On the Use of Discrete Cosine Transforms for Multicarrier Communications

Fernando Cruz-Roldan, Maria Elena Dominguez-Jimenez, Gabriela Sansigre Vidal, Pedro Amo-L6pez, Manuel Blanco-Velasco, and Angel Bravo-Santos

Abstract—In this correspondence, the conditions to use any kind of discrete cosine transform (DCT) for multicarrier data transmission are derived. The symmetric convolution-multiplication property of each DCT implies that when symmetric convolution is performed in the time domain, an element-by-element multiplication is performed in the corresponding discrete trigonometric domain. Therefore, appending symmetric redundancy (as prefix and suffix) into each data symbol to be transmitted, and also enforcing symmetry for the equivalent channel impulse response, the linear convolution performed in the transmission channel becomes a symmetric convolution in those samples of interest. Furthermore, the channel equalization can be carried out by means of a bank of scalars in the corresponding discrete cosine transform domain. The expressions for obtaining the value of each scalar corresponding to these one-tap per subcarrier equalizers are presented. This study is completed with several computer simulations in mobile broadband wireless communication scenarios, considering the presence of carrier frequency offset (CFO). The obtained results indicate that the proposed systems outperform the standardized ones based on the DFT.

Index Terms—Carrier-frequency offset (CFO), discrete cosine transform (DCT), discrete multitone modulation (DMT), multicarrier modulation (MCM), multicarrier transceiver, orthogonal frequency-division multiplexing (OFDM).

I. INTRODUCTION

A general block diagram to implement multicarrier modulation (MCM) is shown in Fig. 1. At the transmitter, the incoming data are processed by an A*-point inverse transform ($T_I^*$), with N being the number of subchannels or subcarriers. At the receiver, a discrete transform ($T_c$) is also performed. Discrete multitone modulation (DMT) and orthogonal frequency-division multiplexing (OFDM) are particular forms of multicarrier modulation (MCM) [1], and they have been the modulation choice for fixed and nomadic broadband communications (see e.g. [2]-[5]). In mobile communications, OFDM has also been adopted in WiMAX [6], LTE downlink [7], and in the physical layer of 802.1 lp that describes the wireless access in vehicular environments (vehicle-to-vehicle (V2V) communications) [8]. In DMT and OFDM, the $N \times A^*$ matrices $T_a$ and $T_c$ are carried
based on the use of discrete trigonometric transforms (DTTs), mainly employing any kind of DCT at both the transmitter (T) and the receiver (R). The first one consists in deriving the conditions to use any kind of DCT for multicarrier data transmission using matrices. Let us consider the system of Fig. 1, in which a left prefix and also a right suffix are introduced as redundancy into each data symbol to be transmitted: x = [x_L x_R]. The goal of these prefix x_L and suffix x_R is to obtain a channel matrix H perfectly diagonalizable by DCTs. Then, using the diagonalization properties of DCTs shown in [9], the frequency-domain equalizer at the receiver is reduced to a simple structure. The above matrix formulation was employed in [13], [17] to obtain the conditions for using the DCT Type-II even (DCT2e) and DCT Type-III (even and odd) in the MCM context.

The second way of formulating the use of DCTs for multicarrier data transmission is based on the interpretation of the symmetric convolution. That is, if DCTs are used as block transforms in the system of Fig. 1, the idea consists in some way forcing the linear convolution performed by the channel to become a symmetric convolution in the time-domain, or equivalently, an element-by-element operation in the frequency domain. This leads to defining the kind of symmetric convolution in the time-domain or equivalently, an element-by-element operation in the corresponding DCT domain. In this regard, Martucci shows in [16] that DCTs also have the convolution-multiplication property, and the result of symmetric convolution in time-domain is the same as that obtained by taking an inverse DCT of the pointwise product of the forward DCTs. Therefore, this work complements the results obtained in previous studies for DCT2e [13], [14] and DCT3 [17]. Since there are objective reasons to use DCT as an alternative to DFT for multicarrier communications (see e.g. [13]), the goal of our study is focused on deriving the conditions to design the system of Fig. 1 with the eight different types of DCTs. One key contribution of this work is to define the kind of redundancy to be introduced into each data symbol to be transmitted, along with the symmetry to be imposed on the channel impulse response.

