On the Degree of Team Cooperation in CD Grammar Systems

Fernando Arroyo , Juan Castellanos , and Victor Mitrana ⋆

Abstract. In this paper, we introduce a dynamical complexity measure, namely the *degree of team cooperation*, in the aim of investigating "how much" the components of a grammar system cooperate when forming a team in the process of generating terminal words. We present several results which strongly suggest that this measure is trivial in the sense that the degree of team cooperation of any language is bounded by a constant. Finally, we prove that the degree of team cooperation of a given cooperating/distributed grammar system cannot be algorithmically computed and discuss a decision problem.

1 Introduction

A cooperating grammar system, as introduced in [8] with motivations related to two level grammars, is a set of usual Chomsky grammars, which rewrite, in turn, the same sentential form. Initially, this is a common axiom. At each moment, a grammar is active, that means it is authorized to rewrite the common string, and the others are inactive. The conditions under which a component can become active or disabled and leaves the sentential form to the other components are specified by the cooperation protocol. The language of terminal strings generated in this way is the language generated by the system.

A rather intensive study of cooperating grammar systems has been started after relating them in [3] with artificial intelligence notions, such as the blackboard models in problem solving [10]. Along these lines, more conditions for components enabling and disabling were considered, namely step limitations (a component can work a prescribed number of steps, at least or at most a prescribed number), and the maximal competence strategy, similar in some extent

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to the stopping condition in [8]: a component must work as long as it can (in [8] a component must work until it introduces a non-terminal which cannot be rewritten by the same component). CD grammar systems working under the last mentioned strategy characterize one of the most important language classes in the L-systems area, namely the $ET0L$ language family [3]. The same strategy of cooperation is considered in [1], where modular grammars were introduced with motivations from the regulated rewriting area.

An important part of the theory of grammar systems is the theory of cooperation protocols; the focus is not on the generative capacity, but on the functioning of the systems, and on its influence on the generative capacity and on other specific properties. For a survey, the reader may want to consult [6].

In order to increase the power of such mechanisms, a simple and natural idea is to allow several components to become active during a time unit, see [7]. The sets of grammars which become active at each unit time, are called *teams*.

In [9], teams which rewrite strings in a synchronized manner, are considered: at each step when the team is working, each member of the team uses one of its rules. The teams considered in [9] differ essentially from the other types of teams already considered in [5,7,11], where the size of teams is a prescribed constant. In [9], the number of components in teams is not prescribed, moreover, at different steps the team that processes the sentential form is dynamically formed by components having the same *level of excitation*. More precisely, all components that can rewrite each nonterminal appearing in the sentential form constitute a team. This strategy increases considerably the computational power of CD grammar systems. Thus, important classes, e.g. $ET0L$ and the class of matrix grammars, are strictly included in the class of languages generated by teams in CD grammar systems [9]. It is worth mentioning work [2], where hybrid CD grammar systems with teams of different derivation modes, possibly of variable size and/or formed automatically were considered.

In this paper, we introduce a dynamical complexity measure, namely the *degree of team cooperation*, in the aim of investigating "how much" cooperate the components of a grammar system forming a team in the process of generating terminal words. We present several results that suggest that this measure is trivial in the sense that the degree of team cooperation of any language is bounded by a constant. These results are:

(i) The degree of team cooperation of every $ET0L$ language is 1.

(ii) Every language having the degree of team cooperation equal to 1 belongs to the class of languages generated by random context grammars with forbidding contexts only.

(iii) Every language generated by random context grammars with forbidding contexts only has the degree of team cooperation at most 2.

Finally, we consider a few computability and decidability issues. More precisely, we prove that the degree of team cooperation of a given CD grammar system is not algorithmically computable. We also show that deciding whether or not a team plays in a CD grammar system is algorithmically equivalent to the emptiness problem for the language generated by teams in a CD grammar system.

