Free convection boundary layers driven by exothermic surface reactions: critical ambient temperatures

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Abstract—A model for the free convection boundary-layer flow near a forward stagnation point driven by heating from a surface on which there is a catalytic reaction is discussed. The governing equations are made dimensionless so as to highlight the ambient temperature with the main emphasis being to determine critical ambient temperatures. The basic model is then reduced to a standard free convection problem by a transformation of variables from which bifurcation diagrams (plots of a dimensionless surface temperature against a dimensionless ambient temperature) can be constructed. These show a hysteresis bifurcation, the position of which can be readily deduced. A feature of the present formulation is the occurrence of disjoint bifurcation diagrams whereby the upper solution branch becomes separated from the lower solution branches. This aspect is also discussed in detail.

1. INTRODUCTION

There are many chemical reactions with important practical applications which proceed only very slowly, or not at all, except in the presence of a catalyst. A common configuration for such reactions is for the reactants (usually, but not exclusively, in the gaseous phase) to be made to flow over the solid catalyst, with the reaction taking place on the surface of the catalyst. The reaction is maintained by a fresh supply of reactants being brought to the catalyst surface by the flow. A review of the chemical aspects of surface, or heterogeneous, reactions has been given recently by Gray and Scott (1990), and Scott (1991). A full discussion of catalysis and a description of many of its practical applications is given by Bond (1987). Examples of such catalytic surface reactions, which are of importance in the chemical industry, are provided by the work (both experimental and theoretical) on methane/ammonia and propane oxidation over platinum by L. D. Schmidt and co-workers (Williams et al., 1991a,b; Song et al., 1991a,b,c). These processes are modelled by boundary-layer flows, mostly stagnation-point flows (see also the work on hydrogen-air combustion by Ikeda et al., 1993) with an exothermic reaction being allowed for on the boundary surface.

The detailed modelling of these, often complex, reaction systems usually also includes the effect of the reaction in the bulk (homogeneous reaction). Though this
effect can, in some cases, play a significant role in the overall combustion process, there are many operating conditions for which it plays only a minor role, with response of the combustion system being dominated by the surface (or heterogeneous) reactions. Even in cases where the homogeneous reaction cannot be ignored, a catalytic surface reaction is needed for bulk reaction to be sustained.

In order to gain some insight into this process, we decided to isolate the effect of the catalytic reaction and in a series of recent papers, Chaudhary and Merkin (1994a, b, c) we derived a model for free convection boundary layers which are driven purely by the heat supplied to the surrounding fluid by an exothermic surface reaction. We modelled the reaction by the single first order Arrhenius kinetics

\[ A \rightarrow B \quad \text{rate} \quad k_0 a e^{-E/RT}, \]

where \( a \) is the concentration of reactant \( A \), \( T \) is the temperature, \( E \) is the activation energy, and \( R \) is the gas constant. Heat is released by the reaction at a rate \( Qk_0ae^{-E/RT} \), which is taken from the surface into the surrounding fluid and a convective fluid motion is set up by the resulting buoyancy forces.

In Chaudhary and Merkin (1994a) we considered the two-dimensional boundary-layer flow near a stagnation point, examining in detail the possible steady states that can arise. We found the parameter ranges over which multiple steady states are possible and we were able to determine these ranges explicitly. We discussed some aspects of the time-evolution of this model in Chaudhary and Merkin (1994b). The analogous problem of the convective flow along a vertical surface was treated in Chaudhary and Merkin (1994c). Here we found that the system could still exhibit critical behaviour, i.e. undergo a rapid transition from a slow reaction state to a rapid reaction state, but now this was found to occur at a finite distance from the leading edge rather than for specific values of dimensionless parameters, as was the case for the stagnation-point flow.