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### Table I

**Definitions for the Discrete Cosine Transforms Even and Odd**

<table>
<thead>
<tr>
<th>DCT even</th>
<th>DCT odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{1e})_{k,j} = 2^{-1/2} \cos (\frac{2\pi k j}{2^N}) , 0 \leq k,j &lt; 2^N - 1</td>
<td>(C_{1o})_{k,j} = 2a, \cos (\frac{2\pi k j}{2N}) , 0 \leq k,j &lt; N - 1</td>
</tr>
<tr>
<td>\alpha_0 = 1/2, \alpha_j = 1,j \neq 0</td>
<td>\alpha_0 = 1/2, \alpha_j = 1, j \neq 0</td>
</tr>
<tr>
<td>(C_{2e})_{k,j} = 2 \cos (\frac{(2^k+1)\pi j}{2^N}) , 0 \leq k,j &lt; 2^N - 1</td>
<td>(C_{1o})_M = 26a, \cos (\frac{2\pi k j}{2^N}) , 0 \leq k,j &lt; N - 1</td>
</tr>
<tr>
<td>\alpha_0 = 1/2, \alpha_j = 1, j \neq 0</td>
<td>\alpha_0 = 1/2, \alpha_j = 1,j \neq 0</td>
</tr>
<tr>
<td>\alpha_0 = 1/2, \alpha_j = 1, j \neq 0</td>
<td>(C_{3o})_{k,j} = 2a, \cos (\frac{2\pi k j}{2N}) , 0 \leq k,j &lt; N - 1</td>
</tr>
<tr>
<td>(C_{4e})_{k,j} = 2 \cos (\frac{(2^{k-1}+1)\pi j}{2^N}) , 0 \leq k,j &lt; 2^N - 1</td>
<td>(C_{4o})_M = 2 \cos (\frac{2\pi k j}{2^N}) , 0 \leq k,j &lt; 2^N - 1</td>
</tr>
</tbody>
</table>

![Fig. 1. Block diagram of a transforms-based multicarrier system over a channel with additive noise.](image-url)
Let us consider four different types of symmetry in a sequence [16]: whole-sample symmetry (WS), half-sample symmetry (HS), or half-sample antisymmetry (WA). Fig. 2 shows examples of each one of the above sequences. In [16], the convolution-multiplication properties of the eight types of DCTs, the eight types of DSTs, and also forty different types of symmetric convolution are reported. Table II summarizes some of the results included in [16] for general digital signal processing; $s_n$ and $S_b$ mean the symmetric extension operators applied to each sequence to be convolved, and $T_a$ and $T_e$ are the DTTs that must be taken in the first sequence and in the right-half samples of the second one, respectively. Finally, $T^{-1}$ is the appropriate inverse transform which relates the convolution-multiplication domains.

The interpretation of Table II in the MCM context is as follows. The symmetries in $s_n$ establish the prefix and the suffix that have to be inserted in the transmitted sequence $x$. On the other hand, since we assume that $N > 2^v + 1$, only the first symmetry in $S_b$ is considered to obtain $h$. The third column in Table II indicates the domain(s) or transform(s) where the symmetric convolution is solved by an element-by-element multiplication. Observe that in the MCM context the inverse of $T_a$ is related to the block transform used as transmitter, and $T_e$ is indicating the DTT to obtain the $N_d$ coefficients from the $N$-Vl fromor $(7 + N + 1) \times 1$ column vector $h_{2\nu}$, which is the filter-right-half $h' = [h_0, h_1, \cdots, h_N]$ extended by zero-padding to $N + N_1$ samples. Finally, the fourth column of Table II indicates the inverse of the transform used at the receiver.

