A computer program for the analysis of the dynamic bending–torsion coupling in bridges using a mini-computer

R. PICON
ETS Ingenieros Industriales, Universidad de Sevilla, Seville, Spain.

E. ALARCON
ETS Ingenieros Industriales, Universidad Politécnica de Madrid, Madrid 3, Spain

The analysis of modes and natural frequencies is of primary interest in the computation of the response of bridges. In this article the transfer matrix method is applied to this problem to provide a computer code to calculate the natural frequencies and modes of bridge-like structures. The Fortran computer code is suitable for running on small computers and results are presented for a railway bridge.

INTRODUCTION

The analysis of modes and natural frequencies is of primary interest in the computation of the response of bridges. In the case of simple beams in which the load is applied eccentrically the torsion modes are very important (Fig. 1). If the applied loads are through the torsion centre and parallel to one of the principal inertia axes of the section, there is a displacement in the force direction and vibrations can be easily studied.

If the loads have a general direction then the displacement can be obtained adding the components in the two principal directions. In general the load is not at the torsion centre and then, there is a complex phenomenon of bending and torsion (Fig. 2). The method of working is the use of an equivalent system: a force through the torsion centre and a torsion moment in it.

COUPLING THROUGH SHAPE FUNCTIONS

The energy method is very well adapted to the use of orthogonal shape functions. If \( y \) and \( \theta \) are the displacement and twist in the torsion centre, the response can be approached setting:

\[
y(x, t) = \sum_{i=1}^{m} Y_i(x) \cdot q_i(t) \quad (1)
\]

\[
\theta(x, t) = \frac{1}{e_0} \sum_{i=m+1}^{m} Y_i(x) \cdot q_i(t) \quad (2)
\]

This leads to a matrix system as:

\[
Aq' + Bq = 0 \quad (3)
\]

in which typical elements are:

\[
a_{ii} = \int_0^L m Y_i Y_i' \, dx \quad (4)
\]

\[
(k_{ii}) = \int_0^L E I \, Y_i Y_i' \, dx \quad (5)
\]
contains the characteristic features in every node, then

$$Z'_i = U_{pi} Z'_i$$

(8)

where $U_{pi}$ is the point matrix and the superior index refer to the 'right' and 'left' words.

In every field:

$$Z'_{i+1} = U_{ij} Z'_i$$

(9)

If one defines the transfer matrix in section $i$ as:

$$Z'_i = U'_i Z'_i$$

(10)

then

$$U_i = U_{ji} U_{pi}$$

(11)

For the simple beam case, the field matrix is:

$$U_j^f = \begin{pmatrix} 1 & L_i & L_i^2/2EI_i & L_i^3/6EI_i \\ 0 & 1 & L_i/EI_i & L_i^2/2EI_i \\ 0 & 0 & 1 & L_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(12)

If there is only torsion:

$$U_j^T = \begin{pmatrix} 1 & L_i/GJ_i \\ 0 & 1 \end{pmatrix}$$

(13)

where $G$ is the shear modulus and $J$ the polar moment of inertia of the bar cross-section.

As the coupling is a dynamic one it will only appear in the point matrices, so in general:

$$U_{ei} = \begin{pmatrix} 1 & L_i & L_i^2/2EI_i & L_i^3/6EI_i & 0 & 0 \\ 0 & 1 & L_i/EI_i & L_i^2/2EI_i & 0 & 0 \\ 0 & 0 & 1 & L_i & 0 & 0 \\ 0 & 0 & 0 & 1 & L_i/GJ_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(14)

To establish the point matrix it is worth remembering that coupling appears because of the lack of coincidence of the torsion centre and the centroid.

The equilibrium is:

$$Q'_i = Q'_i + p^2 m_i \gamma_i^m$$

(15)
\[ M_i' = M_i' - p^2 m_i y_i^e e_i - p^2 J_i \theta_i \]  
\[ (16) \]

where \( y_i^e = \) centroid displacement, \( p = \) natural circular frequency, \( e_i = \) eccentricity, \( J_i = \) polar mass moment.

