Quasi-steady expansion of plasma ablated from laser-irradiated pellets

J. Sanz, A. Liñán, M. Rodríguez, and J. R. Sanmartín
Escuela Técnica Superior de Ingenieros Aeronauticos, Universidad Politécnica de Madrid, Madrid, Spain
(Received 14 August 1980; accepted 1 July 1981)

The ablative, quasi-steady expansion of the spherical coronal plasma produced by irradiating an overdense pellet by a high-intensity laser pulse, is studied for large ion charge number $Z_i$. The entire structure of the flow and its changes as the laser power is increased, are determined. The instantaneous power $W$ required to generate a given ablation pressure $P_a$ and pellet radius $r_p$ at any time is determined in terms of $Z_i$, ion mass $m_i$, and critical density; the mass ablation rate is also found. If the time law $P_a(t)$ and (consequently) $r_p(t)$ for a desired optimal compression of the pellet are determined independently, the results allow one to obtain the laser power history $W(t) = W(P_a(r_p(t))$, $Z_i$, $m_i, n_i)$. For a critical radius much larger than $r_p$, most of the energy flows outward; this seems to invalidate known simple estimates of the relation $W(P_a)$. The abrupt system of the problem; the finite electric field, the flow and the changes as the laser power is increased, are determined. The instantaneous power $W$ required to generate a given ablation pressure $P_a$ and pellet radius $r_p$ at any time is determined in terms of $Z_i$, ion mass $m_i$, and critical density; the mass ablation rate is also found. If the time law $P_a(t)$ and (consequently) $r_p(t)$ for a desired optimal compression of the pellet are determined independently, the results allow one to obtain the laser power history $W(t) = W(P_a(r_p(t))$, $Z_i$, $m_i, n_i)$. For a critical radius much larger than $r_p$, most of the energy flows outward; this seems to invalidate known simple estimates of the relation $W(P_a)$.

I. INTRODUCTION

In order to achieve thermonuclear reactions by means of short, intense laser pulses, a pellet must be compressed to very high densities, and have its core heated to a high temperature. This is only possible if the pressure at the pellet surface (the ablation pressure $P_a$) changes in time in a special manner, dependent on the type of pellet. Clearly, the laser pulse should be appropriately shaped.

As a complement to the results of thorough numerical simulations, it would be convenient to have available simple, analytical, scaling laws, relating $P_a$ (and may be the mass ablation rate) to the laser power $W$, and to other parameters such as the critical density $n_c$ (or equivalently the laser wavelength), the pellet radius $r_p$, the ion mass $m_i$, and charge number $Z_i$ in the expanding corona of the plasma; this requires a careful analysis of the energy flux in the corona.

The planar problem has recently been studied, allowing for unsteady effects and different ion and electron temperatures, $T_e$ and $T_i$, and using classical heat conduction, absorption at $n_c$, and a laser pulse linear in time, for which a self-similar motion develops: the evolution of the behavior of both the coronal plasma and the dense pellet as the pulse rise-time shortened was thoroughly analyzed.

A steady, spherical corona was considered by Gitomer et al., who assumed $T_e = T_i$, absorption at $n_c$, and classical heat conduction; unsteady effects were allowed for in related numerical computations.

A scale law was given by Caruso for the limit $r_p/\rho_c \to \infty$, $r_p$ being the critical radius, where $n = n_c$. In this paper we re-examine the work of Ref. 8 and complete it in four respects:

(i) We properly take into account that for the conditions of interest (a not too steep pulse initially, and $n_c$ much less than the pellet density $n_e$) there exists a well-defined ablation surface; this closes the problem in such a way that, instead of families of solutions as in Ref. 8 we arrive at a single universal relation between $P_a$ and $W$, and the parameters $r_p$, $n_c$, $m_i$, and $Z_i$, the goal of the paper.

(ii) We find that for $W$ less than a limiting value $W[R, n_c, m_i, Z_i]$, which we determine, the results of Ref. 8 are invalid. Our solution for $W < W^*$ is found to agree in the limit $W/W^* \to 0$ (when heat conduction is restricted to a thin layer) with planar results from Ref. 5.

(iii) We find a value $W[R, n_c, m_i, Z_i]$, at which the faraway plasma experiences a discontinuous transition in the way $T_i$ decays with radius $r$ (from $T_i \to \gamma^{3/4}$ for $W < W^*$, to $T_i \to \gamma^{3/4}$ for $W > W^*$, through the transition law $T_i \to \gamma^{3/4}$ for $W = W^*$). This phenomenon, similar to one found in planar, unsteady flow, is of particular relevance for some effects not considered here, such as the appearance of a two-temperature electron population, the breakdown of quasi-neutrality, etc.