The most natural physical variable in any bifurcation analysis is, perhaps, the ambient temperature \( T_a \), since this is the most readily controlled experimentally. The response of the system is then described by a bifurcation diagram, i.e. a plot of some characteristic measure of the solution such as the surface temperature \( T_s \) against \( T_a \). Such bifurcation diagrams usually have the typical S-shape from which the critical ambient temperature \( T_{a,c} \) can be determined. For values of \( T_a \) less than \( T_{a,c} \) the surface temperature lies on the lower solution branch and the reaction proceeds very slowly. However, for \( T_a > T_{a,c} \) the lower solution branch is no longer accessible, and very much higher values of \( T_s \) are attained on the upper solution branch. Thus, as \( T_a \) passes through \( T_{a,c} \), the system exhibits a large increase in its reactivity.

A major difficulty associated with the traditional way in which the equations for combustion systems are made dimensionless is the identification of the critical ambient temperature, as has been shown by B. F. Gray and co-workers (Gray and Wake, 1988; Burnell et al., 1989; Gray et al., 1990, 1991; Gray and Merkin, 1990, 1993). The problem is that the ambient temperature usually appears in more than one of the dimensionless parameters and so changes in just one of these parameters, while holding the others fixed in a theoretical bifurcation analysis, cannot be identified directly with
changes only in $T_a$. In our previous work on our free convection model we adopted the standard (Frank–Kamenetskii) approach to the non-dimensionalisation of the system. This led naturally to a series of dimensionless parameters all involving $T_a$. Here we wish to concentrate on evaluating critical ambient temperatures and to do so we follow the approach suggested by Gray et al. (1991). This enables us to make the equations dimensionless in such a way that $T_a$ appears in only one dimensionless parameter, which we then use as our bifurcation parameter.

A further consequence of adopting the approach suggested by Gray et al. (1991), which is not seen when using the Frank–Kamenetskii variables, is the occurrence of disjoint bifurcation diagrams. Here the critical temperature on the upper solution branch (extinction point) approaches zero as the parameters of the system are varied, and for further changes in the parameters would require it to continue into, physically unacceptable, negative (absolute) temperatures. This point is addressed in some detail by Gray et al. (1991) and Sadiq and Merkin (1994). We find this situation to hold also in our model.

2. EQUATIONS

We consider the two-dimensional steady free convection boundary-layer flow near a stagnation point. For simplicity we make the standard Boussinesq approximation. This leads to the equations (Chaudhary and Merkin, 1993)

\begin{align}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \tag{2a} \\
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= g \beta (T - T_a) S(x) + v \frac{\partial^2 u}{\partial y^2}, \tag{2b} \\
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{\nu \partial^2 T}{\sigma \partial y^2}, \tag{2c} \\
\frac{\partial a}{\partial x} + v \frac{\partial a}{\partial y} &= \frac{\nu \partial^2 a}{S_c \partial y^2}, \tag{2d}
\end{align}

where $x$ and $y$ are coordinates measuring distance along the surface and normal to it, $u$ and $v$ are the corresponding velocity components, $T$ and $a$ are the temperature and concentration of reactant A, respectively, $\nu$ is the kinematic viscosity, $\sigma$ and $S_c$ are Prandtl and Schmidt numbers, respectively, $g$ is the acceleration due to gravity, and $\beta$ is the coefficient of thermal expansion. The boundary conditions to be applied are

\[ u = 0, \quad v = 0, \quad k_c \frac{\partial T}{\partial y} = -Q k_0 a e^{-E/R T}, \]
\[
D \frac{\partial a}{\partial y} = k_0 ae^{-E/RT} \quad \text{on} \quad y = 0,
\]
\[
u \rightarrow 0, \quad T \rightarrow T_a, \quad a \rightarrow a_0 \quad \text{as} \quad y \rightarrow \infty,
\]

where \(a_0\) is the (constant) concentration of the reactant \(A\) in the ambient fluid and \(k_c\) and \(D\) are thermal conductivity and mass diffusivity, respectively. The function \(S(x)\) is the sine of the angle between the outward normal to the body and the downward vertical and will involve some length scale \(l\) (say) for the body. For flow near a stagnation point we will have

\[
S(x) = x/l
\]

To make Eqns (2), (3) dimensionless, we start with the variable for the temperature suggested by Gray et al. (1989, 1991), putting

\[
\phi = \frac{RT}{E}, \quad \phi_a = \frac{RT_a}{E}.
\]

