The Minimum Number of Points Taking Part in \( k \)-Sets in Sets of Unaligned Points

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Abstract: The study of \( k \)-sets is a very relevant topic in the research area of computational geometry. The study of the maximum and minimum number of \( k \)-sets in sets of points of the plane in general position, specifically, has been developed at great length in the literature. With respect to the maximum number of \( k \)-sets, lower bounds for this maximum have been provided by Erdős et al., Edelsbrunner and Welzl, and later by Toth. Dey also stated an upper bound for this maximum number of \( k \)-sets. With respect to the minimum number of \( k \)-set, this has been stated by Erdos el al. and, independently, by Lovasz et al. In this paper the authors give an example of a set of \( n \) points in the plane in general position (no three collinear), in which the minimum number of points that can take part in, at least, a \( k \)-set is attained for every \( 1 \leq k < n/2 \). The authors also extend Erdos’s result about the minimum number of points in general position which can take part in a \( k \)-set to a set of \( n \) points not necessarily in general position. That is why this work complements the classic works we have mentioned before.

Key words: \( k \)-set, convex hull, intersection of convex polygons.

1. Introduction

The search of upper and lower bounds on the number of halving lines or \( k \)-sets in a set of \( n \) points located in the plane in general position is a problem widely reflected in the literature. Recall that a halving line in a set of \( n \) points \( \{p_1, ..., p_n\} \) is a line that joins two points of \( \{p_1, ..., p_n\} \) leaving the same number of points of \( \{p_1, ..., p_n\} \) in each half-plane (\( n \) is an even number) and a \( k \)-set is a subset of \( \{p_1, ..., p_n\} \) with \( k \) points that can be separated of the other points of \( \{p_1, ..., p_n\} \) by a straight line.

With respect to the maximum number of \( k \)-sets, lower bounds for this maximum have been given by Erdős et al. [1], and also independently by Edelsbrunner and Welzl [2]. They established a lower bound of the order \( O(n \log k) \) for the maximum number of \( k \)-sets. Later, Tóth [3] discovered a construction of a set of \( n \) points with \( O(n^{1/2} \log k) \) \( k \)-sets for every \( n \) and \( k < n/2 \). Attending to upper bounds of this maximum number of \( k \)-sets, Dey [4] stated an upper bound of the order \( O(n^{1/3} k) \). Nowadays, this is the best upper bound for this number.

With respect to the minimum number of halving lines and \( k \)-sets, it is known that the minimum number of halving lines is \( \frac{n}{2} \) [5] and the minimum number of \( k \)-sets is \( 2k + 1 \) [1, 6] (the authors refer to the latter fact as “Result 2” throughout the paper).

The problem of establishing the minimum number of points that can intervene in at least one \( k \)-sets of a given set of \( n \) points was also posed by Erdős et al. [1]. They proved that this minimum is also \( 2k + 1 \) (hereafter “Result 1”), and gave an example where this minimum is attained: \( 2k + 1 \) points are the
vertices of a regular polygon, and the remaining points lie close enough to the centre of the polygon (this example also attains the minimum number of \( k \)-sets).

In this paper the authors present an example of a set of \( n \) points in the plane where the minimum of \( 2k+1 \) points taking part in a \( k \)-set is attained for every \( k < \frac{n}{2} \) (Subsection 2.1). Furthermore, the authors prove that a similar example to the presented in Subsection 2.1 cannot be found for the minimum number of \( k \)-sets (Section 3). So the authors conclude that the only intersection of these convex hulls.

The authors also generalize Result 1 to sets of points that are not necessarily in general position, but do not consist of a set of points on a line (Subsection 2.2).

Throughout the paper \( k \) and \( n \) are positive integers, the following definitions also apply:

**Definition 1:** Consider a set \( A \) of points in the plane and the convex hulls of all possible subsets of \( A \) with \( t \) points. The authors define \( C_{A,t} \) as the intersection of these convex hulls.

**Remark:** The following properties for \( C_{A,t} \) hold [7]:

1. If the points of \( A \) are in general position, then \( C_{A,t} \) does not consist only of a segment;
2. If \( t < \frac{|A|}{2} + 1 \), then \( C_{A,t} \) is the empty set, where \( |A| \) is the cardinal of \( A \);
3. If the points of \( A \) are not collinear, then \( C_{A,\lfloor \frac{|A|}{2} \rfloor + 1} \subset \{p\} \) for some point \( p \).

