Finite Temperature Effects in Free-Electron-Lasers Amplifiers

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Introduction

The Free-Electron Laser (1),(2), exhibits novel important features such as tuneability in a range of wavelengths not covered by atomic lasers. These characteristics make it a prime candidate for numerous applications. The experiments consist of a relativistic electron beam passing through certain types of devices (such as helical wigglers or an undulating wall) which induce a perpendicular velocity on the electrons, thus allowing an interaction with the transverse components of an electromagnetic wave. The most commonly used type of undulator by far is a helical wiggler and this will be the one assumed in this work unless otherwise stated. Different configurations such as oscillators or amplifiers are possible.

The first theoretical analyses of the single particle Free-Electron -Laser were performed within a quantum mechanical framework. The quantization of the fields is not an essential feature of the Free-Electron-Lasers as demonstrated in (3) where non-linear differential equations relating the phase and the amplitude of the fields were derived, much in the same way as in non-linear optics problems, and the same parametric dependence as in the quantum mechanical analysis was obtained.

Hasegawa (4) has interpreted the stimulated Raman and Compton regimes for the Free-Electron-Laser instability as three and two wave scattering respectively in the beam frame. Sprangle (5) has formulated the problem by using a non-linear wave equation in the laboratory frame and considering limiting cases according to how the ponderomotive compared to the space charge potential.

Bernstein and Hirshfield (6) incorporated new features in the analysis. They included equilibrium potentials which were exact to all orders of the pump except for the radial dependence and used a new set of basis vectors which simplified the problem. Furthermore, by recalling the Traveling-Wave Tube (16), which although obeying different physics possesses the feature of the propagation of several modes while only one grows exponentially, they formulated a boundary-value problem, thus enabling them to calculate the coupling loss. Their treatment is capable of including kinetic effects although they did not pursue this aspect far. Later analysis by them included a guide magnetic field with a cold fluid model.

The presence of the guide magnetic field poses new problems: not only do the electron orbits exhibit new features but also new resonances may arise. The stability of

the different types of orbits induced by the guide magnetic field with a wiggler having no radial dependence was studied by Friedland (7). In general, in a Free-Electron-Laser, the gain increases with the transverse component of the electron velocity. It is then natural to consider substantial values of the transverse electron velocity. However, as pointed out by Diamant (8), this results in substantial departures from the axis and radial corrections to the wiggler potentials have to be incorporated.

The physics of the Free-Electron-Laser near magnetoresonance has been the subject of recent controversy. McMullin and Davidson (11) have shown two competing effects: the Cyclotron Maser and the Free-Electron -Laser instability. However, one consequence of magnetoresonance is that the electrons exhibit large component of the perpendicular velocity, and as it was pointed out above radial corrections to the wiggler have to be included.

At present, the role of theoretical research in Free-Electron- Laser appears as follows:

(i) On the tenuous and high energy beam side (Compton), better efficiency enhancement mechanisms, such as tapered wigglers.

(ii) On the dense, low energy beam situation (Raman), a

better understanding of the effect of energy spread and the effect of the guide magnetic field.

Beam quality is a critical parameter in the operation of a dense beam Free-Electron-Laser. If the mediating plasma wave Landau damps, the gain is severely reduced. Within the Bernstein and Hirshfield framework, the Raman regime would arise whenever two of the roots of the dispersion relation have imaginary parts, and the Compton regime mechanism would be represented by spatial interference of the different modes. In practice both situations may arise and compete with each other.

Our analysis starts by introducing a distribution function in the perturbed current from (6). The beam is assumed to be infinite in cross section and a wiggler with no radial dependence is considered. This approximation is justified for small pump parameters (8). The momentum integrals are analytically continued by adding the residues from the Landau type resonant terms. The spread in the distribution function alters the coupling of the negative energy plasma wave to one of the electromagnetic modes. The numerical results show a mismatch between both coupled waves which causes a decrease in the gain together with a broadening and a shift of the spectrum towards lower frequencies. The modes

hermitian character for real entries of the frequency and wave vector and the presence of new resonances which explain the stability properties of the equilibrium orbits. A comparison is made between their results and ours. In a later article Friedland and Fruchtman (10) rederived a new dispersion tensor featuring this natural response electron frequency. However, their dispersion tensor is again asymmetric and their constant upper limit in frequencies is only the result of ignoring the perturbed transverse current in Maxwell equations. The rest of the features are common in our and their calculations: the guide magnetic field results in an enhancement of the ponderomotive potential and therefore in an increase in gain and broadening of the spectrum for the electrons executing orbits with small transverse velocities. However for electron executing orbits with large perpendicular velocity components, their results showing a very broad spectrum and very large gain are questionable since for this range of perpendicular velocities, radial corrections to the wiggler potential should have been incorporated.

The stability of the orbits as studied by Friedland (7) was presented in terms of a certain critical guide magnetic field. An equivalent formulation is to consider for a

certain configuration of wiggler and guide magnetic fields, a minimum energy in the momentum integrals (12). When inverting the Laplace transform using the method of analytic continuation by deformation of contour, branch points are encountered and consequently branch cut contributions have to be added to the usual contribution from the poles. By examining the dispersion tensor the leading contributions to these branch cuts can be evaluated. For zero pump parameter, they represent two trains of waves which for the case of a beam with small energy spread, central momentum close to the minimum momentum and operation close to magnetoresonance can be approximated to yield the two cyclotron modes.

2. Theory

2.1 Formalism

2.1.1 Fluid Model

A self-consistent, cold fluid, strong pump formalism for the Free- Electron-Laser was developed by Bernstein and Friedland (9). A brief description of this formalism follows. The main purpose here is that their dispersion tensor was uncorrectly derived; it lacks fundamental physical properties such as some symmetry properties and the presence of resonances which explain the stability of the orbits.

Consider a relativistic electron beam governed by the following equations:

(i) Continuity Equation

$$\frac{\partial N}{\partial t} + \nabla \cdot (N \vec{v}) = 0 \quad (2.1.1)$$

(ii) Momentum Equation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) (m \gamma \vec{v}) = -|e| \left(\vec{E} + \vec{v} \times \frac{\vec{B}}{c} \right) \quad (2.1.2)$$

(iii) Energy Equation

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) (m c^2 \gamma) = -|e| \vec{E} \cdot \vec{v} \quad (2.1.3)$$

Assuming a beam of infinite cross section, the Maxwell equations are:

$$\frac{\partial^2 \vec{A}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi N |e|}{c} \vec{v}_\perp, \quad (2.1.4)$$

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi N |e|. \quad (2.1.5)$$

Consider an equilibrium magnetic field of the form

$$\vec{B} = B_0 \hat{e}_z + B_w (\hat{e}_x \cos k_0 z + \hat{e}_y \sin k_0 z). \quad (2.1.6)$$

Introduce a new set of basis vectors

$$\hat{e}_1 = -\hat{e}_x \sin k_0 z + \hat{e}_y \cos k_0 z$$

$$\hat{e}_2 = -\hat{e}_x \cos k_0 z - \hat{e}_y \sin k_0 z \quad (2.1.7)$$

$$\hat{e}_3 = \hat{e}_z$$

By using these coordinates, a possible solution to (2.1.1) to (2.1.5) is :

$$\vec{V}_0 = V_{03} \hat{e}_3 + V_{01} \hat{e}_2$$

$$N_0 = \text{const.}$$

$$V_{01} = \frac{K_0 c V_{03} \tilde{\gamma}_0}{K_0 V_{03} \tilde{\gamma}_0 - \Omega} \quad (2.1.8)$$

$$\text{where } \tilde{\gamma}_0 = \frac{|e| B_0}{m c K_0} \text{ and } \Omega = \frac{|e| B_0}{m c}$$

After linearizing the equations, one obtains the following dispersion tensor, which differs from that of Bernstein and Hirshfield and which can be written in the form

$$\tilde{\epsilon} = \tilde{\Theta} + \Omega \tilde{\Psi} \quad (2.1.9)$$

where

$$\Theta_{11} = 1 - \frac{(K_0^2 + K^2) c^2}{\omega^2} - \frac{\omega_p^2}{\gamma_0 \omega^2},$$

$$\Theta_{22} = 1 - \frac{(K_0^2 + K^2) c^2}{\omega^2} - \frac{\omega_p^2}{\gamma_0 \omega^2} \left[1 + \frac{v_{01}^2}{c^2} \frac{(K^2 c^2 - \omega^2)}{(K v_{03} - \omega)^2} \right],$$

$$\Theta_{33} = 1 - \frac{\omega_p^2}{\gamma_0 (K v_{03} - \omega)^2} \left(1 - \frac{v_{03}^2}{c^2} \right),$$

$$\Theta_{12} = -\Theta_{21} = -\frac{2i K_0 K c^2}{\omega^2},$$

$$\Theta_{13} = -\Theta_{31} = 0,$$

$$\Theta_{23} = \Theta_{32} = -\frac{\omega_p^2}{\gamma_0 \omega^2} \frac{v_{01}}{v_{03}} \omega \frac{(K v_{03} - \omega v_{03}^2/c^2)}{(K v_{03} - \omega)^2},$$

$$\Psi_{11} = \frac{\omega_p^2}{\gamma_0^3 \omega^2} \frac{[\gamma_0 K_0 v_{03} - \omega (1 + v_{01}^2/v_{03}^2)]}{[(K v_{03} - \omega)^2 - \omega_0^2]},$$

$$\Psi_{22} = \frac{\omega_p^2}{\gamma_0^3 \omega^2} \frac{[K v_{03} (1 + v_{01}^2/v_{03}^2) - \omega] (\gamma_0 K_0 v_{03} - \omega)}{(K v_{03} - \omega)^2 [(K v_{03} - \omega)^2 - \omega_0^2]},$$

$$\Psi_{33} = \frac{\omega_p^2}{\gamma_0^3 (K v_{03} - \omega)^2} \frac{v_{01}^2}{v_{03}^2} \frac{(\gamma_0 K_0 v_{03} - \omega)}{[(K v_{03} - \omega)^2 - \omega_0^2]},$$

$$\Psi_{12} = -\Psi_{21} = \frac{-i \omega_p^2}{\gamma_0^2 \omega^2} \frac{[K v_{03} (1 + v_{01}^2/v_{03}^2) - \omega]}{[(K v_{03} - \omega)^2 - \omega_0^2]},$$

$$\Psi_{13} = -\Psi_{31} = \frac{-i \omega_p^2}{\gamma_0^2 \omega^2} \frac{v_{01}}{v_{03}} \frac{\omega}{[(K v_{03} - \omega)^2 - \omega_0^2]},$$

$$\Psi_{23} = \Psi_{32} = \frac{\omega_p^2}{\gamma_0^3 \omega} \frac{v_{01}}{v_{03}} \frac{[K v_{03} (1 + v_{01}^2/v_{03}^2) - \omega] [\gamma_0 K_0 v_{03} - \omega]}{(K v_{03} - \omega)^2 [(K v_{03} - \omega)^2 - \omega_0^2]},$$

where $\omega_p^2 = 4\pi N_0 e^2 / m$.