To sum up, Table III shows the interpretation of Table II in the MCM context. Special attention must be paid to the length of the $T^*$ transform since in some cases it must be $N + 1$ ($C_4$ or $C_8$ in the second or the last row of Table III respectively), whereas for the rest of cases this length must be $N$.

### III. CONSISTENCY WITH PREVIOUS RESULTS

#### A. DCT-Ile-Based MCM

In [13], the discrete cosine transform type-II even (DCT2e or $C_{2e}$)-based multicarrier modulator transceiver is derived. Basically, it can be obtained from the block diagram of Fig. 1 as follows:

(i) Inverse transform $T^{*1}$: IDCT2e (N x N matrix $C^{*}\$).

(ii) Parallel-to-serial converter, including a prefix

$$(x_p)_n = x_n + e_n, \quad n = 0, \cdots, N - 1,$$

and also a suffix:

$$(x_s)_n = x_n + e_n, \quad n = 0, \cdots, N - 1.$$

(iii) A front-end-prefilter $w$ that imposes the symmetry condition $h_k = h_{N - k}$, $k = 1, \cdots, N$ in the equivalent impulse response $h = [h_0, h_1, \cdots, h_N]$. For channels with long memory (longer than $2\nu + 1$), this front-end prefilter also has to produce an effective shortened impulse response. Once $h$ is obtained, a filter-right-half is defined as $h' = [h_{0}, h_{1}, \cdots, h_{N}]$.

(iv) Selection of the corresponding $N$ samples of interest, and serial-to-parallel converter.

(v) Direct transform $T_e$: DCT2e $(N \times N$ matrix $C_{2e})$.

(vi) EQ block: Frequency-domain equalizer, in which the numbers $d_i$ are given by [13, p. 917]

$$e^{*} \left( C_{2e} e^i \right)$$

for $e = {e^1, e^2, \ldots, e^{N-1}}$, $N = \sum_{i=1}^{M} a_i 2^{i-1}$, $a_i \in \{0, 1\}$, $i = 1, \cdots, M$. The numbers $a_i$ are calculated as follows:

$$a_i = \sum_{j=1}^{i} \left( \left[ \frac{e^{j-1}}{2^{j-1}} \right] \right) \mod 2$$

for $i = 1, \cdots, M$.

### II. DISCRETE COSINE TRANSFORM FOR MULTICARRIER MODULATION

The list of possible DCTs for using in MCM data transmission can be widened to the eight different DCTs reported in [9] and defined in Table I. In this section, there is an explanation of how to do it, considering the symbol data length satisfies the following condition: $N > (2^v + 1)$, which is the case of interest to avoid significant guard overhead.

As in [13], [17], only symmetric channel filters $h (h_{-k} = h_k, k = 0, \cdots, N)$ are considered throughout this work. This requirement can be met in practice by means of the front-end filter $w$, which is commonly used in DFT-based MCMs. Remind that frequency domain channel equalization is easy to be performed in OFDM/MIMO when the cyclic prefix length is at least the order $L$ of the transmission channel impulse response. However, this requirement is often restrictive, especially at high sampling rates, where the channel order can extend into many hundreds of samples. In order to overcome this problem, the pre-filter $w$ is placed at the receiver in cascade with the channel to produce an effective shortened impulse response, optimally squeezing the channel energy into a time-frame of less than $L + 1$ samples [19]. Different solutions to meet the symmetry condition are proposed in [13, p. 915].