**Definition 2:** Consider a set \( A \) of points in the plane, two points \( p, q \in A \) and the convex hulls of all possible subsets of \( A \) with \( t \) points such that \( p \) and/or \( q \) belongs to the subset. The authors define \( C_{A,t}^{p,q} \) as the intersection of these convex hulls.

2. Minimum Number of Points Taking Part in \( k \)-Sets of \( A \)

2.1 Example for a Set of \( n \) Points and Every \( k < \frac{n}{2} \)

In order to give the example of a set of \( n \) points, with even \( n \), with the minimum number of points taking part in at least one \( k \)-set for every \( k < \frac{n}{2} \), the authors shall need some previous results. Throughout this Subsection it is assumed that the points of every set are in general position:

**Proposition 1:** Let \( A \) be a set of \( n \) points. The points of \( A \) included in \( C_{A,n-k} \) cannot belong to any \( k \)-set.

**Proof:** If one of these points belonged to a \( k \)-set, then a straight line would separate it from \( n-k \) points of \( A \). Therefore, this point would not be included in at least one convex hull of \( n-k \) points and could not belong to \( C_{A,n-k} \), a contradiction.

**Remark:** Conversely, the points of \( A \) that are not included in \( C_{A,n-k} \) belong to at least a \( k \)-set. Consequently the authors wish to find an example of a set \( A \) of \( n \) points such that \( n-(2k+1) \) points belong to \( C_{A,n-k} \) for every \( k \) in the range \( 1 \leq k < \frac{n}{2} \).

**Lemma 1:** Let \( U \) and \( V \) be the sets \( U = \{p_1, ..., p_t\} \), \( V = \{p_1, ..., p_t, p_{t+1}, p_{t+2}\} \), where \( t \) is an odd number. If the points \( p_{t+1} \) and \( p_{t+2} \) belong to \( C_{U,\lceil \frac{t+2}{2} \rceil} \), then these points also belong to \( C_{V,\lceil \frac{t+2}{2} \rceil} \). Furthermore, \( C_{V,\lceil \frac{t+2}{2} \rceil} \) has a non empty interior set (\( \lceil \rfloor \) stands for the floor).

**Proof:** Consider a set of \( \lceil \frac{t+2}{2} \rceil + 2 = \lceil \frac{t}{2} \rceil + 3 \) points of \( V \). If these points do not include both \( p_{t+1} \) and \( p_{t+2} \), then they will contain at least \( \lceil \frac{t}{2} \rceil + 2 \) points of \( U \). Thus, the convex hull of the \( \lceil \frac{t}{2} \rceil + 3 \) points considered must contain the convex hull of \( \lceil \frac{t}{2} \rceil + 2 \) points of \( U \).
Consequently, the first convex hull contains the segment joining \( p_{r+1} \) and \( p_{r+2} \) by the hypothesis of the lemma.

Now, if the set of \( \left\lceil \frac{r+2}{2} \right\rceil \cdot 2 \) points of \( V \) considered contains both \( p_{r+1} \) and \( p_{r+2} \), then the segment joining \( p_{r+1} \) and \( p_{r+2} \) is included in the convex hull. This segment is therefore in \( C_{r+2} \) and consequently \( C_{r+2} \) is not a finite set. But the set \( C_{r+2} \) does not consist only of this segment, because the points are in general position. Hence, \( C_{r+2} \) has non empty interior set.

**Lemma 2:** Consider a set of \( n \) points \( A = \{p_1, \ldots, p_n\} \) and its subset \( B = \{p_1, \ldots, p_{2k+1}\} \). If \( C_{2k+1} \) contains the \( n - (2k+1) \) points of \( A - B \), then \( C_{n-k} \) also contains these \( n - (2k+1) \) points of \( A \).

**Proof:** Consider a subset of \( n - k \) points taken from \( A \). If this subset does not contain all of the last \( n - (2k+1) \) points of \( A \) (\( P_{2k+2}, \ldots, P_n \)), then there are at least \( k+2 = \left\lceil \frac{2k+1}{2} \right\rceil + 2 \) points in subset \( B \), so their convex hull contains the last \( n - (2k+1) \) points of \( A \) by assumption, then \( P_{2k+2}, \ldots, P_n \) are in \( C_{n-k} \).