This dispersion tensor is hermitian for real values of the frequency and wave vector. Note also the characteristic frequency ω_0 appearing in the denominators of the entries in

$\tilde{\Psi}$ and defined by

$$\omega_0^2 = \frac{1}{\gamma_0^2} (c K_0 \gamma_0 v_{03} - \Omega) \left[c K_0 \gamma_0 v_{03} - \Omega \left(1 + \frac{v_{01}^2}{v_{03}^2} \right) \right]$$

2.1.2. Kinetic Theory without a Guide Magnetic Field

The next step in order of difficulty when studying the Free-Electron-Laser is to start with the relativistic Vlasov equation and to analyze the effect of the energy spread without a guide magnetic field. The formalism with a guide magnetic field will be presented in a later section. However, the kinetic effects associated with the guide magnetic field are important only in conditions close to magnetoresonance. The competing processes of three-wave coupling and Landau damping and the coupling loss mechanism can be studied without considering a guide magnetic field. The basic features of Bernstein and Hirshfield's analysis up to the point where they calculate the perturbed current will be outlined. At this point our analysis will be shown in detail in order to derive the dispersion relation and solve the boundary-value problem.

Assume an equilibrium field composed of a helical wiggler with no radial dependence and without guide magnetic field:

$$\vec{B}_0 = - \frac{B_w}{K_0} (\hat{x} \cos K_0 z + \hat{y} \sin K_0 z) \quad (2.1.2.1)$$

where B_w is the wiggler field strength and K_0 is the wiggler field wave number.

Use the first-order linearized Vlasov equation

$$\frac{\partial f^{(1)}}{\partial t} + \dot{z}_0 \frac{\partial f^{(1)}}{\partial z} = |e| \frac{\vec{E}^{(1)} \cdot \vec{u}_0}{mcu} \frac{\partial f^{(0)}}{\partial u}, \quad (2.1.2.2)$$

where the superscript (1) represents the first-order perturbed quantities, $\dot{z}_0$ is the zeroeth -order axial velocity, $\vec{u}_0 = \frac{c}{\gamma} \dot{z}_0$ and $u = |\vec{u}|$, together with Maxwell equations

$$\frac{\partial^2 \vec{A}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{J}_\perp, \quad (2.1.2.3)$$

$$\frac{\partial^2 \Phi}{\partial z \partial t} = 4\pi J_3, \quad (2.1.2.4)$$

where $\vec{A}$ and Φ are the vector and scalar potentials and $\vec{J}_\perp$ and J_3 are the perpendicular and parallel components of the current respectively.

Assume that all the perturbed quantities vary harmonically with angular frequency ω ,

$$\vec{E}^{(1)}(z, t) = \text{Re } \vec{a}(z) e^{-i\omega t}, \quad f^{(1)}(z, t) = \text{Re } F(z) e^{-i\omega t}, \quad (2.1.2.5)$$

The current density $\vec{J}$ is

$$\vec{J} = -|e| \int d^3u \vec{u} \frac{cf}{\gamma} = -|e| \int d\alpha d\beta du \frac{\vec{u} cfu}{\gamma u_{z0}} \quad (2.1.2.6)$$

where α and β denote

$$\alpha = u_x - |e| A_x / mc^2, \quad \beta = u_y - |e| A_y / mc^2,$$

From (6), the first-order part for a distribution function of the form

$$f^{(1)} = n_0 \delta(\alpha) \delta(\beta) g(u), \quad (2.1.2.7)$$

$$F(\alpha, \beta, u, 0) = n_0 \delta(\alpha) \delta(\beta) G(u)$$

is then

$$\begin{aligned} \vec{j}^{(1)} = & -|e| \int du \left\{ -i \left(\frac{|e| u g}{m \omega \gamma u_{03}} \right) \vec{a}_1 - i \left(\frac{|e| u^2 g}{m \omega \gamma u_{03}^3} \right) \hat{e}_2 \hat{e}_2 \cdot \vec{a}_1 \right. \\ & + \left(\frac{\gamma_0 \hat{e}_2}{u_{03}} + \hat{e}_3 \right) \exp\left(\frac{i \omega \gamma z}{u_{03} c}\right) \left[\frac{\gamma}{u_{03}} + \right. \\ & \left. \left. \frac{|e|}{mc} \frac{dg}{du} \int dz' \vec{a}_1(z') \cdot \left(\frac{\gamma_0 \vec{a}_2}{u_{03}} + \hat{e}_3 \right) \cdot \exp\left(\frac{-i \omega \gamma z'}{u_{03} c}\right) \right] \right\} \end{aligned} \quad (2.1.2.8)$$

Taking $\hat{e}_1^0, \hat{e}_2$, and $\hat{e}_3$ components of (2.1.2.3) and (2.1.2.4)

gives

$$\frac{d^2 a_1}{dz^2} + \left(\frac{\omega^2}{c^2} - k_0^2 \right) a_1 - 2k_0 \frac{da_2}{dz} =$$

$$a_1 \frac{\omega_0^2}{c^2} \int du \frac{u}{\gamma u_{03}} g \quad (2.1.2.9)$$

$$\begin{aligned}
& \frac{d^2 a_2}{dz^2} + \left(\frac{\omega^2}{c^2} - k_0^2 \right) a_2 + 2 K_0 \frac{da_1}{dz} \\
& = a_2 \frac{\omega_p^2}{c^2} \int du \frac{u}{\gamma u_{03}} g \left(1 + \frac{v_{01}^2}{v_{03}^2} \right) + \\
& \frac{i\omega}{c} \frac{\omega_p^2}{c^2} \int du \frac{v_{01}}{v_{03}} \frac{dg}{du} \int_0^z dz' \exp \left[\frac{i\omega}{v_{03}} (z - z') \right] \cdot \left[a_3(z') + \frac{v_{01}}{v_{03}} a_2(z') \right] \\
& + \frac{i\omega}{c} 4\pi n_0 |e| \int du \frac{u}{\gamma} \frac{v_{01}}{v_{03}} G \exp \left(\frac{i\omega}{v_{03}} z \right) ,
\end{aligned}$$

(2.1.2.10)

$$a_3 = \frac{i\omega}{c} \frac{\omega_p^2}{\omega^2} \int du \frac{dg}{du} \int_0^z dz' \exp \left[\frac{i\omega}{v_{03}} (z - z') \right] \cdot \left[a_3 + \frac{v_{01}}{v_{03}} a_2 \right]$$

$$\frac{i\omega}{c} 4\pi n_0 |e| \int du \frac{u}{\gamma} G \exp \left(\frac{i\omega}{v_{03}} z \right) . \quad (2.1.2.11)$$

Equations (2.1.2.9), (2.1.2.10) and (2.1.2.11) constitute a system of integro-differential equations of the convolution type. They can be solved by the standard method of Laplace inversion:

$$\vec{\tilde{a}}(k) = \int_0^\infty dz e^{-ikz} \vec{a}(z) \quad (2.1.2.12)$$

The transformed equations are:

$$-k^2 \tilde{a}_1 + \left(\frac{\omega^2}{c^2} - k_0^2 \right) \tilde{a}_1 - 2ik k_0 \tilde{a}_2 - \tilde{a}_1 \frac{\omega_p^2}{c^2} \int du \frac{u}{\gamma u_{03}} = a_1'(0) + ik a_1(0) - 2k_0 a_2(0), \quad (2.1.2.13)$$

$$-k^2 \tilde{a}_2 + \left(\frac{\omega^2}{c^2} - k_0^2 \right) \tilde{a}_2 + 2ik k_0 \tilde{a}_1 - \tilde{a}_2 \frac{\omega_p^2}{c^2} \int du \frac{u}{\gamma u_{03}} g\left(1 + \frac{v_{01}^2}{v_{03}^2}\right) - \frac{\omega}{c} \frac{\omega_p^2}{c^2} \int du \frac{v_{01}}{v_{03}} \frac{dg}{du} \frac{v_{03}}{kv_{03} - \omega} \left(\tilde{a}_3 + \frac{v_{01}}{v_{03}} \tilde{a}_2 \right) = \frac{\omega}{c} 4\pi m_0 |e| \int du \frac{u}{\gamma} \frac{v_{01}}{v_{03}} G \frac{v_{03}}{kv_{03} - \omega}$$

$$+ a_2'(0) + ik a_2(0) + 2k_0 a_1(0), \quad (2.1.2.14)$$

$$\tilde{a}_3 - \frac{\omega}{c} \frac{\omega_p^2}{\omega^2} \int du \frac{dg}{du} \frac{v_{03}}{kv_{03} - \omega} \left(\tilde{a}_3 + \frac{v_{01}}{v_{03}} \tilde{a}_2 \right) =$$

$$\frac{c}{\omega} 4\pi m_0 |e| \int du \frac{u}{\gamma} G \frac{v_{03}}{kv_{03} - \omega} \quad (2.1.2.15)$$

This system can be written symbolically in the form

$$\tilde{\epsilon}(\mathbf{k}) \cdot \tilde{\vec{\alpha}}(\mathbf{k}) = \vec{\mathbf{I}}(\mathbf{k}) \quad (2.1.2.16)$$

With the normalizations: $x = \frac{kc}{\omega}$, $x_0 = \frac{k_0 c}{\omega}$, $\omega_p' = \frac{\omega_p}{\omega}$, the dispersion tensor becomes

$$\epsilon_{11} = 1 - (x_0^2 + x^2) - \omega_p'^2 \int du \frac{u}{\gamma u_{03}} g, \quad ,$$

$$\epsilon_{12} = -\epsilon_{21} = -2ix_0 x, \quad \epsilon_{13} = \epsilon_{31} = 0, \quad ,$$

$$\epsilon_{22} = 1 - (x_0^2 + x^2) - \omega_p'^2 \int du \frac{u}{\gamma u_{03}} g \left(1 + \frac{z_0^2}{u_{03}^2} \right) + \omega_p'^2 \int du \frac{z_0^2}{u_{03}} \frac{dg}{du} \frac{1}{\gamma - x u_{03}}, \quad ,$$

$$\epsilon_{33} = 1 + \omega_p'^2 \int du \frac{dg_0}{du} \frac{u_{03}}{\gamma - x u_{03}}, \quad ,$$

$$\epsilon_{23} = \epsilon_{32} = \omega_p'^2 z_0 \int du \frac{dg}{du} \frac{1}{\gamma - x u_{03}}$$

where

$$u_{03} = \sqrt{u^2 - z_0^2}.$$

2.1.3 Kinetic theory with a guide magnetic field

The formalism for the kinetic theory with a guide magnetic field was developed in (14). What follows is a brief summary.