The torsion centre displacement is:

\[ y_i = y_i^e - e_i - \theta_i \]  
\[ (17) \]

and then

\[ Q_i' = Q_i' + p^2 m_i y_i - p^2 J_i \theta_i - p^2 m_i e_i^2 \theta_i \]

The point matrix is:

\[ U_p = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
p^2 m_i & 0 & 0 & 1 & -p^2 m_i e_i & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
(-p^2 m_i e_i) & 0 & 0 & 0 & -p^2 (m_i e_i^2 + J_i) & 1
\end{pmatrix} \]  
\[ (18) \]

The \( U_p \) and \( U_m \) products is the transfer matrix \( U_i \) for the \( i \) knot (shown in equation 19).

\[ U_i = \begin{pmatrix}
1 + p^2 m_i L_i^2 & L_i & L_i & 0 & 0 & 0 \\
6 E_i & 2 E_i & 6 E_i & 0 & 0 & 0 \\
p^2 m_i L_i & 1 & L_i & L_i & 0 & 0 \\
2 E_i & E_i & 2 E_i & 0 & 0 & 0 \\
p^2 m_i L_i & 0 & 1 & L_i & 0 & 0 \\
6 E_i & 6 E_i & 6 E_i & 0 & 0 & 0 \\
\end{pmatrix} \]  
\[ (19) \]

It is worth noting that if \( e_i = 0 \) there is no coupling and both phenomenon are independent.

**COMPUTER PROGRAM**

Using the previous theory a computer program to be used in a Hewlett–Packard 21 MX has been prepared.

There are a variable number of masses, with a maximum of 200. The program allows \( K_i = 1 \) different elements of uniform cross-section. (Numbers in brackets refer to program variables.)

Automatically masses are lumped at the ends of every element. Then mass and constant characteristics of every element are read (D1, D2, D3, D4, D5, D6).

Using the data, transfer matrices are formed, starting with the last mass-element unity (a fictitious element of zero length is added for preventing the symmetry of the procedure). Matrix products gives the transfer matrix which relates the ends of every element.

The transfer matrices formation is completed by the HAMA subroutine, whose argument define the mass-element unity.

The only remaining task is to apply the boundary conditions from which the frequency determinant is obtained. The value of the last (RE) is the final product of the program and, when this value is zero, we know that it is a natural mode of vibration.

Though the program is for the analysis of natural modes with coupling, it is also possible to compute bending frequencies (by the residuals 1 to 6) or torsion frequencies (by residuals 7 to 9).

In these cases it is only necessary to put the eccentricity (EX) equal to zero and to define with a non-zero number (1, for instance) the necessary data to circumvent a divide by zero.

If one is studying non-circular sections it will be necessary to define the adequate connection factor (FACT).

In the circular case FACT = 1.

As an application we have considered the following boundary conditions \( i.e. \) hinged ends for bending and built-in ends for torsion. It is a simple matter to establish other kinds of conditions.

The following step in the program is the change of sign detection in (RE) to get the number of modes established in the input (NFN).

The INPUT-OUTPUT units are defined by an internal subroutine (RMPAR) which establishes the correspondence between the LOG and INPUT units and between the NPR and the OUTPUT. Variables to define input and output are provided through the keyboard in the same order as they are executed in the program.

\[ \text{Figure 5. Single track bridge: bending} \]
Input data

If K1 = 1

1. Header card, columns 1–80
2. Column 5: K1(1)
   Columns 9–10: number of the sought residue, after boundary conditions (see program listing)
3. Columns 8–9: number of masses
   Columns 10–20: number of sought natural modes.
4. Beam length (LT), beam mass (MW), frequency increment (INW), initial frequency (WINI) and correction factor for the inertia polar moment (FACT) in 10 columns fields with point.
5. Inertia bending moment (I), radius of gyration of the masses (RGIR), called D2 in the reading of data), polar moment (MIP). Young's modulus (E), shear modulus (G) and eccentricity (EX) in 10 columns fields with point.

If K1 = 2, everything is the same but the 5th and following cards in which we define the properties of masses and bars in 10 columns fields with point, with a maximum of 7 fields every card.

