9} & 0.279363 \cdot 10^{-9}i & 0.7737179471 & 0.5773502588i \end{pmatrix}.$$

*Algorithm II.* Starting with the 4th-section of the shift right matrix, and making 20 iterations of $\mathcal{T}_\Phi(D)$ we obtain the following matrix

$$\begin{pmatrix} 0.5773497186i & -0.2 \cdot 10^{-9} & -0.4182428884i & -1.3 \cdot 10^{-10} + 2.09 \cdot 10^{-43}i \\ 0.6666666668 & 0.5773497190i & 10^{-9} - 5.2 \cdot 10^{-43}i & 3.2 \cdot 10^{-42} - 0.3487499858i \\ 0 & 0.7888106377 & -1.5 \cdot 10^{-42} + 0.5773497186i & -3.3 \cdot 10^{-10} - 10^{-41}i \\ 0 & 0 & 0.7737179434 - 3.1 \cdot 10^{-54}i & -1.4 \cdot 10^{-41} + 0.5773497189i \end{pmatrix}.$$

These two matrices agree with 8 digits of precision (Algorithm I) and 6 digits of precision (Algorithm II) with the 4th order Jacobi matrix of this measure.

**Example III.** Let $T$ be the Sierpinski triangle as above. Consider the invariant measure for the same IFSS with probabilities $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{5}$, $p_3 = \frac{7}{10}$



*Algorithm I.* Applying $\mathcal{T}_\Phi$ 7 times starting with the identity matrix we obtain an approximation of the 4th-section of the Hessenberg matrix of the measure $\mu$:

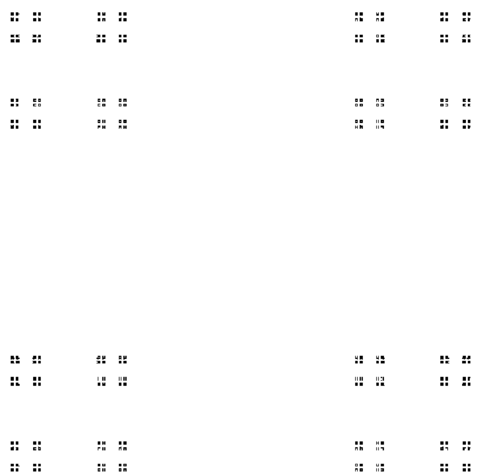
$$\begin{pmatrix} 0.0992 + 1.2029i & -0.2046 - 0.1459i & -1.799 \cdot 10^{-6} - 0.3176i & -0.0123 + 0.0555i \\ 0.5538 + 1.3 \cdot 10^{-10}i & 0.1439 + 0.8415i & 0.0208 - 0.0718i & -0.0396 - 0.3027i \\ 5.68 \cdot 10^{-10} + 1.7 \cdot 10^{-21}i & 0.6848 + 5.36 \cdot 10^{-10}i & 0.0390 + 0.7027i & 0.0117 - 0.0461i \\ 5.39 \cdot 10^{-9} + 8.09 \cdot 10^{-10}i & 7.12 \cdot 10^{-9} - 2.64 \cdot 10^{-10}i & 0.7116 - 2.39 \cdot 10^{-10}i & 0.07365 + 0.6745i \end{pmatrix}.$$

*Algorithm II.* Starting with the 4th-section of the shift right matrix, and making 7 iterations of $\mathcal{T}_\Phi(D)$ we obtain the following matrix

$$\begin{pmatrix} 0.099218+1.202963\mathrm{i} & -0.204629-0.145941\mathrm{i} & -0.0000179-0.317680\mathrm{i} & -0.012314+0.055542\mathrm{i} \\ 0.5538131313 & 0.143933+0.841541\mathrm{i} & 0.020889-0.0718614\mathrm{i} & -0.039695-0.302772\mathrm{i} \\ 0 & 0.684812+2.05958\cdot 10^{-12}\mathrm{i} & 0.0390029+0.702786\mathrm{i} & 0.011747-0.046155\mathrm{i} \\ 0 & 0 & 0.711680+1.54964\cdot 10^{-12}\mathrm{i} & 0.0736565+0.674541\mathrm{i} \end{pmatrix}.$$

In this case the precision is worse for both algorithms. It seems that it is due to the lack of symmetry of this measure, because the probabilities are different for every similarity.

**Example IV.** Let $C$ be the plane Cantor set.



Consider the uniform measure $\mu$ on this set.