Consider the relativistic Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{c}{\gamma} \vec{u} \cdot \frac{\partial f}{\partial \vec{p}} - \frac{|e|\hbar}{m} \left(\vec{E} + \frac{c}{\gamma} \vec{u} \times \vec{B} \right) \cdot \frac{\partial f}{\partial \vec{u}} = 0 \quad (2.1.3.1)$$

where $\vec{u} = \gamma \frac{\vec{p}}{c}$ is the dimensionless momentum, $\gamma = (1 + u^2)^{1/2}$ is the relativistic factor and m is the electron rest mass.

Consider an equilibrium field composed of a helical wiggler without radial dependence of strength B_w and wavenumber K_0 , and a guide magnetic field B_0 . The equilibrium vector potential is then

$$\vec{A}_0 = \frac{B_0}{2} (x \hat{y} - y \hat{x}) - \frac{B_w}{K_0} (x \cos K_0 z + y \sin K_0 z) \quad (2.1.3.2)$$

Two constants of motion are

$$\vec{\alpha}_\perp = \vec{u}_\perp + \vec{\beta}_\perp(z, t) - \frac{e}{c} \hat{z} \times \vec{r}_\perp \quad (2.1.3.3)$$

where $\vec{\beta}_\perp = -\frac{|e|\hbar}{mc^2} \vec{A}_\perp$

The change of coordinates (u_x, u_y, u_z) to $(u, \vec{\alpha}_\perp)$ then simplifies the Vlasov equation, which becomes

$$\frac{\partial f}{\partial t} + \frac{cu_z}{\gamma} \frac{\partial f}{\partial z} - \frac{|e|}{mc} \vec{E} \cdot \frac{\vec{u}}{u} \frac{\partial f}{\partial u} = 0 \quad (2.1.3.4)$$

The next step is to calculate the perturbed current,

$$\begin{aligned} \vec{j}(z, t) &= -|e|c \int d^3u \frac{\vec{u}}{\gamma} f = \\ &= -|e|c \int du \int d^2\alpha_\perp \mathcal{J} \frac{\vec{u}}{\gamma} f, \end{aligned} \quad (2.1.3.5)$$

where $\mathcal{J}$ is the Jacobian of the variable transformation.

Linearization of (2.1.3.5) yields

$$\begin{aligned} \vec{j}^{(1)} &= -n_0 |e|c \int du \int d^2\alpha_\perp \frac{u}{\gamma} \left\{ \left(\frac{f^{(1)}}{n_0} + g \cdot \mathcal{J}^{(1)} \right) \right. \\ &\quad \left. \left(\hat{z} + \frac{v_{0\perp}}{v_{0z}} \hat{e}_2 \right) + g_0 \left(\frac{\vec{u}_\perp^{(1)}}{u_{0z}} - \frac{u_z^{(1)}}{u_{0z}^2} \vec{u}_{0\perp} \right) \right\}, \end{aligned} \quad (2.1.3.6)$$

where

$$\vec{u}_\perp^{(1)} = \frac{-i|e|}{mc\omega} \vec{a}(z) + \frac{\omega}{c} \hat{z} \times \vec{\rho}^{(12)}(u, z) \quad (2.1.3.7)$$

$$u_3^{(1)} = - \frac{v_{01}}{v_{03}} \hat{e}_2 \cdot \vec{u}_1^{(1)}, \quad (2.1.3.8)$$

and $\vec{f}^{(1)}$ is the first order perturbed Jacobian.

The vector $\vec{f}^{(1)}$, from the equations of motion, has components

$$\begin{aligned} f_1(z, \vec{u}) = & \frac{1}{i\omega} \frac{|e|}{mc^2} \frac{1}{\omega_0 u_{03}} \int_0^z dz' \sin \left[\frac{\omega_0}{v_{03}} (z - z') \right] \cdot \\ & \exp \left[\frac{i\omega}{v_{03}} (z - z') \right] \times \left\{ (-i\omega + v_{03} \frac{d}{dz'}) c a_1(z') + \right. \\ & + \frac{1}{\gamma} (c K_0 u_{03} - \omega) \left[c \left(1 + \frac{v_{01}^2}{v_{03}^2} \right) a_2(z') + \right. \\ & + v_{01} \frac{c^2}{v_{03}^2} \frac{i\omega}{c} \int_0^{z'} dz'' \exp \left[\frac{i\omega}{v_{03}} (z' - z'') \right] \cdot \\ & \left. \left[a_3(z'') + \frac{v_{01}}{v_{03}} a_2(z'') \right] \right\}, \end{aligned} \quad (2.1.3.9)$$

$$\begin{aligned} f_2(z, \vec{u}) = & \frac{1}{i\omega} \frac{|e|}{mc} \frac{1}{(K_0 c u_{03} - \omega)} \left\{ -c a_1(z) + \right. \\ & \left. \frac{1}{v_{03} \omega_0} (-i\omega + v_{03} \frac{d}{dz}) \int_0^z dz' \sin \left[\frac{\omega_0}{v_{03}} (z - z') \right] \cdot \right. \end{aligned} \quad (2.1.3.10)$$

$$\begin{aligned} & \exp \left[\frac{i\omega}{v_{03}} (z - z') \right] \left[(-i\omega + v_{03} \frac{d}{dz'}) c a_1(z') + \right. \\ & \left. \frac{1}{\gamma} (c K_0 u_{03} - \omega) \left(1 + \frac{v_{01}^2}{v_{03}^2} \right) c a_2(z') \right] \end{aligned}$$

$$+ \frac{1}{\gamma} (c k_0 n_{o3} - \Omega) v_{o1} \frac{c^2}{v_{o3}^2} \frac{i\omega}{c},$$

$$\cdot \int_0^{\gamma} dz' \exp \left[\frac{i\omega}{v_{o3}} (z - z') \right] \left[a_3(z') + \frac{v_{o1}}{v_{o3}} a_2(z') \right] \Bigg\},$$

$$\omega_o^2 = \frac{1}{\gamma^2} (c k_0 n_{o3} - \Omega) \left[c k_0 n_{o3} - \Omega \left(1 + \frac{v_{o1}^2}{v_{o3}^2} \right) \right], \quad (2.1.3.11)$$

$$\frac{1}{2} = n_o g_o.$$

This is the same frequency which appeared before in the course of the cold fluid calculation.

Substituting all the above into Maxwell equations,

$$\frac{d^2 \vec{a}_1}{dz^2} + \frac{\omega^2}{c^2} \vec{a}_1 = - \frac{4\pi i \omega}{c^2} \vec{j}_1^{(1)}, \quad (2.1.3.12)$$

$$a_3 = - \frac{4\pi i}{\omega} j_3^{(1)}, \quad (2.1.3.13)$$

and taking $\hat{e}_1, \hat{e}_2$, and $\hat{z}$ components, there results the system of three equations

$$\frac{d^2 a_1}{dz^2} + \left(\frac{\omega^2}{c^2} - k_o^2 \right) a_1 - 2k_o \frac{da_2}{dz} =$$

$$a_1 \frac{\omega_p^2}{c^2} \int du \int d^2 \alpha_1 \frac{u}{\gamma u_{o3}} g_o -$$

$$- i 4\pi n_{o1} \frac{\omega \Omega}{c^2} \int du \int d^2 \alpha_1 \frac{u}{\gamma u_{o3}} g_o f_2(z), \quad (2.1.3.14)$$

$$\begin{aligned}
& \frac{d^2 a_2}{dz^2} + \left(\frac{\omega^2}{c^2} - K_0^2 \right) a_2 + 2 K_0 \frac{da_1}{dz} = \\
& a_2 \frac{\omega_p^2}{c^2} \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma u_{03}} g_0 \left(1 + \frac{v_{0\perp}^2}{v_{03}^2} \right) + \\
& \cdot i 4\pi n_0 |e| \frac{\omega}{c^2} \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma u_{03}} g_0 \left(1 + \frac{v_{0\perp}^2}{v_{03}^2} \right) f_{\perp}(z) \\
& + \frac{i\omega}{c} \frac{\omega_p^2}{c^2} \int du \int d^2 \alpha_{\perp} \frac{v_{0\perp}}{v_{03}} \frac{\partial g_0}{\partial u} \int_0^z dz' \exp \left[\frac{i\omega}{v_{03}} (z-z') \right] \cdot \\
& \cdot \left[a_3(z') + \frac{v_{0\perp}}{v_{03}} a_2(z') \right] + \\
& i 4\pi n_0 |e| \frac{\omega}{c} \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma} g_0 \frac{v_{0\perp}}{v_{03}} g^{(2)}(z) \quad (2.1.3.15) \\
& + \frac{i\omega}{c} 4\pi n_0 |e| \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma} \frac{v_{0\perp}}{v_{03}} g^{(1)}(z=0) \exp \left[\frac{i\omega}{v_{03}} z \right],
\end{aligned}$$

$$\begin{aligned}
a_3 &= \frac{i\omega}{c} \frac{\omega_p^2}{\omega^2} \int du \int d^2 \alpha_{\perp} \frac{\partial g_0}{\partial u} \int_0^z dz' \exp \left[\frac{i\omega}{v_{03}} (z-z') \right] \cdot \\
& \left[a_3(z') + \frac{v_{0\perp}}{v_{03}} a_2(z') \right] + \\
& \frac{i}{\omega} 4\pi n_0 |e| c \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma} g_0 g^{(1)}(z) + \\
& + \frac{ic}{\omega} 4\pi n_0 |e| \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma} g^{(1)}(z=0) \exp \left[\frac{i\omega}{v_{03}} z \right]
\end{aligned} \quad (2.1.3.16)$$