The input order is: inertia bending moments (I), polar mass moments (MIMPA), polar cross-section moments (MIP), elasticity modulus (E), shear modulus (G), eccentricity (EX), element length (L) and mass (M).

If there are 20 masses there will be 3 cards (7, 7 and 6) with bending inertias, then 3 polar mass inertias, etc.

Output

The program produces the following output:
Echo-check of first card and number of masses, listing of frequencies and residuals.

Example

The example presented here (Fig. 2) corresponds to the characteristics of a prestressed precast bridge of Barton3. The Figure is of one-track bridge but a double one will be built with two of them.

The span is 13 m and we have taken the train-mass as 10 t/ml. There is also a ballast layer of 40 cm.

The residuals for the first two frequencies have been plotted in Figs. 5, 6, 7, 8, 9. Number of lumped masses was 5 and it is worth noting that an increase to 20 masses affects results very slightly (about 32).

REFERENCES

PROGRAM MTR

* PROGRAM TO COMPUTE THE NATURAL FREQUENCIES OF A CONTINUOUS SYSTEM USING THE TRANSFER MATRIX METHOD.

LEGEND

E: MODULUS OF ELASTICITY.
G: SHEAR MODULUS.
EX: EXCENTRICITY. DISTANCE FROM THE CENTRE OF GRAVITY TO THE CENTRE OF TORSION.
MIP: POLAR MOMENT OF INERTIA OF THE MEMBER CROSS SECTION.
MIPMA: POLAR MOMENT OF INERTIA OF THE MASSES.
RGIR: RADIUS OF INERTIA OF THE MASSES. (D2 WHEN DATA INPUT)
I: FLEXURAL MOMENT OF INERTIA OF THE MEMBER CROSS SECTION.
RE: RESIDUE.
LT: TOTAL LENGTH OF THE BEAM.
LM: TOTAL MASS OF THE BEAM.
W: CIRCULAR FREQUENCY.
INW: INCREMENT OF FREQUENCY.
INI: INITIAL FREQUENCY.
LM: LUMPED MASSES.
L: LENGTH OF THE BEAM SEGMENTS.
NM: NUMBER OF LUMPED MASSES.
FACT: CORRECTING FACTOR FOR THE MIP. FACT=1 WHEN CIRCULAR CROSS SECTION.
K1: PARAMETER THAT DEPENDS ON THE INPUT FORM.
K2: PARAMETER TO SELECT THE RESIDUE.
HFN: NUMBER OF MODES TO BE COMPUTED.
LOG: INPUT UNIT.
NFR: OUTPUT UNIT.

INTEGER A(40)
REAL E(200),EX(200),MIP(200),I(200),RE,LT,MY
1,INW,M(200),L(200),G(200),UN(6,6),UC(6,6)
2,UMU(6,6),M(6,6),MIPMA(200)
COMMON W,M,L,E,MIP,EX,I,G,UN,MIPMA,FACT

SET INPUT AND OUTPUT PARAMETERS.

DIMENSION IPAR(5)
CALL RMPAR(IPAR)
LOG=IPAR(1)
NFR=IPAR(2)
IF(LOG .LE. 0) LOG=1
IF(NFR .LE. 0) NFR=6

INITIAL VALUES OF THE CONSTANTS.

C=1.
NMD=0.
READ TITLE.

READ CHARACTERISTICS AND SET SEGMENTS AND PARTIAL MASSES.

READ(LOG,43)K1,K2
43 FORMAT(ZI5)
READ(LOG,10)NM,NFN,LT,MY,INW,WINI,FACT
10 FORMAT(ZI10/5F10.2)
GO TO (40,41),K1
41 READ(LOG,21) (I(J),J=1,NN)
I(NM)=1.
READ(LOG,21) (MIPJ(J),J=1,NN)
READ(LOG,21) (MIP(J),J=1,NN)
MIP(NM)=1.
READ(LOG,21) (E(J),J=1,NN)
READ(LOG,21) (G(J),J=1,NN)
READ(LOG,21) (E(J),J=1,NN)
READ(LOG,21) (L(J),J=1,NN)
GO TO 80
21 FORMAT(7F10.2)
L(NM)=0.
READ(LOG,21) (M(J),J=1,NN)
GO TO 80
40 NM=NM-1
DO 3 J=1,NN
3 L(J)=LT/(NM-1)
L(NM)=0.
M(1)=MY/(2.*(NM-1))
M(NM)=M(1)
DO 4 J=2,NN
4 M(J)=MV/(NM-1)
READ(LOG,21) D1,D2,D3,D4,D5,D6
DO 76 J=1,NN
I(J)=D1
MIP(J)=M(J)*D2**2
MIP(J)=D3
E(J)=D4
G(J)=D5
76 EXJ= D6