An important conclusion of our work is that when $r_p/\rho_c$ is large, at the end of the compression, most of the energy absorbed at $r_p$ flows outward; this result seems to invalidate simple estimates of the relation $P_a(W)$ in spherical geometry.11,12

II. STATEMENT OF THE PROBLEM

We consider a single ion-species plasma, and assume that the flow is quasi-neutral ($Z_i \rho_i = n_e \approx n$), spherically symmetric, and that the ratio $n_e/n_i$ is small. To lowest order in an asymptotic expansion in that parameter, the density in the corona will go to infinity at the pellet surface, and the flow may be assumed steady (the pellet/corona characteristic velocity and therefore the characteristic time ratio is of order $(n_e/n_i)^{1/2}$ since the momentum fluxes in pellet and corona, being produced by $P_a$, are comparable). The mass flow rate, being independent of $r$, and the pressure, being equal to $P_a$, should remain finite, hence, the ion and electron velocities ($v_i$ and $v_e$) and temperatures, should vanish at the pellet. Clearly, the collision fre-
We use classical values for the ion-electron energy relaxation time \( t_i = \frac{T_e}{T_i} \) and the electron heat conductivity \( (K_e = \frac{RT_e}{m_e}) \), and neglect ion conduction and the weak variations of the Coulomb logarithm in both \( t_i \) and \( K_e \).

In steady flow, mass conservation may be written in the form

\[
\nu v^2 = \mu, \quad (1)
\]

where \( \mu \) is a constant and \( 4\pi n_i m_i / Z_i \) is the mass flow rate. Using quasi-neutrality and neglecting electron inertia, the equation for momentum conservation in the ion-electron fluid reads

\[
m_i \nu \frac{d\theta}{d\tau} = -\frac{d}{d\tau}[\nu (Z_i T_e + P)] + F, \quad (2)
\]

where \( n \) is Boltzmann’s constant. Finally, the electron and ion entropy equations are

\[
n T_e \nu \frac{d\theta}{d\tau} \left( \frac{K T_e}{n} \right) = \frac{K_i}{\rho} \frac{d}{d\tau}(\nu T_e^2) + \frac{3}{2} \frac{K T_e}{K T_e^{1/2}} + W_0 (r - r_e) \frac{d^2 r}{d\tau^2},
\]

\[
n \frac{Z_i}{\rho} \nu \frac{d\theta}{d\tau} \left( \frac{K T_e}{n} \right) = \frac{3}{2} \frac{K T_e}{K T_e^{1/2}}, \quad (3)
\]

absorption occurs at the critical radius given by

\[
n(r_e) = n_e.
\]

If \( Z_i \) is large, the ion pressure term may be dropped in (2), which, using (1), becomes

\[
\frac{d}{d\tau}\left( \frac{1}{2} \frac{m_i \nu^2}{Z_i T_e} \right) = h Z_i T_e \left( \frac{2}{\nu} - \frac{1}{T_e} \frac{d T_e}{d\tau} \right) \left( 1 - \frac{Z_i}{m_i} \frac{d T_e}{d\tau} \right)^{-1}. \quad (2a)
\]

As seen from (4), the energy exchange term may also be dropped in (3) when \( Z_i \) is large; using (1) and (2'), and integrating from \( r \to r_e \) to an arbitrary radius, we get

\[
\mu \left( \frac{Z_i}{\rho} \nu \frac{d\theta}{d\tau} \left( \frac{K T_e}{n} \right) \right) = Z_i K T_e^{1/2} \frac{d T_e}{d\tau} + \frac{W_0 (r - r_e)}{4\pi}, \quad (3')
\]

where \( H \) is the Heaviside function

\[
H = 0 \quad (r < r_e), \quad H = 1 \quad (r > r_e);
\]

to obtain (3'), which is the energy equation for the ion-electron fluid, we used the conditions \( \nu = T_e = \frac{d T_e}{d\tau} = 0 \) at \( r = r_e \). Equation (5) becomes

\[
\nu = \frac{\nu_0 (r)}{n_e}. \quad (5)
\]

We must then solve Eqs. (2') and (3'), subject to the boundary conditions

\[
T_e = 0 \quad \text{at} \quad r = r_e, \quad (6)
\]

\[
\frac{m_i \nu}{Z_i} \frac{d^2 r}{d\tau^2} = P_a \quad \text{at} \quad r = r_e, \quad (7)
\]

where the pressure, \( nh T_e \), was rewritten by means of (1), in the form \( \frac{m_i \nu}{Z_i} T_e \). An additional boundary condition is that as \( r \to \infty \), \( T_e \) becomes neither negative nor multivalued; this, together with the requirement of vanishing density (and pressure) at infinity, is seen to imply

\[
T_e \to 0 \quad \text{as} \quad r \to \infty. \quad (8)
\]