This system of integro differential equations can be solved by the standard method of Laplace transformation:

$$\tilde{a}(K) = \int_0^\infty dz e^{-iKz} a(z) \quad (2.1.3.17)$$

After Laplace transforming, one obtains

$$\begin{aligned} & -K^2 \tilde{a}_1 + \left(\frac{\omega^2}{c^2} - K_0^2\right) \tilde{a}_1 - 2iK \cdot K \tilde{a}_2 - \\ & \tilde{a}_1 \frac{\omega_p^2}{c^2} \int du \int d^2\alpha_1 \frac{u}{\gamma u_{03}} g_0 + \\ & i4\pi n_0 |e| \frac{\omega \Omega}{c^2} \int du \int d^2\alpha_1 \frac{u}{\gamma u_{03}} g_0 \tilde{p}_2 = \\ & a_1(0) + iK a_1(0) - 2K_0 a_2(0) , \end{aligned} \quad (2.1.3.18)$$

$$\begin{aligned} & -K^2 \tilde{a}_2 + \left(\frac{\omega^2}{c^2} - K_0^2\right) \tilde{a}_2 + 2iK \cdot K \tilde{a}_1 - \\ & \tilde{a}_2 \frac{\omega_p^2}{c^2} \int du \int d^2\alpha_1 \frac{u}{\gamma u_{03}} g_0 \left(1 + \frac{v_{01}^2}{v_{03}^2}\right) - \\ & -i4\pi n_0 |e| \frac{\omega \Omega}{c^2} \int du \int d^2\alpha_1 \frac{u}{\gamma u_{03}} g_0 \tilde{p}_1 \frac{[K v_{03} (1 + v_{01}^2/v_{03}^2) - \omega]}{(K v_{03} - \omega)} \end{aligned}$$

$$\begin{aligned}
& - \frac{\omega}{c} \frac{\omega_p^2}{c^2} \int du \int d^2 \alpha_{\perp} \frac{v_{0\perp}}{v_{03}} \frac{\partial g_0}{\partial u} \frac{v_{03}}{K v_{03} - \omega} \cdot \left(\tilde{a}_3 + \frac{v_{0\perp}}{v_{03}} \tilde{a}_2 \right) \\
& = \frac{\omega}{c} 4\pi n_0 |e| \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma} \frac{v_{0\perp}}{v_{03}} g^{(1)}(\beta=0) \frac{v_{03}}{K v_{03} - \omega} \\
& + a'_2(0) + i K a_2(0) + 2 K_0 a_2(0) ,
\end{aligned}$$

$$\begin{aligned}
& \tilde{a}_3 - \frac{\omega}{c} \frac{\omega_p^2}{\omega^2} \int du \int d^2 \alpha_{\perp} \frac{\partial g_0}{\partial u} \frac{v_{03}}{K v_{03} - \omega} \left(\tilde{a}_3 + \frac{v_{0\perp}}{v_{03}} \tilde{a}_2 \right) \quad (2.1.3.19) \\
& - 4\pi i n_0 |e| \omega \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma v_{03}} \frac{v_{0\perp}}{v_{03}} g_0 \tilde{p}_1 \frac{1}{(K v_{03} - \omega)} \\
& = \frac{c}{\omega} 4\pi n_0 |e| \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma} g^{(1)}(\beta=0) \frac{v_{03}}{K v_{03} - \omega}
\end{aligned}$$

(2.1.3.20)

where use has been made of the relation

$$a_j^{(1)} = \frac{v_{0\perp}}{v_{03}} \frac{\omega}{(kv_{03} - \omega)} \frac{\tilde{p}_1}{\gamma v_{03}} \quad (2.1.3.21)$$

After Laplace transforming (2.1.3.9) and (2.1.3.10), one obtains

$$\begin{aligned} \tilde{f}_2(K, \vec{u}) = & \frac{1}{i\omega} \frac{|e|}{m\gamma} \frac{1}{[(kv_{03} - \omega)^2 - \omega_0^2]} \cdot \\ & \left\{ -i(kv_{03} - \omega) \tilde{a}_1 + \omega \frac{v_{0\perp}}{v_{03}} \frac{(cK_0 u_{03} - \omega)}{\gamma(\omega - kv_{03})} \tilde{a}_3 \right. \\ & \left. - \frac{1}{\gamma} (cK_0 u_{03} - \omega) \frac{[kv_{03}(1 + v_{0\perp}^2/v_{03}^2) - \omega]}{(kv_{03} - \omega)} \tilde{a}_2 \right\}, \end{aligned}$$

(2.1.3.22)

$$\begin{aligned} \tilde{p}_2(k, \vec{a}) &= \frac{1}{i\omega} \frac{|e|}{m\gamma} \frac{1}{[(kv_0 - \omega)^2 - \omega_0^2]} \cdot \\ &\left\{ \frac{\gamma \omega_0^2}{(ck_0 u_{03} - \Omega)} \tilde{a}_1 - i\alpha \frac{v_{01}}{v_{03}} \tilde{a}_3 - \right. \\ &\left. i \left[kv_0 \left(1 + \frac{v_{01}^2}{v_{03}^2} \right) - \omega \right] \tilde{a}_2 \right\}. \end{aligned} \quad (2.1.3.23)$$

The system of equations can now be written

$$\tilde{\epsilon}(k) \cdot \tilde{\vec{a}}(k) = \tilde{\vec{I}}(k)$$

where

$$\begin{aligned} \epsilon_{11} &= 1 - \frac{(k_0^2 + k^2)c^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} \int d\alpha \int d^2\alpha_{\perp} \frac{u}{\gamma u_{03}} g_0 + \\ &\frac{\omega_p^2}{\omega^2} \Omega \int d\alpha \int d^2\alpha_{\perp} \frac{1}{\gamma^2} \frac{u}{u_{03}} g_0 \frac{[kv_0 - \Omega/\gamma (1 + v_{01}^2/v_{03}^2)]}{[(kv_0 - \omega)^2 - \omega_0^2]}, \\ \epsilon_{22} &= 1 - \frac{(k_0^2 + k^2)c^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} \int d\alpha \int d^2\alpha_{\perp} \frac{u}{\gamma u_{03}} \left(1 + \frac{v_{01}^2}{v_{03}^2} \right) \\ &+ \frac{\omega}{kc} \frac{\omega_p^2}{\omega^2} \int d\alpha \int d^2\alpha_{\perp} \frac{v_{01}^2}{v_{03}^2} \frac{\partial g_0}{\partial u} \frac{kv_0}{\omega - kv_0} \\ &+ \frac{\omega_p^2}{\omega^2} \Omega \int d\alpha \int d^2\alpha_{\perp} \frac{u}{\gamma^3 u_{03}} g_0 \frac{[kv_0 (1 + v_{01}^2/v_{03}^2) - \omega]^2 (ck_0 u_{03} - \Omega)}{[(kv_0 - \omega)^2 - \omega_0^2] (kv_0 - \omega)^2} \end{aligned}$$

$$\begin{aligned}
\varepsilon_{33} &= 1 + \frac{\omega}{kc} \frac{\omega_p^2}{\omega^2} \int du \int d^2 \alpha_{\perp} \frac{\partial g_0}{\partial u} \frac{ck v_{03}}{(\omega - ck v_{03})} \\
&+ \frac{\omega_p^2}{\omega^2} \Omega \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma^3 u_{03}} \frac{v_{0\perp}}{v_{03}^2} g_0 \frac{(ck_0 u_{03} - \Omega)}{[(ck v_{03} - \omega)^2 - \omega_0^2] (ck v_{03} - \omega)^2}, \\
\varepsilon_{22} = -\varepsilon_{21} &= -\frac{2i ck k_0 c^2}{\omega^2} - \frac{i \omega_p^2}{\omega^2} \Omega \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma^2 u_{03}} g_0 \frac{ck v_{03} (1 + v_{0\perp}^2 / v_{03}^2) - \omega}{[(ck v_{03} - \omega)^2 - \omega_0^2]}, \\
\varepsilon_{13} = -\varepsilon_{31} &= -i \frac{\omega_p^2 \Omega}{\omega} \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma^2 u_{03}} \frac{v_{0\perp}}{v_{03}} g_0. \quad (2.1.3.25)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{[(ck v_{03} - \omega)^2 - \omega_0^2]}, \quad \varepsilon_{23} = \varepsilon_{32} = \frac{\omega}{kc} \frac{\omega_p^2}{\omega^2} \int du \int d^2 \alpha_{\perp} \frac{v_{0\perp}}{v_{03}} \frac{\partial g_0}{\partial u} \frac{ck v_{03}}{\omega - ck v_{03}} \\
& + \frac{\omega_p^2}{\omega^2} \Omega \int du \int d^2 \alpha_{\perp} \frac{u}{\gamma^3 u_{03}} \frac{v_{0\perp}}{v_{03}} g_0 \frac{[ck v_{03} (1 + v_{0\perp}^2 / v_{03}^2) - \omega] [ck_0 u_{03} - \Omega]}{[(ck v_{03} - \omega)^2 - \omega_0^2] (ck v_{03} - \omega)^2}
\end{aligned}$$

This tensor is hermitian for real frequency and wave number, corresponding to the case of no dissipation. It reduces to the cold fluid case derived in section 2.1.1, through the substitution

$$g_0(u, \alpha_{\perp}) = \frac{u_{03}}{u} \delta^2(\alpha_{\perp}) \delta(u - u_0) \quad (2.1.3.26)$$

2.2 Analytic Continuation. Branch cuts.

The equilibrium condition (2.1.8) leads to the relation

$$u^2 = u_{03}^2 + \left[\frac{K_0 c u_{03} \zeta_0}{K_0 c u_{03} - \omega} \right]^2. \quad (2.2.1)$$

This relation is shown in Figure 2.2.1. Friedland (7) showed the existence of two types of stable orbits. This aspect will be analyzed in section 3.1. For our present purpose it is sufficient to say that for type C orbits, which are the ones with a small component of the perpendicular velocity, there is a minimum momentum given by

$$u_{\min} = \left(\zeta_0 \Gamma + \frac{\omega}{K_0 c} \right) \frac{(\Gamma^2 + 1)^{1/2}}{\Gamma} \quad (2.2.2)$$

where $\Gamma = \left(\frac{\omega}{c K_0 \zeta_0} \right)^{1/3}$. When $\Gamma \rightarrow 0$, $u \rightarrow \zeta_0$, in agreement with $u_{\min}$

$$u^2 = u_{03}^2 + \zeta_0^2 \quad (2.2.3)$$

for the case of zero guide magnetic field.