This measure is self-similar for the following IFSS

$$\Phi = \left\{\varphi_1(z) = \frac{1}{4}z + \frac{1+\mathrm{i}}{2},\ \varphi_2(z) = \frac{1}{4}z + \frac{1-\mathrm{i}}{2},\right.$$
$$\left.\varphi_3(z) = \frac{1}{4}z + \frac{-1+\mathrm{i}}{2},\ \varphi_4(z) = \frac{1}{4}z + \frac{-1-\mathrm{i}}{2};\ p_i = \frac{1}{4}\right\}$$

*Algorithm I.* Applying $\mathcal{T}_\Phi$ 10 times starting with the identity matrix we obtain an approximation of the 5th-section of the Hessenberg matrix of $\mu$:

$$\begin{pmatrix} 0 & 0 & 0 & -0.5534617900 & 0 \\ 0.7302967432 & 0 & 0 & 0 & -0.1728136409 \\ 0 & 0.7720611578 & 0 & 0 & 0 \\ 0 & 0 & 0.8042685429 & 0 & 0 \\ 0 & 0 & 0 & 0.6168489579 & 0 \end{pmatrix}.$$

*Algorithm II.* Starting with the 5th-section of the shift right matrix, and making 10 iterations of $\mathcal{T}_\Phi(D)$ we obtain the following matrix

$$\begin{pmatrix} 0.0+0.0i & 0.0+0.0i & 0.0+0.0i & -0.5534617900+0.0i & 0.0+0.0i \\ 0.7302967435 & 0.0+0.0i & 0.0+0.0i & 0.0+0.0i & -0.1728136412+0.0i \\ 0.0 & 0.7720611574 & 0.0+0.0i & 0.0+0.0i & 4.0\times 10^{-11}+0.0i \\ 0.0 & 0.0 & 0.8042685430 & 0.0+0.0i & 0.0+0.0i \\ 0.0 & 0.0 & 0.0 & 0.6168489588 & 0.0+0.0i \end{pmatrix}.$$

These two matrices agree with 8 digits of precision with the 5th order Jacobi matrix of this measure.

Note that both algorithms work for self-similar measures and allow to approximate the Hessenberg or Jacobi matrix associated with such measures with similar results. Using any of these methods we can approximate the first elements of the orthogonal polynomials sequence.

In the following two examples we consider sums of measures which are not self-similar. The first example corresponds to a sum of shifts and the second one to the sum of Chebyshev polynomials on different intervals. These examples serve to illustrate the algorithm in Theorem 4.

**Example V.** Consider $\mu_1$ the normalized Lebesgue measure on the unit circle and $\mu_2$ the normalized Lebesgue measure on the circle of center (0, 0) and radius $r$. Then

$$D^{(1)} = S_R \quad \text{and} \quad D^{(2)} = rS_R.$$

Applying the algorithm in Theorem 4 to the sum measure,

$$\mu = (1-p)\mu_1 + p\mu_2,$$

we obtain the Hessenberg matrix $D = (d_{jk})_{j,k=0}^{\infty}$, which turns out to be a shift matrix whose only non-trivial entries are

$$d_{n+1,n} = \sqrt{\frac{1-p+p\,r^{2(n+1)}}{1-p+p\,r^{2n}}}, \qquad n = 0, 1, 2, 3 \ldots.$$

In this case the auxiliary vectors which we construct in the process are

$$v_n^{(1)} = \frac{1}{\sqrt{1-p+p\,r^{2n}}}\, e_n, \qquad v_n^{(2)} = \frac{r^n}{\sqrt{1-p+p\,r^{2n}}}\, e_n, \qquad n = 0, 1, 2, \ldots$$

where $\{e_k\}_{k=0}^{\infty}$ are the vectors of the canonical basis of $\ell^2$.

**Example VI.** Let $a \in \mathbb{R}$. Consider the Chebyshev polynomial of the second kind corresponding to the distributions $\mathrm{d}\mu_1 = \sqrt{1-(x-a)^2}dx$ and $\mathrm{d}\mu_2 = \sqrt{1-(x+a)^2}dx$ on the intervals $[a-1, a+1]$ and $[-a-1, -a+1]$, respectively. Let $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ be the sum measure. We can construct the corresponding Jacobi matrix.