This gives a temperature scale \(E/R\) and leads naturally to a free convection velocity scale \(U_s\) and Grashof number \(G_r\) as

\[
U_s = \left( \frac{g\beta E l}{R} \right)^{1/2}, \quad G_r = \frac{g\beta E l^3}{R v^2}.
\]

We then put

\[
u = U_s \bar{u}, \quad v = U_s G_r^{-1/4} \bar{v}, \quad x = l \bar{x}, \quad y = l G_r^{-1/4} \bar{y}, \quad a = a_0 \bar{a}.
\]

Equations (2), (3) then become, on dropping the bars for convenience,

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]
\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = (\phi - \phi_a)S(x) + \frac{\partial^2 u}{\partial y^2},
\]
\[
u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} = \frac{1}{\sigma} \frac{\partial^2 \phi}{\partial y^2},
\]
\[
u \frac{\partial a}{\partial x} + v \frac{\partial a}{\partial y} = \frac{1}{S_c} \frac{\partial^2 a}{\partial y^2},
\]

subject to the boundary conditions

\[
u = 0, \quad v = 0, \quad \frac{\partial \phi}{\partial y} = -\lambda ae^{-1/\phi},
\]
Critical ambient temperatures

\[ \frac{\partial a}{\partial y} = \alpha \lambda ae^{-\phi} \quad \text{on} \quad y = 0, \quad (8a) \]

\[ u \to 0, \quad \phi \to \phi_a, \quad a \to 1 \quad \text{as} \quad y \to \infty. \quad (8b) \]

The parameters \( \lambda \) and \( \alpha \) are given by

\[ \lambda = \frac{Qk_0 a_0 Rl}{k_c E G_r^{1/4}}, \quad \alpha = \frac{k_c E}{D a_0 Q R}. \quad (9) \]

Thus the system is specified by Eqns (7), (8) and involves the dimensionless parameters \( \phi_a, \lambda \) and \( \alpha \) (as well as \( \sigma \) and \( S_c \)). Note that the ambient temperature \( T_a \) appears only in the single parameter \( \phi_a \).

For flow near a stagnation point, where \( S(x) \) is given by (4), the system can be reduced to a set of ordinary differential equations by the transformation

\[ u = xf'(y), \quad v = -f(y), \quad a = a(y). \quad (10) \]

This results in the ordinary differential equations

\[ f'''' + \phi - \phi_a + ff'' - f'^2 = 0, \quad (11a) \]

\[ \frac{1}{\sigma} \phi'' + f \phi' = 0, \quad (11b) \]

\[ \frac{1}{S_c} a'' + fa' = 0 \quad (11c) \]

with boundary conditions

\[ f = f' = 0, \quad \phi' = -\lambda ae^{-1/\phi}, \quad a' = \alpha \lambda ae^{-1/\phi} \quad \text{on} \quad y = 0, \quad (12a) \]

\[ f' \to 0, \quad \phi \to \phi_a, \quad a \to 1 \quad \text{as} \quad y \to \infty \quad (12b) \]

(where primes denote differentiation with respect to \( y \)).

Thus we require a solution of Eqns (11) subject to boundary conditions (12). This will give us values of the dimensionless surface temperature \( \phi_s \), related to \( T_s \) via \( T_s = E \phi_s / R \). In particular, we wish to construct bifurcation diagrams, plots of \( \phi_s \) against \( \phi_a \) (ambient temperature parameter), for different values of the other parameters, from which the critical ambient temperatures can be determined.
3. SOLUTION

To determine the solution of Eqns (11), (12), we exploit an invariance property of the equations that has been noted for the previous formulation of the problem in Chaudhary and Merkin (1994a). Suppose

\[ \phi_{\tau} \equiv \phi(0) = \phi_a + \phi_0, \]  

(13a)

then a rescaling of Eqns (11) via

\[ f = \phi_0^{1/4} F, \quad \phi = \phi_a + \phi_0 H, \quad a = 1 - (1 - a_0) A, \quad \eta = \phi_0^{1/4} \eta, \]  

