Diffeomorphism-invariant Covariant Hamiltonians of a pseudo-Riemannian Metric and a Linear Connection

J. Muñoz Masqué†, M. Eugenia Rosado María‡

†Instituto de Física Aplicada, CSIC
C/ Serrano 144, 28006-Madrid, Spain
‡Departamento de Matemática Aplicada
Escuela Técnica Superior de Arquitectura, UPM
Avda. Juan de Herrera 4, 28040-Madrid, Spain

jaime@iec.csic.es, eugenia.rosado@upm.es

e-print archive:
Abstract

Let $M \to N$ (resp. $C \to N$) be the fibre bundle of pseudo-Riemannian metrics of a given signature (resp. the bundle of linear connections) on an orientable connected manifold $N$. A geometrically defined class of first-order Ehresmann connections on the product fibre bundle $M \times_N C$ is determined such that, for every connection $\gamma$ belonging to this class and every $\text{Diff}_N$-invariant Lagrangian density $\Lambda$ on $J^1(M \times_N C)$, the corresponding covariant Hamiltonian $\Lambda^\gamma$ is also $\text{Diff}_N$-invariant. The case of $\text{Diff}_N$-invariant second-order Lagrangian densities on $J^2M$ is also studied and the results obtained are then applied to Palatini and Einstein-Hilbert Lagrangians.
1 Introduction

In Mechanics, the Hamiltonian function attached to a Lagrangian density \( \Lambda = L(t, q^i, \dot{q}^i)dt \) on \( \mathbb{R} \times TQ \) is given by \( H = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \), but—as it was early observed in [16]—this is not an invariant definition if an arbitrary fibred manifold \( t: E \to \mathbb{R} \) is considered (thus generalizing the notion of an absolute time) instead of the direct product bundle \( \mathbb{R} \times Q \to \mathbb{R} \); e.g., see [7], [23], [25] for this point of view. In this case, an Ehresmann connection is needed in order to lift the vector field \( \partial/\partial t \) from \( \mathbb{R} \) to \( E \), and the Hamiltonian is then defined by applying the Poincaré-Cartan form attached to \( \Lambda \) to the horizontal lift of \( \partial/\partial t \).

In the field theory—where no distinguished vector field exists on the base manifold—the need of an Ehresmann connection is even greater, in order to attach a covariant Hamiltonian to each Lagrangian density; e.g., see [24, 4.1], [23], and the definitions below.

Let \( p: E \to N \) be an arbitrary fibred manifold over a connected manifold \( N \), \( n = \dim N \), \( \dim E = m+n \), oriented by \( v_n = dx^1 \wedge \cdots \wedge dx^n \). Throughout this paper, Latin (resp. Greek) indices run from 1 to \( n \) (resp. \( m \)). An Ehresmann connection on a fibred manifold \( p: E \to N \) is a differential 1-form \( \gamma \) on \( E \) taking values in the vertical sub-bundle \( V(p) \) such that \( \gamma(X) = X \) for every \( X \in V(p) \) (e.g., see [23], [24], [32], [34]). Once an Ehresmann connection \( \gamma \) is given, a decomposition of vector bundles holds \( T(E) = V(p) \oplus \ker \gamma \), where \( \ker \gamma \) is called the horizontal sub-bundle determined by \( \gamma \). In a fibred coordinate system \( (x^j, y^\alpha) \) for \( p \), an Ehresmann connection can be written as

\[
\gamma = (dy^\alpha + \gamma_i^\alpha dx^i) \otimes \frac{\partial}{\partial y^\alpha}, \quad \gamma_i^\alpha \in C^\infty(E).
\]
According to [24], the covariant Hamiltonian $\Lambda^\gamma$ associated to a Lagrangian density on $J^1 E$, $\Lambda = L v_n$, $L \in C^\infty(J^1 E)$, with respect to $\gamma$ is the Lagrangian density defined by,

\[ \Lambda^\gamma = \left( (p^1_0)^* \gamma - \theta \right) \wedge \omega_\Lambda - \Lambda, \]

where, $p^1_0: J^1 E \to J^0 E = E$ is the projection mapping, $\theta = \theta^\alpha \otimes \partial / \partial y^\alpha$, $\theta^\alpha = dy^\alpha - y^\alpha_i dx^i$ is the $V^\alpha$-valued 1-form on $J^1 E$ associated with the contact structure, written on a fibred coordinate system $(x^i, y_\alpha^i)$, and $\omega_\Lambda$ is the Legendre form attached to $\Lambda$, i.e., the $V^*(p)$-valued $p^1$-horizontal $(n-1)$-form on $J^1 E$ given by

\[ \omega_\Lambda = (-1)^{i-1} \frac{\partial L}{\partial y^\alpha_i} i_{\partial / \partial x^i} v_n \otimes dy^\alpha, \]

where $(x^i, y^\alpha^i; y^\alpha_i)$ is the coordinate system induced from $(x^i, y^\alpha)$ on the 1-jet bundle and $p^1: J^1 E \to N$ is the projection on the base manifold. Locally,

\[ \Lambda^\gamma = \left( (\gamma^\alpha_i + y^\alpha_i) \frac{\partial L}{\partial y^\alpha_i} - L \right) dx^1 \wedge \cdots \wedge dx^n. \]

From (1) we obtain the following decomposition of the Poincaré-Cartan form attached to $\Lambda$ (e.g., see [17], [23], [27]): $\Theta_\Lambda = \theta \wedge \omega_\Lambda + \Lambda = (p^1_0)^* \gamma \wedge \omega_\Lambda - \Lambda^\gamma$.

A diffeomorphism $\Phi: E \to E$ is said to be an automorphism of $p$ if there exists $\phi \in \text{Diff} N$ such that $p \circ \Phi = \phi \circ p$. The set of such automorphisms is denoted by $\text{Aut}(p)$ and its Lie algebra is identified to the space $\text{aut}(p) \subset \mathfrak{X}(E)$ of $p$-projectable vector fields on $E$. Given a subgroup $\mathcal{G} \subseteq \text{Aut}(p)$, a Lagrangian density $\Lambda$ is said to be $\mathcal{G}$-invariant if $(\Phi(1))^* \Lambda = \Lambda$ for every $\Phi \in \mathcal{G}$, where $\Phi(1): J^1 E \to J^1 E$ denotes the 1-jet prolongation of $\Phi$. Infinitesimally, the $\mathcal{G}$-invariance equation can be reformulated as $L_X \Lambda = 0$ for every $X \in \text{Lie}(\mathcal{G})$, $X(1)$ denoting the 1-jet prolongation of the vector field $X$.

When a group $\mathcal{G}$ of transformations of $E$ is given, a natural question arises:

- Determine a class—as small as possible— of Ehresmann connections $\gamma$ such that $\Lambda^\gamma$ is $\mathcal{G}$-invariant for every $\mathcal{G}$-invariant Lagrangian density $\Lambda$.

Below we tackle this question in the framework of General Relativity, i.e., the group $\mathcal{G}$ is the group of all diffeomorphisms of the ground manifold $N$ acting in a natural way either on the bundle of pseudo-Riemannian metrics $p_M: M = M(N) \to N$ of a given signature $(n^+, n^-)$, $n^+ + n^- = n$, or on the product bundle $p: M \times_N C \to N$, where $p_C: C = C(N) \to N$ is the bundle of linear connections on $N$. Namely, we solve the following two problems:
(P): Determine a class—as small as possible—of Ehresmann connections $\gamma$ such that for every $\text{Diff}_N$-invariant first-order Lagrangian density $\Lambda$ on the bundle $J^1(M \times NC)$, the corresponding covariant Hamiltonian $\Lambda^\gamma$ is also $\text{Diff}_N$-invariant.

Similarly to the problem (P), we formulate the corresponding problem on $J^2M$ as follows:

(P2): Determine a class of second-order Ehresmann connections $\gamma^2$ on $M$ such that for every $\text{Diff}_N$-invariant second-order Lagrangian density $\Lambda$ on the bundle $J^2M$, the corresponding covariant Hamiltonian $\Lambda^\gamma^2$—defined in (42)—is also $\text{Diff}_N$-invariant.

Essentially, a class of first-order Ehresmann connections on the bundle $M \times NC$ is obtained, defined by the conditions $(C_M)$ and $(C_C)$ below (see Propositions 3.4 and 3.5), solving the problem (P). This class of connections also helps to solve (P2) by means of a natural isomorphism between $J^1M$ and $M \times NC_{\text{sym}}$, where $C_{\text{sym}}$ denotes the sub-bundle of symmetric connections on $N$ (cf. Theorem 4.1). Finally, this approach is applied to Palatini and Einstein-Hilbert Lagrangians ([3], [4]), obtaining results compatible with their usual Hamiltonian formalisms.

2 Invariance under diffeomorphisms

2.1 Preliminaries

2.1.1 Jet-bundle notations

Let $p_k: J^kE \rightarrow N$ be the $k$-jet bundle of local sections of an arbitrary fibred manifold $p: E \rightarrow N$, with projections $p^k_l: J^kE \rightarrow J^lE$, $p^k_I(j^k_xs) = j^l_I(x)$, for $k \geq l$, $j^k_xs$ denoting the $k$-jet at $x$ of a section $s$ of $p$ defined on a neighbourhood of $x \in N$.

A fibred coordinate system $(x^i, y^\alpha)$ on $V$ induces a coordinate system $(x^i, y^\alpha_I)$, $I = (i_1, \ldots, i_N) \in \mathbb{N}^n$, $0 \leq |I| = i_1 + \cdots + i_N \leq r$, on $(p^r_0)^{-1}(V) = J^rV$ as follows: $y^\alpha_I(j^r_xs) = (\partial^{|I|}(y^\alpha \circ s)/\partial x^I)(x)$, with $y^\alpha_0 = y^\alpha$.

Every morphism $\Phi: E \rightarrow E'$ whose associated map $\phi: N \rightarrow N'$ is a diffeomorphism, induces a map

\[
\Phi^{(r)}: J^rE \rightarrow J^rE',
\]

\[
\Phi^{(r)}(j^r_xs) = j^r_{\phi(s)}(\Phi \circ s \circ \phi^{-1}).
\]
If $\Phi_t$ is the flow of a vector field $X \in \text{aut}(p)$, then $\Phi_t^{(r)}$ is the flow of a vector field $X^{(r)} \in \mathfrak{X}(J^r E)$, called the infinitesimal contact transformation of order $r$ associated to the vector field $X$. The mapping

$$\text{aut}(p) \ni X \mapsto X^{(r)} \in \mathfrak{X}(J^r E),$$

is an injection of Lie algebras, namely, one has

$$(\lambda X + \mu Y)^{(r)} = \lambda X^{(r)} + \mu Y^{(r)},$$
$$[X, Y]^{(r)} = [X^{(r)}, Y^{(r)}],$$
$$\forall \lambda, \mu \in \mathbb{R}, \forall X, Y \in \text{aut}(p).$$

In particular, for $r = 1$,

$$X = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E),$$
$$X^{(1)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, \quad v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + y_i^\beta \frac{\partial v^\alpha}{\partial y^\beta} - y_k^\alpha \frac{\partial u^k}{\partial x^i}.$$  

2.1.2 Coordinates on $M(N)$, $F(N)$, $C(N)$

Every coordinate system $(x^i)$ on an open domain $U \subseteq N$ induces the following coordinate systems:

1) $(x^i, y_{jk})$ on $(p_M)^{-1}(U)$, where $p_M: M \rightarrow N$ is the bundle of metrics of a given signature, and the functions $y_{jk} = y_{kj}$ are defined by,

$$g_x = \sum_{i \leq j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \forall g_x \in (p_M)^{-1}(U).$$  

2) $(x^i, x^j_i)$ on $(p_F)^{-1}(U)$, where $p_F: F(N) \rightarrow N$ is the bundle of linear frames on $N$, and the functions $x^j_i$ are defined by,

$$u = ((\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x) \cdot (x^j_i(u)), \quad x = p_F(u), \forall u \in (p_F)^{-1}(U),$$

or equivalently,

$$u = (X_1, \ldots, X_N) \in F_x(N), \quad X_j = x^j_i(u) \left(\frac{\partial}{\partial x^i}\right)_x, \quad 1 \leq j \leq n.$$  

3) $(x^i, A^j_{kl})$ on $(p_C)^{-1}(U)$, where $p_C: C \rightarrow N$ is the bundle of linear connections on $N$, and the functions $A^j_{kl}$ are defined as follows. We first recall some basic facts. Connections on $F(N)$ (i.e., linear connections of $N$) are the splittings of the Atiyah sequence (cf. [2]),

$$0 \rightarrow \text{ad}F(N) \rightarrow T_{GL(n, \mathbb{R})}F(N) \xrightarrow{(p_F)_*} TN \rightarrow 0,$$

where

a) $\text{ad}F(N) = T^*N \otimes TN$ is the adjoint bundle,
b) \( T_{Gl(n, \mathbb{R})}(F(N)) = T(F(N))/Gl(n, \mathbb{R}) \), and

c) \( \text{gau}F(N) = \Gamma(N, \text{ad}F(N)) \) is the gauge algebra of \( F(N) \).