The singularities in the denominators of the momentum integrals in the dispersion tensor can be expressed

$$\omega - k v_0 = l \omega_0 \quad (2.2.4)$$

where $l = 0, \pm 1$

In terms of the momentum and to zero order in the pump,

$$u_{rs} = \frac{l \omega_0 c (k + l k_0) \pm \omega \sqrt{c^2 (k + l k_0)^2 + l^2 \omega_0^2 - \omega^2}}{c^2 (k + l k_0)^2 - \omega^2} \quad (2.2.5)$$

The resonances are single pole singularities. When $\text{Im}(k) > 0$ and one of the poles crosses the real axis (Fig.2.2.2), according to the Landau prescription, the value of the residue at the pole should be added. There is the possibility of one pole passing through the segment

Fig. 2.2.1

Fig. 2.2.2

Fig. 2.2.3

$(0, u_{\min})$. If one sense of analytic continuation is already chosen (e.g. clockwise), the pole slipping through the segment encounters a jump of $\pi i \frac{dg_0}{du}$ compared to an infinitely close point which had gone through the horizontal axis at a place further from the origin than $u_{\min}$ (Fig. 2.2.2). There is therefore a branch point at $u_{\min}$ in the complex plane. The possibility of the poles slipping through portions of the real axis outside the range of integration was pointed out by Bernstein (17) in the general theory of plasma waves in a neutral plasma. When inverting the Laplace transform, besides the usual residues at the single poles, there are additional branch-cut contributions (Fig. 2.2.3), i.e.,

$$\vec{a}(z) = i \sum_{j=1}^N \left[\lim_{k \rightarrow k_j} (k - k_j) \frac{\tilde{G}(k) \cdot \vec{I}(k)}{D(k)} \right] e^{i k_j z} \\ + \frac{1}{2\pi} \sum_{l=0, \pm 1} \int_{\text{Branch cuts}} \alpha_k \text{ Discontinuity } \left[\frac{\tilde{G}(k) \cdot \vec{I}(k)}{D(k)} \right] e^{i k z} \quad (2.2.6)$$

where $\tilde{Q}(k)$ is the cofactor matrix and $D(k)$ is the dielectric function.

These branch cuts are the image of the segment $(0, u_{\min})$ in the complex u plane under the mapping

$$k_{e, \text{res}} = \frac{1}{V_{30}(u)} [\omega - l \omega_0(u)].$$

There are therefore three branch cuts corresponding to three resonance conditions. These branch cuts correspond to the segment $(k_{e, \text{res}}(u_{\min}), \infty)$, where

$$k_{e, \text{res}}(u_{\min}) = \frac{1}{\left[\beta_0 r + \frac{\omega}{k_0 c} \right]_c} \cdot \left\{ \left[\left(\beta_0 r + \frac{\omega}{k_0 c} \right)^2 \left(\frac{r^2 + 1}{r^2} + 1 \right) \right]^{\frac{1}{2}} - l (c k_0 \beta_0 r)^{\frac{1}{2}} \cdot \left[c k_0 \beta_0 r - \omega r^2 \right]^{\frac{1}{2}} \right\}$$

for $l=0, \pm 1$ and where $r = \left(\frac{\omega}{c k_0 \beta_0} \right)^{\frac{1}{3}}$

Now

$$\frac{Q_{21} I_1 + Q_{22} I_2}{D(k)} = \frac{-\epsilon_{33} [\epsilon_{12} I_1 + \epsilon_{11} I_2] + O(\beta_0^2)}{\epsilon_{33} Q' + O(\beta_0^2)} \quad (2.2.7)$$

The discontinuity in x space starts at $x_{min} = \sqrt{1 + \frac{\omega^2}{x_0^2 \omega^2}} x_i$; then none of the individual terms of the dispersion tensor becomes negligible and (2.2.7) becomes approximately

$$\frac{-\epsilon_{12} I_1 - \epsilon_{11} I_2}{Q'}$$

When evaluating the branch cut contributions, one has to notice that the leading contribution which would correspond to $\beta_0 = 0$ possesses poles arising from the magnetoresonant denominators

$$\left(\frac{x u}{\gamma} - 1\right)^2 - \frac{1}{\gamma^2} (x_0 u - \omega)^2$$

(2.2.8)

Then to zero order in the pump, the only branch cuts will correspond to the two magnetoresonance conditions and will represent a packet of waves, with indices of refraction satisfying the dispersion relation

$$1 - \frac{xu}{\gamma} = \pm \frac{1}{\gamma} (x_0 u - \eta) \quad (2.2.9)$$

The second component of the branch-cut contribution to the electric field is

$$a_{2,\beta c} \approx \frac{1}{2\pi} \frac{\omega}{c} \sum_{\ell=\pm 1} \int_{\sqrt{1+\frac{\eta^2}{x_0^2 \omega^2}}}^{\infty} dx \operatorname{dis} \left[\frac{\tilde{G}(x) \cdot \vec{I}(x)}{Q'(x)} \right] e^{iKz} \quad (2.2.10)$$

The discontinuity can be approximated by

$$\operatorname{dis} \left[\frac{\tilde{G}(x) \cdot \vec{I}(x)}{Q'(x)} \right] \approx - \frac{\tilde{G}(x) \cdot \vec{I}(x) \operatorname{dis} Q'(x) + Q'(x) \operatorname{dis} [\tilde{G}(x) \cdot \vec{I}(x)]}{Q'^2(x)} \quad (2.2.11)$$

$$\operatorname{dis} Q' \approx 2 \left[(1 - (x^2 + x_0^2)) \operatorname{dis} (\epsilon_{12}) + 2ix_0 x \operatorname{dis} (\epsilon_{32}) \right] \quad (2.2.12)$$

The discontinuity is proportional to $\frac{\omega_p^2}{\omega^2} \frac{\Omega}{\omega}$. This parameter is small for the present generation of Free-Electron-Lasers. It is clear, then, that the branch-cut contributions will be significant only when $Q'(x)$ is close to zero, and consequently

$$\text{dis} \left[\frac{\tilde{G}(x) \cdot \vec{I}(x)}{Q'(x)} \right] \approx - \frac{\tilde{G}(x) \cdot \vec{I}(x) \text{dis } Q'(x)}{Q'^2(x)}$$

(2.2.13)

Then, on using the expansion

$$\frac{1}{\left(1 - \frac{xu}{\delta}\right)^2 - \frac{1}{\delta^2} \left(x_0 u - \frac{\Omega}{\omega}\right)^2} = \frac{\delta^2}{2 \left(x_0 u - \frac{\Omega}{\omega}\right)} \left[\frac{(x-x_0)u + \Omega/\omega + \delta}{[(x-x_0)^2 - 1] (u-u_2^-)(u-u_2^+)} \right. \\ \left. - \frac{(x+x_0)u - \Omega/\omega + \delta}{[(x+x_0)^2 - 1] (u-u_1^+)(u-u_1^-)} \right], \quad (2.2.14)$$

we find

$$\text{dis}_{+1} \left[\frac{\tilde{G}(x) \cdot \vec{I}(x)}{Q'(x)} \right] \approx \frac{\Omega}{\omega} \frac{\omega_p^2}{\omega^2} \frac{a(x)}{\sqrt{(x+x_0)^2 + \Omega^2/\omega^2 - 1} [1 - (x-x_0)^2]} [1 - (x+x_0)^2]$$

(2.2.15)

where

$$a(x) = 2i x_0 x (1+x-x_0) - [1-(x-x_0)^2] i (1+x).$$

(2.2.16)

The previous argument concerning the resonant denominator leads to a further simplification of the branch cut corresponding to the $\ell=+1$ resonant condition (B.C.) :

$$B.G_{+1} \approx \frac{ix_0}{\eta} \frac{\eta}{\omega} \frac{\omega_p^2}{\omega^2} \int_0^{u_{min}} du \frac{1+\eta/\omega \delta}{u^2 \delta} \cdot$$

$$\frac{g_0(u) x (1+x-x_0) e^{\frac{ix\omega}{c} z}}{\sqrt{(x+x_0)^2 + \eta^2/\omega^2 - 1} [1-(x-x_0)^2] [1-(x+x_0)^2]^2} \quad (2.2.17)$$

where

$$x = -x_0 + \frac{\delta + \eta/\omega}{u} \quad (2.2.18)$$

The denominator exhibits the resonance

$$x = 1 + x_0 .$$

(2.2.19)

This leads to

$$u = \frac{\omega / \omega (1 + 2x_0) \pm \sqrt{\omega^2 / \omega^2 + 4x_0(1 + x_0)}}{4(1 + x_0)} .$$

(2.2.20)

The physical solution corresponds to the positive radical. For moderate momentum spread, Δ , the integral will be significant only when $u \approx u_{\min}$. This leads to

$$\frac{\omega^2}{\omega^2} \approx \frac{4(1 + x_0)x_0}{4(2 + 3x_0 + x_0^2)} \approx \frac{x_0}{2}$$

or

$$k_0 v_{z0} \approx \frac{\omega}{\gamma}$$

(2.2.21)

This is the magnetoresonance condition. It is inaccessible in type C orbits, but this latter fact is a consequence of

considering higher order expansions in the pump. In our present calculation, the interpretation of (2..2.20) is that the branch cut contribution $l=1$ will be significant when the central momentum approaches u_{min} , and the cyclotron mode interacts with EM+.