(13b)

where \( a_0 = a(0) \), leads to the standard free convection problem

\[ F''' + H + FF'' - F''^2 = 0, \]  

(14a)

\[ H'' + \sigma FH' = 0, \]  

(14b)

\[ A'' + S_c FA' = 0 \]  

(14c)

subject to the boundary conditions

\[ F(0) = F'(0) = 0, \quad H(0) = 1, \quad A(0) = 1, \]  

(15a)

\[ F' \to 0, \quad H \to 0, \quad A \to 0 \quad \text{as} \quad \eta \to \infty \]  

(15b)

(primes now denote differentiation with respect to \( \eta \)). The solution of the system (14), (15) provides values for

\[ H'(0) = -c_0(\sigma), \quad A'(0) = -c_1(\sigma, S_c), \]  

(16)

where the values of \( c_0 \) and \( c_1 \) are positive and will be known for given values of \( \sigma \) and \( S_c \). Note that \( A = H \) and \( c_0 = c_1 \) when \( S_c = \sigma \).

If we now apply (13) in boundary conditions (12a) we obtain

\[ \phi_0^{5/4} c_0 = \lambda a_0 e^{-1/\phi}, \]  

(17a)

\[ (1 - a_0)\phi_0^{1/4} c_1 = \alpha \lambda a_0 e^{-1/\phi}. \]  

(17b)

Combining (17) gives

\[ a_0 = 1 - \gamma \phi_0, \]  

(18)
Critical ambient temperatures

Figure 1. Plots of (a) $c_0$ against $\sigma$, (b) $c_1$ against $\sigma$ for $S_c = 0.2, 1.0, 5.0$, (c) $c_1$ against $S_c$ for $\sigma = 0.2, 1.0, 5.0$; obtained from the numerical solution of Eqns (14), (15).
Figure 2. A typical sequence of bifurcation diagrams, plots of $\phi_s$ and $\phi_u$, for $\sigma = S_c = 1.0$, $\alpha = 1$ and (a) $\lambda = 0.5$, (b) $\lambda = 1.0051$, (c) $\lambda = 1.5$, (d) $\lambda = 2.5977$, (e) $\lambda = 3.5$. 
Critical ambient temperatures

where \( \gamma = \gamma(\sigma, S_c) = \alpha c_0 / c_1 \). Then applying (18) in either of (17a,b) gives finally

\[
(\phi_s - \phi_a)^{5/4} = \frac{\lambda}{c_0} \left(1 - \gamma(\phi_s - \phi_a)\right) e^{-1/\phi_s}.
\]  

Expression (19) gives a relation between \( \phi_s \) and \( \phi_a \) for given values of the other parameters, \( \lambda, \alpha, \sigma, \) and \( S_c \). Note that from (18) (or (19)), we must have, for \( \alpha_0 \geq 0, \)

\[
\phi_s - \phi_a \leq \frac{1}{\gamma}.
\]  

Before proceeding to discuss the implications of expression (19) for the critical ambient temperatures of the system, it is worth noting that the standard free convection
problem (14), (15) can be solved numerically without difficulty by a standard two-point
boundary value problem solver; the results are shown in Figs 1a–c. In Fig. 1a
we give a plot of $c_0$ against $\sigma$, while in Figs 1b and 1c we give plots of $c_1$ against $\sigma$
and $S_c$, respectively, for representative values of either $S_c$ or $\sigma$.