PRINT TITLE AND HEAD LINES.

WRITE(NPR,20)A,NM
20 FORMAT(2X,40A2,/,2X,"NUMBER OF MASSES=",13/,,/,,7X,"FREQUENCY"
1(RDS/SG)",9X,"RESIDUE"/7X,19("*"),5X,16("*"))

FORM THE FIRST TRANSFER MATRIX USING THE INITIAL FREQUENCY.

FORM THE NEXT MATRIX. MULTIPLY MATRICES AND TRANSFER PRODUCT TO
C MATRIX "U". REPEAT UNTIL THE FINAL MATRIX IS OBTAINED.

DO 7 LL=2, NM
NN=NM-LL+1
CALL HAMA(NH)
DO 8 J=1,6
DO 9 K=1,6
9 UMU(J,K)=UN(J,K)
8 CONTINUE
DO 11 J=1,6
DO 12 K=1,6
P=0.
DO 13 N=1,6
PP=U(J,N)*UMU(N,K)
13 P=P+PP
MT(J,K)=P
12 CONTINUE
11 CONTINUE
DO 14 J=1,6
DO 15 K=1,6
U(J,K)=MT(J,K)
15 CONTINUE
14 CONTINUE
7 CONTINUE

C COMPUTE RESIDUE DEPENDING ON BOUNDARY CONDITIONS.
C PRINT FREQUENCY AND RESIDUE.
GO TO (47,48,49,50,51,52,53,55,56,57), K2
C FLEXURAL CANTILEVER BEAM. -1-
47 RE=U(3,3)*U(4,4)-U(3,4)*U(3,4)
GO TO 97
C FLEXURAL SIMPLY-SUPPORTED BEAM. -2-
48 RE=U(1,2)+U(3,4)-U(2,2)*U(1,4)
GO TO 97
C FLEXURAL FIXED ENDS BEAM. -3-
49 RE=U(1,3)+U(2,4)-U(2,3)*U(1,4)
GO TO 97
C FLEXURAL FREE ENDS BEAM. -4-
50 RE=U(3,1)*U(4,2)-U(4,1)*U(3,2)
GO TO 97
C FLEXURAL SIMPLY-SUPPORTED AND FIXED BEAM. -5-
51 RE=U(1,3)+U(3,4)-U(2,3)*U(1,4)
GO TO 97
C FLEXURAL SIMPLY-SUPPORTED AND FREE BEAM. -6-
52 RE=U(3,2)*U(4,4)-U(4,2)*U(3,4)
GO TO 97
C TORSIONAL FREE ENDS BEAM. -7-
0:79  C  53  RE=U(6.5)
0:80  
0:81  GO  TO  97
0:82  C
0:83  TORSIONAL  FIXED  ENDS  BEAM.  -8-
0:84  C
0:85  55  RE=U(5.6)
0:86  GO  TO  97
0:87  C
0:88  TORSIONAL  CANTILEVER  BEAM.  -9-
0:89  C
0:90  56  RE=U(6.6)
0:91  GO  TO  97
0:92  C
0:93  COUPLING.  FLEXURAL  SIMPLY-SUPPORTED.  TORSIONAL  FIXED  ENDS.  -10-
0:94  C
0:95  57  RE=U(1,2)+U(3,4)+U(5,6)+U(1,6)+U(1,6)+U(1,4)+U(1,4)+U(3,6)+U(5,2)
0:96  1U(1,6)+U(3,4)+U(5,2)+U(1,4)+U(3,2)+U(5,6)+U(1,2)+U(3,6)+U(5,4)
0:97  97  WRITE(NFR,500)W.RE
0:98  500  FORMAT(6X,E13.6,10X,E13.6)
0:99  C
0:100  DETECT  CHANGES  OF  SIGN.  INCREASE  FREQUENCY  AND  GO  BACK  IN  THE
0:101  ITERATING  PROCESS.
0:102  C
0:103  B=RE/C
