ASYMPTOTIC DESCRIPTION OF FLUTTER AMPLITUDE SATURATION BY NONLINEAR FRICTION FORCES

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ABSTRACT

The computation of the non-linear vibration dynamics of an aerodynamically unstable bladed-disk is a formidable numerical task, even for the simplified case of aerodynamic forces assumed to be linear. The nonlinear friction forces effectively couple different travelling waves modes and, in order to properly elucidate the dynamics of the system, large time simulations are typically required to reach a final, saturated state. Despite of all the above complications, the output of the system (in the friction microslip regime) is basically a superposition of the linear aeroelastic unstable travelling waves, which exhibit a slow time modulation that is much longer than the elastic oscillation period. This slow time modulation is due to both, the small aerodynamic effects and the small nonlinear friction forces, and it is crucial to determine the final amplitude of the flutter vibration. In this presentation we apply asymptotic techniques to obtain a new simplified model that captures the slow time dynamics of the amplitudes of the travelling waves. The resulting asymptotic model is very reduced and extremely cheap to simulate, and it has the advantage that it gives precise information about the characteristics of the nonlinear friction models that actually play a role in the saturation of the vibration amplitude.

INTRODUCTION

The detailed computation of the non-linear vibration dynamics of an aerodynamically unstable bladed-disk is extremely CPU costly. In the case of a time domain simulation, both, the fluid equations for the aerodynamic flow, and the nonlinear FEM for the displacements of the structure have to be solved simultaneously, iterating at each time step in order to satisfy both sets of equations. Even in the frequent case of small displacements where the aerodynamic forces can be linearized, the computations are quite costly because the nonlinear friction forces effectively couple different travelling waves modes and, in order to properly simulate the evolution of the system, the time integration has to include a very large number of elastic oscillatory cycles.

Despite of all the above complications, the output of the system, in the microslip regime, shows always a clear structure with two very different time scales. The resulting vibration amplitude typically takes the form of a fast oscillation, corresponding to the purely elastic oscillation, with a superimposed modulation that takes place in a much longer time scale (see, e.g. [1, 2], and the sketch Fig. 1).

The slow time modulation shown in the sketch in Fig. 1 is due first to the slow exponential growth of the flutter instability, and then to the saturation of the vibration amplitude by the nonlinear microslip friction forces; both are weak effects that appear only after many elastic oscillation periods.

The direct evaluation of this long time behavior vibration of the system is a very expensive computation (see, e.g., the recent detailed simulations reported in [3] and [4]), but it is also of high importance from the industrial point of view, since the determin-
nation of the final amplitude of the flutter induced vibration is crucial for the correct estimation of the blade fatigue life.

Reduced models are therefore required to evaluate the system vibrational response at a much lower computational cost. With this idea in mind, Sinha and Griffin [5, 6] used a simplified mass-spring system to qualitatively analyze the nonlinear friction stabilizing effect of an unstable rotor stage. In this paper, a new, quantitatively accurate simplified model is derived using asymptotic techniques. In the resulting model the fast linear oscillation of the blades is filtered out, and it describes only the slow time dynamics of the system.

This asymptotic methodology can be applied to a detailed FEM with nonlinear friction elements, and the resulting model is extremely reduced, and very cheap to simulate. This asymptotic model has also the advantage that it gives precise information about the characteristics of the nonlinear friction elements that actually play a role in the saturation of the vibration amplitude.

The derivation procedure of this asymptotic model will be illustrated using the bladed disk conceptual model considered in [2] and [1]. And the resulting reduced model will be numerically simulated for different values of the parameters in order to explore the different dynamical regimes that can be present in the nonlinear friction saturation of an aerodynamically unstable rotor.

**NOMENCLATURE**

- **P**: Transformation matrix from TW amplitudes to sectro displacements
- **\( \tilde{\eta} \)**: Rescaled aerodynamic frequency correction
- **\( \tilde{\xi} \)**: Rescaled critical damping ratio
- **\( \xi \)**: Critical damping ratio
- **\( F_a \)**: Aerodynamic force on the airfoil
- **N**: Number of sectors
- **\( X_j \)**: Complex amplitude of the displacement of airfoil \( j \)
- **\( y_j \)**: Microslip DOF displacement of \( j \)-sector
- **\( F_f \)**: Nonlinear friction force
- **\( x_j \)**: Airfoil displacement of \( j \)-sector
- **\( \tilde{x}_j \)**: Nondimensional \( j \)-sector airfoil displacement
- **\( \tilde{y}_j \)**: Nondimensional \( j \)-sector microslip DOF displacement
- **\( A_k \)**: Complex amplitude of the \( k \) TW
- **\( m_a, k_a \)**: Mass and stiffness of the airfoil
- **IBPA**: Interblade phase angle
- **\( \omega_a \)**: Blade alone natural frequency of the airfoil
- **TW**: Travelling wave
- **\( \eta \)**: Aerodynamic correction of the frequency
- **\( m_f \)**: Mass of the microslip nonlinear spring