We think of \( \text{gau}F(N) \) as the ‘Lie algebra’ of the gauge group \( \text{Gau}F(N) \).
Moreover, \( p_C: C \rightarrow N \) is an affine bundle modelled over the vector bundle \( \otimes^2 T^* N \otimes TN \). The section of \( p_C \) induced tautologically by the linear connection \( \Gamma \) is denoted by \( s_{\Gamma}: N \rightarrow C \). Every \( B \in \mathfrak{gl}(n, \mathbb{R}) \) defines a one-parameter group \( \varphi_B^t: U \times \text{Gl}(n, \mathbb{R}) \rightarrow U \times \text{Gl}(n, \mathbb{R}) \) of gauge transformations by setting (cf. [5]), \( \varphi_B^t(x, \Lambda) = (x, \exp(tB) \cdot \Lambda) \). Let us denote by \( \tilde{B} \in \text{gau}(p_F)^{-1}(U) \) the corresponding infinitesimal generator. If \( (E_i^j) \) is the standard basis of \( \mathfrak{gl}(n, \mathbb{R}) \), then \( \tilde{E}_i^j = \sum_{h=1}^n x_h \partial / \partial x_h \), for \( i, j = 1, \ldots, n \), is a basis of \( \text{gau}(p_F)^{-1}(U) \). Let \( \tilde{E}_i^j = \tilde{E}_j^i \) \( \text{mod} \mathbb{G} \) be the class of \( E_i^j \) on \( \text{ad}F(N) \). Unique smooth functions \( A_{ij}^k \) on \( (p_C)^{-1}(U) \) exist such that,

\[
\begin{align*}
\Gamma \left( \frac{\partial}{\partial x^j} \right) &= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma) \tilde{E}_k^i \\
&= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma)x_h^k \frac{\partial}{\partial x_h^i},
\end{align*}
\]

for every \( \Gamma \) and \( A_{jk}^i(\Gamma(x)) = \Gamma_{jk}^i(x) \), where \( \Gamma_{jk}^i \) are the Christoffel symbols of the linear connection \( \Gamma \) in the coordinate system \( (x^i) \), see [20, III, Propostion 7.4].

### 2.2 Natural lifts

Let \( f_M: M \rightarrow M \), cf. [30] (resp. \( \tilde{f}: F(N) \rightarrow F(N) \), cf. [20, p. 226]) be the natural lift of \( f \in \text{Diff}N \) to the bundle of metrics (resp. linear frame bundle); namely \( f_M(g_x) = (f^{-1})^* g_x \) (resp. \( \tilde{f}(X_1, \ldots, X_N) = (f \cdot X_1, \ldots, f \cdot X_N) \)), where \( (X_1, \ldots, X_N) \in F_x(N) \); hence \( p_M \circ f_M = f \circ p_M \) (resp. \( p_F \circ \tilde{f} = f \circ p_F \)), and \( f_M: M \rightarrow M \) (resp. \( \tilde{f}: F(N) \rightarrow F(N) \)) have a natural extension to jet bundles \( f_M^{(r)}: J^r(M) \rightarrow J^r(M) \) (resp. \( \tilde{f}^{(r)}: J^r(F(N)) \rightarrow J^r(F(N)) \)) as defined in the formula (3), i.e.,

\[
\begin{align*}
f_M^{(r)}(j_x^r g) &= j_f^r((f_M \circ g \circ f^{-1}) \ (\text{resp. } \tilde{f}^{(r)}(j_x^r s) = j_f^r((\tilde{f} \circ s \circ f^{-1})).
\end{align*}
\]

As \( \tilde{f} \) is an automorphism of the principal \( \text{Gl}(n, \mathbb{R}) \)-bundle \( F(N) \), it acts on linear connections by pulling back connection forms, i.e., \( \Gamma' = \tilde{f}(\Gamma) \) where \( \omega_{\Gamma'} = (\tilde{f}^{-1})^* \omega_{\Gamma} \) (see [20, II, Proposition 6.2-(b)], [5, 3.3]). Hence, there exists a unique diffeomorphism \( \tilde{f}_C: C \rightarrow C \) such that,

1) \( p_C \circ \tilde{f}_C = f \circ p_C \), and
2) $\tilde{f}_C \circ s_\Gamma = s_{\tilde{f}_G}$ for every linear connection $\Gamma$.

If $f_t$ is the flow of a vector field $X \in \mathfrak{X}(N)$, then the infinitesimal generator of $(f_t)_M$ (resp. $\tilde{f}_t$, resp. $(\tilde{f}_t)_C$) in $\text{Diff}M$ (resp. $\text{Diff}F(N)$, resp. $\text{Diff}C$) is denoted by $X_M$ (resp. $\tilde{X}$, resp. $\tilde{X}_C$) and the following Lie-algebra homomorphisms are obtained:

$$
\begin{align*}
\begin{cases}
\mathfrak{X}(N) \to \mathfrak{X}(M), & X \mapsto X_M \\
\mathfrak{X}(N) \to \mathfrak{X}(F(N)), & X \mapsto \tilde{X} \\
\mathfrak{X}(N) \to \mathfrak{X}(C), & X \mapsto \tilde{X}_C
\end{cases}
\end{align*}
$$

If $X = u^i \partial/\partial x^i \in \mathfrak{X}(N)$ is the local expression for $X$, then

1) From [30, eqs. (2)–(4)] we know that the natural lift of $X$ to $M$ is given by,

$$
X_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M),
$$

and its 1-jet prolongation,

$$
X_M^{(1)} = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial^2 u^h}{\partial x^i \partial x^j} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^i} y_{hi} \right) \frac{\partial}{\partial y_{ij}} - \sum_{i \leq j} \left( \frac{\partial^2 u^h}{\partial x^i \partial x^j} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^i} y_{hi} \right) \frac{\partial}{\partial y_{ij,k}}.
$$

2) From [10, Proposition 3] (also see [20, VI, Proposition 21.1]) we know that the natural lift of $X$ to $F(N)$ is given by,

$$
\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^i} x^j \frac{\partial}{\partial x^j},
$$

and its 1-jet prolongation,

$$
\tilde{X}^{(1)} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^i} x^j \frac{\partial}{\partial x^j} + v^i_{jk} \frac{\partial}{\partial x^j_{jk}},
$$

$$
v^i_{jk} = \frac{\partial u^i}{\partial x^k} x^j_{j,k} - \frac{\partial u^i}{\partial x^k} x^i_{j,k} + \frac{\partial^2 u^i}{\partial x^k \partial x^l} x^j_{j,l}.
$$

3) Finally,

$$
\tilde{X}_C = u^i \frac{\partial}{\partial x^i} - \left( \frac{\partial^2 u^i}{\partial x^j \partial x^k} A^i_{jk} + \frac{\partial u^i}{\partial x^k} A^i_{jl} + \frac{\partial u^i}{\partial x^j} A^i_{lk} \right) \frac{\partial}{\partial A^i_{jk}},
$$
\[ \tilde{X}_C^{(1)} = u^i \frac{\partial}{\partial x^i} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jkh}}, \]

(7) \[ w^i_{jk} = -\frac{\partial^2 u^i}{\partial x^j \partial x^k} + \frac{\partial u^i}{\partial x^j} A^i_{jk} - \frac{\partial u^i}{\partial x^k} A^i_{jk}, \]

(8) \[ w^i_{jkh} = -\frac{\partial^2 u^i}{\partial x^j \partial x^l \partial x^k} + \frac{\partial^2 u^i}{\partial x^j \partial x^l} A^i_{jk} - \frac{\partial^2 u^i}{\partial x^k \partial x^l} A^i_{jk} - \frac{\partial^2 u^i}{\partial x^l} A^i_{jk} \]

Let \( p: M \times_N C \to N \) be the natural projection.

We denote by \( \tilde{f} = (f_M, \tilde{f}_C) \) (resp. \( \tilde{X} = (X_M, \tilde{X}_C) \in \mathcal{X}(M \times_N C) \)) the natural lift of \( f \) (resp. \( X \)) to \( M \times_N C \). The prolongation to the bundle \( J^1(M \times_N C) \) of \( X \) is as follows:

\[ \bar{X}^{(1)} = \left( X^{(1)}_M, \tilde{X}^{(1)}_C \right) \]

\[ = u^i \frac{\partial}{\partial x^i} + \sum_{i \leq j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \leq j} v_{ijk} \frac{\partial}{\partial y_{ijk}} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jkh}}, \]

where

(10) \[ v_{ij} = \frac{\partial u^h}{\partial x^j} y_{ij} - \frac{\partial u^h}{\partial y_{hi}} y_{ij}, \]

(11) \[ v_{ijk} = \frac{\partial^2 u^h}{\partial x^i \partial x^j} y_{ijk} - \frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{ijk} - \frac{\partial u^h}{\partial x^i} y_{ijk} - \frac{\partial u^h}{\partial y_{i,k}} y_{ijk} \]

and \( w^i_{jk}, w^i_{jkh} \) are given in the formulas (7), (8), respectively.

### 2.3 Diff\(N\)- and \(\mathfrak{X}(N)\)-invariance

A differential form \( \omega_r \in \Omega^r(J^1(M \times_N C)) \), \( r \in \mathbb{N} \), is said to be Diff\(N\)-invariant— or invariant under diffeomorphisms— (resp. \(\mathfrak{X}(N)\)-invariant) if the following equation holds: \((f^{(1)})^* \omega_r = \omega_r, \forall f \in \text{Diff} \, N \) (resp. \( L_{\mathfrak{g}(1)} \omega_r = 0 \), \( \forall X \in \mathfrak{X}(N) \)). Obviously, “Diff\(N\)-invariance” implies “\(\mathfrak{X}(N)\)-invariance” and the converse is almost true (see [14, 28]). Because of this, below we consider \(\mathfrak{X}(N)\)-invariance only.