0:104  IF(B>82,84,83
0:105  83  W=W+1NW
0:106  C=RE
0:107  GO  TO  19
0:108  82  HM0=HM0+1
0:109  IF(HM0=HM0)83,18,18
0:110  84  RE=1
0:111  GO  TO  92
0:112  18  STOP
0:113  END
0:114  C
0:115  FORM  THE  TRANSFER  MATRICES.
0:116  C
0:117  517  SUBROUTINE  HAMA(N)
0:118  REAL  N(200),L(200),KIP (200),I(200),MIFMA(200)
0:119  COMMON  U,M,L,E(200),KIP,EX(200),I,E(200),UN(6,6),MIFMA,FACT
0:120  UN(1,1)=1.+(W**2*M(N)*L(N)**3)/(6.*E(N)*I(N))
0:121  UN(1,2)=L(N)
0:122  UN(1,3)=(L(N)**2)/(2.*E(N)*I(N))
0:123  UN(1,4)=(L(N)**3)/(6.*E(N)*I(N))
0:124  UN(1,5)=U**2*M(N)*E(N)*L(N)**3)/(6.*E(N)*I(N))
0:125  UN(1,6)=0
0:126  UN(2,1)=(W**2*M(N)*L(N)**2)/(2.*E(N)*I(N))
0:127  UN(2,2)=1.
0:128  UN(2,3)=(L(N)**2)/(2.*E(N)*I(N))
0:129  UN(2,4)=(L(N)**2)/(2.*E(N)*I(N))
0:130  UN(2,5)=(W**2*M(N)*E(N)*L(N)**2)/(2.*E(N)*I(N))
0:131  UN(2,6)=0.
0:132  UN(3,1)=(W**2*M(N)*L(N))
0:133  UN(3,2)=0.
0:134  UN(3,3)=1.
0:135  UN(3,4)=L(N)
0:136  UN(3,5)=(W**2*M(N)*E(N)*L(N))
0:137  UN(3,6)=0.
0:138  UN(4,1)=(W**2*M(N))
0.239 \quad \text{UN}(4, 2) = 0.
0.240 \quad \text{UN}(4, 3) = 0.
0.241 \quad \text{UN}(4, 4) = 1.
0.242 \quad \text{UN}(4, 5) = (w^{**2}M(N) * EX(N)).
0.243 \quad \text{UN}(4, 6) = 0.
0.244 \quad \text{UN}(5, 1) = (-1 * \text{M}(N) * \text{EX}(N)) / (\text{FACT} * \text{G}(N) * MIF(N)).
0.245 \quad \text{UN}(5, 2) = 0.
0.246 \quad \text{UN}(5, 3) = 0.
0.247 \quad \text{UN}(5, 4) = 0.
0.248 \quad \text{UN}(5, 5) = -((w^{**2}L(N)) * (M(N) * EX(N)) * 2).
0.249 \quad 1 + \text{MIFMA}(N) / (\text{FACT} * \text{G}(N) * MIF(N)) = 1.
0.250 \quad \text{UN}(5, 6) = L(N) / (\text{FACT} * \text{G}(N) * MIF(N)).
0.251 \quad \text{UN}(6, 1) = (-1 * w^{**2}M(N) * EX(N)).
0.252 \quad \text{UN}(6, 2) = 0.
0.253 \quad \text{UN}(6, 3) = 0.
0.254 \quad \text{UN}(6, 4) = 0.
0.255 \quad \text{UN}(6, 5) = (-1 * W^{**2}M(N) * EX(N)) * 2 + \text{MIFMA}(N).
0.256 \quad \text{UN}(6, 6) = 1.
0.257 \quad \text{return}
0.258 \quad \text{end}
0.259 \quad \text{end}