An Efficient Nonlinear Transformation for the Numerical Computation of the Singular Integrals Appearing in the 2-D Boundary Element Method

E. ALARCON(#), M. DOBLARE(+), J. SANZ-SERNA(*)

(#) Departamento de Ingeniería de Construcción. U. P. Madrid (Spain)
(+) Departamento de Ingeniería Mecánica. Universidad de Zaragoza (Spain)
(*) Departamento de Matemática Aplicada y Computación. Universidad de Valladolid (Spain)

Summary

We discuss several methods, based on coordinate transformations, for the evaluation of singular and quasisingular integrals in the direct Boundary Element Method. An intrinsic error of some of these methods is detected. Two new transformations are suggested which improve on those currently available.

Introduction

An essential aspect of the implementation of the direct Boundary Element Method (B.E.M.) is the accurate computation of the integrals appearing in the coefficients of the resulting algebraic system. Special techniques are required, due to the inherent singularities (weak or Cauchy type) of the integrands. If constant or linear straight boundary elements are used, it is feasible to perform the integrations analytically [1]. This is not possible in general with curved elements. Semianalytical techniques can be employed with an acceptable accuracy. These techniques require an initial analytical effort to find the coefficient of the singularity and depend on the type of the singularity, so that different segments of code must be employed for different problems, with a lost of systematization [2][3].

An alternative consists in using quadrature rules tailored to the specific type of the singularity [3][4]. Again different quadratures would be required for different problems. The commonest alternative in standard B.E. codes computes singular terms by imposing special conditions (constant potentials or rigid body movements). In some cases (axisymmetric problems), it is difficult or even impossible to program these special "strain" cases.

This aspect of the B.E.M. has become even more critical by the recent inclusion in B.E. problems of adaptive techniques [5][6] with approximation functions of high degree. This imply more approximation nodes (i.e. more singularities) and nonstandard placement of the nodes within the elements.

Furthermore, quasisingular integrals appear when the collocation point is very close to the element of integration, which happens when computing variables at an internal point very close to the boundary or when two adjacent elements have very dissimilar lengths. This problem is aggravated in the p-adaptive version, where the collocation point can be very near the edges of the element.

In order to solve these problems, different authors [7][8][9] have recently suggested the use of nonlinear coordinate transformations of the integrand in order to smooth it. The aim of the present wiork is to study these transformations. We prove that some of them, although asymptotically correct as the number of Gauss points increases, incorporate a hidden error source, that can be easilly removed. Also two consistent nonlinear transformations are sugested which show a very efficient behaviour.

Nonlinear transformations for the computation of 2-D singular integrals

We begin by revising the transformations suggested in [7][8][9]. The three of them transform the integration variable (η) into a new variable (ξ). The transformations in [8] and [9], subdivide the standard integration interval (-1,1) into two subintervals $(-1,\eta_s)$ $(\eta_s,1)$ and each of them is then mapped onto the interval [ξ: (-1,1)]). The transformation [11] keeps the initial interval so that [η: (-1,1)]) transforms into [ξ: (-1,1)]).The following transformations are then defined:

$$\eta(\xi) = a\xi^3 + b\xi^2 + c\xi + d \quad \text{for transformations [7] and [8]}$$

$$\eta(\xi) = a\xi + b.tg[\phi(\xi)] + c \quad \text{for transformation [9]} \tag{1}$$

with $\phi(\xi) = n.\frac{\pi}{4}.(\xi S + 1)$

and S = +1, If the integration is performed along the right hand side of η_s.
S = -1, If the integration is performed along the left hand side of η_s.

The boundary conditions that are imposed to the two above transformations are defined as:

CUBIC [7]	BICUBIC [8]	BITANGENT [9]
$\eta(\xi = S) = S$	$\eta(\xi = S) = S$	$\eta(\xi = S) = S$
$\eta(\xi = -S) = \eta_s$	$\eta(\xi = -S) = \eta_s$	$\eta(\xi = -S) = \eta_s$
$\left(\frac{\partial\eta(\xi)}{\partial\xi}\right)_{\xi=-S} = 0$	$\left(\frac{\partial\eta(\xi)}{\partial\xi}\right)_{\xi=-S} = 0$	$\eta(\xi = -S\xi_a) = -\eta_s + S\varepsilon$
$\left(\frac{\partial^2\eta(\xi)}{\partial\xi^2}\right)_{\xi=-S} = 0$	$\eta(\xi = -S\xi_a) = -\eta_s + S\varepsilon$	

(2)

with ξ_a the abcissa of the last Gauss integration point, and ε the distance from η_s, to $\eta(-S\xi_a)$. In the case of [8], accurate results are obtained for ε values of

$$\varepsilon = \frac{1 - |\xi_a|}{R} \qquad R = 100 \text{ to } R = 1000 \tag{3}$$

while the bitangent transformation is much more stable for smaller Rs.

In the case of a logarithmic singularity, the integrand may be always expressed as:

$$I = a.\ln r + I_r \tag{4}$$

with I_r a regular funcion. Let us consider the effect of a transformation with a null Jacobian at the singularity. Taylor expansion near the singularity $\eta(\xi) = \eta_s + A_1(\xi - \xi_s)^2 + A_2(\xi - \xi_s)^3 + \ldots$ leads after transformation to the new integrand

$$I'(\xi) = a.\ln r(\xi).2A_1\,(\xi - \xi_s) + \ldots + I_r(\xi).2A_1(\xi - \xi_s) + \ldots. \tag{5}$$

which is a regular funcion. This is the reason of the good performance of all the transformations in (2): they impose the condition of null jacobian at the singularity (Telles and Cerrolaza) or this condition is approximately satisfied when using small ε (Alcantud).