A linear frame \( (X_1, \ldots, X_N) \) at \( x \) is said to be orthonormal with respect to \( g_x \in M_N \) (or simply \( g_x \)-orthonormal) if \( g_x(X_i, X_j) = 0 \) for \( 1 \leq i < j \leq n \), \( g(X_i, X_i) = 1 \) for \( 1 \leq i \leq n^+ \), \( g(X_i, X_i) = -1 \) for \( n^+ + 1 \leq i \leq n \).
As \( N \) is an oriented manifold, there exists a unique \( p \)-horizontal \( n \)-form \( \mathbf{v} \) on \( M \times N \) such that, \( \mathbf{v}_{(g_x, g_v)}(X_1, \ldots, X_N) = 1 \), for every \( g_x \)-orthonormal basis \((X_1, \ldots, X_N)\) belonging to the orientation of \( N \). Locally \( \mathbf{v} = \rho \mathbf{v}_n \), where \( \rho = \sqrt{(-1)^n \det(y_{ij})} \) and \( \mathbf{v}_n = dx^1 \wedge \cdots \wedge dx^n \). As proved in [30, Proposition 7], the form \( \mathbf{v} \) is \( \text{Diff}_N \)-invariant and hence \( \mathfrak{X}(N) \)-invariant. A Lagrangian density \( \Lambda \) on \( J^1(M \times N) \) can be globally written as \( \Lambda = \mathcal{L} \mathbf{v} \) for a unique function \( \mathcal{L} \in C^\infty(J^1(M \times N)) \) and \( \Lambda \) is \( \mathfrak{X}(N) \)-invariant if and only if the function \( \mathcal{L} \) is. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

**Proposition 2.1.** A function \( \mathcal{L} \in C^\infty(J^1(M \times N)) \) is \( \mathfrak{X}(N) \)-invariant if and only if the following system of partial differential equations hold:

\[
\begin{align*}
0 &= X^i(\mathcal{L}), \quad \forall i, \\
0 &= X^i_h(\mathcal{L}), \quad \forall h, i, \\
0 &= X^i_{hk}(\mathcal{L}), \quad \forall h, i \leq k, \\
0 &= X^i_{jkh}(\mathcal{L}), \quad \forall i, j \leq k, h,
\end{align*}
\]

where

\[
X^i = \frac{\partial}{\partial x^i}, \quad \forall i,
\]

\[
X^i_h = -y_{hi} \frac{\partial}{\partial y_{j1}} - y_{hj} \frac{\partial}{\partial y_{i1}} - y_{ij} \frac{\partial}{\partial y_{h1}} - y_{jk} \frac{\partial}{\partial y_{ih}} - \sum_{s \leq j} y_{sj} \frac{\partial}{\partial y_{sj}},
\]

\[
+ A^i_{jk} \frac{\partial}{\partial A^j_{hk}} - A^r_{jh} \frac{\partial}{\partial A^r_{ji}} - A^r_{hk} \frac{\partial}{\partial A^r_{ik}}.
\]

\[
+ A^i_{jkh} \frac{\partial}{\partial A^{jkh}_{hj}} - A^s_{jkh} \frac{\partial}{\partial A^{s}_{ji}} - A^s_{hkr} \frac{\partial}{\partial A^{s}_{kr}} - A^r_{jkh} \frac{\partial}{\partial A^{r}_{ji}}, \quad \forall h, i,
\]

\[
X^i_{hk} = -y_{hi} \frac{\partial}{\partial y_{iv}} - y_{hv} \frac{\partial}{\partial y_{ki}} - y_{hv} \frac{\partial}{\partial y_{ki}} - y_{hv} \frac{\partial}{\partial y_{ki}} - \sum_{s \leq j} y_{sj} \frac{\partial}{\partial y_{sj}},
\]

\[
+ A^i_{jks} \frac{\partial}{\partial A^j_{hs}} - A^s_{jks} \frac{\partial}{\partial A^s_{kj}} - A^s_{hkr} \frac{\partial}{\partial A^s_{kr}}.
\]

\[
+ A^i_{jks} \frac{\partial}{\partial A^j_{hs}} - A^s_{jks} \frac{\partial}{\partial A^s_{kj}} - A^s_{hkr} \frac{\partial}{\partial A^s_{kr}}, \quad \forall h, i \leq k,
\]

\[
X^i_{jkh} = \frac{\partial}{\partial A^j_{kh}} + \frac{\partial}{\partial A^j_{hj}} + \frac{\partial}{\partial A^i_{kj}} + \frac{\partial}{\partial A^i_{hj}} + \frac{\partial}{\partial A^i_{kh}} + \frac{\partial}{\partial A^i_{hk}}, \quad \forall i, h \leq j \leq k.
\]

Moreover, the vector fields \( X^i, X^i_h, X^i_{hk}, X^i_{jkh} \) are linearly independent and they span an involutive distribution on \( J^1(M \times N) \) of rank \( n(n+3)/3 \). Hence, the number of functionally invariant Lagrangians on \( J^1(M \times N) \) is

\[
\frac{1}{6} (5n^4 + 3n^3 - 5n^2 + 3n).
\]
Proof. According to the formula (9), $\mathcal{L}$ is invariant if and only if,

$$
0 = u^i \frac{\partial \mathcal{L}}{\partial x^i} + \sum_{i<j} u_{ij} \frac{\partial \mathcal{L}}{\partial y_{ij}} + \sum_{i<j} u_{ijk} \frac{\partial \mathcal{L}}{\partial A_{jk}} + w^i_{jk} \frac{\partial \mathcal{L}}{\partial A_{jk}^i} + w^i_{jkh} \frac{\partial \mathcal{L}}{\partial A_{jk}^i} = 0,
$$

$\forall u^i \in C^\infty(N)$,

and expanding on this equation by using the formulas (10), (11), (7), and (8) we obtain

$$
0 = u^i \frac{\partial \mathcal{L}}{\partial x^i} + \frac{\partial u^h}{\partial x^i} \left( -y_{ih} \frac{\partial \mathcal{L}}{\partial y_{ij}} - y_{ijk} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} - y_{ij,k} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} \right)
$$

$$
- \sum_{s<j} y_{sj,h} \frac{\partial \mathcal{L}}{\partial y_{sj,i}} + A_{jk}^i \frac{\partial \mathcal{L}}{\partial A_{jk}^i} - A_{jr}^i \frac{\partial \mathcal{L}}{\partial A_{kr}^r} - A_{jk}^i \frac{\partial \mathcal{L}}{\partial A_{jk}^i}
$$

$$
+ A_{jk}^i \frac{\partial \mathcal{L}}{\partial A_{jk}^i} - A_{jr}^i \frac{\partial \mathcal{L}}{\partial A_{jr}^r} - A_{jk}^i \frac{\partial \mathcal{L}}{\partial A_{kr}^r}
$$

$$
+ \frac{\partial^2 u^h}{\partial x^i \partial x^k} \left( -y_{ih} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} - y_{ijk} \frac{\partial \mathcal{L}}{\partial y_{ij,k}} - \frac{\partial \mathcal{L}}{\partial A_{ik}} \right)
$$

$$
+ A_{js}^i \frac{\partial \mathcal{L}}{\partial A_{js}^j} - A_{js}^j \frac{\partial \mathcal{L}}{\partial A_{js}^j} - A_{jk}^i \frac{\partial \mathcal{L}}{\partial A_{kr}^r}
$$

$$
- \frac{\partial^3 u^i}{\partial x^h \partial x^i \partial x^j} \frac{\partial \mathcal{L}}{\partial A_{jk}^i}.
$$

This equation is equivalent to the system of the statement as the values for $u^h$, $\partial u^h/\partial x^i$, $\partial^2 u^h/\partial x^i \partial x^j$ ($i \leq j$), and $\partial^3 u^h/\partial x^i \partial x^j \partial x^k$ ($i \leq j \leq k$) at a point $x \in N$ can be taken arbitrarily. Moreover, assume a linear combination holds

$$
\lambda_a X^a + \lambda_b^c X^b + \sum_{b \leq c} \lambda_{bc}^c X^c + \sum_{b \leq c \leq d} \lambda_{bcd}^d X^{bcd} = 0,
$$

$\lambda_a, \lambda_b^c, \lambda_{bc}^c, \lambda_{bcd}^d \in C^\infty(J^1(M \times N C))$.

By applying (15) to $x^a$ (resp. $y_{ab}$) we obtain $\lambda_a = 0$ (resp. $\lambda_b^c = 0$); again by applying (15) to $A_{bc}^a$, $b \leq c$ (resp. $A_{bc}^a$, $c \leq b$) and taking the expressions of the vector fields (13) and (14) into account, we obtain $\lambda_{bc}^a = 0$, $b \leq c$ (resp. $\lambda_{bc}^a = 0$, $c \leq b$). Hence, (15) reads $\sum_{b \leq c \leq d} \lambda_{bcd}^d X^{bcd} = 0$, and by applying it to $A_{bc}^a$ and taking the expressions of the vector fields (14) into account, we finally obtain $\lambda_{bcd}^d = 0$. The distribution

$$
\mathcal{D}_{M \times N C} = \left\{ X^{(1)}_{(j_2g,j_1x)} : X \in \mathcal{X}(N), (j_1 g, j_2 x) \in J^1(M \times N C) \right\}.
$$
in $T(J^1(M \times N C))$, where $\bar{X}^{(1)}$ is defined in (9), is involutive as

$$\left[ \bar{X}^{(1)}, \bar{Y}^{(1)} \right] = [X, Y]^{(1)}, \quad \forall X, Y \in \mathfrak{X}(N),$$

and it is spanned by $X^i, X^h_i, X^j_k, X^j_k^h$, as proved by the formulas above. The rest of the statement follows from the following identities:

$$\# \left\{ X^i; X^h_i; X^j_k; h \leq j \leq k : h, i, j, k = 1, \ldots, n \right\} = n + n^2 + n(n+1)/2 + n(n+2)/3 = n(n+3)/3,$$

$$\dim J^1(M \times N C) - n(n+3)/3 = \frac{1}{6} (5n^4 + 3n^3 - 5n^2 + 3n).$$

\[ \square \]

3 Invariance of covariant Hamiltonians

3.1 Position of the problem

On the bundle $E = M \times N C$, an Ehresmann connection can locally be written as follows:

$$\gamma = \sum_{i \leq j} \left( dy_{ij} + \gamma_{ijk} dx^k \right) \otimes \frac{\partial}{\partial y_{ij,k}} + \left( dA^j_{ik} + \gamma^i_{jkl} dx^l \right) \otimes \frac{\partial}{\partial A^i_{jkl}},$$

$$\gamma_{ijk}, \gamma^i_{jkl} \in C^\infty(M \times N C).$$

In particular, for a Lagrangian density $\Lambda$ on $J^1(M \times N C)$ we obtain

$$\Lambda^\gamma = \left( \sum_{i \leq j} \left( \gamma_{ijk} + y_{ijk} \right) \frac{\partial L}{\partial y_{ijk,k}} + \left( \gamma^i_{jkl} + A^i_{jkl} \right) \frac{\partial L}{\partial A^i_{jkl,l}} - L \right) dx^1 \wedge \ldots \wedge dx^n,$$

or equivalently, $\mathcal{L}^\gamma = D^\gamma(\mathcal{L}) - \mathcal{L}$, where

$$D^\gamma = \sum_{i \leq j} \left( \gamma_{ijk} + y_{ijk} \right) \frac{\partial}{\partial y_{ijk,k}} + \left( \gamma^i_{jkl} + A^i_{jkl} \right) \frac{\partial}{\partial A^i_{jkl,l}}.$$

Remark 3.1. The horizontal form $(p_0^1)^* \gamma - \theta = (\gamma^i + y_i^\alpha) dx^i \otimes \partial/\partial y^\alpha$ can also be viewed as the $p_0^1$-vertical vector field

$$D^\gamma = (\gamma^i + y_i^\alpha) \frac{\partial}{\partial y_0^\alpha},$$

taking the natural isomorphism $V(p_0^1) \cong (p_0^1)^*(p^*T^*N \otimes V(p))$ into account (cf. [23], [24], [32], [34]).
According to the previous formulas, this means: If the system \((12)\) holds for a Lagrangian function \(L\), then it also holds for the covariant Hamiltonian \(L^\gamma\).