Using the same type of arguments, the $l=-1$ branch cut can be written as:

$$B.C._{-1} \approx \frac{i x_0}{\pi} \frac{n}{\omega} \frac{\omega_p^2}{\omega^2} \int^{u_{min}} du \frac{1 + n/\omega \gamma}{u^2 \gamma} \frac{g_0(u) \times (1 + x - x_0) e^{i x \frac{\omega}{c} z}}{\sqrt{(x+x_0)^2 + n^2/\omega^2 - 1} [1 - (x-x_0)^2]^2 [1 - (x+x_0)^2]} \quad (2.2.22)$$

3. Analysis

3.1 The dispersion relation

In this section, the effects of energy spread for the limiting case of zero guide magnetic field will be analyzed. The effect of the guide magnetic field will be shown later in section 3.3.1.

The dispersion relation for zero guide magnetic field can be written as:

$$\epsilon_{33} [\epsilon_{11} \epsilon_{22} + \epsilon_{12}^2] = \epsilon_{23}^2 \epsilon_{11}$$

$$\epsilon_{33} Q'(x) = \epsilon_{23}^2 \epsilon_{11} \quad (3.1.1) \text{ where}$$

This dispersion relation was solved numerically for the following set of values: $\omega_p = 5.026 \times 10^{10} \text{ r/sec}$,

$$\beta_0 = 0.058, \quad K_0 = 5.027 \text{ cm}^{-1}, \quad \gamma = 2.5$$

and several values of momentum spread. An initial frequency

ω was given as an input and the real and imaginary parts of the wave vector were calculated. There are six roots which when uncoupled correspond to two plasma waves, two backward and two forward propagating electromagnetic waves.

$Q(x)$ save terms of order $\omega_p'^2$ can be approximated by $Q(x)$ where

$$Q(x) = [1 - (x_0^2 + x^2)]^2 - 4x_0^2 x^2, \quad (3.1.2)$$

which has the four roots

$$x = \begin{cases} -1 \pm x_0 \\ 1 \pm x_0 \end{cases}$$

The longitudinal factor ϵ_{33} can be written $\epsilon_{33} = 1 + \omega_p'^2 \chi$ where χ denotes the susceptibility

$$\chi = \int_0^\infty du \frac{dg}{du} \frac{u_{03}}{\gamma - xu_{03}} \quad (3.1.3)$$

For a Gaussian distribution, χ is shown in Figures (3.1.1) and (3.1.2). Qualitatively, it has similar features to the usual plasma dispersion function.

For x real and $x > 1$,

$$u_{res} = \sqrt{\frac{1 + \beta_0^2 x^2}{x^2 - 1}} \quad (3.1.4)$$

$$\begin{aligned} \text{Re } \mathcal{K}(x) = & \frac{1}{2u_{res}(x^2 - 1)} \left[\int_{\beta_0}^{\infty} du \frac{dg}{du} \frac{(\gamma + x u_{03})}{u - u_{res}} \right. \\ & \left. - \int_{\beta_0}^{\infty} du \frac{dg}{du} \frac{(\gamma + x u_{03})}{u + u_{res}} \right] \end{aligned} \quad (3.1.5)$$

$$\text{Im } \mathcal{K}(x) = \frac{\pi}{2u_{res}(x^2 - 1)} \left. \frac{dg}{du} \right|_{u=u_{res}} (\gamma + x u_{03}) \quad (3.1.6)$$

The first term is clearly the dominant one. The dispersion relation was solved numerically by analytically continuing the integrals whenever $\text{Im}(k) > 0$.

It is illustrative to compare solutions to the dispersion relation for the thermal and cold fluid cases. In both cases there are six roots. The significant difference between them, is that both plasma waves become more separated and Landau damped. The indices of refraction for both waves satisfy when uncoupled, the dispersion relation

$$X_{p1, p2} = \frac{c}{v_{z0}} \left(1 \pm \frac{\omega'_p}{\gamma_0^{3/2}} \right) \pm f(\Delta) \quad (3.1.7)$$

where f is an increasing function of momentum spread which goes to zero at $\Delta = 0$ and v_{z0} is the central parallel velocity. From (3.1.2), the uncoupled forward electromagnetic waves satisfy approximately

$$X_{EM \pm} = 1 \pm X_0 \quad (3.1.8)$$

Therefore, the coupling occurs at lower frequency and the spectrum broadens as Δ increase.

When the slow space-charge wave couples to EMt, we note the following: (i) the roots corresponding to the two coupled waves are no longer complex conjugates since the dispersion relation is no longer a sixth order polynomial. (ii) the

real parts of the coupled roots are shifted and the growth rate is diminished due to Landau damping. (iii) The shift of the real parts is minimum at the frequency corresponding to the highest growth rate.

The spatial growth rate as a function of frequency is shown in Figure (3.1.3). The spectrum shifts towards lower frequencies when Δ increases. At about 2% spread it starts to develop a tail. Both the maximum growth rate and its corresponding wave vector decrease with momentum spread (fig.3.1.4); for a momentum spread of 5%, $Re(k)$ at maximum decreases by about 10%. The growth rate is sensitive to the particular distribution function employed. Fig. 3.1.5 compares the growth rate window for a Gaussian and a Lorentzian distribution function at 1% energy spread. The reason for the smaller growth rate for the Lorentzian is that there is a greater Landau damping due to a steeper slope in the tail of the distribution function.

It is instructive to track how the two plasma waves interact with the distribution function. One electromagnetic wave couples to the the negative energy plasma wave while they interact with electrons in the region of positive slope of the distribution function. However, the growth rate diminishes due to the fact the the slow plasma wave has negative energy. This can be observed numerically since

Fig. 3.1.1

Fig. 3.1.2

$$\frac{dx}{d\omega} = \frac{c}{\omega} \frac{v_{ph} - v_g}{v_{ph} v_g} \quad (3.1.9)$$

and so $\text{sign}(\frac{\Delta x}{\Delta \omega}) = \text{sign}(v_{ph} - v_g)$.

At larger frequencies, once the EM+ wave and the negative energy plasma wave have decoupled, both plasma waves are Landau damped (fig.3.1.6). At about 2% momentum spread there is a small bump in the damping rate for the damped coupled mode. For frequencies for which there is a tail in the growth rate of one of the modes, the damping of the other mode does not go monotonically to zero, but instead it goes through a local minimum and then continues growing. When beam temperature diminishes, the two frequency windows move towards higher values; the curves become thinner and taller. There is no other region of coupling; at larger frequencies the EM+ wave crosses over the positive energy plasma wave without coupling.

Fig. 3.1.3

Fig. 3.1.4

Fig. 3.1.5

K_j (cm^{-1})

Fig. 3.1.6

Fig. 3.2.3

3.2 Boundary-value problem.

Although a traveling wave tube (16) is different from a Free-Electron -Laser, from the physics standpoint, the geometries of the experiments are similar. In previous analyses of traveling-wave tubes, three forward and one backward propagating modes were found and the propagation of the modes was linked to the boundary conditions. Bernstein and Hirshfield (6), by recalling this geometric resemblance, formulated a boundary problem for the Free-Electron-Laser. We have extended this formalism to include an arbitrary equilibrium distribution and also a perturbed distribution function which can represent physically the extra perpendicular energy spread in the beam induced by the non-adiabaticity when entering the interaction region. The motivation here is to investigate how the coupling loss changes with temperature. When evaluating the mode coefficients, one must bear in mind that we are concerned with quantities of order one. Therefore, although terms of order ω_p^2 and β_0^2 were not negligible for the purpose of growth rate calculations, they can be ignored when studying how the power is shared among the different modes.

After inverting the Laplace transform by means of the residue theorem, we have

$$a_2(z) = \sum_{j=1}^6 B_j e^{iK_j z} \quad (3.2.1)$$

where

$$B_j = \sum_{l=1}^3 \frac{C_{2l}(K_j) I_l(K_j)}{\frac{dD}{dK} \big|_{K_j}} \quad (3.2.2)$$

$D = \det(\tilde{\epsilon})$, G_{ij} is the cofactor matrix, and

$$\begin{aligned} I_1(K_j) &= [a'_1(0) + iK_j a_1(0) - 2K_0 a_2(0)] \frac{c^2}{\omega^2}, \\ I_2(K_j) &= \frac{c}{\omega} 4\pi n_0 |e| \int_{\gamma_0}^{\infty} du \frac{u}{\gamma} \frac{V_{01}}{V_{03}} G \frac{V_{03}}{K V_{03} - \omega} + \\ &\quad [a'_2(0) + iK_j a_2(0) + 2K_0 a_1(0)] \frac{c^2}{\omega^2}, \\ I_3(K_j) &= \frac{c}{\omega} 4\pi n_0 |e| \int_{\gamma_0}^{\infty} du \frac{u}{\gamma} G \frac{V_{03}}{K V_{03} - \omega} \end{aligned} \quad (3.2.3)$$

In order to model the amplifier experiment, let us assume that the two backward propagating electromagnetic modes have zero amplitudes. These conditions ($B_1=0$, and $B_2=0$) leave $a_1(0)$ and $a_2(0)$ as the only unknowns. The polarization of the incident wave was chosen as $a_1(0)=0$ and $a_2(0)=1$.

The expression (3.2.1) was evaluated numerically for different momentum spreads and for the same set of values employed in the solution of the dispersion relation. For the root with imaginary part greater than zero, the momentum integral was analytically continued by adding the residue at the pole. The formalism allows for the use of an arbitrary perturbation at the entrance of the interaction region $G(u)$.

The fast plasma wave has negligible amplitude ($B_4 \approx 0$). When temperature increases, B_3 remains close to 0.5 but B_6 progressively increases approaching 0.5 whereas B_5 approaches zero.