In a similar form, the integrand of a Cauchy Principal Value Integral (CPVI) in two dimensions may be written as

$$I = \frac{a}{r} + I_r \tag{6}$$

with I_r regular. It is then obvious that it is sufficient to investigate the performance of the transformations of the integral

$$I = \int_{-1}^{1} \frac{d\eta}{\eta - \eta_s} \quad \text{con} \quad -1 < \eta_s < 1 \tag{7}$$

Let us now consider a double nonlinear transformation on each side of the singularity (as in [8],[9]), with Taylor ~~developments~~ expansions around η_s of the form:

$$\begin{aligned} \eta_1 &= \eta_1(\xi) = \eta_s + a_1(\xi - 1) + b_1(\xi - 1)^2 + c_1(\xi - 1)^3 + \ldots \\ \eta_2 &= \eta_2(\overset{\xi}{\cancel{\eta}}) = \eta_s + a_2(\xi + 1) + b_2(\xi + 1)^2 + c_2(\xi + 1)^3 + \ldots \end{aligned} \tag{8}$$

Taking these expressions to (7) and using the definition of CPVI it is possible to get

$$I = \int_{-1}^{\eta_1^{-1}(\eta_s - \varepsilon)} \frac{d\xi}{\xi - 1} + \int_{\eta_2^{-1}(\eta_s + \varepsilon)}^{1} \frac{d\xi}{\xi + 1} + I_r = I_s + I_r \tag{9}$$

with I_r again a regular integral. If we now perform the numerical integration (with the same quadrature rule for both sides) of I_s along the interval ξ:(-1,1) the result is zero by symmetry. However, the true value of I_s is

$$I_s = \ln \frac{|\,\eta_1^{-1}(\eta_s - \varepsilon) - 1\,|}{|\,\eta_2^{-1}(\eta_s + \varepsilon) + 1\,|} \quad \text{and in the limit } \varepsilon \to 0 \qquad \lim_{\varepsilon \to 0} I_s = \ln \frac{|a_2|}{|a_1|} \tag{10}$$

If both $a_i = 0$, (10) becomes the logarithmic of the modulus of the ratio between b_i etc. With this, it is clear that the analytical value of I_s will be the same as the numerical one when the dominant coefficients of both transformations are the same. If this is not the case, there will be an intrinsic error in the numerical computation of (7) due to the exclusion of the term (10). The transformations [7][8] and [9] include this error in their formulation, but the authors in [8] and [9] present results of the computation of CPVI with a very acceptable accuracy. We must then conclude that in these cases the term (10) must be small.

For transformation [8], the ratio of the dominant terms is given by

$$\frac{b_2}{b_1} = \frac{8\varepsilon + (1+\eta_s)(\xi_a-1)^3}{-8\varepsilon + (1+\eta_s)(\xi_a-1)^3} \tag{11}$$

Therefore, the modulus of this ratio tends to 1 when ξ_a tends to 1, and this happens very rapidly when the order of the Gauss-Legendre quadrature is increased. This explains the good performance of this transformation for high-order quadratures, despite the intrinsic error included. Note that larger values of ε imply a faster reduction on the intrinsical error. However, for large ε (R<100) this effect is offset by the poor performance in the integration of the regular part due to the important distorsion introduced by the transformation away from the singularity.

With respect to the bitangent formula, the ratio of the dominant coefficients is

$$\frac{a_2}{a_1} = \frac{\varepsilon \mathrm{tg}\, n\frac{\pi}{2} + (\xi_a-1)(1-\eta_s)\mathrm{tg} n\frac{\pi}{2} + (1-\eta_s)\mathrm{tg}[n\frac{\pi}{4}(\xi_a-1)]}{-\varepsilon\, \mathrm{tg}\, n\frac{\pi}{2} + (\xi_a-1)(1+\eta_s)\mathrm{tg} n\frac{\pi}{2} + (1+\eta_s)\mathrm{tg}[n\frac{\pi}{4}(\xi_a-1)]} \tag{12}$$

and a similar discussion applies. In this case it is possible to increase substantially the value of ε without important distorsion of the regular part. Finally (12) also explains why values of n closer to 1 give rise to better results, becouse the value of $\varepsilon \mathrm{tg} n\pi/2$ is also larger.

Suggested transformations

It is now clear that a transformation should satisfy the following requirements: 1) zero Jacobian at the singularity; 2) If two transformations are used they should have the same dominant terms (alternatively a single transformation may be used if it transforms η_s in $\xi_s = 0$); 3) The regular terms are not greatly distorted.

There is only one single cubic transformation satisfying 1)-2), namely $\eta(\xi) = \eta_s(1-\xi^2) + \xi^3$. However, 3) is not satisfied. In fact this transformation does not perform well in practice. An alternative is to use a single transformation of degree 4. Conditions 1)-2) leave one free parameter that can be chosen to get an almost monotonic transformation, thus implying 3). We have tested the transformation

$$\eta(\xi) = \eta_s(1-\xi^4) + \xi^3 \tag{13}$$

which has shown a very good behaviour, even in the case $\eta_s > 0.75$ where a maximum appears within the integration interval (Fig.1a).