If \(X \in \{X^i, X^i_h, X^i_{hk}, X^i_{jkh}\}\), then \(X(L) = X(D^\gamma(L))\), as \(L\) is assumed to be invariant and hence \(X(L) = 0\). Therefore

\[
X(L^\gamma) = X(D^\gamma(L)) = [X, D^\gamma(L)],
\]

and we conclude the following:

**Proposition 3.2.** The property \((P)\) holds for an Ehresmann connection \(\gamma\) on \(M \times_N C\) if and only if the vector field \(D^\gamma\) transforms the sections of the distribution \(D_{M \times N C}\) into themselves, namely, \([D^\gamma, \Gamma(D_{M \times N C})] \subseteq \Gamma(D_{M \times N C})\).

The problem thus reduces to compute the brackets \([X^i, D^\gamma]\), \([X^i_h, D^\gamma]\), \([X^i_{hk}, D^\gamma]\), and \([X^i_{jkh}, D^\gamma]\). We have

\[
[X^h, D^\gamma] = \sum_{i \leq j} \frac{\partial \gamma_{ijk}}{\partial y_{ij,k}} \frac{\partial}{\partial x^h} + \frac{\partial \gamma_{jkl}}{\partial x^h} \frac{\partial}{\partial A^{j}_{k,l}},
\]

\[
[X^c_{da}, D^\gamma] = X^c_{da}, \quad \forall b, c \leq d \leq a,
\]

\[
[X^i_h, D^\gamma] = \sum_{a \leq b} Y^i_h (\gamma_{abc}) \frac{\partial}{\partial y_{ab,c}} + Y^i_h (\gamma_{abc}) \frac{\partial}{\partial A^d_{ab,c}} + X^i_h - Y^i_h,
\]

\[
[X^i_{hk}, D^\gamma] = \sum_{a \leq b} Y^i_{hk} (\gamma_{abc}) \frac{\partial}{\partial y_{abc}} + Y^i_{hk} (\gamma_{abc}) \frac{\partial}{\partial A^d_{abc}} + X^i_{hk} - Y^i_{hk},
\]

where

\[
Y^i_h = -y_{hi} \frac{\partial}{\partial y_{ii}} - y_{nj} \frac{\partial}{\partial y_{ij}} + A^i_{jk} \frac{\partial}{\partial A^h_{jk}} - A^r_{jh} \frac{\partial}{\partial A^r_{ji}} - A^r_{hk} \frac{\partial}{\partial A^r_{ik}},
\]

\[
Y^i_{hk} = -\frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}},
\]
and the following formula has been used:

$$\frac{\partial y_{rs,k}}{\partial y_{ij,h}} = \delta^k_h \left( \delta^r_i \delta^s_j + \delta^r_j \delta^s_i - \delta^i_j \delta^r_s \delta^s_i \right).$$

### 3.2 The class of the Ehresmann connections defined

Let $p: M \times N C \to N$, $pr_1: M \times N C \to M$, $pr_2: M \times N C \to C$ be the natural projections. By taking the differential of $pr_1$ and $pr_2$, a natural identification is obtained $T(M \times N C) = TM \times TN TC$. Hence

$$V(p) = V(p_M) \times_N V(p_C) = pr_1^* V(p_M) \oplus pr_2^* V(p_C)$$

and two unique vector-bundle homomorphisms exist

$$\gamma_M: pr_1^* TM \to pr_1^* V(p_M), \quad \gamma_C: pr_2^* TC \to pr_2^* V(p_C),$$

such that,

$$\gamma(X) = (\gamma_M (pr_1 X), \gamma_C (pr_2 X)), \quad \forall X \in T(M \times N C),$$

$$\gamma_M(Y) = Y, \quad \forall Y \in pr_1^* V(p_M),$$

$$\gamma_C(Z) = Z, \quad \forall Z \in pr_2^* V(p_C).$$

If $\gamma$ is given by the local expression of the formula (16), then

$$\gamma_M = \sum_{i \leq j} (dy_{ij} + \gamma_{ijk} dx^k) \otimes \frac{\partial}{\partial y_{ij}}, \quad \gamma_C = \left( dA^i_{jk} + \gamma_{jkl}^i dx^l \right) \otimes \frac{\partial}{\partial A^i_{jk}},$$

$$\gamma_{ijk}, \gamma_{jkl}^i \in C^\infty(M \times N C).$$

### 3.2.1 The first geometric condition on $\gamma$

Let $q: F(N) \to M$ be the projection given by

$$q(X_1, \ldots, X_N) = g_x = \varepsilon_h w^h \otimes w^h,$$

where $(w^1, \ldots, w^n)$ is the dual coframe of $(X_1, \ldots, X_N) \in F_x(N)$, i.e., $g_x$ is the metric for which $(X_1, \ldots, X_N)$ is a $g_x$-orthonormal basis and $\varepsilon_h = 1$ for $1 \leq h \leq n^+$, $\varepsilon_h = -1$ for $n^+ + 1 \leq h \leq n$. As readily seen, $q$ is a principal $G$-bundle with $G = O(n^+, n^-)$.

Given a linear connection $\Gamma$ and a tangent vector $X \in T_x N$, for every $u$ in $p^{-1}(x)$ there exists a unique $\Gamma$-horizontal tangent vector $X_{u}^{hr} \in T_u(FN)$.
such that, \((p_F)_*X^{hr}_u = X\). The local expression for the horizontal lift is known to be ([20, Chapter III, Proposition 7.4]),

\[
\left( \frac{\partial}{\partial x^i} \right)^{hr} = \frac{\partial}{\partial x^j} - \Gamma^i_{jk} x^k \frac{\partial}{\partial x^i}.
\]

**Lemma 3.3.** Given a metric \(g_x \in p^{-1}_M(x)\), let \(u \in p^{-1}_F(x)\) be a linear frame such that \(q(u) = g_x\). The projection \(q_*(X^{hr}_u)\) does not depend on the linear frame \(u\) chosen over \(g_x\).

**Proof.** In fact, any other linear frame projecting onto \(g_x\) can be written as \(u \cdot A\), \(A \in G\). As the horizontal distribution is invariant under right translations (see [20, II, Proposition 1.2]), the following equation holds:

\[
(R_A)_* (X^{hr}_u) = X^{hr}_{u \cdot A}.
\]

Hence

\[
q_* (X^{hr}_{u \cdot A}) = q_* ((R_A)_* (X^{hr}_u)) = (q \circ R_A)_* (X^{hr}_u) = q_* (X^{hr}_u).
\]

□

**Proposition 3.4.** An Ehresmann connection \(\gamma\) on \(M \times_N C\) satisfies the following condition:

\[
(C_M): \quad \gamma_M ((g_x, \Gamma_x), X) = X - q_* \left( ((p_M)_*(X))^{hr}_u \right),
\]

\(\forall X \in T_g M, \ u \in q^{-1}(g_x), \) (which does not depend on the linear frame \(u \in q^{-1}(g_x)\) chosen, according to Lemma 3.3) if and only if the following equations hold:

\[
\gamma_{klj} = - (y_{al} A^a_{jk} + y_{ak} A^a_{jl}),
\]

where the functions \(\gamma_{klj}\) (resp. \(y_{ij}\), resp. \(A^i_{jk}\)) are defined in the formula (16) (resp. (4), resp. (6)).

**Proof.** Letting \((x^i_j)_{i,j=1}^n = (x^i_j)_{i,j=1}^n \), the dual coframe of the linear frame \(u = (X_1, \ldots, X_N) \in F_x(N)\) given in (5) is \((w^1, \ldots, w^n)\), \(w^h = \chi^h_k(u) (dx^k)_x\), \(1 \leq h \leq n\), and the projection \(q\) is given by

\[
q(u) = g_x = \sum_{h=1}^n \varepsilon_h \chi^h_k(u) \chi^h_i(u) \left( dx^k \right)_x \otimes \left( dx^l \right)_x.
\]
Therefore the equations of the projection (21) are as follows:

\[ x^i \circ q = x^i, \]
\[ y_{kl} \circ q = \sum_{h=1}^{n} \varepsilon_h \chi^h_k \chi^h_l. \]

Hence

\[ q^* \left( \frac{\partial}{\partial x^a_b} \right)_u = \sum_{k \leq l} \varepsilon_h \left\{ \frac{\partial \chi^h_k}{\partial x^a_b} \chi^h_l + \chi^h_k \frac{\partial \chi^h_l}{\partial x^a_b} \right\} (u) \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}. \]

Taking derivatives with respect to \( x^a_b \) on the identity \( \chi^h_i \chi^i_r \delta^r_h = \delta^h_i \), multiplying the outcome by \( \chi^i_k \), and summing up over the index \( i \), the following formula is obtained:

\[ \frac{\partial \chi^h_k}{\partial x^a_b} = -\chi^h_a \chi^b_k. \]

Replacing this equation into the expression for \( q^* \left( \frac{\partial}{\partial x^a_b} \right)_{u} \) above, we have

\[ q^* \left( \frac{\partial}{\partial x^a_b} \right)_u = -\sum_{k \leq l} \left\{ \chi^b_k (u) y_{al} (g_x) + \chi^b_l (u) y_{ak} (g_x) \right\} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}. \]

From (22), evaluated at \( u \in q^{-1}(g_x) \), we deduce

\[ q^* \left( \frac{\partial}{\partial x^3} \right)^{h \Gamma}_u = \left( \frac{\partial}{\partial x^3} \right)_{g_x} \Gamma^a_{jc}(x) x^c_b (u) q^* \left( \frac{\partial}{\partial x^a_b} \right)_{g_x} \]
\[ = \left( \frac{\partial}{\partial x^3} \right)_{g_x} \Gamma^a_{jc}(x) x^c_b (u) \left\{ \chi^b_k (u) y_{al} (g_x) + \chi^b_l (u) y_{ak} (g_x) \right\} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x} \]
\[ = \left( \frac{\partial}{\partial x^3} \right)_{g_x} + \sum_{k \leq l} \left\{ \Gamma^a_{jk}(x) y_{al} (g_x) + \Gamma^a_{jl}(x) y_{ak} (g_x) \right\} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}. \]

The condition \( (CM) \) holds automatically whenever \( X \in V(p_M) \). Hence, \( (CM) \) holds if and only if it holds for \( X = (\partial/\partial x^3)_{g_x} \), namely,

\[ \sum_{k \leq l} \gamma_{klj} (g_x, \Gamma_x) \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x} = \gamma_M \left( g_x, \Gamma_x, \left( \frac{\partial}{\partial x^3} \right)_{g_x} \right) \]
\[ = \left( \frac{\partial}{\partial x^3} \right)_{g_x} - q^* \left( \frac{\partial}{\partial x^3} \right)_u \]
\[ = -\sum_{k \leq l} \left\{ \Gamma^a_{jk}(x) y_{al} (g_x) + \Gamma^a_{jl}(x) y_{ak} (g_x) \right\} \left( \frac{\partial}{\partial y_{kl}} \right)_{g_x}, \]

thus proving the formula (23) in the statement. \( \square \)
3.2.2 The canonical covariant derivative

As is known (e.g., see [20, III, section 1], [23, pp. 157–158]) every connection $\Gamma$ on a principal $G$-bundle $P \to N$ induces a covariant derivative $\nabla^\Gamma$ on the vector bundle associated to $P$ under a linear representation $\rho: G \to \text{Gl}(m, \mathbb{R})$ with standard fibre $\mathbb{R}^m$. In particular, this applies to the principal bundle of linear frames, thus proving that every linear connection $\Gamma$ on $N$ induces a covariant derivative $\nabla^\Gamma$ on every tensorial vector bundle $E \to N$.