Let us next describe the example satisfying the required conditions:

**Example 1**

Let \( A = \{p_1, \ldots, p_n\} \) be a set of \( n \) points (\( n \) is an even number) defined in the following way: \( p_1, p_2, \ldots, p_{n/2} \) are not in a line, and for \( k = 1, \ldots, n/4 \), \( P_{2k+2} \) are in \( C_{2k+3} \) such way that \( P_{2k+3} \) are in general position (this can always be done, since \( C_{2k+3} \) has non empty interior set by Lemma 1). Finally, \( p_n \) is located in \( C_{n/2} \) (Fig. 1).

This configuration of points satisfies the condition that for every \( k = 1, \ldots, \frac{n-4}{2} \), \( P_{2k+2}, \ldots, P_n \) belong to \( C_{n-k} \). The authors already know that \( P_{2k+2}, P_{2k+3} \) belong to \( C_{2k+1} \). Hence, to prove the assertion it is enough to see that \( C_{t+k} \supset C_{t+k+1} \) for \( t > k \).

This relation will be true for all \( t > k \) if the authors see it for \( t = k + 1 \). The following inclusion is obvious:

\[
C_{\{p_1, \ldots, p_{2k+1}\}, \left\lceil \frac{2k+3}{2} \right\rceil + 2} \subset C_{\{p_1, \ldots, p_{2k+1}\}, \left\lceil \frac{2k+3}{2} \right\rceil + 2}.
\]

On the other hand, consider a selection of \( \left\lceil \frac{2k+3}{2} \right\rceil + 2 \) points from the sequence \( P_1, \ldots, P_{2k+1} \). Assuming that \( P_{2k+2} \) and/or \( P_{2k+3} \) are included, this selection contains at most \( \left\lceil \frac{2k+1}{2} \right\rceil + 2 \) points from the sequence \( P_1, \ldots, P_{2k+1} \). Therefore, the convex hull of the \( \left\lceil \frac{2k+3}{2} \right\rceil + 2 \) points is contained within a convex hull of \( \left\lceil \frac{2k+1}{2} \right\rceil + 2 \) points from \( P_1, \ldots, P_{2k+1} \). This result follows from the fact that \( P_{2k+2} \) and \( P_{2k+3} \) are in every convex hull of \( \left\lceil \frac{2k+3}{2} \right\rceil + 2 \) points taken from the sequence \( P_1, \ldots, P_{2k+1} \).

Thus \( C_{\{p_1, \ldots, p_{2k+3}\}, \left\lceil \frac{2k+3}{2} \right\rceil + 2} \supset C_{\{p_1, \ldots, p_{2k+1}\}, \left\lceil \frac{2k+1}{2} \right\rceil + 2} \).

This completes the desired inclusion.

For \( k = \frac{n-2}{2} \), it is also true that the point \( P_{2k+2} = p_n \) is in \( C_{\{p_1, \ldots, p_{2k+1}\}, \left\lceil \frac{2k+1}{2} \right\rceil + 2} \).
according to the construction of \( A \).

Thus, according to Lemma 2 there are \( n-(2k+1) \) points in \( C_{A,n-k} \) for \( k=1,\ldots,\frac{n}{2}-1 \). Therefore, by Proposition 1 this is an example of a set of \( n \) points that attains the minimum of \( 2k+1 \) points taking part in \( k \)-sets for every \( k=1,\ldots,\frac{n}{2}-1 \).

Remarks:
(1) For odd \( n \), the previous example can be modified to obtain an example of a set of \( n \) points with the minimum number of \( 2k+1 \) points belonging to at least one \( k \)-set for every \( k<\frac{n}{2} \). The authors just avoid placing the last point in the last intersection.

(2) As Fig. 1 shows, \( C_{\{p_1,\ldots,p_n\}+\frac{2k+1}{2}} \) is a triangle such that \( p_{2k}, p_{2k+1} \) are two of its vertices.