<table>
<thead>
<tr>
<th>Table II SOME TYPES OF SYMMETRIC CONVOLUTION [16]</th>
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</thead>
<tbody>
<tr>
<td>Row</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>8</td>
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<td>9</td>
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<tr>
<td>10</td>
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</tbody>
</table>

Fig. 2. Examples of symmetry types.
Fig. 3. Symmetries in x and h to turn the linear convolution into a symmetric
^rZP
\text{type-I (DCT1)} of the filter

Next, we demonstrate that both (1), (2) are equivalent to the result
are given by (1), which can be expressed

show that
et al. Al-Dhahir
Type-I even (C_{rZP}^i), whereas in [13],
e> +1) -point DCT

the front-end prefilter \(w\) has to obtain an equivalent impulse response
to be introduced simultaneously as prefix and suffix into \(x\). In addition,
a respond to row 2 in Table III. This means the extension HS (see

transmitter and the receiver sides, respectively. These transforms cor­

results in Table III perfectly match with those derived in [13], except for
Accordingly, the re­
h with symmetry WS (see first symmetry of
Sb).

Remark that in this case
is extended by zero-padding to

Fig. 3 illustrates an example of the symmetries to be imposed on \(x\)
position and
ith
\(x\) 1 column vector that has a one in the

\(N\) x 1 column vector \(H_{eqiv}\) is the first column of \(H_{eqiv}\)
defined in [13] as

\[ [h_0 + h_1, h_i + h_{i+1}, h_{i-1} + h_i, h_0, 0, 0]^T. \]

Next, we demonstrate that both (1), (2) are equivalent to the result
shown in Table III, i.e., \(d_i\) corresponds to the discrete cosine transform
type-I (DCT1) of the filter \(^rZP\)

\[ d_i = \begin{pmatrix} y_{rZP} \end{pmatrix}_{j,k=0} = (C_{tZP})_{i+1}, \quad i = 0, \ldots, iV - 1, \]

\[ \text{Therefore, it suffices to prove that} \]

\[ (C_{2e}^{\text{equiv}})_{j,k} = (C_{2e}^{\text{equiv}})_{j,k} \cdot 1, \ldots, iV. \]

So we just make the following computations:

\[ (C_{2e} h_{eqiv})_k = 2 C_{2e} \cdot 2W \cdot h_0 \]

\[ 2 C_{2e} \cdot 2W \cdot h_0 + E_{k=1}^N \cos \left( \frac{\pi k}{N} \right) \cdot \frac{\cos \left( \frac{\pi k}{N} \right)}{12N} \]

Notice that the coefficients of each one-tap per subcarrier equalizer are
obtained from the \((N+1)\) x 1 column vector \(C_{tZP}\). Therefore, the
above results confirm what appears in [13]; moreover, they provide a
direct and simpler way to get the coefficients of the one-tap per subcar­
rier equalizers.

B. DCT-III-Based MCM

The results obtained in [17] can also be deduced from Table III as
follows:

1) For IDCT3e at the transmitter and DCT3e at the receiver. It corre­
ponds to row 5 in Tables II and III. A whole-point symmetry (WS)
on the left and a whole-point antisymmetry (WA) on the right have
to be applied to the original symbol. Again in accordance with [16]
for the FEQ block, \(d_i\) corresponds to the \(N\)-point DCT3e of \(h_{rZP}'\).

2) For IDCT3o at the transmitter and DCT3o at the receiver. It corre­
ponds to row 7 in Tables II and III. A whole-point symmetry (WS)
on the left and a half-point antisymmetry (HA) on the right have
to be applied to the original data symbol. For the FEQ block, \(d_i\)
corresponds to the \(V\)-point DCT3o of \(h_{rZP}'\).
In this section, we evaluate and compare by computer simulations the performance of the DFT-MCM and the DCT4e-MCM introduced earlier, also considering the impact of CFO on the bit error rate. A comparison between the DFT-MCM systems and the rest of DCT-MCM has not been included in this work because the simulation results show identical bit error rate (BER) performances for every kind of DCT (I, II, III and IV, even and odd) in all the communication scenarios herein considered.