The bundles $(p_C)^*E$, where $E$ is a tensorial vector bundle, are endowed with a canonical covariant derivative $\nabla^E$ completely determined by the formula:

$$(\nabla^E)_X (f\xi) (\Gamma_x) = ((Xf) \xi) (\Gamma_x) + f (\Gamma_x) \left( \nabla^\Gamma_{(pc)_x} X \xi \right) (x),$$

for all $X \in T_{\Gamma_x}C$, $f \in C^\infty(C)$, and every local section $\xi$ of $E$ defined on a neighbourhood of $x$. The uniqueness of $\nabla^E$ follows from (24) as the sections of $E$ span the sections of $(p_C)^*E$ over $C^\infty(C)$, see [8, 0.3.6]. Below, we are specially concerned with the cases $E = TN$ and $E = \wedge^2 T^*N \otimes TN$.

3.2.3 The 2-form associated with $\gamma_C$

As $p_C: C \to N$ is an affine bundle modelled over $\otimes^2 T^*N \otimes TN$, there is a natural identification

$$V(p_C) \cong (p_C)^* (\otimes^2 T^*N \otimes TN)$$

and consequently, an Ehresmann connection $\gamma_C$ on $C$ can also be viewed as a homomorphism $\gamma_C: T_C \to \otimes^2 T^*N \otimes TN$. If $\gamma_C$ is locally given by

$$(25) \quad \gamma_C = (dA_{jk}^i + \gamma_{jkl}^i dx^l) \otimes \frac{\partial}{\partial A_{jk}^i}, \quad \gamma_{jkl}^i \in C^\infty(C),$$

then

$$\gamma_C = (dA_{jk}^i + \gamma_{jkl}^i dx^l) \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l},$$

and $\gamma_C$ induces a 2-form $\tilde{\gamma}_C$ taking values in $(p_C)^*(T^*N \otimes TN)$ as follows:

$$\tilde{\gamma}_C(X, Y) = c_1^1 ((p_C)_*(Y) \otimes \gamma_C(X)) - c_1^1 ((p_C)_*(X) \otimes \gamma_C(Y)),$$

$\forall X, Y \in T_{\Gamma_x}C$,

where

$$c_1^1: TN \otimes T^*N \otimes T^*N \otimes TN \to T^*N \otimes TN,$$

$$c_1^1 (X_1 \otimes w_1 \otimes w_2 \otimes X_2) = w_1(X_1) w_2 \otimes X_2,$$

$X_1, X_2 \in T_xN, w_1, w_2 \in T_x^*N.$
If $\gamma_C$ is given by (25), then from the very definition of $\tilde{\gamma}_C$ the following local expression is obtained:

$$\tilde{\gamma}_C = (dA^c_{lh} + (\gamma^c_{la} - \gamma^c_{ah}) \, dx^a) \wedge dx^l \otimes dx^h \otimes \frac{\partial}{\partial x^c}.$$  

### 3.2.4 The second geometric condition on $\gamma$

Let $\text{alt}_{12} : \otimes^2 T^*N \otimes TN \to \wedge^2 T^*N \otimes TN$ be the operator alternating the two covariant arguments.

The vector bundle $(p_C)^* (\wedge^2 T^*N \otimes TN)$ admits a canonical section

$$\tau_N : C \to \wedge^2 T^*N \otimes TN,$$

$$\tau_N(\Gamma_x) = T^\Gamma_x, \forall \Gamma_x \in C,$$

where $T^\Gamma_x$ is the torsion of $\Gamma_x$. Locally,

$$\tau_N = \sum_{j<k} (A^c_{jk} - A^c_{kj}) \, dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^c}.$$  

From the previous formulas the next result follows:

**Proposition 3.5.** Let $\gamma$ be an Ehresmann connection on $M \times N_C$, let $\nabla^{(1)} = \nabla^{E_1}$ with $E_1 = TN$, let $R^{\nabla^{(1)}}$ be its curvature form, and finally, let $\nabla^{(2)} = \nabla^{E_2}$ with $E_2 = \wedge^2 T^*N \otimes TN$.

\((C_C)\) Assume the component $\gamma_C$ of $\gamma$ is defined on $C$. Then, the equations

(26) $$\tilde{\gamma}_C = R^{\nabla^{(1)}},$$

(27) $$\text{alt}_{12} \circ \gamma_C = \nabla^{(2)} \tau_N,$$

are locally equivalent to the following ones:

(28) $$\gamma^h_{str} - \gamma^h_{rst} = A^h_{rm} A^m_{st} - A^h_{sm} A^m_{rt},$$

(29) $$\gamma^h_{rst} - \gamma^h_{srt} = A^h_{tm} (A^m_{rs} - A^m_{sr}) + A^m_{ts} \left( A^h_{mr} - A^h_{mr} \right) + A^m_{tr} \left( A^h_{sm} - A^h_{ms} \right).$$

### 3.3 Solution to the problem (P)

**Theorem 3.6.** If the connection $\gamma$ on $M \times N_C$ satisfies the conditions $\text{(C_M)}$ and $\text{(C_C)}$ introduced above, then the vector field $D^\gamma$ satisfies the property stated in Proposition 3.2 and, accordingly the covariant Hamiltonian with respect to $\gamma$ of every $\mathfrak{X}(N)$-invariant Lagrangian is also $\mathfrak{X}(N)$-invariant.
Proof. When $\gamma_M$ satisfies the condition $(C_M)$ the brackets (18), (19), and (20) are respectively given by

$$[X^h, D\gamma] = \frac{\partial \gamma^i_{jkl}}{\partial x^h} \frac{\partial}{\partial A^i_{jkl}},$$

(30)

$$[X^i_h, D\gamma] = \left(Y^i_h (\gamma_{bcr}) - \delta^i_a \gamma^i_{bcr} + \delta^i_c \gamma^a_{bhr} + \delta^i_r \gamma^a_{bch}\right) \frac{\partial}{\partial A^a_{bcr}},$$

(31)

$$[X^k_h, D\gamma] = \left(-\frac{\partial \gamma^d_{abc}}{\partial A^h_{ik}} + \delta^i_h \left(\delta^k_d A^a_{bh} - \delta^k_b A^d_{ah} - \delta^k_a A^d_{hb}\right) \frac{\partial}{\partial A^d_{abc}}.ight.$$

In addition, if $\gamma_C$ satisfies the condition $(C_C)$, then taking derivatives with respect to $x^h$ in (28) and (29) we obtain

$$\frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h}, \quad \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h},$$

and renaming indices we deduce

$$\frac{\partial \gamma^i_{jjk}}{\partial x^h} = \frac{\partial \gamma^i_{kjj}}{\partial x^h} = \frac{\partial \gamma^i_{kkj}}{\partial x^h} = \frac{\partial \gamma^i_{jkk}}{\partial x^h} (j < k < l).$$

From (30) we obtain

$$[X^h, D\gamma] = \sum_{j<k<l} \frac{\partial \gamma^i_{jkl}}{\partial x^h} X^i_{jkl} + \frac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{jlk}}{\partial x^h} X^i_{jlk} + \frac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{kjl}}{\partial x^h} X^i_{kjl} + \frac{1}{6} \frac{\partial \gamma^i_{jjk}}{\partial x^h} X^i_{jjk},$$

and consequently the values of $[X^h, D\gamma]$ belong to the distribution $D_{M \times NC}$.

Moreover, as $\gamma_C$ is assumed to be defined on $C$, we have

$$Y^i_h (\gamma_{bcr}) = (\delta^i_h A^i_{jkh} - \delta^i_k A^i_{jhh} - \delta^i_j A^i_{hkh}) \frac{\partial \gamma^a_{bcr}}{\partial A^a_{jkh}}.$$
Taking derivatives with respect to $A_{jk}^s$, the equations (28) y (29) yield

$$\frac{\partial \gamma_{bc}}{\partial A_{jk}^s} - \frac{\partial \gamma_{cb}}{\partial A_{jk}^s} = \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b} - \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b} + \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b} - \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b} + \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b} + \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b} - \delta_{i}^{s} \delta_{a}^{k} A_{rk}^{b},$$

$$\frac{\partial \gamma_{rc}}{\partial A_{jk}^s} - \frac{\partial \gamma_{cr}}{\partial A_{jk}^s} = \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c} - \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c} + \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c} - \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c} + \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c} + \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c} - \delta_{i}^{s} \delta_{a}^{k} A_{rb}^{c}.$$

From these expressions, the following symmetries of indices are obtained:

$$(T_{h}^{a})_{bb} = (T_{h}^{a})_{cb} = (T_{h}^{a})_{bc} (b < c),$$

$$(T_{h}^{a})_{bc} = (T_{h}^{a})_{cb} = (T_{h}^{a})_{cc} (b < c),$$

$$(T_{h}^{a})_{bcd} = (T_{h}^{a})_{dc} = (T_{h}^{a})_{cd} = (T_{h}^{a})_{dcb} = (T_{h}^{a})_{cbb} (b < c < d),$$

and from (31) we obtain

$$[X_{h}^{a}, D^{\gamma}] = \sum_{b < c < d} (T_{h}^{a})_{bcd} X_{a}^{bcd} + \frac{1}{2} \sum_{b < c} (T_{h}^{a})_{bcb} X_{a}^{bbc} + \frac{1}{6} (T_{h}^{a})_{bbb} X_{a}^{bbb}.$$ 

Hence $[X_{h}^{a}, D^{\gamma}]$ also takes values into the distribution $\mathcal{D}_{M \times S C}$.

The proof for the third bracket is similar to the previous two cases but longer. Letting

$$(T_{h}^{ik})_{rbc} = -\frac{\partial \gamma_{rb}}{\partial A_{ik}^s} - \frac{\partial \gamma_{rb}}{\partial A_{ik}^s} + \delta_{i}^{h} \left( \delta_{a}^{k} A_{rk}^{b} - \delta_{a}^{k} A_{rh}^{b}, A_{h}^{b} \right) + \delta_{i}^{h} \left( \delta_{a}^{k} A_{rk}^{b} - \delta_{a}^{k} A_{rh}^{b}, A_{h}^{b} \right),$$

the following symmetries are obtained:

$$(T_{h}^{ik})_{bb} = (T_{h}^{ik})_{cb} = (T_{h}^{ik})_{bc} (b < c),$$

$$(T_{h}^{ik})_{bc} = (T_{h}^{ik})_{cb} = (T_{h}^{ik})_{cc} (b < c),$$

$$(T_{h}^{ik})_{bcd} = (T_{h}^{ik})_{dc} = (T_{h}^{ik})_{cd} = (T_{h}^{ik})_{dcb} = (T_{h}^{ik})_{cbb} (b < c < d).$$

Hence

$$[X_{h}^{ik}, D^{\gamma}] = \sum_{b < c < d} (T_{h}^{ik})_{bcd} X_{a}^{bcd} + \frac{1}{2} \sum_{b < c} (T_{h}^{ik})_{bcb} X_{a}^{bbc} + \frac{1}{6} (T_{h}^{ik})_{bbb} X_{a}^{bbb},$$

and the proof is complete. \qed
**Theorem 3.7.** The Ehresmann connections on $C$ satisfying the equations (26) and (27) are the sections of an affine bundle over $C$ modelled over the vector bundle $(p_C)^* (S^3 T^* N \otimes T N)$. Consequently, there always exist Ehresmann connections on $M \times_N C$ fulfilling the conditions $(C_M)$ and $(C_C)$ introduced above.