To gain some initial insight into the nature of the solution, we present a sequence
of bifurcation diagrams in Figs 2a–e, determined from expression (19) for the case
$\sigma = S_c = 1.0$ (where $c_0 = c_1 = 0.42143$) for increasing values of the parameter $\lambda$.
These diagrams depend only on the parameter $\gamma$ and we take $\gamma = 1.0$ (equivalent
here to taking $\alpha = 1.0$). There are two important features to note about this sequence
of diagrams. First, there is the occurrence of a hysteresis point at $\lambda = \lambda_H$ (say)
($\lambda_H = 1.0051$ in this case), with the curve being monotone for $\lambda < \lambda_H$. As $\lambda$
is increased from $\lambda_H$, multiplicity of solution appears, with the diagram having a typical
S-shape. However, a value of $\lambda$ is reached at $\lambda = \lambda_0$ (say) (here $\lambda_0 = 2.5977$) where
the upper critical point (extinction point) touches the $\phi_a = 0$ axis. For values of
$\lambda > \lambda_0$ this would have to be continued into the (physically unacceptable) $\phi_a < 0$
region. As a consequence the bifurcation diagram becomes disjoint for $\lambda > \lambda_0$
with the upper solution branch becoming separated from the lower solution branches. We
now examine both these features in detail.

3.1. Critical points, hysteresis

To find the critical points $(\phi_{a,c}, \phi_{s,c})$ we need to solve Eqn (19) together with the
condition that $d\phi_a/d\phi_s = 0$ at $(\phi_{a,c}, \phi_{s,c})$. This leads, after a little calculation, to the
extra relation

$$\gamma \phi_{s,c}^3 - (5 + 4\gamma + \gamma \phi_{a,c}) \phi_{s,c}^2 + 4(1 + 2\gamma \phi_{a,c}) \phi_{s,c} - 4 \phi_{a,c}(1 + \gamma \phi_{a,c}) = 0. \quad (21)$$

Equations (21) can be used in (18) to obtain

$$a_0 = \frac{4\gamma \phi_{s,c}^2 (\phi_{s,c} - \phi_{a,c})}{4(\phi_{s,c} - \phi_{a,c}) - 5\phi_{s,c}^2} \quad (22a)$$

with (20) and the requirement that $a_0 \geq 0$ giving the condition

$$\phi_{a,c} \leq 1/5. \quad (22b)$$

For given values of the other parameters, the critical ambient temperature $\phi_{a,c}$ is
then determined by solving Eqns (19) and (21). It is straightforward to show that
Eqn (21), regarded as a cubic equation for $\phi_{s,c}$, has at least one positive root and this
largest root is always such that condition (20) is violated. It is also straightforward
to show that the turning points of Eqn (21) (when they exist) occur at positive values
of $\phi_{s,c}$. Hence we require the two smaller (positive) roots of Eqn (21). We next need
to consider the existence of these two real roots, and to do this, we determine the
values of $\gamma$ and $\phi_{a,c}$ at which these roots are co-incident, giving a hysteresis point
at $\phi_{a,c} = \phi_{a,h}, \phi_{s,c} = \phi_{s,h}$. Thus to find $\phi_{a,h}$, we have to solve Eqns (19), (21)
Critical ambient temperatures

(with $\phi_{a,c}$ and $\phi_{s,c}$ replaced by $\phi_{a,h}$ and $\phi_{s,h}$, respectively) together with the further equation

$$3\gamma\phi_{s,h}^2 - 2(5 + 4\gamma + \gamma \phi_{a,h})\phi_{s,h} + 4(1 + 2\gamma \phi_{a,h}) = 0.$$  \hspace{1cm} (23)

It is readily shown that, for small $\gamma$

$$\phi_{a,h} \sim \frac{1}{5} - \frac{4}{125} \gamma + \cdots, \quad \phi_{s,h} \sim \frac{2}{5} - \frac{16}{125} \gamma + \cdots, \quad \lambda_h \sim \frac{c_0e^{5/2}}{5^{3/4}} \left(1 + \frac{2}{5} \gamma + \cdots\right). \hspace{1cm} (24)$$

For $\gamma$ large, we have, from Eqns (21), (23), that

$$\phi_{a,h} \sim \left(2\sqrt{5} - 4\right)\gamma^{-1/2}, \quad \phi_{s,h} \sim \phi_{a,h} + \left(5 - 2\sqrt{5}\right)\gamma^{-1} \quad (25 \text{a})$$

with $\lambda_h$ then determined from (19)

$$\lambda_h/c_0 \sim 0.95298\gamma^{-5/4} \exp\left(2.11803\gamma^{1/2}\right) \quad (25 \text{b})$$

as $\gamma \to \infty$.