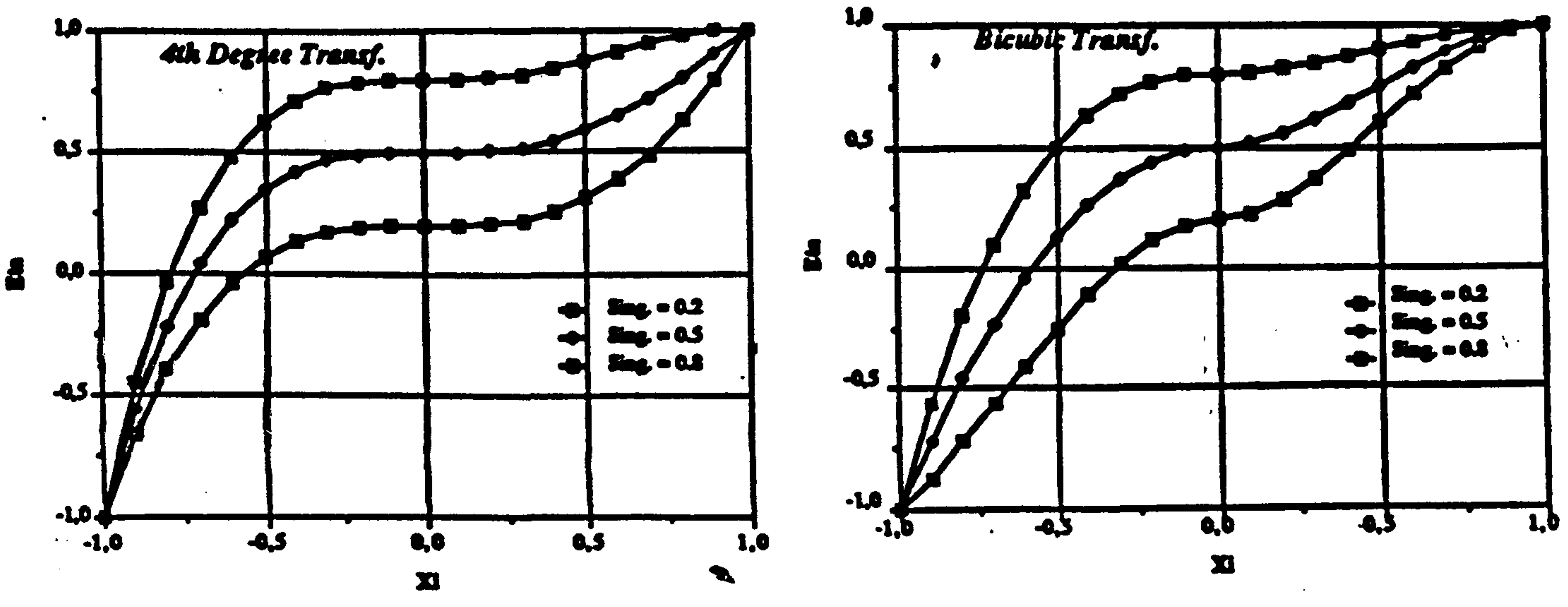

Fig. 1. Proposed nonlinear transformations

Another possibility is to use a bicubic transformation as [8][9] but with equal dominant terms. Subject to condition 1), these are the form

$$\eta_1(\xi) = \eta_s - \frac{1+\eta_s}{4}(\xi - 1)^2 + \frac{B}{2}(\xi^2 - 1)$$

$$\eta_2(\xi) = \eta_s + \frac{1-\eta_s}{4}(\xi + 1)^2 + \frac{B}{2}(1 - \xi^2)$$

where B is a free parameter. The transformation is monotonic if $B \geq 0$ and $B \leq \frac{3}{4}(1 - |\eta_s|)$. Good results have been obtained with $B = \frac{3}{4}[1 - |\eta_s|]$.

Results

Fig. 2a shows the results obtained when integrate an integral with a logarithmic singularity, using the different transformations included here, with 8 evaluations (4 Gauss points for each interval for the bicubic and bitangent transformations and 8 for the rest).

From this, it can be observed that good results are obtained in this case with any of these transformations, as was expected due to the condition of null jacobian at the singularity that all of them "include". However, a greater stability may be seen when using bicubic transformations, due perhaps to the lower degree of the jacobian of the transformation.

In the same way, Fig. 2b shows the results corresponding to the use of these transformations (except [7] which gives rise to error 2 or 3 order greater, as was explained before) for the numerical computation of an integral with a singularity of type 1/r. As it was expected, there appears an important error for transformations [8][9], which is avoided when using the transformations proposed here. These last ones are very similar, although the 4th degree one gives slightly better results. It is remarkable the high accuracy got with only 8 evaluations, with maximum errors of about 0,1 %, which are reduced to 0,0002 % when using 16 evaluations.