(3) It is not possible to obtain a similar example where the minimum number of \( k \)-sets in a set of \( n \) points is attained for every \( k<\frac{n}{2} \), because this example would contradict the lower bound on the number of \( \leq k \)-sets given by Lovasz [6] that is \( \left\lfloor \frac{k+1}{2} \right\rfloor \). As a matter of fact, it is easy to see that the number of \( k \)-sets in the present example is \( 4k-1 \) for every \( k<\frac{n}{2} \), \( 2k+1 \) being the minimum number of \( k \)-sets.

2.2 Case of Points That Are Not in General Position

This Subsection generalises Result 1 by proving that for every \( k<\left\lfloor \frac{n}{2} \right\rfloor \) and every set of \( n \) points, the minimum number of points taking part in \( k \)-sets is \( 2k+1 \), provided that the \( n \) points are not collinear. A previous lemma is given:

Lemma 3: For a set \( A=\{p_1,\ldots,p_n\} \), if \( C_{A,n-k} \) contains \( l \) points of \( A \), say \( p_1,\ldots,p_l \), then these points must be located in \( C_{\{p_1,\ldots,p_l\},n-k-(l-1)} \) \( (l<n-k+1) \).

Proof: If there is some point of \( p_1,\ldots,p_l \) that is not located in the proposed intersection, then there exists a convex hull \( C \) of \( n-k-(l-1) \) points of \( p_{l+1},\ldots,p_n \) that does not contain every point of \( p_1,\ldots,p_l \). But if such is the case, at least one point of \( p_1,\ldots,p_l \), for example \( p_1 \) is located at a vertex along the boundary of the convex hull of \( p_1,\ldots,p_l \) and the \( n-k-(l-1) \) points aforementioned. This implies that the convex hull of the following points of \( A \), \( p_2,\ldots,p_l \) and the \( n-k-(l-1) \) points defining \( C \), does not contain \( p_1 \), a contradiction because \( p_1 \in C_{A,n-k} \).

Hence \( p_1,\ldots,p_l \) are in \( C_{\{p_1,\ldots,p_l\},n-k-(l-1)} \).

Remark: If \( l=n-2k+1 \), then \( n-k-(l-1)=k \) with \( k<\frac{l+l+1}{2} \), so the set \( C_{\{p_1,\ldots,p_{n-k-(l-1)} \}} \) is empty. In this case \( p_1,\ldots,p_l \) cannot be included in the set. Consequently the maximum number of points of \( A \) that can be located in \( C_{A,n-k} \) is \( n-2k \). This maximum is always attained if the \( n \) points of \( A \) are arranged in a line.

Next, it is can be seen that this is the only case in which the maximum number of points in \( C_{A,n-k} \) is attained.

Proposition 2: If the maximum of \( n-2k \) points of \( A \) inside \( C_{A,n-k} \) is attained, then the \( n \) points of \( A \) are in a straight line \( (k<\left\lfloor \frac{n}{2} \right\rfloor) \).

Proof: If there are \( n-2k \) points of \( A=\{p_1,\ldots,p_n\} \), say \( p_1,\ldots,p_{n-2k} \), included in \( C_{A,n-k} \), then by Lemma 3 the authors find that \( p_1,\ldots,p_{n-2k} \) must belong to \( C_{\{p_1,\ldots,p_{n-2k}\},k+1} \).

If \( p_{n-2k+1},\ldots,p_n \) are not collinear, then they have \( C_{\{p_1,\ldots,p_{n-2k}\},k+1} \subseteq \{p\} \) (since \( k+1=\frac{p_{n-2k+1},\ldots,p_n}{2} \)).

Hence, because \( p_1,\ldots,p_{n-2k} \) are in \( C_{\{p_1,\ldots,p_{n-2k}\},k+1} \), the authors necessarily have that \( n-2k=1 \) and thus \( k=\frac{n-1}{2}=\left\lfloor \frac{n}{2} \right\rfloor \), in contradiction with the condition \( k<\left\lfloor \frac{n}{2} \right\rfloor \). Consequently, \( p_{n-2k+1},\ldots,p_n \) are in a line, and \( C_{\{p_1,\ldots,p_{n-2k}\},k+1} \) is included in this line. This implies that \( p_1,\ldots,p_{n-2k} \) are also in the line, so all \( n \) points of \( A \) are aligned.