We have

$$\beta_j = \frac{i \epsilon_{33} [\epsilon_{12} I_1 + \epsilon_{11} I_2]}{\alpha D / \alpha K} \quad (3.2.4)$$

$$\frac{\alpha D}{\alpha K} = \frac{\alpha \epsilon_{33}}{\alpha K} Q' + \epsilon_{33} \frac{\alpha Q'}{\alpha K} + O(\epsilon_0^2) \quad (3.2.5)$$

For small pump parameters

$$\frac{\alpha D}{\alpha K} \approx \frac{\alpha \epsilon_{33}}{\alpha K} Q' + \epsilon_{33} \frac{\alpha Q'}{\alpha K} \quad (3.2.6)$$

Thus, for the slow plasma wave x_4 , $\epsilon_{33}(x_4) \approx 0$ and so

$$\beta_4 \approx \frac{i \epsilon_{33} [\epsilon_{12} I_1 + \epsilon_{11} I_2]}{\alpha \epsilon_{33} / \alpha K Q'} \approx 0 \quad (3.2.7)$$

For x_3 , eliminating $a'_{1,2}(0)$ and substituting, we find

$$B_3 \approx \frac{i [\epsilon_{12} I_1 + \epsilon_{21} I_2]}{\alpha Q'/dK} \approx \frac{1}{2} \quad (3.2.8)$$

save terms of order $\omega_{p'}^2$.

In the cold beam limit in the complex roots regime, B_5 and B_6 are complex conjugates, then $\text{Re}(B_5) = \text{Re}(B_6) = 0.25$. For both roots, $\epsilon_{33} \approx 0$ but also $Q' \approx 0$ and so B_5 and B_6 remain finite. In a thermal beam there is a mismatch between the roots and one tends to satisfy $\epsilon_{33} \approx 0$ but not $Q' \approx 0$ and vice-versa. Consequently, one amplitude approaches zero and the other 0.5 (Fig.3.2.1).

The above results in the cold fluid limit, differ from those of Bernstein and Hirshfield (16). This disparity stems from considering different pump regimes. Bernstein and Hirshfield's solution to the cubic dispersion relation is only valid within what Sprangle (5) calls the strong pump regime. Their numerical results are perfectly consistent with this assumption as they consider pump parameters close to 0.5. Within this regime, the two modes with non-zero

Fig. 3.2.1

imaginary parts and the (when uncoupled) positive-energy plasma wave are closely coupled together. Therefore, $\epsilon_{33}(x_4) \neq 0$ and (3.2.7) does not hold. Within the strong pump regime, each of the three coupled modes' coefficients is $1/6$ and consequently the coupling loss is $1/36$.

The pump parameter analyzed in this section ($\beta_0 = 0.050$) is well within the weak pump regime. In this regime, $\epsilon_{33}(x_4) = 0$ and the two coupled mode coefficients are $1/4$. Therefore, in the cold fluid model, the coupling loss will decrease from $1/16$ to $1/36$ when the pump parameter increases. However, for large pump parameters, one must bear in mind that the simplified canonical model of a wiggler potential without radial dependence is no longer valid and therefore the above results cited above are questionable.

Fig.(3.2.1) shows how the situation evolves from a three mode process into a two mode one, when temperature increases. It is not that the positive energy wave becomes Landau damped, but rather that they start with a very small amplitude at the entrance of the interaction region when the temperature is significant. Fig.(3.2.2) shows the inverse of the coupling loss as function of momentum spread. For values of Δ less than 2% there is a small increase when the roots

Fig. 3.2.2

Fig. 3.2.4

become slightly more separated. For values greater than 2% it decreases monotonically. It is thus possible to extend the region of permissible temperatures for a Free-Electron-Laser by recognizing that the decrease in the coupling loss can compensate for the Landau damping. This regime would be an intermediate one between Compton and Raman operation (Fig.3.2.3).

As was the case for the cold beam analyzed in (5), the longitudinal component of the electric field is orders of magnitude lower than the transverse ones. Outside the frequency range for which there are complex conjugate roots, the gain due to pure spatial interference of the modes was only a factor of 4 at 100 wiggler wavelengths for a momentum spread of 2%.

The radiation power oscillates as a function of distance. At the frequency corresponding to the central emission, these oscillations become milder when the momentum spread increases (Fig. 3.2.4). There is a characteristic length at which, finite temperature effects are important; it is given approximately by $l_{\Delta} = u_0 \frac{2\pi}{k_0 \Delta}$. At distances smaller than l_{Δ} , the amplitude of the power oscillations is such that the cold and thermal beam cases are undistinguishable. In Figure 3.2.5, the power at 50 wiggler wavelengths corresponding to

Fig. 3.2.5

Fig. 3.2.6

a momentum spread of 2% can be distinguished from the cold beam case, but the distinction between the cold beam and the $\Delta = 1\%$ beam is not possible. However at 100 wiggler wavelengths the difference between the cold and the $\Delta = 2\%$ beams is clear (fig.3.2.5).

3.3 The effect of the guide magnetic field

3.3.1 The effect of the guide magnetic field on the dispersion relation

In order to have stability of the orbits it is required that $\omega_p^2 > 0$. Together with the equilibrium condition

$$\frac{v_{z0}}{v_{g0}} = \frac{K_0 c \beta_0}{K_0 c u_{03} - \Omega} , \quad (3.3.2)$$

this results in two possible regions of stability:

$$u_{03} > \frac{\beta_0 \Omega^{1/3} c^{2/3} \beta_0^{2/3}}{K_0^{1/3}} + \frac{\Omega}{K_0 c} \quad (3.3.3)$$

for the type C orbit, and

$$u_{03} < \frac{\Omega}{K_0 c} \quad (3.3.4)$$

for the type A orbit. These results were obtained by Friedland using other considerations. The stability in

Fig. 3.3.1

both types of orbits can be inferred by examining Fig.

2.2.1. In both cases, $(\frac{d\gamma_3}{d\gamma} > 0)$ and consequently $(\frac{d\gamma_1}{d\gamma} < 0)$, where $\gamma_3 = 1 / \sqrt{1 - v_{03}^2/c^2}$

and
$$\gamma_1 = \frac{1}{\sqrt{1 - v_{01}^2/c^2}}$$

Therefore, a small increase in the energy results in a radially inward motion and the orbits are stable.

When solving the dispersion relation, two different types of behavior, depending upon whether the electrons follow one type of orbit or the other, are encountered. In type A orbits (fig.3.3.2), the spectrum is much broader and the growth rate much larger than in type C orbits. ^(fig.3.3.1) The results for the type A orbits are questionable since the large perpendicular components of the velocity cause substantial departures, and consequently the radial dependence of the wiggler should be included.

The guide magnetic field for type C orbits causes a broadening of the spectrum and larger growth rates. This effect is independent of the increase in B_0 caused by an increase in $v_{\perp 0}$, from the equilibrium relation. Freund et al. (15) interpret it as an enhancement of the

Fig. 3.3.2

Fig. 3.3.3

ponderomotive potential. Figure 3.3.3 shows the dependence of the growth rate on the guide magnetic field for the same $\frac{V_{te}}{V_{30}}$ ratio. Figure 3.3.3 serves the additional purpose of comparing our results to those of Bernstein and Friedland. The only significant difference is that there is not a common upper limit on the frequency window. This result, which appears again in (10), may be due to their having neglected the self consistent transverse current in Maxwell's equations.

As pointed out before when evaluating the determinant of the dispersion tensor, only terms with magnetoresonant denominators to the first power appear. Then,

$$\det \tilde{\xi} = \det \tilde{\theta} + \frac{n}{\omega} \frac{\omega_p^2}{\omega^2} (\text{terms with } R^{-1}) \\ + \frac{n^2}{\omega^2} \frac{\omega_p^4}{\omega^4} (\text{term with } R^{-1})$$

(3.3.5)

where $\tilde{\theta}$ is the dispersion tensor without the guide magnetic field, and R denotes the resonant denominator. the

leading contribution will then arise from the terms proportional to $\frac{n}{\omega}$ and with zeroth order of the pump parameter. Then

$$\det \tilde{\epsilon} \approx \det \tilde{\theta} + \frac{n}{\omega} \frac{\omega_p^2}{\omega^2} I(x)$$

(3.3.6)

where for the cold fluid model

$$I(x) = \frac{[1 - (x^2 + x_0^2)](x_0 u - \frac{n}{\omega}) - 2x_0 x (x u - \gamma)}{\gamma [(x u - \gamma)^2 - (x_0 u - \frac{n}{\omega})^2]}$$

(3.3.7)

$I(x)$ exhibits two resonances corresponding to the two magneto resonant conditions. Due to the smallness of $\frac{\omega_p^2}{\omega^2} \frac{n}{\omega}$, $\det \tilde{\epsilon} \approx \det \tilde{\theta}$ except close to magnetoresonance. There, because of the asymptotic behavior, the corrections become significant and two new roots to the dispersion relation appear.

In the kinetic theory

$$I(x) = \int_{u_{\min}}^{\infty} \frac{[1 - (x^2 + x_0^2)](x_0 u - \eta/\omega) - 2x_0 x (xu - \delta)}{\gamma^3 \left[\left(x \frac{u}{\delta} - 1 \right)^2 - \frac{1}{\gamma^2} \left(x_0 u - \frac{\eta}{\omega} \right)^2 \right]} g_0(u) du$$

(3.3.8)

Using the expansion (2.2.14), (3.3.8) for the left magnetoresonance condition ($l=-1$) can be written

$$I(x_L) \approx - \int_{u_{\min}}^{\infty} \frac{\{ [1 - (x^2 + x_0^2)](x_0 u - \eta/\omega) - 2x_0 x (xu - \delta) \}}{4(x_0 u - \frac{\eta}{\omega}) \gamma \sqrt{(x - x_0)^2 + \frac{\eta^2}{\omega^2} - 1} (u - u_2^+)} \cdot$$

$$\cdot \left[\delta + (x + x_0)u - \frac{\eta}{\omega} \right] g_0(u) du$$

(3.3.9)

where

$$u_1^+ = \frac{-\eta/\omega(x + x_0) + \sqrt{(x + x_0)^2 + \eta^2/\omega^2 - 1}}{[(x + x_0)^2 - 1]}$$

(3.3.10)

$I(x_L)$ is shown in Figure 3.3.4, for a cold fluid model

and for different values of momentum spread. The two former roots with zero imaginary parts which were present in the cold fluid model do not appear now. The only possible roots can be encountered through analytic continuation of $I(x)$ for $\text{Im}(x) > 0$. Due to the smallness of $\text{Re}(I\omega)$ and $\text{Im}(I(x))$ for $\text{Im}(x) = 0$ for temperatures of the order of 1%, they would have a very large imaginary part. They correspond to heavily damped waves which would travel a very short distance in the amplifier experiment.