**ROTOR MODEL**

The rotor model is that considered in [2] and [1], which corresponds to a simplified mechanical representation of a high aspect ratio low-pressure turbine rotor-blade. The model represents a wheel of rotor blades attached by means of a fir-tree to a infinitely stiff disk. The model includes two DOFs per sector: the first DOF of the \( j^{th} \) sector, \( x_j \), corresponds to a bending mode of the airfoil, and the second DOF, \( y_j \), is associated with the micro displacements at the fir-tree (see the sketch in Fig. 2).

![Figure 2](image.png)

The governing equations for the \( j^{th} \) sector are given by

\[
m_a \ddot{x}_j + k_a (x_j - y_j) = F_a(\ldots, x_j, \ldots, y_j, \ldots),
\]

\[
m_f \ddot{y}_j + k_a (y_j - x_j) = F_f(y_j),
\]

for \( j = 1, \ldots, N \), where \( N \) is the total number of sectors of the wheel.

The mass and stiffness of the bending mode of the airfoil are \( m_a \) and \( k_a \), and, in the limit of no displacements at the fir-tree (\( y_j = 0 \)), the blade alone natural frequency of the airfoil, \( \omega_a^2 = k_a/m_a \), is recovered. The term \( F_a \) represents the aerodynamic forces on the airfoil, which, in general, depend on the displacements and velocities of all airfoils. The mass associated
with the microslip nonlinear spring is denoted by \( m_f \), and the nonlinear friction forces are contained in \( F_f \), which depends on the microslip displacement, \( y_j \).

The nonlinear friction forces in the microslip regime are modeled using a simple Sellgren and Olofsson model [7]. For a load/unload cycle, the nonlinear friction force can be expressed as

\[
F_f(y_j) = F_U + F_c(1 - \left(\frac{y_j - y_U}{y_c}\right)^5),
\]
during the loading part of the cycle. The parameters \( F_c \) and \( y_c \) are the maximum limit force and displacement of the microslip region, and \( F_U \) and \( y_U \) are, respectively, the force and the microslip at the end of the previous unloading part of the cycle. The subsequent unloading part of the cycle can be written as

\[
F_f(y_j) = F_L - F_c(1 - \left(\frac{y_j - y_L}{y_c}\right)^5),
\]
where, again, \( F_L \) and \( y_L \) are the corresponding friction force and displacement at the end of the previous loading part of the cycle. The resulting loading/unloading cycle is sketched in Fig. 3, which shows the typical nonlinear friction hysteresis loop.

The model is completed with the prescription of the aerodynamic forces, \( F_a \), which are taken into account using a linear approximation (which is quite appropriate for the usual situation of small airfoil displacements), and are computed for a fixed frequency equal to the airfoil elastic frequency \( \omega_a \) (this is also justified because, as it will be explained below, the nonlinear friction only produces small variations of the oscillation frequency that, in first approximation, can be neglected).

The linear aerodynamic characteristics used in this paper are given in Fig. 4 for the different travelling wave IBPA (interblade phase angle). Note that there is range of TW with negative critical damping ratio, and, therefore, this model corresponds to an aeroelastically unstable rotor wheel. The asymptotic description of the saturation of this flutter instability by the nonlinear friction forces is main purpose of this paper.

![Figure 3](image.png)

**Figure 3.** Sketch of the friction load vs. displacement loop (L: loading, U: unloading, from [1]).

![Figure 4](image.png)

**Figure 4.** Linear aerodynamic characteristics of the airfoil for the elastic frequency of the airfoils \( \omega_a \). Top: aerodynamic frequency correction vs. IBPA. Bottom: critical damping ratio vs. IBPA. (IBPA: interblade phase angle).