2 Preliminaries

The reader is referred to [14] for basic elements of formal language theory. If V is an alphabet then V^* is the set of all words over V. The empty word is denoted by λ and the set of all nonempty words is $V^+ = V^* - \{\lambda\}$. Denote by $\mid x \mid$ the length of $x \in V^*$ and by $(x)_U$ the string obtained from x by erasing all symbols that are not in U. For a finite set A, $card(A)$ denotes the number of elements in A.

A *cooperating distributed (shortly CD) grammar system* [3] is a construct :

$$\Gamma = (N, T, w_0, P_1, \ldots, P_n),$$

where N, T are disjoint finite alphabets, $w_0 \in (N \cup T)^*$, and $P_i, 1 \leq i \leq n$, are finite sets of context-free rules over $(N \cup T)^*$. These sets are called the *components* of Γ.

A *team* in a *CD* grammar system is a set $\mathcal{T} = \{P_{i_1}, \ldots, P_{i_m}\}, m \geq 1, i_j \in \{1, 2, \ldots, n\}, 1 \leq j \leq m$.

Let $\mathcal{T} = \{P_{i_1}, \ldots, P_{i_m}\}$ be a team and $x \in (N \cup T)^*$. Define the derivation relation :

$$x \Longrightarrow_{\mathcal{T}} y \text{ if and only if } \begin{array}{l} x = x_0 A_1 x_1 \ldots A_m x_m, x_k \in (N \cup T)^*, 0 \leq k \leq m, \\ y = x_0 y_1 x_1 \ldots y_m x_m, \text{ and } A_j \to y_j \in P_{i_j}, 1 \leq j \leq m. \end{array}$$

Observe that $\mathcal{T}$ is a set (though two components may be identical but they are identified by their names that are different) and, thus, the members P_{i_j} can be considered in any sequence. In other words, each member of a team rewrites exactly one nonterminal in an arbitrary order. In the sequel, we consider the following way of constituting dynamically a team. This is the so-called *total level of excitation* in [9].

Let $dom(P_i)$ be the set of all symbols in the left-hand side of the rules from P_i, i.e.,

$$dom(P_i) = \{A \mid A \to x \in P_i\}.$$

For a string $x \in (N \cup T)^*$ the *level of excitation* of $P_i, 1 \leq i \leq n$, with respect to x, is the maximal set of symbols from $dom(P_i)$ that appear in x. The level of excitation of P_i with respect to x is *total* if $(x)_N \in dom(P_i)^*$. The team consisting of all components which have a total level of excitation with respect to x, is denoted $\mathcal{T}_x$. Formally

$$\mathcal{T}_x = \{P_i | (x)_N \in (dom(P_i))^*\}$$

The language generated by teams in the CD grammar system $\Gamma = (N, T, w_0, P_1, \ldots, P_n)$ is

$$L(\Gamma) = \{w \in T^* \mid w_0 \Longrightarrow_{\mathcal{T}_{w_0}} w_1 \Longrightarrow_{\mathcal{T}_{w_1}} w_2 \Longrightarrow \ldots \Longrightarrow_{\mathcal{T}_{w_{m-1}}} w_m = w, m \geq 1\}.$$

Example 1. *Assume that* Γ_1 *is the following CD grammar system :*

$$\Gamma_1 = (\{S, A, B, C, X\}, \{a\}, S, P_1, P_2, P_3, P_4, P_5, P_6)$$

$$\begin{array}{ll} P_1 = \{S \to a, S \to aa, S \to AA\}, & P_2 = \{A \to BB, B \to B\} \\ P_3 = \{B \to C, C \to C\}, & P_4 = \{C \to A, C \to X, A \to A\} \\ P_5 = \{C \to A, C \to X, X \to X, X \to a\}, & P_6 = \{X \to a\} \end{array}$$

The language generated by teams in Γ_1 *is* $L(\Gamma_1) = \{a^{2^n} \mid n \geq 0\}$. *Let us list below a derivation for the word* a^4.