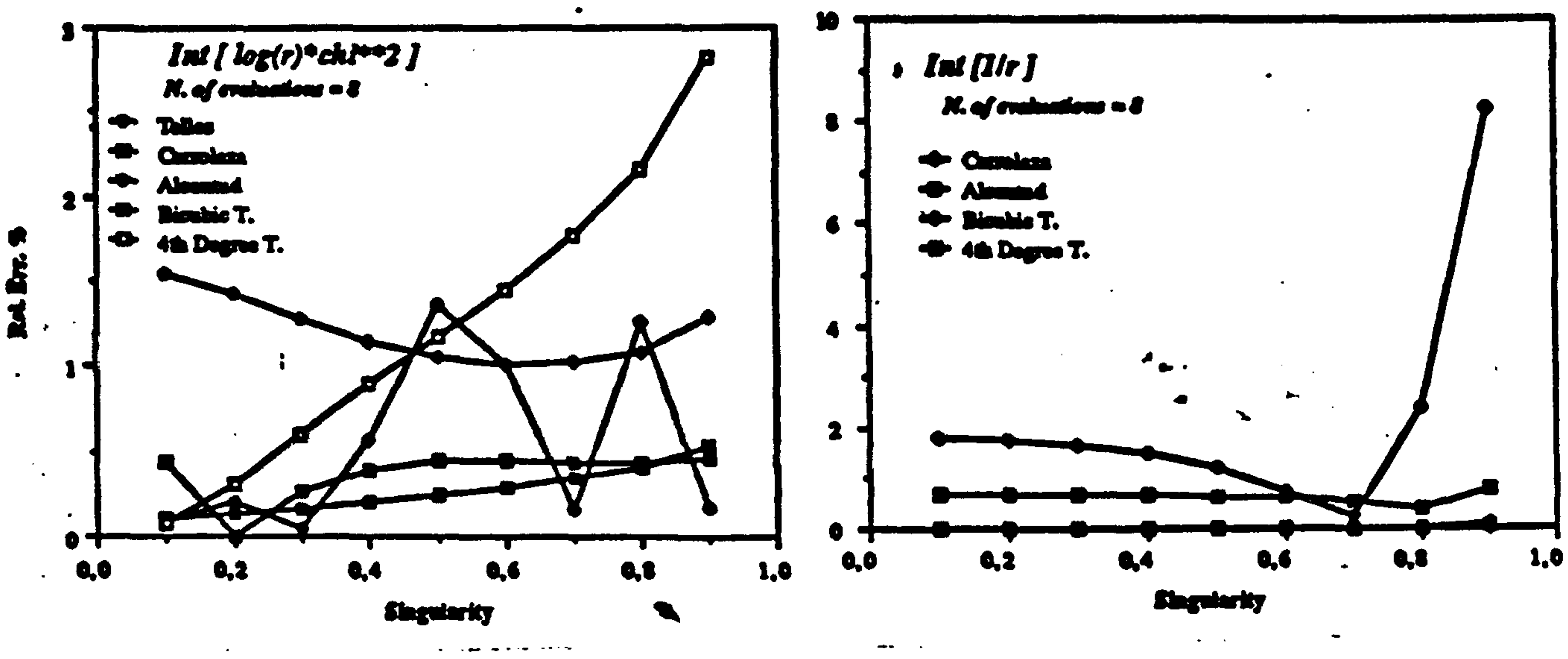

Fig. 2. Computation of singular integrals

Finally, Fig. 3a and 3b show the relative errors obtained when the proposed 4th degree transformation is used to integrate quasisingular integrals with the same type of integrands. The results are again very acurate, and of course less dependent of the position of the closest point of the interval to the collocation point, and now very dependent of the distance between these two points. Anyway, with only 8 evaluations maximum errors of 2% are obtained for distances around 1% of the length of the interval. If the quadrature is improved (16 evaluations) this error is reduce to only 0.4%.

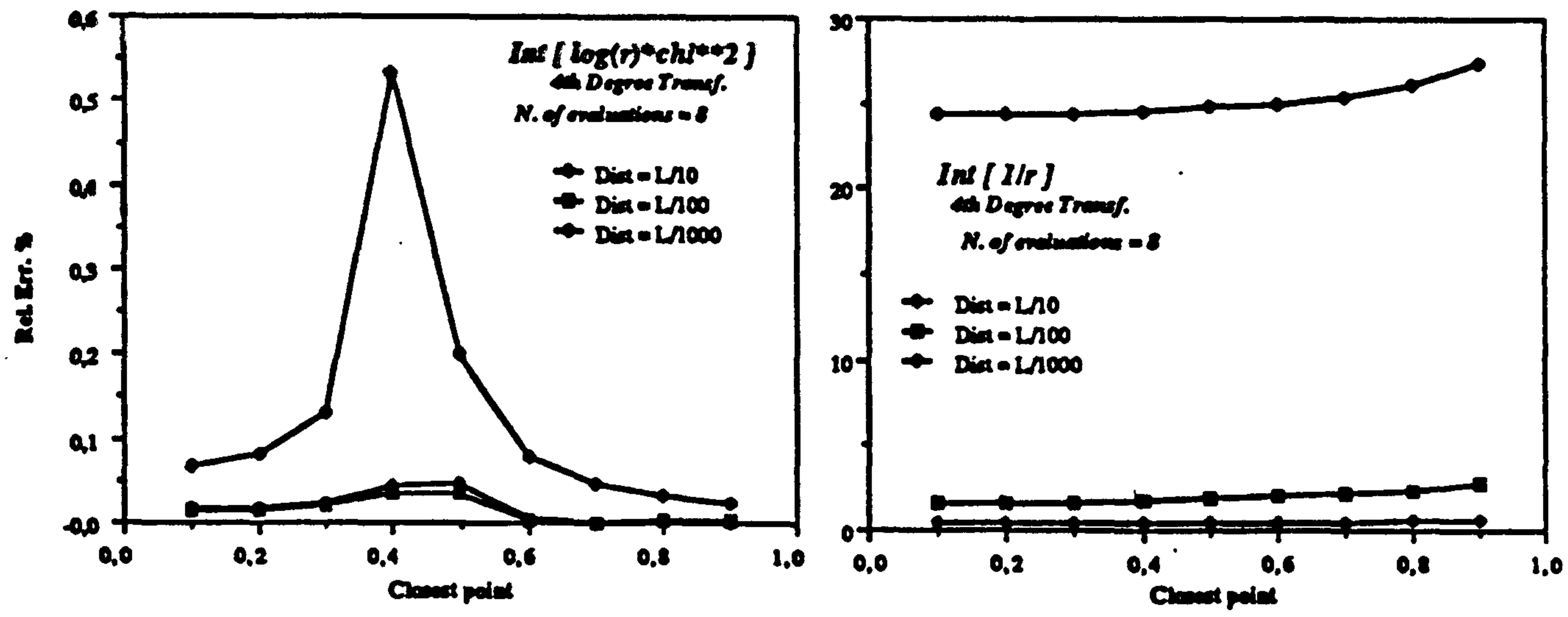

Fig. 3. Computation of quasisingular integrals

Conclusions

The above results along with extensive numerical experimentation suggest the following conclusions:

- All the nonlinear transformations considered here are well adapted to the computation of singular integrals of logarithmic type.
- For singularities of type $1/r$ the transformation suggested in [7] does not work.
- The bicubic [8] and bitangent [9] transformations produce similar results, although this latter is more accurate and robust. They both need high order quadratures rules (8+8 nodes) to get an acceptable accuracy. The best results with [9] are obtained when R is between 50 and 100, while [8] works best with R around 1000. In the bitangent transformation high values of n (near 0.9) are best, except when the singularity is near the edges. The large n produces an unwelcome distorsion.
- A quartic and a bicubic transformations have been suggested that, with fewer evaluations (8), produce good results when applied to the singular and quasisingular B.E.M. integrals.
- For collocation points placed at the edge of the elements it is sufficient to consider the two adjacent elements to this node as a unique interval and work over both elements at the same time with the same transformation between the coordinate s and the integration coordinate ξ.
- The methods discussed here render it possible the accurate computation of B.E. integrals in a type-dependent way, a feature that reduces coding time, coding errors and, perhaps, execution time.