**Proof.** If two Ehresmann connections $\gamma_C, \gamma'_C$ satisfy the equations (26) and (27), then the difference tensor field $t = \gamma'_C - \gamma_C$, which is a section of the bundle $(p_C)^* (S^3 T^* N \otimes T N)$, satisfies the following symmetries:

\[
\begin{align*}
(32) & \quad t(X_1, X_2, X_3) = t(X_3, X_2, X_1), \\
(33) & \quad t(X_1, X_2, X_3) = t(X_2, X_1, X_3),
\end{align*}
\]

according to (28), (29), respectively, for all $X_1, X_2, X_3 \in T_x N$, $\Gamma_x \in C_x (N)$. Hence

\[
t(X_1, X_3, X_2) \overset{(32)}{=} t(X_2, X_3, X_1) \overset{(33)}{=} t(X_3, X_2, X_1) \overset{(32)}{=} t(X_1, X_2, X_3),
\]

thus proving that $t$ is totally symmetric. The second part of the statement thus follows from the fact that an affine bundle always admits global sections, e.g., see [20, I, Theorem 5.7]. □

**Remark 3.8.** The results obtained above also hold if the bundle of linear connections is replaced by the subbundle $C^\text{sym} = C^\text{sym} (N) \subset C$ of symmetric linear connections; the only difference to be observed between both bundles is that in the symmetric cases the equation (27), or equivalently (29), holds automatically.

### 4 The second-order formalism

In this section we consider the problem of invariance of covariant Hamiltonians for second-order Lagrangians defined on the bundle of metrics, i.e., for functions $L \in C^\infty(J^2 M)$, where $M$ denotes, as throughout this paper, the bundle of pseudo-Riemannian metrics of a given signature $(n^+, n^-)$ on $N$.

#### 4.1 Second-order Ehresmann connections

A second-order Ehresmann connection on $p: E \to N$ is a differential 1-form $\gamma^2$ on $J^1 E$ taking values in the vertical sub-bundle $V(p^1)$ such that $\gamma^2(X) = X$ for every $X \in V(p^1)$. (We refer the reader to [29] for the basics on Ehresmann connections of arbitrary order.) Once a connection $\gamma^2$ is given, we have a decomposition of vector bundles $T(J^1 E) = V(p^1) \oplus \ker \gamma^2$, where $\ker \gamma^2$ is called the horizontal sub-bundle determined by $\gamma^2$. In the
coordinate system on $J^1E$ induced from a fibred coordinate system $(x^j, y^\alpha)$ for $p$, a connection form can be written as

$$\gamma^2 = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha} + (dy_i^\alpha + \gamma_{ij}^\alpha dx^j) \otimes \frac{\partial}{\partial y_i^\alpha}, \quad \gamma_j^\alpha, \gamma_{ij}^\alpha \in C^\infty(J^1E).$$

As in the first-order case, the action of the group Aut($p$) on the space of second-order connections is defined by the formula

$$\Phi \cdot \gamma^2 = \left(\Phi^{(1)}\right)_* \circ \gamma^2 \circ \left(\Phi^{(1)}\right)^{-1}_*, \quad \forall \Phi \in \text{Aut}(p).$$

As $\Phi^{(1)}: J^1M \to J^1M$ is a morphism of fibred manifolds over $N$, $(\Phi^{(1)})_*$ transforms the vertical subbundle $V(p^1)$ into itself; hence the previous definition makes sense.

### 4.2 A remarkable isomorphism

**Theorem 4.1.** Let $\Gamma^g$ be the Levi-Civita connection of a pseudo-Riemannian metric $g$ on $N$. The mapping $\zeta_N: J^1M \to M \times_N C^{\text{sym}}$, $\zeta_N(j^1x^g) = (g_x, \Gamma^g_x)$ is a diffeomorphism. There is a natural one-to-one correspondence between first-order Ehresmann connections on the bundle $p: M \times_N C^{\text{sym}} \to N$ and second-order Ehresmann connections on the bundle $p_M: M \to N$, which is explicitly given by,

$$\gamma^2 = \left((\zeta_N^\nu)_* \circ \gamma \circ (\zeta_N^\nu)_*\right)^{-1},$$

where $\gamma: T(M \times_N C^{\text{sym}}) \to V(p)$ is a first-order Ehresmann connection,

$$(\zeta_N^\nu)_*: T(J^1M) \to T(M \times_N C^{\text{sym}})$$

is the Jacobian mapping induced by $\zeta_N$, and $(\zeta_N^\nu)_*: V(p^1_M) \to V(p)$ is its restriction to the vertical bundles.

**Proof.** As a computation shows, the equations of $\zeta_N$ in the coordinate systems introduced in the section 2.1.2, are as follows:

$$x^i \circ \zeta_N = x^i,$$

$$y_{ij} \circ \zeta_N = y_{ij},$$

$$A^h_{ij} \circ \zeta_N = \frac{1}{2}g^{hk}(y_{ik,j} + y_{jk,i} - y_{ij,k}), \quad i \leq j,$$

where $(y^{ij})_{i,j=1}^n$ is the inverse mapping of the matrix $(y_{ij})_{i,j=1}^n$ and the functions $y_{ij}$ are defined in (4). Hence

$$x^i \circ \zeta_N^{-1} = x^i,$$

$$y_{ij} \circ \zeta_N^{-1} = y_{ij},$$

$$y_{ij,k} \circ \zeta_N^{-1} = y_{ik}A^h_{jk} + y_{hk}A^h_{ik}, \quad i \leq j.$$
As the diffeomorphism $\zeta_N$ induces the identity on the ground manifold $N$, it follows that the definition of $\gamma^2$ in (35) makes sense and the following formulas are obtained:

\[
\gamma_{ijkr} = \frac{1}{2} \sum_{a \leq b} \frac{\delta_{ak} \delta_{hi} + \delta_{ai} \delta_{bh}}{1 + \delta_{hi}} (\gamma_{abr} \circ \zeta_N) y^h l (y_{jl,k} + y_{kl,j} - y_{jk,l}) + \frac{1}{2} \sum_{a \leq b} \frac{\delta_{ak} \delta_{hi} + \delta_{ai} \delta_{bh}}{1 + \delta_{hi}} (\gamma_{abr} \circ \zeta_N) y^h l (y_{il,k} + y_{kl,i} - y_{ik,l}) + \sum_{j \leq a} \frac{\delta_{ak} \delta_{hi} + \delta_{ai} \delta_{bh}}{1 + \delta_{hi}} (\gamma_{iar} \circ \zeta_N) y_{hi} + \sum_{a \leq i} \frac{\delta_{ak} \delta_{hi} + \delta_{ai} \delta_{bh}}{1 + \delta_{hi}} (\gamma_{iar} \circ \zeta_N) y_{hj},
\]

where

\[
\gamma = \sum_{i \leq j} (dy_{ij} + \gamma_{ijkr} dx^k) \otimes \frac{\partial}{\partial y_{ij}} + \sum_{j \leq k} (dA_{jk}^i + \gamma_{jkl}^i dx^l) \otimes \frac{\partial}{\partial A_{jk}^i},
\]

or equivalently,

\[
\gamma = \frac{1}{2 - \delta_{ij}} (dy_{ij} + \gamma_{ijkr} dx^k) \otimes \frac{\partial}{\partial y_{ij}} + \frac{1}{2 - \delta_{ik}} (dA_{jk}^i + \gamma_{jkl}^i dx^l) \otimes \frac{\partial}{\partial A_{jk}^i},
\]

assuming $\gamma_{hir} = \gamma_{ihr}$ for $h > i$, and $\gamma_{jkr}^h = \gamma_{jkr}^i$ for $j > k$. Taking the symmetry $A_{jk}^i = A_{kj}^i$ into account, we obtain

\[
\gamma_{ijkr} = \frac{1}{2} (\gamma_{hir} \circ \zeta_N^1) y^h l (y_{jl,k} + y_{kl,j} - y_{jk,l}) + \frac{1}{2} (\gamma_{hjr} \circ \zeta_N^1) y^h l (y_{il,k} + y_{kl,i} - y_{ik,l}) + (\gamma_{jkr} \circ \zeta_N^1) y_{hi} + (\gamma_{ikr} \circ \zeta_N^1) y_{hj}.
\]

Hence

\[
\gamma_{ijkr} \circ \zeta_N^{-1} = \gamma_{hir} A_{jk}^h + \gamma_{hjr} A_{ik}^h + \gamma_{jkr} y_{hi} + \gamma_{ikr} y_{hj}, \quad i \leq j.
\]

Permuting the indices $i, j, k$ cyclically on the previous equation, we have

\[
\gamma_{ijr}^s = -\gamma_{hkr} A_{ij}^h y^{ks} - \frac{1}{2} (\gamma_{ijkr} \circ \zeta_N^{-1} - \gamma_{jkr} \circ \zeta_N^{-1} - \gamma_{ikr} \circ \zeta_N^{-1}) y^{ks},
\]

thus proving that the mapping $\gamma \mapsto \gamma^2$ defined in the statement, is bijective.
4.3 Covariant Hamiltonians for second-order Lagrangians

The Legendre form of a second-order Lagrangian density \( \Lambda = Lv_n \) on the bundle \( p: E \rightarrow N \) is the \( V^*(p^1) \)-valued \( p^3 \)-horizontal \( (n-1) \)-form \( \omega_\Lambda \) on \( J^3E \) locally given by (e.g., see [17, 26, 35]),

\[
\omega_\Lambda = i_{\partial/\partial x^i}v_n \otimes \left( L^i_\alpha dy^\alpha + L^i_{ij} dy^\alpha_j \right),
\]

where

\[
L^i_\alpha = \frac{1}{2-\delta_{ij}} \frac{\partial L}{\partial y^\alpha_j},
\]

\[
L^i = \frac{\partial L}{\partial y^\alpha_i} - \sum_j 2-\delta_{ij} D_j \left( \frac{\partial L}{\partial y^\alpha_j} \right),
\]

and

\[
D_j = \frac{\partial}{\partial x^j} + \sum_{\substack{I \in \mathbb{N}^n, |I| = 0}} \gamma^\alpha_{I+(j)} \frac{\partial}{\partial y^\alpha_I}
\]
denotes the total derivative with respect to the variable \( x^j \).

The Poincaré-Cartan form attached to \( \Lambda \) is then defined to be the ordinary \( n \)-form on \( J^3E \) given by, \( \Theta_\Lambda = (p^3_1)^* \theta^2 \wedge \omega_\Lambda + \Lambda \), where \( \theta^2 \) is the second-order structure form (cf. [33, (0.36)]) and the exterior product of \( (p^3_1)^* \theta^2 \) and the Legendre form, is taken with respect to the pairing induced by duality, \( V(p^1) \times_{p^1} V^*(p^1) \rightarrow \mathbb{R} \). The most outstanding difference with the first-order case is that the Legendre and Poincaré-Cartan forms associated with a second-order Lagrangian density are generally defined on \( J^3E \), thus increasing by one the order of the density.