$$\begin{array}{rl} S & \Longrightarrow_{\{P_1\}} AA \Longrightarrow_{\{P_2,P_4\}} BBA \Longrightarrow_{\{P_2\}} BBBB \Longrightarrow_{\{P_2,P_3\}} \\ CBBB & \Longrightarrow_{\{P_3\}} CCBB \Longrightarrow_{\{P_3\}} CCCB \Longrightarrow_{\{P_3\}} CCCC \Longrightarrow_{\{P_3,P_4,P_5\}} \\ XXCC & \Longrightarrow_{\{P_5\}} XXXC \Longrightarrow_{\{P_5\}} XXXX \Longrightarrow_{\{P_5,P_6\}} \\ aXXX & \Longrightarrow_{\{P_5,P_6\}} aaXX \Longrightarrow_{\{P_5,P_6\}} aaaa \end{array}$$

Example 2. *Let* Γ_2 *be the CD grammar system :*

$$\Gamma_2 = (\{S, A\}, \{a, b\}, S, P_1, P_2, P_3)$$

$$P_1 = \{S \to AA\}, \qquad P_2 = P_3 = \{A \to aAb, A \to ab\}$$

Note that the language generated by teams in Γ_2 *is* $L(\Gamma_2) = \{a^n b^n a^n b^n \mid n \geq 1\}$. *Observe that the sentential form* $a^k A b^k a^k A b^k$ *may be rewritten by the team* $\{P_2, P_3\}$ *into* $a^{k+1} b^{k+1} a^{k+1} A b^{k+1}$, *but the derivation is blocked as in the next step the same team* $\{P_2, P_3\}$ *cannot be applied though it is activated by the sentential form. Furthermore, it is worth mentioning that two components of the aforementioned grammar system are identical and if we remove one of them, then the generated language is completely different which is not the case of usual grammar systems.*

As far as the generative capacity of these devices is concerned, in [9] it was proved that they are strictly stronger then the matrix grammars and at least as powerful as $ET0L$ systems.

Now, we are going to define a dynamical measure, namely the *degree of team cooperation*, with the aim of investigating "how much" the components of a grammar system cooperate when forming a team in the process of generating terminal words. Let $\Gamma = (N, T, w_0, P_1, P_2, \ldots, P_n)$ be a CD grammar system and w be a terminal string and D be the following derivation for w:

$$w_0 \Longrightarrow_{\mathcal{T}_{w_0}} w_1 \Longrightarrow_{\mathcal{T}_{w_1}} w_2 \Longrightarrow \ldots \Longrightarrow_{\mathcal{T}_{w_{m-1}}} w_m = w.$$

The degree of team cooperation in the derivation D for w is

$$Tcoop_\Gamma(w, D) = max\{card(\mathcal{T}_{w_i}) | 0 \leq i \leq m-1\},$$

while the degree of team cooperation in the generation of $x \in T^*$ is

$$Tcoop_\Gamma(x) = \begin{cases} min\{Tcoop_\Gamma(x, D) | D \text{ is a derivation for } x\}, \text{ if } x \in L(\Gamma), \\ 0, \text{ if } x \notin L(\Gamma) \end{cases}$$

Although we used the same name for the two mappings defined above, there is no risk of confusion as they have a different arity.

We further set

$$Tcoop(\Gamma) = max\{Tcoop_{\Gamma}(x) \mid x \in L(\Gamma)\}.$$

Clearly, if Γ is a CD grammar system of degree n, then $1 \leq Tcoop(\Gamma) \leq n$.

For instance, in the case of grammar system from Example 1 we have

$$Tcoop_{\Gamma_1}(a^{2^n}) = \begin{cases} 1, & \text{if } n \in \{0,1\} \\ 3, & \text{if } n \geq 2 \end{cases}$$

whilst in Example 2 we have $Tcoop_{\Gamma_2}(a^n b^n a^n b^n) = 2$, for any $n \geq 1$. Consequently, $Tcoop(\Gamma_1) = 3$ and $Tcoop(\Gamma_2) = 2$.