Finally, for \( v \) not to be multivalued, either the numerator on the right-hand side of (2') must vanish at the isothermal sonic point or this point lies at the critical radius: either

\[
\frac{2}{\nu} - \frac{1}{T_e} \frac{d T_e}{d\tau} = 0, \quad (9a)
\]

or

\[
r = r_e, \quad (9b)
\]

at

\[
v = v_e, \quad T_e = T_e, \quad \eta = \eta_e, \quad \text{and} \quad \eta \to 1, \quad \text{at} \quad \eta \to 1; \quad \text{then}, \quad \eta = \frac{Z_i}{m_i} \frac{P_a}{\nu_0}. \quad (10)
\]

We choose the reference temperature \( T_e \) in such a way that condition (7) reads "\( \theta = 0 \) at \( \eta = 1 \); then, we have

\[
T_e = \frac{Z_i}{m_i} \frac{P_a^2}{\nu_0}, \quad (13)
\]

and Eq. (3') becomes

\[
\frac{5}{2} \theta + \frac{1}{2} \nu_0^2 = \frac{\theta}{\eta^2} \frac{d^2 \theta}{d\eta}, \quad \text{where} \quad \nu_0 = \frac{Z_i}{m_i} \frac{P_a^2}{\nu_0}, \quad (14)
\]

where

\[
H = 0 \quad (\eta < \eta_e), \quad H = 1 \quad (\eta > \eta_e), \quad (15)
\]

\[
\beta = \frac{Z_i}{m_i} \frac{P_a^2}{\nu_0}, \quad (16)
\]

\[
\eta_0 \equiv \frac{\nu_0 (r)}{n_e}. \quad (17)
\]

The boundary conditions are

\[
\theta = 0 \quad \text{at} \quad \eta = 1, \quad (18)
\]

\[
\theta = 0 \quad \text{as} \quad \eta \to 0, \quad (19)
\]

either

\[
\frac{2}{\nu} - \frac{1}{T_e} \frac{d T_e}{d\tau} = 0, \quad (21a)
\]
or
\[ \eta = \eta_0 \]  
(21b)

at \( \eta = (u_0^2 = \theta) \). The equation for \( \theta = T_i/T_r \) is
\[ \beta \eta^2 \left( 3 \frac{d \theta}{d \eta} + 2 \frac{d u^2}{d \eta} \right) = \frac{1}{2} \left( \frac{d u^2}{d \eta} \right)^2 \]
\[ \theta \eta - \eta \left( \frac{d \theta}{d \eta} \right) = 41.7 \frac{\theta - \theta_i}{\theta^2} \]
\[ \theta \eta - 1 \text{ as } \eta - 1. \]

In the next section we solve these equations choosing \( \eta_0 \) as a free parameter in the range \( 1 < \eta_0 < \infty \); for each \( \eta_0 \) we obtain \( \beta, W_t \), and \( u(\eta) \), and then, from (15)-(17), we get \( \eta, \eta_0, \eta_1, \eta_2, \eta_3, \eta_4 \) (or \( \eta_0 \), \( \eta_1 \), \( \eta_2 \), \( \eta_3 \), \( \eta_4 \), \( \eta_5 \) and \( \eta_6 \) leading to the relations
\[ \eta = \text{constant}, \]
\[ \theta = \text{constant}, \]
\[ u = \text{constant}, \]
\[ \beta = \text{constant}, \]
\[ \eta_0 = \text{constant}, \]
\[ \eta_1 = \text{constant}, \]
\[ \eta_2 = \text{constant}, \]
\[ \eta_3 = \text{constant}, \]
\[ \eta_4 = \text{constant}, \]
\[ \eta_5 = \text{constant}, \]
\[ \eta_6 = \text{constant}. \]

**III. GENERAL INTEGRATION OF THE EQUATIONS**

Two points should be noted prior to any further development; (i) For \( H = 0 \) (\( \eta < \eta_0 \)), Eqs. (12) and (14) allow us to introduce phase-plane variables
\[ Y = \eta \beta \eta^{3/2}, \quad M^2 = u^2/\theta \]
(22)
\( (M \text{ is the Mach number based on the isothermal sonic speed}) obeying the equations
\[ dY/d\eta = Y + \frac{5(M^2 + 5)}{4}, \quad \text{(23)} \]
\[ dM^2/d\eta = M^2[8Y - (M^2 + 5)(M^2 + 1)][2Y(M^2 - 1)]^{1/4}, \quad \text{(24)} \]
and eliminating \( \eta \),
\[ dM^2/dY = 2M^2[8Y - (M^2 + 5)(M^2 + 1)] \]
\[ \times [Y(M^2 - 1)(4Y + 5M^2 + 25)]^{1/4}. \quad \text{(25)} \]

(ii) As pointed out in Ref. 8, \( \eta_0 \) cannot be less than \( \eta_1 \); if \( \eta_0 = \eta_1 \), we must have \( \text{see Eq. (21a)} \) \( d\theta/d\eta = 2\theta/\eta > 0 \)
 at \( \eta_1 \), while \( d\theta/d\eta \) should be negative beyond \( \eta_1 \). Hence, (23)-(25) are valid at least up to the sonic radius.