Thus, if \( k<\left\lfloor \frac{n}{2} \right\rfloor \) and the \( n \) points of a set \( A \) are not
in the same line, then the maximum number of points of \( A \) that can be included in \( C_{n-k} \) is \( n-2(k+1) \).

This yields the statement that the authors wanted to prove:

**Corollary:** If \( k < \left\lfloor \frac{n}{2} \right\rfloor \) and the \( n \) points of a set \( A \) are not collinear, then the minimum number of points of \( A \) taking part in some \( k \)-set is \( 2k+1 \).

### 3. Minimum Number of \( k \)-Sets

Remark 2 of Subsection 2.1 states that it is impossible to find an example similar to Example 1 for the minimum number of \( k \)-sets. This section proves that for a set of \( n \) points, the minimum number of \( k \)-sets can be attained for at most one value of \( k \). This minimum is necessarily attained in an example equivalent to the one shown in Erdős et al. Ref. [1] and Lovasz et al. Ref. [6].

**Proposition 3:** For \( k < \frac{n}{2} \), if the minimum number of \( 2k+1 \) \( k \)-sets is attained in a set of \( n \) points in general position \( A = \{p_1, \ldots, p_n\} \), then there is a subset of \( 2k+1 \) points of the set \( A \), say \( B = \{p_1, \ldots, p_{2k+1}\} \) in the boundary of the convex hull of the points of \( A \). The other points are in \( C_{k, \left\lfloor \frac{2k+1}{2} \right\rfloor} \).

**Proof:** If the minimum number of \( 2k+1 \) \( k \)-sets is attained in a set \( A \), then there can be only \( 2k+1 \) points taking part in \( k \)-sets, because a distinct \( k \)-set can be attached to each point belonging to some \( k \)-set [1]. Therefore, the other \( n-(2k+1) \) points must be in \( C_{k, n-k} \) (Proposition 1). But then the number of \( (\leq k) \)-sets in \( A \) is \( (2k+1)k \) and the number of \( (\leq (k-1)) \)-sets is \( (2k+1)(k-1) \). But this is the maximum number of \( (\leq (k-1)) \)-sets when there are just \( m = 2k+1 \) points of the set taking part in them being \( m > 2(k-1)+1 \). Hence, the \( 2k+1 \) points must be in a convex configuration [4]. The other points must be in \( C_{\left\lfloor \frac{2k+1}{2} \right\rfloor} \) because they don’t belong to any \( k \)-set.

To end this section, let us show that Result 2 cannot be generalised to points not in a line in the same way as Result 1:

**Example 2**

Consider a set of eight points, seven in a line and one out of line, as shown in Fig. 2.

This set only has four 3-sets: \( \{1, 2, 3\} \), \( \{1, 2, 8\} \), \( \{5, 6, 7\} \) and \( \{6, 7, 8\} \). This number is less than \( 7 \cdot 2 \).

### 4. Conclusions

This paper complements some of the results contained in Erdős et al. Ref. [1]. One of their findings, referred to as Result 1 in this paper, was that for a set of \( n \) points in general position, the minimum number of points taking part in \( k \)-sets is \( 2k+1 \) if \( k < \frac{n}{2} \). Erdős et al. [1] offered an example of a set of \( n \) points where this minimum is attained for a single value of \( k \).

One improvement offered by the presented paper is an example where the lower bound of \( 2k+1 \)-sets is attained for every \( k < \frac{n}{2} \). According to the notation of Ábrego et al. [8] this is an example of a set with exactly two points in the \( k \)-layer, for every \( k \) with \( 1 < k < \frac{n}{2} \).

![Fig. 2](Fig. 2 A set of points is not in general position with fewer than \( 2k+1 \) \( k \)-sets.)
The other main improvement is the extension of Result 1 to any set of \( n \) points not arranged in a line.

The authors next analysed another theorem of Erdős et al. [1] referred to here as Result 2. This theorem states that the minimum number of \( k \)-sets in a set of \( n \) points in general position is also \( 2k+1 \).

The present paper proves that the example provided for Result 2 in the literature, where the minimum number of \( k \)-sets is attained, is essentially the only possible example.

Finally, the authors provide an example to prove that Result 2 cannot be generalised in the same way as Result 1, for any set of unaligned points.

**References**