For a language L generated by teams in a CD grammar system, we define the degree of team cooperation of L by

$$TCOOP(L) = min\{Tcoop(\Gamma) \mid L = L(\Gamma)\}.$$

3 Is the Degree of Team Cooperation a Trivial Measure?

It is obvious that the degree of team cooperation of any context-free language is 1. A natural problem is whether the converse of this assertion is true, namely whether or not each language having the degree of team cooperation equal to 1 is context-free. The answer is negative which is not surprising. However, it is rather unexpected that the class of languages having the degree of team cooperation equal to 1 is very large. This class includes $ET0L$ which strongly suggests that this measure is trivial, that is there exists a natural number n such that $TCOOP(L) \leq n$ for any language L generated by teams in a CD grammar system. More surprisingly, this number seems to be 1.

We recall the following derivation mode in a CD grammar system. Given a CD grammar system $\Gamma = (N, T, w_0, P_1, P_2, \ldots, P_n)$, we denote by $\Longrightarrow_{P_i}$ the usual one step derivation with respect to P_i. Now we write $x \Longrightarrow^t_{P_i} y$ iff $x \Longrightarrow^*_{P_i} y$ and there is no $z \in (N \cup T)^*$ such that $y \Longrightarrow_{P_i} z$. The language generated by Γ with the t-mode of derivation is denoted by $L(\Gamma, t)$. It is known (see [1,3]) that the class of languages generated by CD grammar systems in the t-mode of derivation is exactly $ET0L$.

Theorem 3. *If L is an ET0L language, then $TCOOP(L) = 1$.*

Proof. Let L be the language generated by the CD grammar system $\Gamma = (N, T, w_0, P_1, P_2, \ldots, P_n)$ with the t-mode of derivation. Without loss of generality, we may assume that each successful derivation in Γ ends in P_n and that P_n is used exactly once in this derivation. We construct a new CD grammar system

$$\Gamma' = (N', T, w_0X, P_0, P'_1, P''_1, P'_2, P''_2, \ldots, P'_n, P''_n),$$

where X is a new symbol, $X \notin (N \cup T)$,

$$N' = N \cup \{X\} \cup \{X_i, X_i' \mid 1 \le i \le n\} \cup \{A_i, A_i' \mid A \in N, 1 \le i \le n-1\} \\ \cup \{A_n \mid A \in N\},$$

and the sets of rules are defined as follows:

$$\begin{aligned} P_0 &= \{X \to X_i \mid 1 \le i \le n\} \cup \{A \to A \mid A \in N\}, \\ P_i' &= \{X_i \to X_i, X_i \to X_i'\} \cup \{A \to A_i \mid A \in N\} \cup \{A_i \to A_i \mid A \in N\} \\ &\quad \cup \{A_j' \to A_i \mid A \in N, 1 \le j \ne i \le n-1\}, \\ &\quad \text{for all } 1 \le i \le n, \\ P_i'' &= \{X_i' \to X_i'\} \cup \{A_i \to h_i(\alpha) \mid A \to \alpha \in P_i,\ h_i \text{ is a morphism defined by} \\ &\quad h_i(Z) = \begin{cases} Z, & \text{if } Z \in T \\ Z_i, & \text{if } Z \in N \end{cases} \} \cup \{A_i \to A_i' \mid A \notin dom(P_i)\} \\ &\quad \cup \{A_i' \to A_i' \mid A \notin dom(P_i)\} \cup \{X_i' \to X_j \mid 1 \le j \ne i \le n\}, \\ &\quad \text{for all } 1 \le i \le n-1, \\ P_n'' &= \{X_n' \to \lambda, X_n' \to X_n'\} \cup \{A_n \to h_n(\alpha) \mid A \to \alpha \in P_n\}, \end{aligned}$$

where h_n is defined similarly to each morphism h_i.