**A. Absorption beyond the sonic radius (\( \eta_2 > \eta_1 \))**

Equations (18) and (19) imply \( Y = 0, M^2 = 0 \) at \( \eta = 1 \).

The solution to (25) near the point \( Y = 0, M^2 = 0 \), which is a nodal point, is \( M^2 = CY^{3/2} \); the constant \( C \) is related to the eigenvalue \( \beta \) through the boundary condition (19). We then get
\[ M^2 = (Y/\beta)^{2/n}, \quad \beta = 1. \quad \text{(26)} \]

At the sonic point \( (M^2 = 1), dM^2/dY \) must remain finite to avoid a multivalued solution; that is, the numerator on the right-hand side of (25) must vanish [condition (21a)], leading to \( Y = 3/2 \). There is a value of \( \beta \) (call it \( \beta_1 \)) that allows solution (26) to reach the (saddle) point \( Y = 3/2, M^2 = 1 \) (see Fig. 2). Starting at this point and integrating backward toward the node we numerically find \( \beta_1 = 11.30 \). Beyond the point \( Y = 3/2 \), the solution \( M^2(Y) \) is valid up to the critical radius, wherever that may be. Once \( M^2(Y) \) is known, Eq. (23) with the boundary condition \( Y = 0 \) at \( \eta = 1 \) yields \( Y(\eta) \), and then \( u(\eta) \) and \( \theta(\eta) \), for \( \eta < \eta_1 \). They are given, together with \( \theta \), in Fig. 2. The sonic point occurs at \( \eta_1 = 1.215 \eta_{ref} \).

A simple estimate of \( \eta_1 \) may be obtained by noting that for \( 1 < \eta < \eta_1 \), we have \( 0 < u^2/\theta < 1 \) and therefore
\[ \theta \eta^{3/2} = 2\theta, \quad \text{yielding, for some intermediate } \alpha, \]
\[ \beta \eta^{3/2} = 5\alpha(\eta - 1)/\eta; \quad \text{at the sonic point we also have} \]
\[ 3\theta = (3\theta \eta^{3/2}) \beta/\eta, \quad \text{so that finally} \]
\[ 1.20 < \eta_{ref} = 1 + 3/5 \alpha < 1.24. \]

For any chosen value of \( \eta_0 > \eta_{ref} \), Fig. 2 yields both \( u(\eta_0) \) and \( \theta(\eta_0) \). Equations (12) and (14) could then be integrated, starting at \( \eta_0 \), if the eigenvalue \( \eta \) were known; \( \eta \text{ is determined by the boundary condition (20).} \)

To carry out the numerical integration it is actually more convenient to start at large \( \eta \) and integrate backward.

Let us then consider the behavior of the solution for large \( \eta \), well beyond \( \eta_0 \). It was found in Ref. 8 that \( \theta \) could decay either as \( \theta - \eta^{-2/3} \) (adiabatic flow) or \( \theta - \eta^{-3/7} \), this second case seeming to be the real one for large enough \( \eta_0 \). We find however that (i) there exists

![Fig. 1. Integral curves in the phase-plane $M^2 - Y$ (dashed curves are schematic).](image)
a third decay law, $\theta \sim \eta^{2/3}$, and (ii) there is a transition value $\eta^*_c$ such that if $\eta_c < \eta^*_c$, then

$$\theta \propto C_2 \eta^{2/3}, \quad \eta^2 - 2W = -5\theta;$$

if $\eta_c > \eta^*_c$,

$$\theta \propto C_2 \eta^{2/7}, \quad \eta^2 - 2W = -4\beta_1 C_2^2/7 - 16\theta;$$

if $\eta_c = \eta^*_c$,

$$\theta \propto \frac{(35/4\beta_1)^{5/6}}{\eta^*}, \quad \eta^2 - 2W^* = -16\theta,$$

where $W^* = W(\eta^*_c)$. We find numerically that $\eta^*_c \approx 2.05$, $W^* = 5.55$. If $\eta_c$ is less than, but close to $\eta^*_c$, $C_1$ is large and the approximation (27) breaks down for $\eta = O(C_1^{15/14})$, the solution taking the form (29) for $\eta = O(C_2^{-1/4})$. Similarly for $\eta_c$ larger than, but close to $\eta^*_c$, $C_2$ is small, and (28) breaks down for $\eta = O(C_2^{-1/4})$, (29) being valid for $\eta_c < \eta < C_2^{-1/4}$. The

regions of validity of (27) and (28) are removed to infinity as $\eta_c - \eta^*_c \to 0^-$ ($C_1 \to \infty$) and $\eta_c - \eta^*_c \to 0^+$ ($C_2 \to 0$), respectively. More details of the transition are discussed in the Appendix.