The block diagram of the system used for the simulations is depicted in Fig. 4. It includes in-phase and quadrature modulators in the presence of CFO (A/T). More than five million binary data were generated and converted into parallel data to be transmitted over one hundred and twenty-eight subcarriers. Before proceeding with the multicarrier modulation, the data at each subcarrier are mapped by QPSK modulation. After mapping them, the parallel data were fed into the inverse block transform at the transmitter, considering the IFFT and the IDCT4e in order to compare their performances. The parameters used for the simulations are summarized in Table IV.

The block diagram of a multicarrier modulation system in the presence of CFO.

### IV. EXAMPLE DESIGN

In this work, the error probability versus 

\[ P_e = \frac{1}{N} \sum_{n=1}^{N} \mathbb{P}(x_n | y_n) \]

where \( x_n \) is the transmitted bit and \( y_n \) is the received bit. The error probability is defined as the probability that a bit is incorrectly decoded.

In Fig. 5 we present the results obtained with the mobile channel and round Doppler spectrum for several frequency offsets. It can be seen that the curves of the error probability versus Doppler spectrum are rounded and tend to increase with increasing Doppler shift. The error probability of the DCT4e-MCM system is lower than that of the DFT-MCM system for all frequency offsets considered. The error probability of the DCT4e-MCM system is lower than that of the OFDM system for all frequency offsets considered. The error floor of the DCT4e-MCM system is reached with a SNR value 15 dB larger than that of the OFDM system.

In Fig. 6 we present the results obtained with the mobile channel and flat Doppler spectrum for several frequency offsets. It can be seen that the curves of the error probability versus Doppler spectrum are flat and tend to increase with increasing Doppler shift. The error probability of the DCT4e-MCM system is lower than that of the DFT-MCM system for all frequency offsets considered. The error floor of the DCT4e-MCM system is reached with a SNR value 15 dB larger than that of the OFDM system.

### TABLE IV

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulation and Demodulation</td>
<td>QPSK</td>
</tr>
<tr>
<td>Subcarrier Number</td>
<td>128</td>
</tr>
<tr>
<td>Channel Model</td>
<td>SUI-5 [20]</td>
</tr>
<tr>
<td>Noise Model</td>
<td>iid AWGN</td>
</tr>
<tr>
<td>Doppler Spectrum</td>
<td>Round, Classic and Flat</td>
</tr>
<tr>
<td>Normalized Carrier Frequency Offsets (A/T)</td>
<td>0.02, 0.1</td>
</tr>
</tbody>
</table>

Fig. 5. Results obtained for DFT-MCM and DCT4e-MCM transmissions in a high speed and multi-path mobile channel in the presence of frequency offset (N = 128, QPSK, and A/T = 0.02, 0.1). The Doppler spectrum is round.

Fig. 6. Results obtained for DFT-MCM and DCT4e-MCM for a multi-path mobile channel with round, flat and classic Doppler spectrum. The frequency offset is A/T = 0.02 and N = 128, QPSK.
results have verified that DCT-MCM outperforms OFDM in different scenarios in the presence of CFO.

V. CONCLUSIONS

In this work, the application of DCTs for multicarrier communications has been addressed. We have shown that the transmitter of each system includes an inverse DCT and a parallel-to-serial converter in which a prefix and a suffix are introduced into the data symbol to be transmitted. The receiver consists of a front-end prefilter plus another DCT. We have obtained the conditions to use any kind of DCT for multicarrier data transmission based on the fact that DCTs have the convolution-multiplication property, that is, a symmetric convolution in the time-domain is transferred in an element-by-element multiplication in the corresponding discrete cosine transform domain. The configuration of both the transmitter and the receiver, the symmetry to be imposed by the front-end prefilter, and the prefix and the suffix to be appended into each data symbol, have been established for the eight different types of DCTs. Furthermore, the values of the coefficients for the one-tap per subcarrier equalizer have also been provided. Finally, the simulation results have verified that DCT-MCM outperforms OFDM in different scenarios in the presence of CFO.

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REFERENCES