Assume that

$$w_0 \Longrightarrow^t_{P_{i_1}} w_1 \Longrightarrow^t_{P_{i_2}} w_2 \Longrightarrow^t_{P_{i_3}} \dots \Longrightarrow^t_{P_{i_k}} w_k \Longrightarrow^t_{P_n} w$$

is a derivation in the t-mode of the terminal word w in Γ. We describe below how this derivation is simulated by teams in Γ'. One starts with $w_0X \Longrightarrow_{\{P_0\}} w_0X_{i_1}$. Now, the team formed by P_{i_1}' only will be iteratively activated until each occurrence of any nonterminal in w_0 is substituted by its copy with the index i_1. In other words,

$$w_0X \Longrightarrow_{\{P_0\}} w_0X_{i_1} \Longrightarrow^*_{\{P_{i_1}'\}} h_{i_1}(w_0)X_{i_1}'. \tag{1}$$

In the next derivation steps, the team $\{P_{i_1}''\}$ is to be activated until the sentential form $h_{i_1}(w_0)X_{i_1}'$ is transformed into $h_{i_1}'(w_1)X_{i_2}$, where each morphism h_j' is defined as follows:

$$h_j' : (T \cup (N \setminus dom(P_j)))^* \to (N' \cup T)^*, \quad h_j'(Z) = \begin{cases} Z, & \text{if } Z \in T, \\ Z_j', & \text{if } Z \in (N \setminus dom(P_j)). \end{cases}$$

The derivation continues with the team $\{P_{i_2}'\}$ that is used for several times until $h_{i_1}'(w_1)$ becomes $h_{i_2}(w_1)$. Hence, $h_{i_1}'(w_1)X_{i_2} \Longrightarrow^*_{\{P_{i_2}'\}} h_{i_2}(w_1)X_{i_2}'$. In conclusion, we have

$$\begin{aligned} w_0X &\Longrightarrow_{\{P_0\}} w_0X_{i_1} \Longrightarrow^*_{\{P_{i_1}'\}} h_{i_1}(w_0)X_{i_1}' \Longrightarrow^*_{\{P_{i_1}''\}} h_{i_1}'(w_1)X_{i_2} \\ &\Longrightarrow^*_{\{P_{i_2}'\}} h_{i_2}(w_1)X_{i_2}'. \end{aligned} \tag{2}$$

Inductively, the derivation continues with

$$h_{i_2}(w_1)X'_{i_2} \Longrightarrow^*_{\{P''_{i_2}\}} h'_{i_2}(w_2)X_{i_3} \Longrightarrow^*_{\{P'_{i_3}\}} \dots \Longrightarrow^*_{\{P'_{i_k}\}} h_{i_k}(w_{k-1})X'_{i_k}$$
$$\Longrightarrow^*_{\{P''_{i_k}\}} h'_{i_k}(w_k)X_n \Longrightarrow^*_{\{P'_n\}} h_n(w_k)X'_n \Longrightarrow^*_{\{P''_n\}} w. \qquad (3)$$

Consequently, $L(\Gamma, t) \subseteq L(\Gamma')$. From the above explanations it immediately follows that $Tcoop_{\Gamma'}(x) = 1$ for all $x \in L(\Gamma, t)$. It now suffices to prove the converse inclusion. To this aim, we analyze the other possible continuations at different steps of the above derivation. Here are two important observations that are very useful in the sequel:

(i) Due to the symbols X_i and X'_i, $1 \leq i \leq n$, there is no possibility to activate teams consisting of more than one component.
(ii) A sentential form containing X_i or X'_i will activate *at most* either the team $\{P'_i\}$ or $\{P''_i\}$, respectively.