For general values of the parameters, the equations have to be solved numerically. This is relatively straightforward ($\phi_{s,h}$ and $\phi_{a,h}$ are determined from Eqns (21), (23) for a given value of $\gamma$ and these results are then used in Eqn (19)). The results are shown in Fig. 3, where we plot $\lambda_h/c_0$ against $\gamma$. For values of $\lambda_h$ and $\gamma$ that

![Figure 3. Plots of $\lambda_0/c_0$ and $\lambda_h/c_0$ against $\gamma$.](image-url)
Figure 4. Graphs of $\phi_{a,h}$ against (a) $\gamma$ and (b) $\log_{10} \gamma$ at which Eqn (21) has equal roots (hysteresis). For values of $\phi_{a,c}$ and $\gamma$ below the curve two roots exist, whereas for values above the curve no roots exist.
Critical ambient temperatures

lie above this curve multiple solutions exist whereas for values below the curve, the bifurcation diagrams are monotone. The corresponding values of $\phi_{a,h}$, plotted against $\gamma$, are shown in Figs 4a,b. The broken line in Fig. 4b corresponds to the asymptotic form (25a). Expressions (24) show that $\phi_{a,h}$ and $\phi_{a,h}$ both decrease linearly with $\gamma$ for $\gamma$ small and that $\lambda_h$ increases linearly with $\gamma$. This is in line with the results shown in Fig. 4a. Expressions (25a) show that $\phi_{a,h}$ and $\phi_{a,h}$ decrease relatively slowly with $\gamma$, whereas $\lambda_h$ increases rapidly with $\gamma$, as suggested by the results shown in Figs 3 and 4b.

3.2. Disjoint bifurcation diagrams

The onset of disjoint bifurcation diagrams, as seen in Figs 2a–e, occurs when the upper critical (extinction) point touches the $\phi_a = 0$ axis at $\phi_{a,c} = \phi_{a,0}$ (say). From Eqn (21), we then have

$$\phi_{a,0} = \frac{1}{2\gamma} \left( 5 + 4\gamma - \sqrt{25 + 24\gamma + 16\gamma^2} \right)$$

with the corresponding $\lambda_0$ then given by expression (19) with $\phi_a$ put to zero. For $\gamma$ small,

$$\phi_{a,0} \sim \frac{4}{5} - \frac{64}{125} \gamma + \cdots, \quad (27a)$$

$$\lambda_0 \sim c_0 \left( \frac{4}{5} \right)^{5/4} e^{5/4} + \cdots. \quad (27b)$$

For $\gamma$ large

$$\phi_{a,0} \sim \frac{1}{\gamma} \left( 1 - \frac{1}{\gamma} + \cdots \right), \quad (28a)$$

$$\lambda_0 \sim c_0 \gamma^{-1/4} e^{\gamma} + \cdots. \quad (28b)$$

A graph of $\lambda_0/c_0$ plotted against $\gamma$ is also shown in Fig. 3, with there being disjoint bifurcation diagrams for values of $\lambda$ and $\gamma$ above this curve. Expressions (28) show that $\phi_{a,0}$ decreases relatively slowly with $\gamma$, whereas $\lambda_0$ increases exponentially with $\gamma$, much faster than $\lambda_h$, as can be seen by results shown in Fig. 3.

4. RESULTS

The discussion of the previous section shows that the critical temperatures of the system depend on just the two parameter groups $\lambda/c_0$ and $\gamma$ (and through these on the basic parameters introduced originally). Consequently, we need only consider how
\( \phi_{a,c} \) varies with \( \lambda/c_0 \) and \( \gamma \). In Fig. 5 we present graphs of \( \phi_{a,c} \) at the lower critical point (ignition point) plotted against \( \lambda/c_0 \) for fixed values of \( \gamma \) (\( \gamma = 0, \gamma = 1 \)). The broken line shows the hysteresis curve. We can see that these curves, which start on the hysteresis curve, follow it closely. We examined these curves for larger values of \( \gamma \) and found that this trend becomes even more pronounced, the larger the value of \( \gamma \) taken. Thus the hysteresis values \( \phi_{a,h} \) at a given value of \( \lambda/c_0 \) should provide a reasonable estimate of the critical temperature parameters \( \phi_{a,c} \) for ignition.