Similarly to the first-order case (see [11, 24]), given a second-order Lagrangian density \( \Lambda \) on \( p: E \rightarrow N \) and a second-order connection \( \gamma^2 \) on \( p: E \rightarrow N \), by subtracting \( (p^3_2)^* \theta^2 \) from \( (p^3_1)^* \gamma^2 \) we obtain a \( p^3 \)-horizontal form, and we can define the corresponding covariant Hamiltonian to be the Lagrangian density \( \Lambda^\gamma^2 \) of third order,

\[
\Lambda^\gamma^2 = ( (p^3_1)^* \gamma^2 - (p^3_2)^* \theta^2 ) \wedge \omega_\Lambda - \Lambda.
\]

Expanding on the right-hand side of the previous equation, we obtain a decomposition of \( \Theta_\Lambda \) that generalizes the classical formula for the Hamiltonian in Mechanics; namely, \( \Theta_\Lambda = (p^3_1)^* \gamma^2 \wedge \omega_\Lambda - \Lambda^\gamma^2 \). With the same notations as in the formulas (34), (40), (41) the following formula is deduced:

\[
L^\gamma^2 = (\gamma^\alpha_i + y^\alpha_i)L^i_\alpha + (\gamma^\alpha_{hi} + y^\alpha_{hi})L^ih_\alpha - L.
\]

Because of the equation (41), \( \Theta_\Lambda \) and \( L^\gamma^2 \) are generally defined on \( J^3E \).
4.4 Invariant covariant Hamiltonians on $J^2M$

Lemma 4.2. If $\gamma$ is a first-order Ehresmann connection on $M \times N \ C^{\text{sym}}$ satisfying the conditions $(C_M)$, then the following equation holds for the second-order Ehresmann connection $\gamma^2$ on $M$ given in the formula (35):

$$\gamma_{abr} \circ \zeta_N = -y_{ab,r}.$$ 

Proof. Actually, from the formulas (23) and (36) we obtain

$$\gamma_{abr} \circ \zeta_N = - (y_{mb} (A^m_{ra} \circ \zeta_N) + y_{ma} (A^m_{rb} \circ \zeta_N))$$

$$= - \frac{1}{2} \left\{ y_{mb} y_{mk} (y_{rk,a} + y_{ak,r} - y_{ra,k}) + y_{ma} y_{mk} (y_{rk,b} + y_{bk,r} - y_{rb,k}) \right\}$$

$$= -y_{ab,r}. \quad \Box$$

Lemma 4.3. If a first-order connection $\gamma$ on $M \times N \ C^{\text{sym}}$ satisfies the condition $(C_C)$ introduced above, then the following formulas for its components hold:

$$(44) \quad \gamma^h_{rst} - \gamma^h_{rpt} = A^h_{sm} A^m_{rt} - A^h_{tm} A^m_{rs}. $$

Proof. As the bundle under consideration is that of symmetric connections, the following symmetry holds: $\gamma^h_{abc} = \gamma^h_{bac}$, and we have

$$\gamma^h_{rts} = \gamma^h_{str} - (A^h_{rm} A^m_{st} - A^h_{sm} A^m_{rt}) \quad \text{[by virtue of (28)]}$$

$$= \gamma^h_{tsr} - (A^h_{rm} A^m_{st} - A^h_{sm} A^m_{rt})$$

$$= (\gamma^h_{rst} + A^h_{rm} A^m_{st} - A^h_{tm} A^m_{rs})$$

$$- (A^h_{rm} A^m_{st} - A^h_{sm} A^m_{rt}) \quad \text{[by virtue of (28)]}$$

$$= \gamma^h_{rst} + (A^h_{sm} A^m_{rt} - A^h_{tm} A^m_{rs}). \quad \Box$$

Proposition 4.4. Let

$$\zeta^2_N = \zeta^{(1)}_N \big|_{J^2M} : J^2M \to J^1(M \times N \ C^{\text{sym}})$$

be the restriction to the closed submanifold $J^2M \subset J^1(J^1M)$ of the prolongation $\zeta^{(1)}_N : J^1(J^1M) \to J^1(M \times N \ C^{\text{sym}})$ of the mapping $\zeta_N$ defined in Theorem 4.1. For every $(j^1_2 g, j^1_2 \Gamma) \in J^1(M \times N \ C^{\text{sym}})$ there exists a unique $j^2_2 g' \in J^2M$ such that, $j^1_2 g = j^1_2 g'$ and $j^1_2 \Gamma g' = j^1_2 \Gamma$ and the mapping $\varkappa : J^1(M \times N \ C^{\text{sym}}) \to J^2M$ defined by $\varkappa(j^1_2 g, j^1_2 \Gamma) = j^2_2 g'$ is a Diff$N$-equivariant retract of $\zeta^2_N$. 
Proof. From the formulas (36) and (37) we obtain
\[
\frac{\partial g'_{ij}}{\partial x^k} = g_{hi} \left( \Gamma^g \right)_{jk}^h + g_{hj} \left( \Gamma^g \right)_{ik}^h,
\]
\[
\left( \Gamma^g \right)_{ij}^k = \frac{1}{2} g^{hk} \left( \frac{\partial g'_{ik}}{\partial x^j} + \frac{\partial g'_{jk}}{\partial x^i} - \frac{\partial g'_{ij}}{\partial x^k} \right).
\]
for every non-singular metric \( g' \) on \( N \). Hence the second partial derivatives of \( g'_{ij} \) are completely determined, namely
\[
\frac{\partial^2 g'_{ij}}{\partial x^k \partial x^l} = \frac{\partial g_{hi}}{\partial x^l} \Gamma^h_{jk} + g_{hi} \frac{\partial \Gamma^h_{jk}}{\partial x^l} + \frac{\partial g_{hj}}{\partial x^l} \Gamma^h_{ik} + g_{hj} \frac{\partial \Gamma^h_{ik}}{\partial x^l}.
\]
Moreover, the Levi-Civita connection of a metric depends functorially on the metric, i.e., \( \phi \cdot \Gamma^g = \Gamma^{\phi \cdot g} \) for every \( \phi \in \text{Diff} \, N \). Hence, by transforming the equations \( j^1_x g' = j^1_x g \) and \( j^1_x \Gamma^g = j^1_x \Gamma^g \) by \( \phi \) we can conclude. □

**Theorem 4.5.** If a first-order Ehresmann connection \( \gamma \) on \( M \times N \) \( C^{sym} \) satisfies the conditions \( (C_M) \) and \( (C_C) \) introduced above, then the covariant Hamiltonian \( \Lambda^\gamma \) attached to every \( \text{Diff} \, N \)-invariant second-order Lagrangian density \( \Lambda \) on \( M \) with respect to the second-order Ehresmann connection \( \gamma^2 \) on \( M \) defined in the formula (35), is defined on \( J^2 \) and it is also \( \text{Diff} \, N \)-invariant.

**Proof.** Given a \( \text{Diff} \, N \)-invariant second-order Lagrangian density \( \Lambda = L \, v \) on \( M \), let \( \Lambda' = L' \, v \) be the first-order Lagrangian density on \( M \times N \) \( C^{sym} \) given by \( \Lambda' = \varkappa^* \Lambda \), which is also \( \text{Diff} \, N \)-invariant as \( \varkappa \) is a \( \text{Diff} \, N \)-equivariant mapping according to Proposition 4.4. Moreover, as \( \varkappa \) is a retract of \( \zeta_N^2 \), we have \( \left( \zeta_N^2 \right)^* \Lambda' = \left( \zeta_N^2 \right)^* \varkappa^* \Lambda = \left( \varkappa \circ \zeta_N^2 \right)^* \Lambda = \Lambda \), i.e., \( \Lambda = \left( \zeta_N^2 \right)^* \Lambda' \). This formula is equivalent to saying \( L = L' \circ \zeta_N^2 \), as the \( n \)-form \( v \) is \( \text{Diff} \, N \)-invariant, and it is even equivalent to \( L = L' \circ \zeta_N^2 \) because \( \zeta_N^2 \) induces the identity on \( N \).

We claim \( L^\gamma = (L')^\gamma \circ \zeta_N^2 \). This formula will end the proof as the mapping \( \zeta_N^2 \) is \( \text{Diff} \, N \)-equivariant and \( (L')^\gamma \) is \( \text{Diff} \, N \)-invariant by virtue of Theorem 3.6.

To start with, we observe that the formula (40) for \( \Lambda \) can be written, in the present case, as follows:
\[
L^{abij} = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y_{ab,ij}},
\]
or equivalently, letting $L^{abij} = \rho^{-1} L^{abij}$,

$$L^{abij} = \frac{1}{2-\delta_{ij}} \frac{\partial L}{\partial y_{ab,ij}}.$$  

Taking the formula in Lemma 4.2 into account, the formula (43) for $\Lambda$ reads as

$$L^{\gamma 2} = \sum_{a \leq b} (\gamma_{abij} + y_{ab,ij}) L^{abij} - L,$$

where $L^{\gamma 2} = \rho^{-1} L^{\gamma 2}$. Hence $L^{\gamma 2}$ is defined over $J^2M$. As $y_{ab,ij} = y_{ab,ji}$, we obtain

$$L^{\gamma 2} = \sum_{a \leq b} \sum_{i \leq j} \left( \frac{1}{2} (\gamma_{abij} + \gamma_{abji}) + y_{ab,ij} \right) \frac{\partial (L' \circ \zeta_N^2)}{\partial y_{ab,ij}} - L' \circ \zeta_N^2$$

$$= \sum_{a \leq b} \sum_{i \leq j} \sum_{k \leq l} \left( \frac{1}{2} (\gamma_{abij} + \gamma_{abji}) + y_{ab,ij} \right) \left( \frac{\partial L'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) \frac{\partial (A_{kl,q}^h \circ \zeta_N^2)}{\partial y_{ab,ij}} - L' \circ \zeta_N^2$$

$$= \sum_{k \leq l} \frac{1}{2} y_{hm} \left( \gamma_{kmql} + \gamma_{kmlq} + \gamma_{lqmk} + \gamma_{lmqk} - \gamma_{klqm} - \gamma_{klmq} \right) \left( \frac{\partial L'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right)$$

$$+ \sum_{k \leq l} \frac{1}{2} y_{hm} \left( y_{km,ql} + y_{lm,qk} - y_{kl,qm} \right) \left( \frac{\partial L'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) - L' \circ \zeta_N^2.$$

Moreover, we have

$$(L')^{\gamma} = \sum_{a \leq b} \left( \gamma_{abc} + y_{abc} \right) \frac{\partial L'}{\partial y_{abc}} + \sum_{a \leq b} \left( \gamma_{ahl} + A_{ahl}^i \right) \frac{\partial L'}{\partial A_{ahl}^i} - L'. $$

Hence

$$(L')^{\gamma} \circ \zeta_N^2 = \sum_{k \leq l} \left( \gamma_{klq}^h \circ \zeta_N + A_{kl,q}^h \circ \zeta_N \right) \left( \frac{\partial L'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) - L' \circ \zeta_N^2$$

$$= \sum_{k \leq l} \left\{ -\frac{1}{2} (\gamma_{klrq} - \gamma_{lrkq} - \gamma_{rklq}) y^{th} 

+ \frac{1}{2} (y_{kr,lq} + y_{lr,kq} - y_{kl,rq}) y^{hr} \right\} \left( \frac{\partial L'}{\partial A_{kl,q}^h} \circ \zeta_N^2 \right) - L' \circ \zeta_N^2.$$

Consequently, the proof reduces to state that the following equation

$$\frac{1}{2} (\gamma_{krql} + \gamma_{krql} - \gamma_{lrqk} + \gamma_{lrqk} - \gamma_{klqr} - \gamma_{klqr}) = -\frac{1}{2} (\gamma_{klrq} - \gamma_{lrqk} - \gamma_{rklq})$$

holds true, or equivalently,

$$0 = (\gamma_{ijkr} - \gamma_{ijrk}) + (\gamma_{irjk} - \gamma_{irkj}) + (\gamma_{rjki} - \gamma_{rjik}) \quad (46).$$
According to the formulas (38) and (23) we obtain
\[
\gamma_{ijkr} \circ \zeta^{-1}_N = \left( \gamma^h_{jk} - A^h_{ra} A^a_{jk} \right) y_{hi} + \left( \gamma^h_{ikr} - A^h_{ra} A^a_{ik} \right) y_{hj} \\
- \left( A^h_{ij} A^a_{ik} + A^h_{ri} A^a_{jk} \right) y_{ah}.
\]
The third term on the right-hand side of this equation is symmetric in the indices \( k \) and \( r \), as \( A^a_{bc} = A^a_{cb} \). Hence
\[
(\gamma_{ijkr} - \gamma_{ijrk}) \circ \zeta^{-1}_N = \left( \gamma^h_{jk} - \gamma^h_{jk} - A^h_{ra} A^a_{jk} + A^h_{ka} A^a_{jr} \right) y_{hi} \\
+ \left( \gamma^h_{ikr} - \gamma^h_{ikr} - A^h_{ra} A^a_{ik} + A^h_{ka} A^a_{ir} \right) y_{hj}.
\]
By composing the right-hand side of the equation (46) and \( \zeta^{-1}_N \), and taking the previous formula and the formulas (28) and (44) into account, we conclude that this expression vanishes indeed. \( \square \)