Now, the strategy of a successful derivation is based on the following three principles:

(I) The symbol X_i of a sentential form cannot be replaced by X'_i until all nonterminals of the sentential form are indexed by i, otherwise the derivation is blocked.
(II) The symbol X'_i of a sentential form cannot be replaced by X_j until all rules from P''_i simulating rules from P_i are applied (this means that the sentential form does not contain any occurrence of A_i with $A \in dom(P_i)$) and all the other nonterminals of the sentential form (that is A_i with $A \notin dom(P_i)$) are substituted by their primed copies.
(III) By our assumption on Γ, as soon as the sentential form activates the team $\{P'_n\}$, the derivation goes to its end by iteratively activating $\{P''_n\}$.

By these explanations, one can easily infer that $L(\Gamma') \subseteq L(\Gamma, t)$ which completes the proof. □

A question that naturally arises is whether Theorem 1 can be extended to a characterization of the class $ET0L$. If this were the case, then there would exist languages having a degree of team cooperation bigger than 1, as it is known that $ET0L$ is strictly included in the class of languages generated by teams in CD grammar systems. We cannot answer this question, however we can indicate a class of languages that contains all languages having a degree of team cooperation equal to 1. This is the class of languages generated by random context grammars with forbidden contexts only [13]. It is known that this class strictly includes $ET0L$ [13] (see also [12,15] for an earlier proof of a stronger form of this statement). If we denote by $TCCD(k)$, $k \geq 1$, and fRC the class of languages having a degree of team cooperation at most k and languages generated by random context grammars with forbidden contexts only, respectively, our result can be stated as follows.

Theorem 4. *$ET0L \subseteq TCCD(1) \subseteq fRC$ and at least one of these inclusions is proper.*

Proof. By the aforementioned considerations, it suffices to prove the inclusion $TCCD(1) \subseteq fRC$. The construction is rather simple, but we first need to briefly recall the definition of a random context grammar with forbidding contexts only. Such a grammar is a construct $G = (N, T, S, P)$, where N, T, S are the classic parameters of a context-free grammar and P is a set of pairs of the form $(A \to x, Q)$ where $A \to x$ is a context-free rule and Q is a set of nonterminals. We say that the rule $(A \to x, Q)$ is applied in the one step derivation $\alpha \Longrightarrow \beta$ in G, if β is obtained from α by applying $A \to x$ as usual in a context-free grammar provided that $(\alpha)_N \cap Q = \emptyset$. In other words, the rule can be applied to α if no symbol from Q appears in α. The generated language by G is defined as usual.

Let $\Gamma = (N, T, w_0, P_1, P_2, \ldots, P_n)$ be a CD grammar system with $Tcoop(\Gamma) = 1$. We construct the random context grammar with forbidding contexts only $G = (N \cup \{S\}, T, S, P)$, where S is a new symbol, $S \notin (N \cup T)$, and P is defined as follows:

$$P = \{S \to w_0\} \cup \bigcup_{i=1}^{n} \{(A \to x, N \setminus dom(P_i)) \mid A \to x \in P_i\}.$$

The fact that G and Γ generate the same language is immediate. □

We finish this section by completing the picture we have emphasized so far with a final result.

Theorem 5. *$fRC \subseteq TCCD(2)$.*

Proof. Let $G = (N, T, S, P)$ be a random context grammar with forbidding contexts only. Assume that $P = \{r_1, r_2, \ldots, r_n\}$ for some $n \geq 1$. We construct the CD grammar system

$$\Gamma = (N', T, SX, P_0, P_1^1, P_1^2, P_1^3, P_2^1, P_2^2, P_2^3, \ldots, P_n^1, P_n^2, P_n^3, P_0'),$$

where the set of nonterminals is

$$N' = \{S, X\} \cup \{X_i, X_i' \mid 1 \leq i \leq n\} \cup \{A_i \mid A \in N, 1 \leq i \leq n\},$$

and the components are defined in the following way:

$$\begin{aligned}
P_0 &= \{X \to X_i \mid 1 \leq i \leq n\} \cup \{S \to S\},\\
P_i^1 &= \{X_i \to X_i, X_i \to X_i'\} \cup \{A \to A_i \mid A \in N\} \cup \{A_i \to A_i \mid A \in N\}\\
&\quad \cup \{A_j \to A_i \mid A \in N, 1 \leq j \neq i \leq n\},\\
&\quad \text{for all } 1 \leq i \leq n,\\
P_i^2 &= \{X_i' \to X_i'\} \cup \{A_i \to h_i(\alpha) \mid r_i = (A \to \alpha, Q) \in P,\ h_i \text{ is a morphism}\\
&\quad \text{defined by } h_i(Z) = \begin{cases} Z, & \text{if } Z \in T \\ Z_i, & \text{if } Z \in N \end{cases}\} \cup \{B_i \to B_i \mid B \in N \setminus (Q \cup \{A\})\},\\
&\quad \text{for all } 1 \leq i \leq n,\\
P_i^3 &= \{X_i' \to X_j \mid 1 \leq j \neq i \leq n\} \cup \{A_i \to A_i \mid A \in N\},\ \text{for all } 1 \leq i \leq n,\\
P_0' &= \{X_i \to \lambda \mid 1 \leq i \leq n\}.
\end{aligned}$$

The argument for proving $L(G) = L(\Gamma)$ is rather similar to that from the proof of Theorem 1. More precisely, if

$$S \Longrightarrow_{r_{i_1}} w_1 \Longrightarrow_{r_{i_2}} w_2 \Longrightarrow_{r_{i_3}} \ldots \Longrightarrow_{r_{i_k}} w_k = w \in T^*,$$

for some $k \geq 1$, is a derivation in G, then the following derivation is possible in Γ:

$$\begin{aligned} SX &\Longrightarrow_{\{P_0\}} SX_{i_1} \Longrightarrow^*_{\{P^1_{i_1}\}} S_{i_1}X'_{i_1} \Longrightarrow_{\{P^2_{i_1}, P^3_{i_1}\}} h_{i_1}(w_1)X_{i_2} \\ &\Longrightarrow^*_{\{P^1_{i_2}\}} h_{i_2}(w_1)X'_{i_2} \Longrightarrow_{\{P^2_{i_2}, P^3_{i_2}\}} h_{i_2}(w_2)X_{i_3} \\ &\Longrightarrow^*_{\{P^1_{i_3}\}} \ldots \Longrightarrow^*_{\{P^1_{i_k}\}} h_{i_k}(w_{k-1})X'_{i_k} \Longrightarrow_{\{P^2_{i_k}, P^3_{i_k}\}} w_kX_j \Longrightarrow^*_{\{P'_0\}} w_k, \end{aligned}$$

for some $1 \leq j \neq i_k \leq n$.
Therefore, $L(G) \subseteq L(\Gamma)$ holds. Furthermore, $Tcoop_\Gamma(x) \leq 2$ for every $x \in L(G)$.

A slightly modified version of the discussion from the second part of the proof of Theorem 1 holds for proving the inclusion $L(\Gamma) \subseteq L(G)$. It is worth mentioning that the construction from the proof of Theorem 1 cannot be used in this case because that construction cannot cope with the situation when x from a rule $(A \to x, Q) \in P$ contains a nonterminal in Q. □

The following problems remain open:

1. Which of the inclusions in the statement of Theorem 2 is proper?
2. Are there languages having a degree of team cooperation larger than 1?
3. If the answer of the previous problem is affirmative, then is the degree of team cooperation a non-trivial measure? If this is not the case, what is the maximal degree of team cooperation?
4. Is the degree of team cooperation a connected measure, that is for any natural n does there exist a language L_n with $TCOOP(L_n) = n$?

4 Computability/Decidability Issues

This section is devoted to some computability and decidability issues. We first investigate the possibility of computing the degree of team cooperation of a CD grammar system.