### ASYMPTOTIC MODEL

Before starting with the asymptotic simplification of the rotor model given by Eqs. (1)-(2), it is convenient to first change to the nondimensional variables:

\[
\tilde{x}_j = \frac{x_j}{x_c}, \\
\tilde{y}_j = \frac{y_j}{y_c}, \\
T = t\omega_a,
\]

where the blade displacement is made nondimensional with the characteristic blade displacement \( x_c = F_c/k_a \) (which generates forces in the fir-tree within the microslip regime), the microslip displacement with its characteristic value \( y_c \), and time with the elastic oscillation frequency of the blade \( \omega_a \).
The resulting nondimensional equations take the form
\[ \ddot{x}_{jTT} + \dot{x}_j - \theta \ddot{y}_j = F_a/(m_a\omega_n^2 x_c), \quad (3) \]
\[ \delta \theta \ddot{y}_{jTT} + \theta \ddot{x}_j - \ddot{x}_j = F_j/F_c, \quad (4) \]

The nondimensional parameter \( \theta \) verifies
\[ \theta = \frac{y_c}{x_c} = \frac{k_a}{(F_c/y_c)} \ll 1, \]

because the stiffness of the microslip nonlinear spring is typically much larger than that of airfoil mode, or, in other words, the microslip displacement is much smaller than the airfoil modal displacement. The second nondimensional parameter
\[ \delta = \frac{m_f}{m_a} \ll 1 \]
is also very small since the inertia of the microslip DOF is typically much smaller than the inertia of the airfoil mode. The right hand side of Eq. (3), \( F_a/(m_a\omega_n^2 x_c) \), is also small since it represents the ratio of the aerodynamic force on the airfoil and the elastic force, and it is of the order of the critical damping ratio and the aerodynamic correction of the frequency, which are typically \( \sim 10^{-2} \) (see the plots in Fig. 4). And, finally, the term \( F_j/F_c \) in Eq. (4) is of order 1 since \( F_c \) is precisely the characteristic value of the friction forces.

Using now multiple scales asymptotic techniques (see, e.g., [8] and [9]), the idea is to obtain a simplified description of Eqs. (3)-(4) in the limit
\[ \theta \ll 1, \]
\[ \delta \ll 1, \]
\[ \frac{F_a}{m_a\omega_n^2 x_c} \ll 1, \]
\[ \frac{F_j}{F_c} \sim 1. \]

To this end, the solution is first expanded as
\[ \tilde{x}_j = x_j^0(T, \tau) + \theta x_j^1(T, \tau) + \ldots, \]
\[ \tilde{y}_j = y_j^0(T, \tau) + \theta y_j^1(T, \tau) + \ldots, \]
where \( \tau = \theta T \) is the slow time scale. Once these expansions are taken into Eqs. (3)-(4), the following equations are obtained at first order,
\[ \ddot{x}_{jTT}^0 + \dot{x}_j^0 = 0, \quad (5) \]
\[ \ddot{y}_j^0 = F_j(y_j^0)/F_c. \quad (6) \]

The first equation can be easily solved to give
\[ \dot{x}_j^0 = X_j(\tau)e^{i\phi} + c.c., \quad (7) \]
which states that, in the fast time scale \( T \), all airfoils oscillate with the elastic frequency, and the amplitude of these oscillations, \( X_j \), is modulated in the slow time scale \( \tau \) (c.c. stands for the complex conjugate). The forcing of the second equation is now known, \( \dot{x}_j^0 \), and, therefore, using the expression for the nonlinear friction force (see Fig. 3), the first order of the microslip DOF can be computed, and it yields a real periodic function in \( T \), which can be written as the following Fourier series
\[ \dot{y}_j^0 = P(T, X_j) = \sum_{k=1}^{\infty} \tilde{P}_k(X_j)e^{ikT} + c.c. \]

The hysteresis cycle in Fig. 3 verifies the symmetry \( y \rightarrow -y \), \( F \rightarrow -F \), and then there is no average value for \( \dot{y}_j^0 \), that is, \( \tilde{P}_0 = 0 \). Moreover, changing the phase of \( X_j \) in Eq. 7 is equivalent to changing the time \( T \), and the same has to happen for the expression of \( \dot{x}_j^0 \), which can be finally expressed as
\[ \dot{y}_j^0 = P(T, X_j) = \sum_{k=1}^{\infty} P_k(|X_j|)e^{ik(T+\phi)} + c.c. \quad (8) \]

where \( |X_j| \) and \( \phi \) are, respectively, the modulus and phase of the amplitude \( X_j \).