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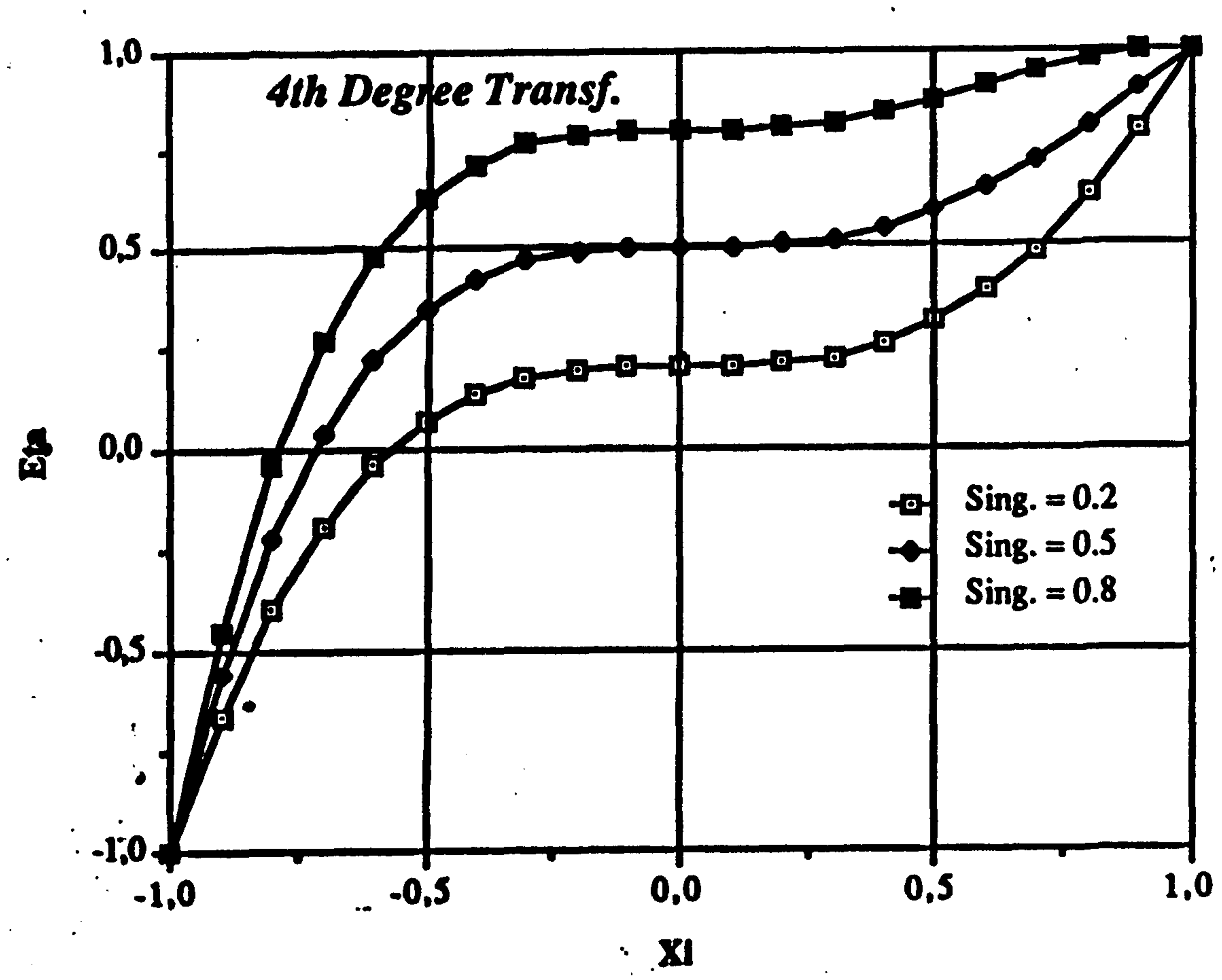
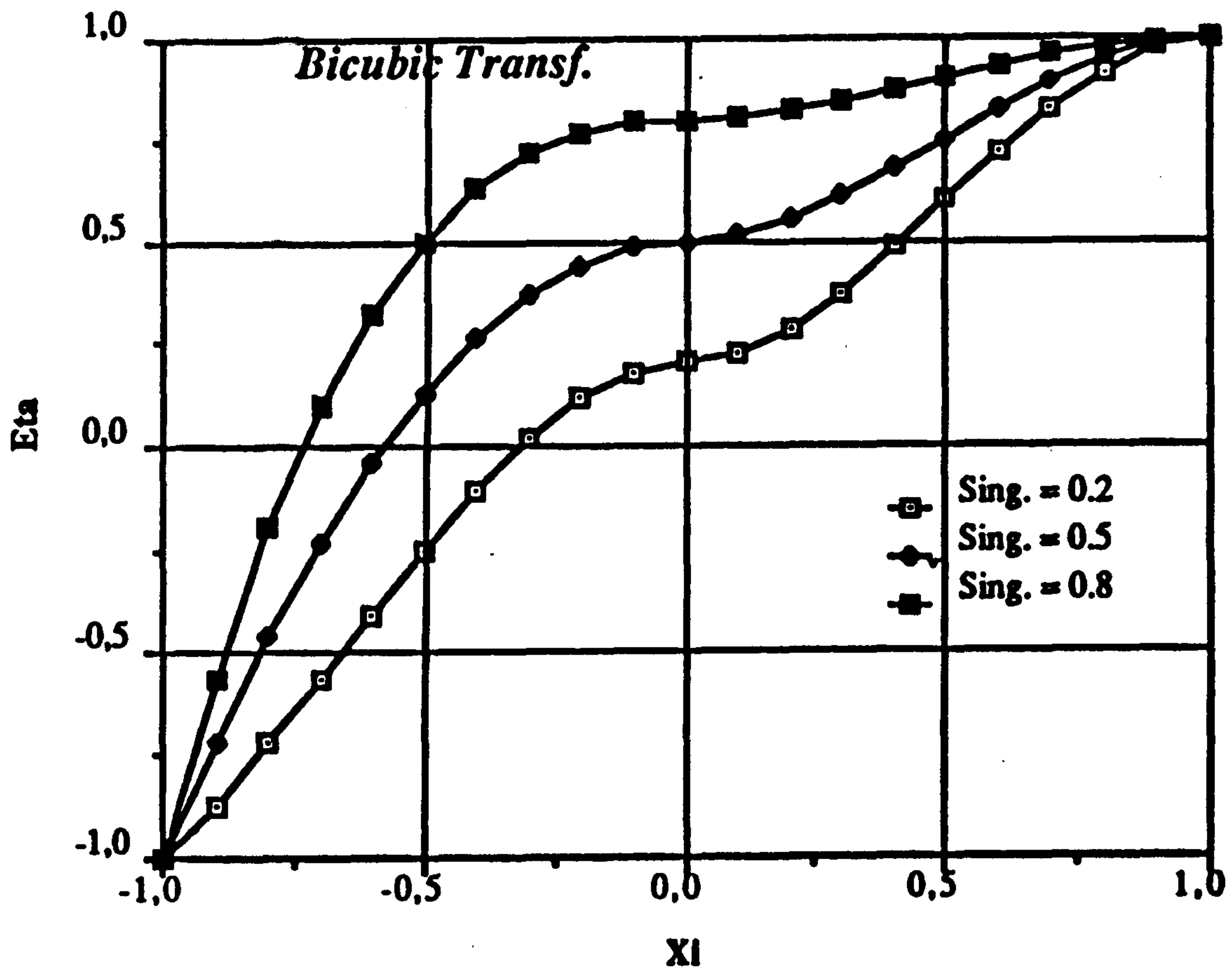