To see how \( \phi_{a,c} \) varies for \( \lambda/c_0 \) large, it is, perhaps, easiest to consider the case when \( \gamma = 0 \). Here

\[
\phi_{s,c} = \frac{2}{5} \left( 1 - \sqrt{1 - 5\phi_{a,c}} \right),
\]

(29a)

which for \( \bar{\lambda} = \lambda/c_0 \) large and \( \phi_{a,c} \) small becomes

\[
\phi_{s,c} \sim \phi_{a,c} + \frac{5}{4} \phi_{a,c}^2 + \cdots.
\]

(29b)

Using this in expression (19) gives, to leading order,

\[
\bar{\lambda} \sim \left( \frac{5}{4} \right)^{5/4} \phi_{a,c}^{5/2} e^{1/\phi_{a,c}}
\]

(30a)

![Figure 5. Graphs of \( \phi_{a,c} \) at the lower critical point (ignition) plotted against \( \lambda/c_0 \) for \( \gamma = 0, \gamma = 1 \). The broken line shows the hysteresis curve.](image-url)
Inverting this, shows that \( \phi_{a,c} \) has an inverse logarithmic behaviour of the form

\[
\phi_{a,c} \sim \frac{1}{\ln \left( 0.7566 \lambda (\ln \lambda)^{5/2} \right)} + \ldots
\]  

(30b)

as \( \lambda \to \infty \). We expect this weak logarithmic decay of \( \phi_{a,c} \) with \( \lambda \) to persist for \( \gamma \neq 0 \). In fact, from (25) we have, for \( \gamma \) large that

\[
\tilde{\lambda}_h \sim \frac{(5 - 2\sqrt{5})^{5/4}}{(2\sqrt{5} - 4)^{7/2}} \phi_{a,h}^{5/2} e^{1/\phi_{a,h}} ,
\]  

(31a)

\[
\phi_{a,h} \sim \frac{1}{\ln \left( 0.1607 \lambda_h (\ln \lambda_h)^{5/2} \right)} + \ldots
\]  

(31b)

which have the same functional forms as (30) when \( \gamma = 0 \). The only difference between the two expressions is in the multiplicative factor (1.322 for \( \gamma = 0 \) and 6.222 for \( \gamma \gg 1 \)), with the expressions for other values of \( \gamma \) lying between these two extreme cases.

Expressions (30b), (31b) show that the critical temperatures on the lower solution branch are insensitive to the values of \( \lambda/c_0 \), at least for relatively large values of this

![Figure 6. Graphs of \( \phi_{n,c} \) plotted against \( \gamma \) for \( \lambda/c_0 = 2, 5, \) and 10. The broken line shows the hysteresis curve.](image)


parameter, with $\phi_{a,c} \log (\lambda/c_0)$ being almost constant. Thus using the measurements of the critical ambient temperature to estimate values for $\lambda$ could well prove unreliable.

In Fig. 6 we show the graphs of $\phi_{a,c}$ plotted against $\gamma$ for fixed values of $\lambda/c_0$. Again the broken line shows the hysteresis values with the upper curve being the lower critical point (ignition) and the lower curve being the upper critical point (extinction). We can see that, as expected, both curves start on the hysteresis curve with the ignition point then decrease only slightly as $\gamma$ is decreased to zero. The upper critical point (extinction) decreases rapidly as $\gamma$ is decreased and, as predicted above, reaches the $\phi_{a,c} = 0$ axis at a non-zero value of $\gamma$. From this we can infer that the ignition point is only weakly sensitive to variations in $\gamma$, whereas the extinction point has a much stronger dependence on $\gamma$ and can be made to disappear altogether at an appropriate value of $\gamma$.

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Critical ambient temperatures