The next order approximation for the airfoil displacements is given by
\[ \ddot{x}_{jTT}^1 + \dot{x}_j^1 = -2 \frac{\partial^2 x_j^0}{\partial \tau \partial T} + \dot{y}_j^0 + \frac{F_a(x_j^0, \ldots, x_N)/(m_a\omega_n^2 x_c)}{\theta}. \]

Using the above expressions (7) and (8) for \( \dot{x}_j^0 \) and \( \dot{y}_j^0 \), and applying solvability conditions to ensure that the fast time dynamics is bounded, the following equations are obtained for the slow time evolution of the amplitudes \( X_j(\tau) \)
\[ 2i \frac{dX_j}{d\tau} = P(|X_j|)e^{i\phi} + \frac{F_a(X_1, \ldots, X_N)/(m_a\omega_n^2 x_c)}{\theta}. \]

All the relevant information about the nonlinear friction loop is contained in a single Fourier coefficient \( P(|X_j|) \). The imaginary part of this complex function is always negative, and gives the damping generated by the nonlinear friction; its real part produces a nonlinear phase shift of the solution.
If the aerodynamic forces are explicitly written and the friction term is expressed as

\[ P_l(|X_l|) = Q(|X_l|)|X_l|, \quad (9) \]

then the complete system for the complex amplitudes of the displacements can be finally expressed in the form

\[
2i \frac{d}{dt} \begin{bmatrix} X_1 \\ \vdots \\ X_j \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} Q(|X_1|) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q(|X_N|) \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_j \\ \vdots \\ X_N \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1 + i \tilde{\xi}_1 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \Re \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_1 + i \tilde{\xi}_1 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \Re \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}. \quad (10)
\]

Here the aerodynamic forces has been rescaled with its characteristic value \( F_{ac} \) in order to make the rescaled frequency correction and damping \(|\tilde{\eta}_1|, |\tilde{\xi}_1| \sim 1\), the aerodynamic coefficient

\[ c_a = \frac{F_{ac}}{(m_o \omega^2 c_x)} \]

measures the ratio between the small aerodynamic forces and the small friction forces, and \( P \) is the transformation matrix from the basis of traveling waves to the basis of physical displacements,

\[
P = \frac{1}{\sqrt{N}} \begin{bmatrix} P_{1e^{i(2\pi l/N)}1} & \cdots & P_{Ne^{i(2\pi N/N)}1} \\ \vdots & \ddots & \vdots \\ P_{1e^{i(2\pi l/N)}N} & \cdots & P_{Ne^{i(2\pi N/N)}N} \end{bmatrix}. \quad (11)
\]

The system (10), in terms of the TW amplitudes, \( A_k \),

\[
X = \begin{bmatrix} X_1 \\ \vdots \\ X_j \\ \vdots \\ X_N \end{bmatrix} = PA = P \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix},
\]

takes the following form

\[
2i \frac{d}{dt} \begin{bmatrix} A_1 \\ \vdots \\ A_k \\ \vdots \\ A_N \end{bmatrix} = P^H \begin{bmatrix} Q(|X_1|) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q(|X_N|) \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_k \\ \vdots \\ A_N \end{bmatrix} -2c_a \begin{bmatrix} \eta_1 + i \xi_1 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \Re \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}. \quad (12)
\]

Both formulations are completely equivalent. The friction forces are diagonal in the displacements basis, but the aerodynamic forces are diagonal in the TW basis.

The numerical integration of the system (12) is much cheaper than that the original system Eqs. (3)-(4) because: (i) the fast elastic oscillation has been removed from system (12), which contains only the evolution of the slow time envelope, and (ii) the numerical integration problems associated with the strong stiffness of the nonlinear spring that models the friction are not present in system (12).

**NUMERICAL SIMULATIONS**

In this section some numerical simulations of the asymptotically simplified system (12) are performed for a set of parameter values similar to those used in [1, 2].

The number of sectors is taken to be \( N = 36 \), the aerodynamic coefficient is set to \( c_a = 1 \), the values used for the normalized aerodynamic coefficients \( \tilde{\eta}_j \) and \( \tilde{\xi}_j \) are those presented in Fig. 4, and the expressions used for the complex friction function \( Q \) are given by

\[
\Re(Q(|X|)) = 0.4(1 + 2|X|), \quad \Im(Q(|X|)) = -0.1|X|(1 + 2|X|),
\]

which are an accurate fit of the values obtained from the Sellgren and Olofsson model [7].