4th Degree Transf.
Sing. = 0.2
Sing. = 0.5
Sing. = 0.8
Eta
Xi
1,0
0,5
0,0
-0,5
-1,0
-1,0
-0,5
0,0
0,5
1,0


Bicubic Transf.
Sing. = 0.2
Sing. = 0.5
Sing. = 0.8
Eta
Xi
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0,5
0,0
-0,5
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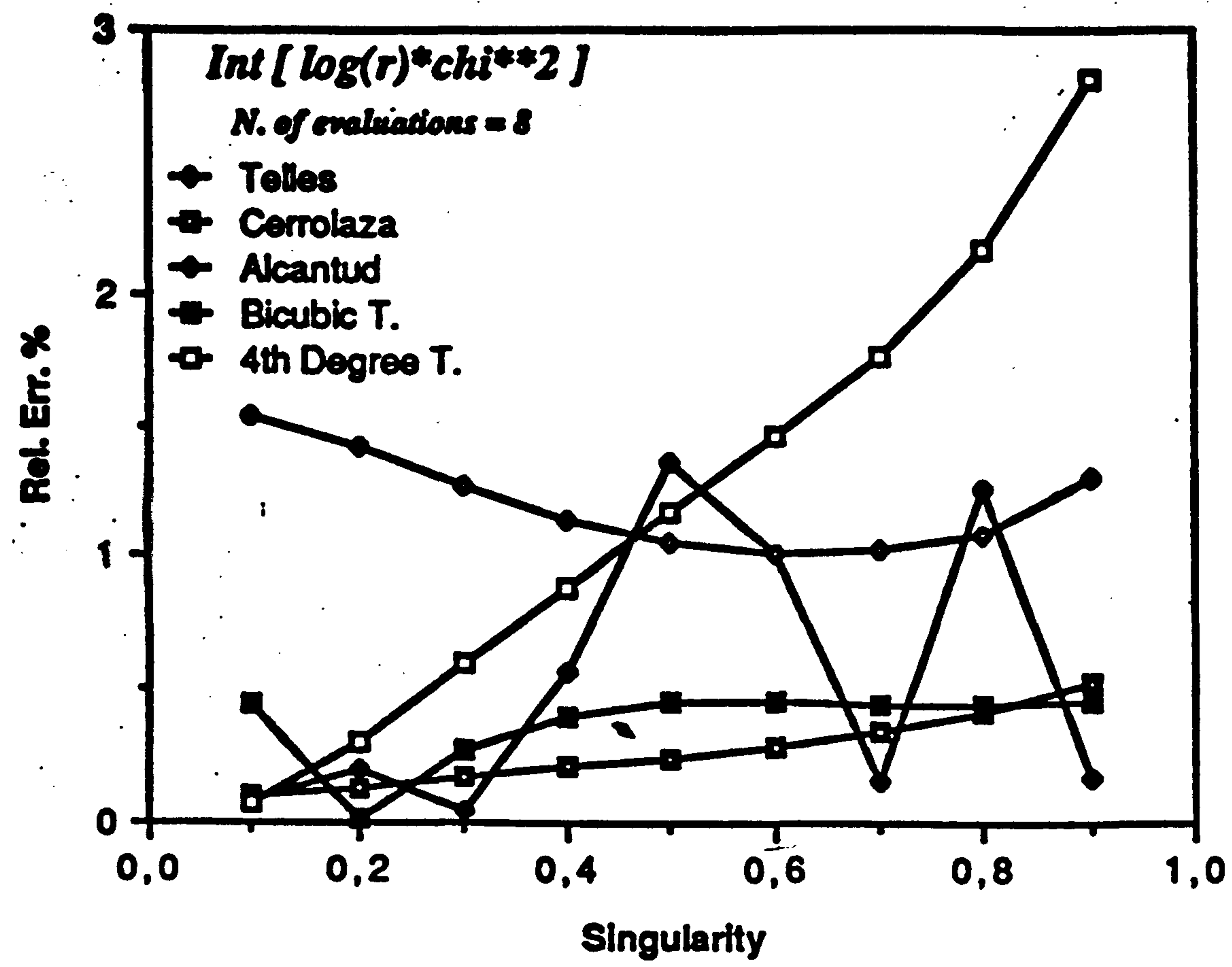
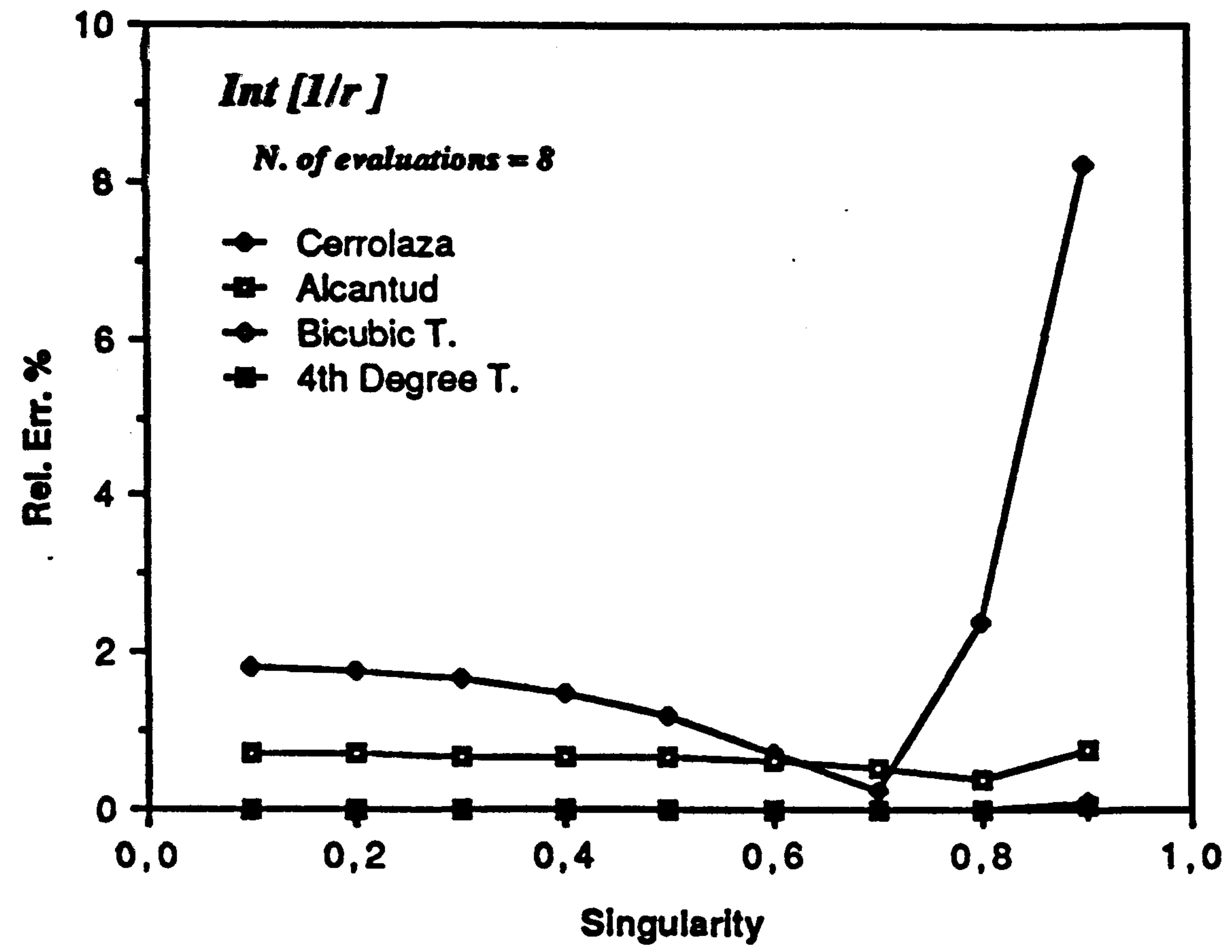
0,0
0,5
1,0


