The logarithmic spiral, autoisoptic curve

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ABSTRACT
In the Line of Investigation that in the department of “Technical Drawing” in the School of Agriculture Engineering of Madrid, we carry out on the study of The Technical Curves and his singularities, we demonstrate an interesting property of the Logarithmic Spiral.

The demonstrated property consists of which the logarithmic spiral is a autoisoptic curve, that is to say that if from a point \( P \) anyone of the spiral tangent straight lines draw up to the previous arc, these form a constant angle \( \alpha \). This demonstration is novel and in addition we get to contribute a method to calculate the angle \( \alpha \) given the equation of the spiral.

1. Introduction

We name Isoptic of a given curve, to the geometric place of the points from which one sees the curve one under constant angle. That is to say: both tangent straight lines we draw to the curve from any point of the isoptic, form always the same angle.

It’s immediate to see, for example, that the isoptic of a circumference is another circumference. When the angle is a right angle the curve names ortoptic. We know that the ortoptic curve of a conic is a circumference.

It is not difficult to demonstrate that the isoptic curve of two circumferences is a Spiral of Pascal.

¿Are there auto-isoptic curves? In this paper we will see that,

1. The Logarithmic Spiral \( \rho = e^{\alpha} \) is autoisoptic.

2. We will give a method to calculate the isoptic angle \( \alpha \) depending on \( a \).
2. The logarithmic spiral is autoisoptic

We have \( a \), a point of the logarithmic spiral \( \rho = e^\alpha \) (Figure 1), corresponding to the value \( \omega = \omega_i \). The tangent in \( A_1 \) will form with the vector radio \( OA_1 \), a angle \( V \) given for the next equation

\[
tgV = \frac{\rho}{\rho'} = \frac{1}{La} = cte
\]  

(1)

If \( \alpha \) is an given constant angle, and we have \( A_2 \), the point in the curve corresponding to \( \omega = \omega_1 + \alpha \). Then we will have:

\[
OA_2 = a^{\omega_i + \alpha} = OA_1 \cdot a^\alpha
\]

(2)

And, analogously, the tangent in \( A_2 \) will form with the vector radio \( OA_2 \), the same angle \( V \). \( T_i \) is the point of intersection of the tangents to the spiral in \( A_i \) and \( A_2 \).

In the point \( A_i \), corresponding to \( \omega_i + \alpha + \alpha \) we'll have:

\[
OA_i = OA_1 \cdot a^\alpha
\]

(3)

And we'll name \( T_i \) to the point of intersection of the tangents in \( A_i \) and \( A_2 \).

Is evidente that the quadrangles \( OA_i T_i A_1 \) and \( OA_2 T_i A_2 \) are similar (both quadrangles have the equal angles and two proportional sides) and the angle \( A_i T_i A_1 \) is equal to the angle \( A_2 T_2 A_1 \), and equal, both, to \( 180^\circ - \alpha \).

But, besides,

\[
T_i OT_2 = \alpha
\]

(4)

and

\[
OT_2 = a^\alpha \cdot OT_1
\]

(5)

That is to say: \( T_i \) and \( T_2 \) describe a spiral of the form \( \rho = b \alpha \) too. The value of \( b \) depend, of course, of the value \( \alpha \). Therefore, if we vary the angle \( \alpha \), always there will be some value where \( T_i, T_2, \ldots \) will be PRECISELY in \( \rho = a^\alpha \), consequently this curve is AUTOISOPTIC.

3. Calculation of the autoisoptic angle of the logarithmic spiral

As we have demonstrated, the tangents from any point of the logarithmic spiral \( \rho = e^\alpha \) to the next spire, form a constant angle (Figure 2).

If \( \omega = 0, \rho = a^0 = 1 \). As the equation (1) is correct, the equation of the tangent to the curve in the point \( A(1,0) \) is:

\[
y = \frac{x - 1}{La}
\]

(6)

Now let's find the point of intersection \( P \) of this tangent with the next spire:

\[
\rho = a^\alpha
\]

(7)

\[
yLa = x - 1
\]

(8)

That is to say, \( \rho \cdot sen\omega La = \rho \cdot cos\omega - 1 \)
If we replace the value $\phi$ in the previous equation, we will obtain:

$$a^\omega \cdot \text{sen} \cdot La = a^\omega \cdot \cos - 1$$

(9)

Therefore:

$$a^\omega = \frac{1}{\cos - La \cdot \text{sen}}$$

(10)

As we want the following spire and $\cos(\omega + 2\pi) = \cos \omega$ and $\text{sen}(\omega + 2\pi) = \text{sen} \omega$, the result will be:

$$a^{\omega + 2\pi} = \frac{1}{\cos - La \cdot \text{sen}}$$

(11)

Then:

$$\omega = \frac{1}{\cos - La \cdot \text{sen}} - 2\pi$$

(12)

We can watch that the previous equation is like $\omega = f(\omega).$ If we try solve it by iteration, that is to say obtaining from an approximate value $\omega = \omega_1,$ $\omega_2 = f(\omega_1)$; ... Then it result that the succession $\omega_1, \omega_2, \omega_3, ...$ diverges. In this case we have to write the equation like that: $\omega = f^{-1}(\omega) = \phi(\omega)$ and now the succession $\omega_1, \omega_2 = \phi(\omega_1), \omega_3 = \phi(\omega_2)$ ... certainly converges.