Fig. 5 shows the resulting time evolution of the amplitudes of the TWs for a random initial with amplitude 0.01. As it can be seen in the inset in Fig. 5, the initial time dynamics is dominated by the linear aerodynamic effects: the amplitudes of the TWs grow or decay exponentially according to their linear stability characteristics given in Fig. 4. When the unstable TWs reach a certain non-negligible amplitude then the nonlinear effects due to
the friction terms come into play, and some of the stable TW are re-activated. After some nonlinear interactions, all TWs decay except for the most unstable one, which corresponds to $k = 7$, and the final state is a friction saturated nonlinear TW with 7 nodal diameters.

The same happens if the size of the initial condition is increased, see Figs. 6 and 7, which correspond, respectively, to random initial conditions with size 0.1 and 1.0. As the amplitude of the random initial condition is increased, the linear, aerodynamically dominated behavior for initial times ceases to exist, but the final state is always a non-linearly saturated TW which correspond to $k = 7$ (most unstable TW) in Fig. 6, and to $k = 6$ (second most unstable TW) in Fig. 7.

All the numerical simulations performed produced final states that were pure, nonlinear TW. In order to explore the possibility of having more complicated final states, the real part of the friction function $Q(|X|)$ was multiplied by 10, which corresponds to increasing dramatically the nonlinear phase shifting produced by the friction. With this friction function the resulting final state from a random initial condition of size 0.01 was much more complicated, with all TW active in a completely unsteady way, see Fig. 8. But this friction term that produces much more phase shifting than damping surely corresponds to a quite nonphysical situation.

**CONCLUSIONS**

A new asymptotic methodology for the analysis of the nonlinear friction saturation of aeroelastically unstable rotors has been introduced in this paper. The method exploits the existing time scale separation between the fast elastic oscillation of the airfoil and the much slower time scale in which the flutter instability and the nonlinear friction come into play. The validity of this asymptotic procedure requires that (i) the motion of the microslip DOFs generates a small perturbation of the elastic airfoil modes, and (ii) the aerodynamic forces are small and produce also only a small correction of the elastic oscillation characteristics of the system. These two conditions hold in a large number of problems of aeroelastic instability saturation by friction forces in realistic bladed disk configurations.

The resulting asymptotic model system is given by Eq. (12), which prescribes the slow time evolution of the system, and it is extremely reduced because it considers only the TWs that actu-
ally take part in the vibratory response of the system. This new, asymptotically reduced model allows us to draw the following final remarks about the combined effect of flutter and nonlinear friction:

1. The asymptotic model has been derived for a conceptual model of the bladed disk, but this methodology can be also applied to a realistic FEM description of the bladed disk in a completely similar way, and it will produce again a drastically reduced model, of the form of Eq. (12), containing only the TWs that take part in the output of the system.

2. The fast elastic oscillation has been removed in the asymptotically reduced model (12), which gives directly the slow time dynamics of the problem, and therefore can be numerically integrated with a much lower CPU cost that the original system. Also the numerical problems associated with the high stiffness of the nonlinear friction (which typically require the use of partially implicit time integration schemes) are not present in the Eq. (12) formulation.

3. The asymptotic analysis of the problem gives the following sequence for the large time dynamics of the system: (i) the airfoil basically oscillates with the elastic frequency, (ii) the microslip DOF at the fir-tree also oscillates, driven by the forcing produced by the airfoil oscillation, and (iii) the resulting microslip DOF generates a small nonlinear effect in the airfoil motion that manifest for large times together with effect of the small linear aerodynamic forces.

4. The asymptotic model (12) gives also precise information about the characteristics of the nonlinear friction that actually have an effect on the vibration of the system. Only the Fourier harmonic of the microslip DOF oscillation given by Eqs. (8) and (9) appears in the asymptotic model Eq. (12); its real part produces a nonlinear phase shift of the solution and its imaginary part generates the nonlinear damping of the friction.

5. The vibration characteristics of bladed disks strongly depend on the implemented nonlinear friction elements and on the number of them that are actually included in the FE model (see the recent paper [3]). With this asymptotic formulation, the analysis of different configurations of nonlinear friction elements is much simpler: it only requires to computation of the function $Q$ in system (12) for each configuration.

6. The numerical computations performed in this paper indicate that the final state is typically a unstable TW nonlinearly saturated by the friction effect (the selected TWs are the most unstable ones). This is very interesting because the nonlinear TWs in a realistic FE model can be computed using phase-lagged boundary conditions, which allow to drastically reduce the computational domain to a single sector.

7. Finally, it is possible to find more complex final states that are unsteady and contain several TWs (see Fig. 8), but this requires to change the nonlinear friction characteristics to produce very strong phase shifts (stronger than the friction dissipation), and this is probably nonphysical and does not correspond to realistic friction effects.

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