5 Palatini and Einstein-Hilbert Lagrangians

Let us compute the covariant Hamiltonian density attached to the Palatini Lagrangian. Following the notations in [20], the Ricci tensor field attached to the symmetric connection \( \Gamma \) is given by \( S^\Gamma(X,Y) = \text{tr}(Z \mapsto R^\Gamma(Z,X)Y) \), where \( R^\Gamma \) denotes the curvature tensor field of the covariant derivative \( \nabla^\Gamma \) associated to \( \Gamma \) on the tangent bundle; hence \( S^\Gamma = (R^\Gamma)_{ij} dx^i \otimes dx^j \), where
\[
(R^\Gamma)_{ij} = (R^\Gamma)^k_{ij}, \\
(R^\Gamma)^k_{ij} = \partial \Gamma^i_j / \partial x^k - \partial \Gamma^i_k / \partial x^j + \Gamma^m_{jk} \Gamma^i_m - \Gamma^m_{jm} \Gamma^i_k - \Gamma^m_{ik} \Gamma^j_m.
\]
The Lagrangian is the function on \( J^1(M \times N C^{\text{sym}}) \) thus given by,
\[
\mathcal{L}_P(j^1_x g, J^1_x \Gamma) = g^{ij}(x)(R^\Gamma)_{ij}(x)
\]
and local expression
\[
\mathcal{L}_P = g^{ij}(A^k_{ij,k} - A^k_{ik,j} + A^m_{ij} A^k_{km} - A^m_{ik} A^k_{jm}).
\]
As a computation shows, for every first-order connection \( \gamma \) on \( M \times N C^{\text{sym}} \) satisfying (44) and taking the formula (2) into account, we obtain \( \mathcal{L}_P = 0 \).
This result is essentially due to the fact that the P-C form of the P density \( \Lambda_P = \mathcal{L}_P \nu = L_P \nu_n \) projects onto \( M \times N C^{\text{sym}} \). In fact, the following general characterization holds:

**Proposition 5.1.** Let \( p: E \to N \) be an arbitrary fibred manifold and let \( \gamma \) be a first-order Ehresmann connection on \( E \). The equation \( L^\gamma = 0 \) holds true for a Lagrangian \( L \in C^\infty(J^1 E) \) if and only if, i) the Poincaré-Cartan form of the density \( \Lambda = L \nu_n \) projects onto \( J^0 E \) and, ii) \( L = \left( (p_0^1)^* \gamma - \theta, dL|_{V(p_0^1)} \right) \).
Proof. The equation \( L^\gamma = 0 \) is equivalent to the equation \( D^\gamma L = L \), where \( D^\gamma \) is the \( p_0^1 \)-vertical vector field defined in the formula (17), and the general solution to the latter is \( L = f(x^i, y^\alpha, \gamma^i_\alpha + y^\alpha_i) \), \( f(x^i, y^\alpha, y^\beta_\alpha) \) being a homogeneous smooth function of degree one in the variables \( (y^\alpha_i) \), \( 1 \leq \alpha \leq m \), \( 1 \leq i \leq n \), according to Euler’s homogeneous function theorem. As \( f \) is defined for all values of the variables \( (y^\alpha_i) \), \( 1 \leq \alpha \leq m \), \( 1 \leq i \leq n \), we conclude that the functions \( L^\alpha_i = \partial L/\partial y^\alpha_i \) must be defined on \( E \). Hence \( L \) is written as \( L = L^\alpha_i(x^j, y^\beta) y^\alpha_i + L_0(x^j, y^\beta) \), but this is exactly the condition for the \( \text{P-C} \) form of \( \Lambda \) to be projectable onto \( J^0 E = E \), as follows from the local expression of this form, namely,

\[
\Theta_\Lambda = \frac{\partial L}{\partial y^\alpha_i} y^\alpha_i \wedge i_{\partial/\partial x^i} v_n + L v_n
\]

Moreover, by imposing the condition \( D^\gamma L = L \) we obtain \( L_0 = L^\alpha_i \gamma^i_\alpha \), or in other words \( L = (\gamma^i_\alpha + y^\alpha_i) \partial L/\partial y^\alpha_i \), which is equivalent to the equation ii) in the statement. \( \Box \)

The corresponding result for the second-order formalism is similar but the computations are more cumbersome. Let us compute the covariant Hamiltonian density attached to the Einstein-Hilbert Lagrangian. As a matter of notation, we set \( S^g(X, Y) = S_\Gamma g(X, Y) \) for the metric \( g \), \( \Gamma^g \) being its Levi-Civita connection, and similarly, \( (R^g)_{ijkl} \). The E-H Lagrangian is thus given by \( \mathcal{L}_{EH} \circ J^2 g = (y^{ij} \circ g)(R^g)^{hi}_{ij} \). As the Levi-Civita connection \( \Gamma^g \) depends functorially on \( g \), \( \mathcal{L}_{EH} \) is readily seen to be Diff\( N \)-invariant; it is in addition linear in the second-order variables \( y_{ij,kl} \). By using the third formula in (36) the following local expression for \( \mathcal{L}_{EH} \) is obtained:

\[
\mathcal{L}_{EH} = \frac{1}{2} y^{ij} y^{hd} \left( y_{dj,hi} - y_{ij,dh} - y_{dh,ij} + y_{hi,dj} \right) + \mathcal{L}'_{EH},
\]

\[
\mathcal{L}'_{EH} = \frac{1}{2} y^{ij} \left\{ y^{hm} y^{mr,j} y^{rd} \left( y_{id,h} + y_{hd,i} - y_{ih,d} \right)
- y^{km} y^{mr,h} y^{rd} \left( y_{id,j} + y_{jd,i} - y_{ij,d} \right)
+ \frac{1}{2} y^{hr} y^{md} \left( y_{id,j} + y_{jd,i} - y_{ij,d} \right) \left( y_{hr,m} + y_{mr,h} - y_{hm,r} \right)
- \frac{1}{2} y^{hr} y^{md} \left( y_{id,h} + y_{hd,i} - y_{ih,d} \right) \left( y_{jr,m} + y_{mr,j} - y_{jm,r} \right) \right\}.
\]
According to (45), for every first-order connection form $\gamma$ on $M \times N C^\text{sym}$ satisfying the conditions $(C_M)$ and $(C_C)$ above, we have

$$\mathcal{L}_{EH}^{\gamma^2} = \sum_{a \leq b} \frac{1}{2-\delta_{ij}} (\gamma_{\alpha ij} + y_{\alpha ij}) \frac{\partial \mathcal{L}_{EH}}{\partial y_{\alpha ij}} - \mathcal{L}_{EH},$$

and as a computation shows,

$$\mathcal{L}_{EH}^{\gamma^2} = \frac{1}{2} y^{ij} (\gamma_{ijd} + \gamma_{jdi} - \gamma_{jdi} - \gamma_{idj} + \gamma_{ijd}) y^{hd}$$

$$+ \frac{1}{2} y^{ij} \left\{ y^{hm} y_{mr,j} y^{rd} (y_{id,j} + y_{jd,i} - y_{ij,d}) \right.$$ 

$$- y^{hm} y_{mr,j} y^{rd} (y_{id,h} + y_{hd,i} - y_{ih,d})$$

$$- \frac{1}{2} y^{hr} y^{md} (y_{id,j} + y_{jd,i} - y_{ij,d}) (y_{hr,m} + y_{mr,h} - y_{hm,r})$$

$$+ \frac{1}{2} y^{hr} y^{md} (y_{id,h} + y_{hd,i} - y_{ih,d}) (y_{jr,m} + y_{mr,j} - y_{jm,r}) \right\}$$

$$= 0,$$

where the formulas (39), (44), (36), and Lemma 4.3 have been used. In this case, the P-C form of the E-H density $\Lambda_{EH} = \mathcal{L}_{EH} \mathbf{v} = \mathcal{L}_{EH} \mathbf{v}_n$,

$$(47) \quad \Theta_{\Lambda_{EH}} = \sum_{k \leq l} \left( I_{EH}^{i,kl} dy_{kl}^{*} + L_{EH}^{ij,kl} dy_{kl,j} \right) \wedge \frac{i_{\partial / \partial x^i}}{v_n} + H v_n,$$

$$H = L_{EH}' - \sum_{k \leq l} I_{EH}^{i,kl} y_{kl,i},$$

$$I_{EH}^{i,kl} = \frac{\partial L_{EH}'}{\partial y_{kl,i}} - \frac{1}{2-\delta_{ij}} \frac{\partial^2 L_{EH}}{\partial y_{ab} \partial y_{kl,ij}},$$

$$I_{EH}^{ij,kl} = \frac{1}{2-\delta_{ij}} \frac{\partial L_{EH}}{\partial y_{kl,ij}},$$

(cf. (40), (41)) is not only projectable onto $J^2 M$ but also on $J^1 M$ (e.g., see [13]), although there is no first-order Lagrangian on $J^1 M$ admitting (47) as its P-C form. This fact is strongly related to a classical result by Hermann Weyl ([39], Appendix II), also see [22], [18]) according to which the only Diff$N$-invariant Lagrangians on $J^2 M$ depending linearly on the second-order coordinates $y_{ab,ij}$ are of the form $\lambda \mathcal{L}_{EH} + \mu$, for scalars $\lambda, \mu$. This also explains why a true first-order Hamiltonian formalism exists in the Einstein-Cartan gravitation theory, e.g., see [37], [38]. In fact, if

$$I_{EH}^{i,kl} = \frac{1}{2-\delta_{ij}} \frac{\partial L_{EH}}{\partial y_{kl,ij}} (\text{hence } I_{EH}^{ij,kl} = \frac{\partial L_{EH}}{\partial y_{kl,ij}})$$

and the momentum functions are defined as follows:

$$p_{kl,i} = L_{EH}^{i,kl} - \frac{\partial L_{EH}'}{\partial y_{kl}},$$

then

$$d \Theta_{\Lambda_{EH}} = dp_{kl,i} \wedge dy_{kl} \wedge i_{\partial / \partial x^i} v_n + dH \wedge v_n,$$
and from the Hamilton-Cartan equation (e.g., see [13, (1)]) we conclude that a metric $g$ is an extremal for $\Lambda_{EH}$ if and only if,

$$0 = \frac{\partial (p_{ab,i} \circ j^1 g)}{\partial x^i} - \frac{\partial H}{\partial y_{ab, i} \circ j^1 g},$$

$$0 = \frac{\partial (y_{ab} \circ g)}{\partial x^i} + \frac{\partial H}{\partial y_{ab, i} \circ j^1 g}.$$

On the other hand, it is no longer true that the covariant Hamiltonians of the non-linear Lagrangians of the form $f(\mathcal{L}_{EH}), f'' \neq 0$, considered in some cosmological models (e.g., see [1], [6], [9], [12], [19], [21], [31]) and those in higher dimensions (e.g., see [15], [36]) vanish. In fact, as a computation shows, one has $f(\mathcal{L}_{EH})^2 = f'(\mathcal{L}_{EH})\mathcal{L}_{EH} - f(\mathcal{L}_{EH}), \forall f \in C^\infty(\mathbb{R})$.

References