Then, to perform the integration for the range $\eta > \eta^*_c$, one chooses some arbitrary value of $C_1$ in (27), and sweeps over a convenient $W$ range until a value is found for which $\theta(\eta)$ and $u(\eta)$ meet the curves in Fig. 2 at a common $\eta$, which will be $\eta_c$. Similar procedures are followed to use (28) and (29).

B. Absorption at the sonic radius ($\eta_c = \eta^*_c$).

The eigenvalue $W$ decreases with decreasing $\eta_c$. When $\eta_c = \eta^*_c = 1.215$, $W$ takes the limiting value $W^* = 1.86$. Since $\eta_c$ cannot be less than $\eta^*_c$, one must have $\eta_c = \eta^*_c < \eta^*_{cl}$ for $W < W^*$. Then, to obtain the solution for

![FIG. 2. Dimensionless squared velocity and temperatures versus dimensionless radius for $1 < \eta < \eta_c$ and $\eta_c > \eta^*_c$.](image)

![FIG. 3. Dimensionless parameters $\beta$, $W$, and critical velocity, and ablation to critical pressure ratio versus critical radius (dashed curves are the asymptotic behavior for large $\eta_c$).](image)
this range, one chooses an arbitrary sonic point $M^2 = 1$, $Y < 3/2$ (see Fig. 1) and integrates toward the node $Y = 0$, $M^2 = 0$; in this way, a value $\beta < \beta_1$ is determined in (26). Once $M(Y)$ is known, Eq. (23) may be solved, yielding $\mu(\eta)$, $\theta(\eta)$ for $\eta < \eta_s$, and a value $\eta_s < \eta_d$ is found. Since now $\eta_c = \eta_s$, the integration beyond $\eta_s$ proceeds as in Sec. III A.

Complete results for $\beta(\eta_a)$, $W(\eta_a)$, and $\mu(\eta_a)$ are given in Fig. 3. Also shown for later reference is the ratio of ablation pressure $P_a$ to critical pressure $P_c = n \mu T_s(r_s)$. From Fig. 3 and Eqs. (15)-(17), $P_a$, $W$, and $\mu$ can be obtained as functions of $\eta_a$. Finally, we arrive at $W(P_a)$ and $\mu(P_a)$, shown in Figs. 4 and 5; for convenience, $P_a(W)$ is shown in Fig. 6.

IV. THE LOW-POWER ($\eta_c \rightarrow 1$) LIMIT

At the beginning of the pulse, the critical surface will lie close to the pellet, that is, $\eta_c \approx 1$. Figure 3 shows that $\beta \rightarrow 0$ as $\eta_c \rightarrow 1$; as seen in Eqs. (12) and (14) heat conduction is restricted, for small $\beta$, to a thin layer where the gradients are large, the flow outside being isentropic:

$$\frac{1}{2} \left( 1 - \frac{\theta}{\mu^2} \right) \frac{d\mu^2}{d\eta} - \frac{\theta}{\eta} = \frac{d\theta}{d\eta},$$

$$\frac{\theta}{2} + \frac{1}{3} u^2 = \bar{W}. \tag{31}$$

The solution to Eqs. (30) and (31) is

$$u^2/3(2\bar{W} - u^2) = 5A\eta^{-4/3}, \tag{32}$$

$$5(\bar{W} - 5\theta) = A\eta^{-4/3}, \tag{33}$$

where $A$ is a constant. It should be noted that the flow velocity can be less than the isentropic sound speed ($u^2 < 5\theta/3$) nowhere in this region because, otherwise, from

$$\frac{d\theta}{d\eta} = \frac{d\mu^2}{d\eta} = \frac{(30\theta/3\eta)u^2}{u^2 - 5\theta/3} \tag{34}$$

FIG. 4. Laser power versus ablation pressure.

FIG. 5. Mass ablation rate versus ablation pressure. To the right of point A ($\eta_c = \eta_d$), the simple law shown holds. Dashed line is the asymptotic behavior for small ablation pressure.
which follows from (30) and (31), we would get a positive $d\delta/d\eta$ value. We thus have a supersonic and isentropic expansion to vacuum, that can only start from sonic or supersonic conditions.