Int [log(r)*chi**2]
N. of evaluations = 8
Telles
Cerrolaza
Alcantud
Bicubic T.
4th Degree T.
Rel. Err. %
0
1
2
3
0,0
0,2
0,4
0,6
0,8
1,0
Singularity


Int [1/r]
N. of evaluations = 8
Cerrolaza
Alcantud
Bicubic T.
4th Degree T.
Rel. Err. %
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2
4
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10
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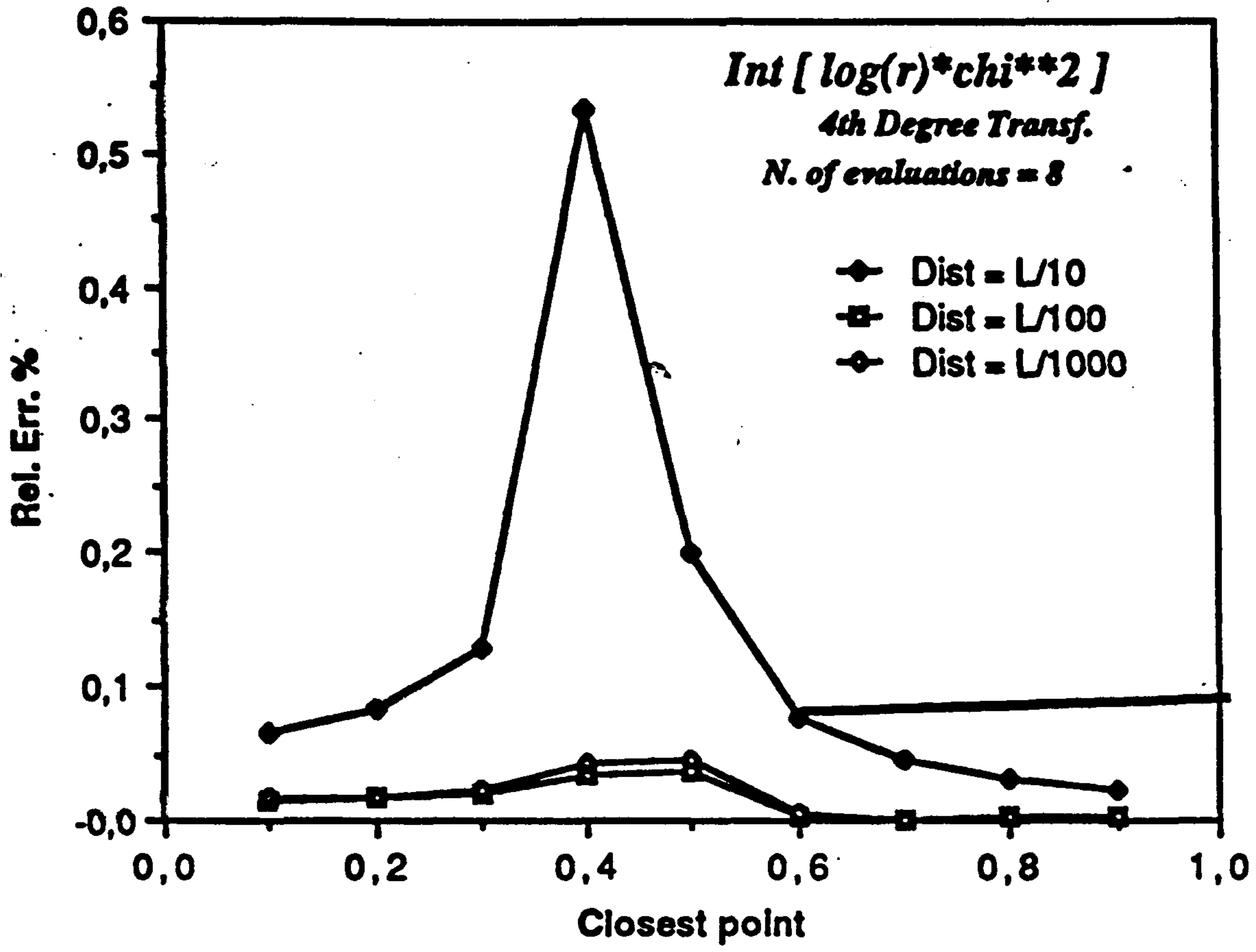