The 6th order Jacobi matrices are

$$D^{(1)} = \begin{pmatrix} a & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & a & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & a & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & a & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & a & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & a \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} -a & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -a & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -a & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -a & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -a & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -a \end{pmatrix}.$$

The 5th-section of the Jacobi tridiagonal of the sum measure will be

$$\begin{pmatrix} 0 & \frac{\sqrt{4a^2+1}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{4a^2+1}}{2} & 0 & \frac{\sqrt{16a^2+1}}{2\sqrt{4a^2+1}} & 0 & 0 \\ 0 & \frac{\sqrt{16a^2+1}}{2\sqrt{4a^2+1}} & 0 & \frac{\sqrt{256a^6+80a^4+44a^2+1}}{2\sqrt{16a^2+1}\sqrt{4a^2+1}} & 0 \\ 0 & 0 & \frac{\sqrt{256a^6+80a^4+44a^2+1}}{2\sqrt{16a^2+1}\sqrt{4a^2+1}} & 0 & \frac{\sqrt{16384a^8+6144a^6+896a^4+100a^2+1}}{2\sqrt{16a^2+1}\sqrt{256a^6+80a^4+44a^2+1}} \\ 0 & 0 & 0 & \frac{\sqrt{16384a^8+6144a^6+896a^4+100a^2+1}}{2\sqrt{16a^2+1}\sqrt{256a^6+80a^4+44a^2+1}} & 0 \end{pmatrix}.$$

Note that the supports of the initial measures are disjoint intervals when the parameter $a > 1$; when $a = 0$ we have the Chebyshev polynomials in the interval $[-1, 1]$, when $0 \le a < 1$ we have a sum of measures with overlapping supports.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## References


[1] A. Atzmon, A moment problem for positive measures on the unit disc, Pacific J. Math. 59 (1975).
[2] M. Barnsley, Fractals Everywhere, Academic Press, Boston, 1988.
[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, in: Mathematics and its Applications, vol. 13, Gordon and Breach, New York, 1978.
[4] John B. Conway, The Theory of Subnormal Operators, in: Mathematical Surveys and Monographs, vol. 36, Rhode Island, Providence, 1985.
[5] S. Elhay, G.H. Golub, J. Kautsky, Jacobi matrices for sums of weight functions, BIT 32 (1992) 143–166.
[6] C. Escribano, M.A. Sastre, E. Torrano, Moments of infinite convolutions of symmetric Bernoulli distributions, J. Comput. Appl. Math. 153 (2003) 191–199.
[7] C. Escribano, M.A. Sastre, E. Torrano, Moment matrix of self-similar measures, Electron. Trans. Numer. Anal. 24 (2006) 79–87.
[8] C. Escribano, M.A. Sastre, E. Torrano, A fixed point theorem for moment matrices of self-similar measures, J. Comput. Appl. Math. 207 (2007) 352–359.
[9] K.J. Falconer, Fractal Geometry, John Wiley and Sons, New York, 1990.
[10] G. Freud, Orthogonal Polynomials, Consultants Bureau, New York, 1961.
[11] W. Gautschi, On generating orthogonal polynomials, SIAM J. Sci. Statist. Comput. 3 (1982) 289–317.
[12] J. Hutchinson, Fractal and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[13] B. Jessen, A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. 38 (1935) 48–88.
[14] P.E.T. Jorgensen, K.A. Kornelson, K.L. Shuman, Iterated Function Systems, Moments, and Transformations of Infinite Matrices, arXiv:0809.2124v1 [math.CA].
[15] B.B. Mandelbrot, The Fractal Geometry of Nature, W. H. Freeman & Co., San Francisco, 1977.

[16] G. Mantica, A stable technique for computing orthogonal polynomials and Jacobi matrices associated with a class of singular measures, Constr. Approx. 12 (1996) 509–530.
[17] G. Mantica, On computing Jacobi matrices associated with recurrent and Möbius iterated function systems, J. Comput. Appl. Math. 115 (2000) 419–431.
[18] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, 1995.
[19] P. Solomyak, H. Xu, On the Mandelbrot set for a pair of linear maps and complex Bernoulli convolutions, Nonlinearity 16 (2003) 1733–1749.
[20] G. Szegö, Orthogonal Polynomials, in: AMS Colloquium Publications, vol. 32, American Mathematical Society, Providence, RI, 1975.
[21] E. Torrano, R. Guadalupe, On the moment problem in the bounded case, J. Comput. Appl. Math. 49 (1993).
[22] E. Torrano, Interpretación matricial de los polinomios ortogonales en el campo complejo, Tesis doctoral, Universidad de Cantabria, Santander, 1987.
[23] V. Tomeo, Subnormalidad de la Hessenberg matrix associated a los P.O. en el caso hermitiano, Tesis doctoral, UPM 2003.