In the thin, nonisentropic layer, the first term on the right-hand side of (30), arising from spherical geometry, may be dropped against the second one. An immediate integration then yields

$$u^2 + \theta = u^2,$$

where the condition $\theta/u = 1$ at $\eta = 1$ has been used. One easily verifies that both $u$ and $\theta$ are of order unity throughout the layer; hence, its width will be of the order of $b_x$. Defining $\eta = (\eta - 1)/b_x$, Eq. (14) becomes

$$\delta^2/\lambda^3 \frac{d\eta}{d\delta} \left( \frac{5\delta}{2} + \frac{x^2}{2} \right) = -\lambda H.$$

Equations (35) and (36) have been studied in Ref. 7. The solution to (35) valid up to terms of order $b_x^2$ is

$$u = \frac{1}{3} \left( 1 + (3 - 2\theta) \right).$$

Since the maximum of $\theta$ must lie at the critical radius, the lower (upper) sign corresponds to $\eta < \eta_c$ ($\eta > \eta_c$), and $\eta_c = 1/4$, $\mu_c = 1/2$ (Ref. 7); the flow is indeed (isothermally) sonic at the critical surface. Using (37) in (38) we get

$$\eta = \int_0^\eta x^{1/2} \left( \frac{5}{2} + \frac{1}{8} \left[ 1 + \left( 1 - 4x \right)^{1/2} \right] - \lambda H \right)^{1/2} dx.$$  

The value $\eta_c = \eta_c(1/4) = 4.66 \times 10^{-3}$ is obtained from (38) with the lower sign and $H = 0$. The function $F(\eta) = 5\eta/2 + \left[ 1 + (1 - 4\eta)^{1/2} \right]/3$ presents a maximum $25/32$ at $\eta = 15/64$. On the other hand, the denominator of (38) must vanish when $\eta \to \infty$; hence, we should have $24/32 = F(1/4) < \lambda < F(15/64) = 25/32$. This inequality, together with (37), leads to the condition that $u^2/\theta$ lie between $5/3$ and unity when $\eta = \infty$. Matching the inner ($\eta \to \infty$) and the outer ($\eta = 0$) solutions for $u^2/\theta$ then yields $u^2/\theta = 5/3$, that is, the Chapman–Jouguet condition (sonic flow) behind the thin conduction layer is satisfied; also, we find $\lambda = 25/32$.

Using (16) and (17), together with $n_x = 1$, $u(\eta_c) = 1/2$, $\lambda = 25/32$, we finally arrive at

$$u = \left( \frac{2b_x}{k_0} \right)^{1/2} \frac{1}{3} \eta^{1/2},$$

$$\lambda = 4(b_x^2)(25/32)(2/\eta_x^0)(1/3)\eta_x^{3/2}.$$

The inner (subsonic) and the outer (supersonic) solutions can be matched smoothly, analyzing an intermediate layer, at the transonic region, where $\eta \gg 1$ but $\eta - 1 \ll 1$; therefore, we shall take $\lambda = \xi(b) \cdot \eta$ of order unity with $\xi(b) \ll 1$, and thus $\eta - 1 = (\xi/b) \xi$ with $\lambda/b = 0$. The dependent variables $\theta$ and $u^2$ and the eigenvalue $\lambda$ differ from their isentropic sonic values, $15/64$, $25/64$, and $25/32$, respectively, in terms of the order of $\lambda$. Defining the expansions in powers of $\lambda$

$$\theta = (1/4)(1 + \xi + \lambda \theta + \cdots),$$

$$u^2 = (25/64)(1 + \xi + \lambda u + \cdots),$$

$$\lambda = (25/32)(1 + \xi + \lambda \xi + \cdots),$$

and, by substitution into Eqs. (12) and (14), we get

$$\theta_0 + u_0^2/3 = 4\lambda W_0/3,$$

for the terms of order $\lambda$; in both equations (the order unity terms reduce to an identity), and

$$\theta_0 + u_0^2/3 + 3\theta_1^0 - 6\lambda \theta_1 - 2\lambda = C_0,$$

$$\theta_0 + u_0^2/3 = 2 \left( \frac{15}{64} \right)^{1/2} \frac{d\theta_1}{dx} + 10W_0,$$

for the terms of order $\lambda^2$, respectively. It can easily be shown, from Eq. (12), that $\lambda = \beta^{1/3}$ (the small parameter of the asymptotic theory of transonic flow). The constants $W_0$ and $C_0$ can be obtained from the matching of (39) and (40), for $\xi = 0$, with the solution (35) (independent of $\lambda$) and valid up to terms of order $\beta$ thereby obtaining $W_0 = C_0 = 0$. The remaining equation
\[
\frac{d\theta}{d\xi} + \left(\frac{15\xi}{2}\right)\frac{64}{15} \theta^2 - 5 \left(\frac{64}{15}\right) \theta^{5/2} \left(\xi - \frac{2\sqrt{2}}{3}\right) = 0, 
\]
which follows from (40) and (41), must have the asymptotic behavior \( \theta_\xi \to \infty \) for \( \xi \to 0 \), and \( d\theta/d\xi \to 0 \) for \( \xi \to \infty \), in order to match correctly with the subsonic and supersonic regions, respectively. These two conditions provide the value of the integration constant and the eigenvalue \( W_\xi \). Solution of (42) can be carried out in terms of the Airy functions giving \( W_\xi = 9.34 \times 10^{15} \), and then \( W = (25/32)[1 + 3.345(\eta - 1)^{9/3} + \cdots] \), in terms of \( \eta_c \) instead of \( \theta \), using the \( \eta_c \) value.