3.3.2 Propagation of the two cyclotron modes.

On using the residue theorem, we have for the case of non-zero guide magnetic field, within the cold fluid model,

$$a_2(z) = i \sum_{j=1}^8 \left[\lim_{K \rightarrow K_j} (K - K_j) \frac{\tilde{Q}(K) \cdot \vec{I}(K)}{D(K)} \right] e^{i K_j z} =$$

$$\sum_{j=1}^8 \frac{B(x_j) e^{i \frac{\omega}{c} x_j z}}{dD/dK} \quad (3.3.2.1)$$

where

$$B(x) = \frac{(\epsilon_{12} \epsilon_{33} - \epsilon_{13} \epsilon_{23}) I_1 + (\epsilon_{11} \epsilon_{33} + \epsilon_{13}^2) I_2}{dD/dK}.$$

Now, for small β_0

$$G_{21} = \theta_{12} \theta_{33} + \omega (\psi_{12} \theta_{33} + \theta_{12} \psi_{33} - \psi_{13} \theta_{23}) \approx \theta_{33} [\theta_{12} + \omega \psi_{12}]$$

(3.3.2.3)

$$G_{22} \approx \theta_{33} [\theta_{11} + \omega \psi_{11}]$$

(3.3.2.4)

When evaluating $D(K) = \det(\tilde{\xi})$, the terms with the magneto-resonant denominator cancel among themselves to third order in R^{-1} , as do the terms with second power magneto-resonant denominators. For the two cyclotron modes,

$$\begin{aligned} D(x_{3,8}) &\approx 2\omega \theta_{33} (\psi_{11} \theta_{11} + \psi_{12} \theta_{12}) \\ &\approx \frac{2\omega}{\omega \gamma^2} \theta_{33} \frac{\omega_p^2 \left\{ [1 - (x_0^2 + x^2)]^{1/2} (x_0 u - 1) - 2 \left(x \frac{u}{\delta} - 1 \right) x_0 x \right\}}{\omega^2 \left(x \frac{u}{\delta} - 1 \right)^2 - \frac{1}{\delta^2} (x_0 u - 1)^2} \end{aligned}$$

(3.3.2.5)

$$B(x_{3,8}) \simeq \frac{i [\pm(1+x) - x_0]}{2 [1 - (x \pm x_0)^2]} \left[\left(x \frac{\omega}{\gamma} - 1 \right)^2 - \frac{1}{\gamma^2} \left(x_0 \omega - \frac{\omega}{\omega_0} \right)^2 \right]$$

(3.3.2.6)

So $B(x_{3,8}) \approx 0$ unless $x \approx 1+x_0$, which are the resonances obtained previously in the kinetic theory through the branch cut contributions. It is clear from the above that the two cyclotron modes do not propagate unless one operates close to magnetoresonance.

Fig. 3.3.4

Fig. 3.3.5

Conclusions

The Bernstein-Hirshfield formalism, incorporates a basic feature in the Free-Electron -Laser theory reminiscent of the traveling wave tube situation, namely the fact that the radiated power is carried by different modes, only one of which grows exponentially with distance. These modes interact with the electron distribution function through Landau damping and can interfere constructively among themselves. These phenomena have been investigated in this work: How does the three-wave interaction responsible for the Free-Electron-Laser gain mechanism compete with Landau damping?. Can the pure interference of the waves compensate for the decrease in gain due to Landau damping?. Furthermore the boundary-value formulation allows one to study another basic problem of the Free-Electron -Laser: the effect of the transition region situated at the entrance of the interaction region. This transition region has been investigated by Friedland by assuming to be adiabatic. This is clearly an approximation for realizable configurations. It is possible to calculate the extra perpendicular energy spread caused by this mechanism. This perturbed distribution

function at the entrance of the interaction region is the equivalent to the ballistic term in the Vlasov theory of plasma waves. The boundary-value problem is thus not a mere generalization of the boundary conditions in the traveling wave tube and it can introduce new physics problems.

The main purpose of this work has been to extend the Bernstein-Hirshfield formalism to accommodate a guide magnetic field and an arbitrary distribution function. One basic feature of the dispersion tensor obtained in this way is its symmetry properties: it is hermitian for real values of the frequency and wavenumber as it corresponds to a non-dissipative situation.

Former theories (even Bernstein and Hirshfield's) considered the Raman and Compton stimulated emission processes in Free-Electron-Lasers as limiting situations in which gain occurred either by an exponentially growing mode (Raman) or by pure spatial interference among the different modes (Compton). In practice there is not such a clear distinction. The Landau damped modes may interfere constructively and both processes can compete.

For a cold fluid, Bernstein and Hirshfield's results showed six modes, two going backwards and four propagating forwards in an oscillator configuration. The addition of a

beam temperature to the boundary-value problem is necessary to sort out which modes propagate and how temperature affects their propagation. It also allows one to investigate how a Raman regime develops into a Compton one. In the amplifier situation and within the Raman regime there are three modes which propagate from the entrance of the interaction region. Only one of them grows exponentially and another decays. The situation is not one in which the positive energy plasma wave is Landau damped convectively, but rather that its power sharing is negligible. It is important to emphasize that all the above occurs within the small pump parameter framework. In regions of large pump parameter the large radial excursion of the electrons from the wiggler axis invalidate the canonical model of a Free-Electron-Laser without radial dependence. At zero temperature, the uncoupled forward propagating electromagnetic mode shares $1/2$ of the total power. While both coupled waves shared $1/4$ each. When temperature increases, one of the mode coefficients approaches $1/2$ and the other zero. As this occurs, the Free-Electron-Laser enters the Compton regime. The gain in this regime would occur through the spatial interference of these two modes when their separation in wave-vector numbers is small. We

have not analyzed this situation in this work which would correspond to design parameters close to the Stanford experiment. The numerical calculations performed in this work showed a very small gain (about 4 at 100 wave-lengths) at zero temperature for initial frequencies outside the frequency window for which there is exponential growth. When temperature increases, the roots become more separated and this Compton-like gain decreases even further.

A crucial feature of the boundary-value formulation is the calculation of the coupling loss. the numerical results show a slight increase in the coupling loss when the roots become more separated, but then a gradual decrease when the decoupling starts to take place. To acquire fine beam qualities is a difficult technological task. When working in wavelength close to the Debye length of the beam, it is possible to take advantage of this decrease in the coupling loss since it alleviates the decrease in gain due to Landau damping.

The polarization of the coupled waves remains practically transverse at all temperatures. The longitudinal component is orders of magnitude smaller. This important fact was pointed out by Bernstein and Hirshfield for a cold fluid. It contradicts former interpretations of the Free-Electron-

Laser gain mechanism in terms of axial bunching of the beam.

There are two characterizations of the temperature effects. One is a spatial one caused by the coupling of the waves and responsible for the coupling loss. This effect is visible from the entrance of the interaction region. There is another one for small temperatures: the power has an oscillatory component. The oscillations become milder when the temperature increases. At distances of the order of ℓ_D , the exponential gain overcomes this oscillatory component and this effect becomes noticeable.

There is another more prosaic aspect of the temperature effects in a Free-Electron -Laser, namely, the dissipation effects. The width of the distribution function causes a mismatch between the two coupled waves resulting in a decrease in the growth rate. Besides this, there is a lowering of the central emission frequency and a broadening of the spectrum characteristic of all processes where dissipation plays a role. The mathematical interpretation is that the two plasma waves become more separated when temperature increases. Thus, the coupling occurs at lower frequencies and the spectrum broadens as temperature increases.

The dissipation effect can acquire a more striking aspect. When the beam is cold, within the region of exponential growth, one of the coupled modes decays. When temperature effects are considered, at frequencies outside the window for exponential growth, the two plasma waves become Landau damped. For temperatures at the transition region between Compton and Raman regimes, both processes superimpose. When temperature diminishes the damping occurs at higher frequencies with much smaller width and at much greater rate.

One important factor related to the interaction between the three wave coupling and the distribution function is the fact that it occurs in the region of positive slope. However, the gain decreases consistent with the negative-energy character of the growing wave. This feature clarifies some previous interpretations given by Sprangle (5). The presence of the distribution function does not add any other frequency window besides the one for a cold fluid model. The possibility of another possible region of growth had been suggested by Fruchtman and Hirshfield (18).

The guide magnetic field has been shown by Friedland (7) to introduce two types of orbits. Bernstein and Friedland (9) also showed that, in type of orbit A (the one with

large perpendicular component of the velocity), the gain is orders of magnitude larger than in type C orbits and the spectrum is much broader. In type C orbit, the effect of the guide magnetic field is, according to Freund et al. (15) to enhance the ponderomotive potential and consequently to enhance the gain. It is worth repeating here that in type A orbit the large component of the transverse velocity causes large departures from the wiggler axis and consequently invalidates the canonical model of the wiggler without radial dependence. These results are therefore questionable.

We have done calculations for a cold fluid, guide magnetic field case. the significance of this work is that our dispersion tensor is hermitian for real values of frequency and wave vector, and it exhibits a resonance frequency which clarifies the stability of the equilibrium orbits. There is not a large numerical difference between the results except for the fact that ours do not show an accumulation point at the upper limit of the frequency window for which there is exponential growth. In a later article, Friedland and Fruchtmann (10) emphasized this point and tried to interpret it physically. Our interpretation is that they neglected the self-consistent transverse component of the current from the outset of their derivations.

Within the cold fluid framework, the guide magnetic field correction to the zero guide magnetic field dispersion function exhibits a resonant character at the two magnetoresonances, and consequently adds two new roots to the dispersion relation. However these two modes do not propagate, unless one operates close to magnetoresonance. When considering a distribution function and for temperatures of the order of $1\frac{1}{2}$, these two roots play no role in the propagation due to their heavy Landau damping.

In type C orbit the momentum integrals in the dispersion tensor have a lower limit of integration different from zero. This feature leads to new contributions, besides the usual residues at the poles, when inverting the Laplace transform. These branch-cut contributions, are significant when operating close to magnetoresonance and represents two trains of waves.

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