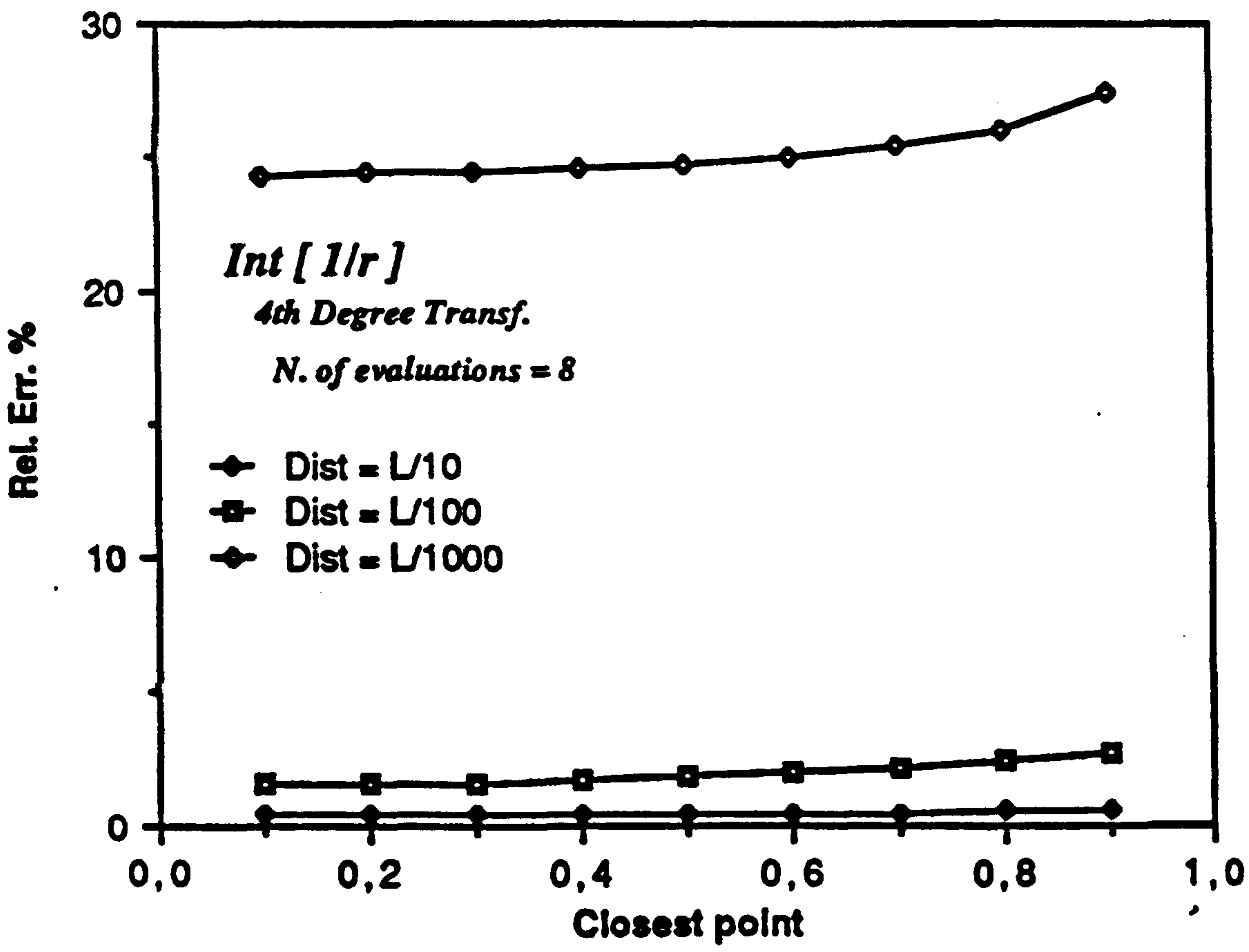
0,6
0,8
1,0
Singularity



Int [log(r)*chi**2]
4th Degree Transf.
N. of evaluations = 8
Dist = L/10
Dist = L/100
Dist = L/1000
Rel. Err. %
0,6
0,5
0,4
0,3
0,2
0,1
-0,0
0,0
0,2
0,4
0,6
0,8
1,0
Closest point



Int [1/r]
4th Degree Transf.
N. of evaluations = 8
Dist = L/10
Dist = L/100
Dist = L/1000
Rel. Err. %
30
20
10
0
0,0
0,2
0,4
0,6
0,8
1,0
Closest point