V. THE HIGH-POWER (\( \eta_c \to \infty \)) LIMIT

Near the end of the compression the pellet radius is expected to become a small fraction of the critical radius, that is, \( \eta_c \to \infty \). In this limit, some simple analytical results may be obtained.

To determine \( \mu(\eta_c) \) and \( \theta(\eta_c) \) for large \( \eta_c \) it will suffice to obtain the asymptotic solution to Eqs. (23)–(25) for large \( \eta_c \). Since \( \theta \) increases up to \( \eta_c \), it is clear that \( Y \to \infty \) as \( \eta \to \infty \) (see Fig. 2). In addition, it is easily shown that \( M^2/Y \to 0 \) as \( Y \to \infty \); in fact, we have roughly \( M^2 \to \infty \). Therefore, in the lowest approximation, Eq. (23) becomes \( dY/d\ln \eta = Y \), which implies \( Y/\eta = \text{const} \). Thus, Eq. (23) yields \( \theta = \text{const} = \theta_\infty \). On the other hand, (25) becomes \( dM^2/dY = 4M^2/(Y^2 - 1) \), whose solution is \( 4Y = M^2 + \text{const} \); using (25) and \( \theta = \theta_\infty \) we arrive at \( \eta^2 - \theta_\infty \ln \eta^2 - \theta_\infty \ln \eta = \text{const} \); the constant being \( -3.94 \) and \( \theta_\infty \) being 0.992, found numerically.

The expression \( \beta^2 = 4\theta_\infty \ln \eta \), used in Ref. 8, is only valid for \( \eta \) unrealistically large; in that case one finds, for the next approximation, \( \theta = \theta_\infty - (2 \ln \eta)/(n\theta_\infty \beta^2) \). It is clear therefore that for \( \eta_c \) large, and neglecting terms of order \( \eta_c^1 \ln \eta \) against unity, we may set

\[
\theta_\infty = \theta_\infty = 0.992, \\
\eta^2 - \theta_\infty \ln \eta^2 - \theta_\infty \ln \eta = \text{const} = -3.94.
\]

(43)

To analyze the range \( \eta > \eta_c \) for \( \eta_c \) large, we define \( \eta = \eta/\eta_c, \theta = \theta/\theta_\infty, u = u/\theta_\infty, u^2 = u^2/\theta_\infty = M^2, \) Then, from (12) and (14), we get

\[
(\frac{M^2 - \theta}{\theta_\infty}) \frac{d\theta}{d\eta} = 4\frac{\beta}{\theta_\infty} - 2\frac{2M^2}{\theta_\infty}.
\]

(44)

\[
(\frac{5M^4 - \theta^2}{\eta^2}) \frac{d\theta}{d\eta} = \frac{2W}{\theta_\infty^2} + \frac{4}{7M^2} \beta^{9/2} \frac{d(M^2/\theta)}{d\eta}.
\]

(46)

Notice that \( M^2 = \text{const} \) for \( \eta_c \) large. We formally consider \( M^2 \) and \( \eta_c \) independent parameters and expand the solution to (44) and (46) in powers of \( \eta_c \), retaining the entire dependence on \( M^2 \); \( \theta = \theta_0(\eta) + \eta^2 \theta_1(\eta) + \cdots, u^2 = u_0^2(\eta) + \eta^2 u_1^2(\eta) + \cdots, W/\eta_0 = W_0 + \eta^2 W_1 + \cdots \), obtaining successively

\[
\frac{2}{7} \beta \frac{\theta_0^{9/2}}{\eta^2} \frac{d(M^2/\theta)^{9/2}}{d\eta} + W = 0, \quad \theta_0(1) = 1, \quad \theta_0(\infty) = 0,
\]

\[
\frac{M^2 - \theta_0^2}{\eta_0^2} \frac{d\theta_0}{d\eta} = \frac{4\theta_0}{\eta_0^2} - \frac{2\theta_0}{\eta_0}, \quad u_0^2(1) = 1.
\]

The equations for \( \theta_0, \theta_1, \) etc., which are of first order must satisfy two boundary conditions each; this allows us to determine \( W_0, W_1, \) etc. Proceeding straightforwardly we arrive at

\[
\theta_0(1) = \theta_0(\infty) = 0.
\]

(47)

(48)

(49)

VI. DISCUSSION

We have studied the ablative, steady expansion of the spherical coronal plasma produced by irradiating an overdense pellet by a high-intensity laser pulse. We assumed quasi-neutrality, absorption at the critical density, classical heat conduction, and large ion charge number, and let the ion and electron temperatures differ from each other. We have found a universal relation, shown in Fig. 4 (and Fig. 5 too) between ablation pressure \( P_a \), laser power \( P \), pellet radius \( r_p \), critical density \( n_c \), ion mass \( m_i \), and charge number \( Z_i \). We also found the mass ablation rate in terms of these parameters (Fig. 5).