If we write:

$$a^{\omega + 2\pi} = \frac{1}{\cos(\omega) + 1} \frac{\cos - La \cdot \text{sen}}{\cos(\omega) + 1} = \frac{1}{\sqrt{(La)^2 + 1}} \frac{\cos - La \cdot \text{sen}}{\sqrt{(La)^2 + 1}}$$

(13)

Being:

$$\text{sent} = \frac{1}{\sqrt{(La)^2 + 1}}$$

(14)

and

$$\cos t = \frac{La}{\sqrt{(La)^2 + 1}}$$

(15)

Then:

$$a^{\omega + 2\pi} = \frac{1}{\sqrt{(La)^2 + 1}} \frac{\cos - La \cdot \text{sen}}{\sqrt{(La)^2 + 1}}$$

(16)

And

$$\text{sent}(t - \omega) = \frac{1}{a^{\omega + 2\pi}} \sqrt{(La)^2 + 1}$$

(17)

Therefore:

$$\omega = t - \arcsen \frac{1}{a^{\omega + 2\pi}} \sqrt{(La)^2 + 1}$$

(18)

If we solve this equation by iteration we obtain $\omega_1.$ And $\rho = a^{\omega + 2\pi}.$ That is to say:

$$x_p = \rho \cdot \cos \omega_p$$

(19)

$$y_p = \rho \cdot \text{sen} \omega_p$$

(20)

These equations give us the point $A(x_p, y_p).$

Now we are going to calculate the value of the autoisoptic angle $\alpha.$ If $M$ is the corresponding point to $\omega = \pi - \alpha$, the tangent straight line in $M$ will pass for $p$ and then we will have:

$$\frac{y_p - y_M}{x_p - x_M} = \tan(V - \alpha)$$

(21)

$$\frac{y_p - a^{\omega \cdot \pi} \cdot \cos(\pi - \alpha)}{x_p - a^{\omega \cdot \pi} \cdot \text{sen}(\pi - \alpha)} = \tan(V - \alpha)$$

(22)

Consequently:

$$a = V - \arctan \frac{x_p - a^{\omega \cdot \pi} \cdot \text{sen} \alpha}{y_p - a^{\omega \cdot \pi} \cdot \cos \alpha}$$

(23)

Equation that is solved by iteration.

4. Deduction of the auto-ortoptic logarithmic spiral

We have $\rho = a^\omega$ (Figure 3). For $\omega = \frac{\pi}{2} \Rightarrow \rho = a^{\frac{\pi}{2}}.$

Figure 3. Graphical base for the deduction of the Auto-ortoptic Logarithmic Spiral

If $\tan V = \frac{\rho}{\rho'} = \frac{1}{La},$ then the tangent in $(1,0)$ is:

$$y = \frac{1}{La}(x - 1)$$

and the tangent in $(0, a^{\frac{\omega}{2}})$ is:

$$y - a^{\frac{\omega}{2}} = -La \cdot x$$

Let’s find the point of intersection $P$ of the tangent straight lines:

$$\frac{1}{La}(x - 1) = a^{\frac{\omega}{2}} - La \cdot x$$

(24)

$$x - 1 = a^{\frac{\omega}{2}} La - (La)^2 x$$

(25)
\[ x[1+(La)^2] = \frac{a}{a^2 - La + 1} \quad (26) \]

\[ x = \frac{\frac{a}{a^2 - La + 1}}{1 + (La)^2} \quad (27) \]

\[ y = \frac{1}{La} \left( \frac{\frac{a}{a^2 - La + 1} - 1}{1 + (La)^2} \right) = \frac{\frac{a}{a^2 - La + 1} - 1}{1 + (La)^2} \quad (28) \]

Consequently the value for \( y \) is:

\[ \frac{\frac{a}{a^2 - La + 1}}{1 + (La)^2} \quad (29) \]

Then, from the equations (27) and (28) we can deduce:

\[ x^2 + y^2 = \left( \frac{\frac{a}{a^2 - La + 1}}{1 + (La)^2} \right)^2 + \left( \frac{\frac{a}{a^2 - La}}{1 + (La)^2} \right)^2 = \frac{\alpha La + 2a^2 La + 1 - a^2 - 2a^2 La + (La)^2}{1 + (La)^2} = \frac{a^2[1 + (La)^2] + 1 + (La)^2}{1 + (La)^2} = \frac{a^2 + 1}{1 + (La)^2} \quad (30) \]

Therefore:

\[ \sqrt{x^2 + y^2} = \rho = \sqrt{a^2 + 1} \quad (31) \]

On the other hand:

\[ \tan \alpha = \frac{y}{x} = \frac{\frac{a}{a^2 - La}}{\frac{a}{a^2 - La} + 1} \quad (32) \]

\[ \alpha = \arctg \frac{\frac{a}{a^2 - La}}{\frac{a}{a^2 - La} + 1} + 2\pi \quad (33) \]

(This if it is the following spire)

It will be necessary to verify, to obtain the auto-ortóptica, the following thing: \( \rho - a^* = 0 \)

For \( \alpha = 1.05 \) we have \( \rho - a^* > 0 \) (we save the calculations)

For \( \alpha = 1.06 \) we have \( \rho - a^* < 0 \)

If we write:

\[ La = \frac{Lp}{\alpha} = \frac{\sqrt{a^2 + 1} \sqrt{1 + (La)^2}}{a^2 - La + 2 \alpha \sqrt{a^2 - La} + 2 \alpha} \quad (34) \]

For iteration we obtain (for example from \( \alpha = 1.05 \)) with the calculator: \( a = 1.056879 \).

That is to say, \( \rho = 1.056879^\circ \) is AUTO-ORTOPTIC.

5. Conclusions

1. The Logarithmic Spiral \( \rho = a^* \) is autoisoptic.

2. We have given a method to calculate the isoptic angle \( \alpha \) depending on \( \rho \).

3. We have calculated \( a \) for the spiral auto-ortóptico (\( \alpha = 90^\circ \)).

4. The content of this paper we have not seen it in the existing bibliography relating to logarithmic spirals.

References