For a given target, one may determine, by neglecting the mass ablation rate, the time law for the ablation pressure, \( P_a(t) \), that generates an optimal compression; this will be called the inner problem and has been studied elsewhere. Its solution yields the pellet radius as a function of time, \( r_p(t) \), too. The present paper solves what we will call the outer problem: Considering the expansion flow to be quasi-steady (a good approximation for \( n_c \) much less than pellet density), we determine for given, instantaneous, \( P \) and \( r_p \), the laser power required \( W(P, r_p, n_c, Z_i, m_i) \). Hence, for any pellet, we can, using Fig. 4, obtain the laser power history that generates an optimal compression

\[
W(t) = W(P_a(t), r_p(t), n_c, Z_i, m_i).
\]

Our solution provides the small mass ablation rate (Fig. 5), which can then be used to obtain a corrected solution for the inner problem, if desired; for powers high enough to yield \( r/v_r > 1.215 \) (\( r_e \) is the critical radius), the mass ablation rate takes a particularly simple form (see Fig. 5).

For \( r/v_r < 1.215 \), we find the flow sonic at \( r_e \). In the model considered here, in which absorption occurs at a surface, the sonic condition leads to infinite acceleration; in reality, absorption may occur in a thin layer around the critical surface, and large accelerations will result. A consequence of this, is a fast drop in density in that region. The ions, which are being heated by collisions with electrons, will then experience a sudden cooling. This is clearly seen in Fig. 7, where \( T^2 \) drops fast beyond the critical surface, while above \( r^2 \) stays close to unity. Note that \( P_a \sim n_e^2/\tau \) always. This result invalidates the hypothesis of Carusio\(^3\) that, for \( r/v_r > 1 \), \( n_e \) would be an ignorable parameter. We find that for \( r/v_r > 1 \) most of the energy flows outward; this unexpected result explains the failure of Caruso’s hypothesis, that leads to pressures much larger than those found here. Estimates of ablation pressure,\(^4\) assuming flux limited conduction, give values of \( P_a \), \( (r/v_e)^{7/3} \) times larger than found here.

**ACKNOWLEDGMENTS**

This research is partly based on a doctoral thesis by J. Sanz.

This work was performed under the auspices of the Junta de Energía Nuclear of Spain.

**APPENDIX**

The heat to internal-energy-convection flow ratio

\[
\phi = \frac{|RT^2/4dT_r/4V|}{2n_b^2 T_e v}
\]

is a measure of the nonisentropic character of the flow. It is easily verified, using Eqs. (27)-(29), that as \( \eta \to 1 \),

\[
\phi \sim \eta^{-7/3} \to 0, \quad \mathcal{W} \to \mathcal{W}_e,
\]

\[
\phi \sim \eta^{-7/3} \to 0, \quad \mathcal{W} \to \mathcal{W}_e.
\]

Thus at \( \mathcal{W} = \mathcal{W}_e \), heat conduction goes discontinuously from negligible to dominant at vanishing density.

For \( \eta > \eta_0 > 1 \), the Mach number \( u^2/\delta \) is negligible and Eq. (12) becomes

\[
\frac{dU_\|^2}{d\eta} = \frac{4\delta}{\eta} - \frac{2d\delta}{d\eta}.
\]

(A1)

Let us define phase-plane variables

\[
\tilde{M}_\| = (2\mathcal{W} - u^2)/\delta, \quad Y = \beta n_0 e^{\theta/2};
\]

then, from Eqs. (A1) and (14), we obtain

\[
\frac{dM_\|}{dY} = \frac{(5 - \tilde{M}_\|)(1 - \tilde{M}_\|^2/2) - 4Y}{Y^2 + 5Y(5 - \tilde{M}_\|^2/4}.
\]

(A2)
The asymptotic solutions given in Eqs. (27)-(29) are represented by the points \((Y=0, M^2 = 5), (Y \to \infty, M^2/Y = 4/7), (Y = 35/4, M^2 = 12)\); the first two are saddle points of \((A2)\) and the third is a node. For \(W < W^*\), the solution, in the \((Y, M^2)\) plane starts at \(Y = 0, M^2 = 5 (\text{large } \eta)\) and goes over to the node \(Y = 35/4, M^2 = 12 (\text{as } \eta \text{ decreases})\). Similarly, for \(W > W^*\), the solution comes from infinity \((M^2/Y = 4/7)\) toward the node as \(\eta \text{ decreases}\).