Theorem 6. *Given a CD grammar system Γ, $Tcoop(\Gamma)$ fails to be algorithmically computable.*

Proof. Let

$$x = (x_1, x_2, \ldots, x_n)$$
$$y = (y_1, y_2, \ldots, y_n)$$

be an instance of the Post Correspondence Problem (PCP) over the alphabet V. Let further $G = (N, V \cup \{c\}, S, P)$, $c \notin (N \cup V)$, be a context-free grammar generating the language $L = \{vccw \mid v, w \in V^+, v \neq w^R\}$. We now construct the CD grammar system $\Gamma = (N \cup \{S_0, X, Y, Z\}, V \cup \{c\}, S_0, P_1, P_2, P_3, P_4)$, where

$$P_1 = P \cup \{S_0 \to S, S_0 \to X\},$$
$$P_2 = \{X \to x_i X y_i^R \mid 1 \leq i \leq n\} \cup \{X \to x_i Y Z y_i^R \mid 1 \leq i \leq n\},$$
$$P_3 = \{Y \to c, Z \to Z\}, \qquad P_4 = \{Z \to c, Y \to Y\}.$$

Clearly, $L(\Gamma) = L(G)$ if and only if $Tcoop(\Gamma) = 1$ if and only if PCP for the instance (x, y) has no solution. Therefore, one cannot compute $Tcoop(\Gamma)$. □

Another problem of interest is to investigate the possibility of computing the degree of team cooperation in the generation of a terminal word in a given CD grammar system. We do not have a solution, but a related problem is: Given a CD grammar system Γ and a terminal word x, is it decidable whether or not a team $\mathcal{T}$ is ever activated in a derivation of x in Γ? A first step towards a solution to this problem is presented in what follows.

Let $\Gamma = (N, T, w_0, P_1, \ldots, P_n)$ be a CD grammar system and let $\mathcal{T} = \{P_{i_1}, \ldots, P_{i_m}\}$ be a team in Γ. We say that $\mathcal{T}$ *"plays"* if there are $x, y \in (N \cup T)^*$, such that

$$w_0 \Longrightarrow^* x \Longrightarrow_{\mathcal{T}} y.$$

We say that a team $\mathcal{T}$ *plays successfully* if there are x, y, z with $x, y \in (N \cup T)^*, z \in T^*$, such that

$$w_0 \Longrightarrow^* x \Longrightarrow_{\mathcal{T}} y \Longrightarrow^* z.$$

Proposition 7. *If the problem "Does a team play successfully?" is decidable, then the emptiness problem for languages generated by teams in CD grammar systems is decidable.*

Proof. Let $\Gamma = (N, T, w_0, P_1, \ldots, P_n)$ be a CD grammar system with $T = \{a_1, a_2, \ldots, a_t\}$; we construct the CD grammar system

$$\Gamma' = (N', T, \bar{w}_0 Z, P'_1, \ldots, P'_n, P'_{n+1}, P'_{n+2})$$

with $N' = N \cup N'' \cup \{Z, Z', Y, \Lambda\}$, where $N'' = \{A_i \mid 1 \leq i \leq t\}$, Z, Z', Y, Λ are new symbols, and the components P'_j are defined as follows:

$$P'_j = \bar{P}_j \cup \{A_i \to A_i \mid 1 \leq i \leq t\} \cup \{\Lambda \to \Lambda, Z \to Z, Z \to Z'\}, 1 \leq j \leq n,$$

where $\bar{P}_j$ is P_j in which the terminals a_i are renamed by A_i and λ is renamed by Λ. Further,

$$P'_{n+1} = \{A_i \to a_i \mid 1 \leq i \leq t\} \cup \{\Lambda \to \lambda, Z' \to Z', Z' \to Y\}, \quad P'_{n+2} = \{Y \to \lambda\}.$$

If $x \in (N \cup T)^*$, then $\bar{x}$ denotes the word obtained from x by renaming each terminal symbol a_i by A_i, $1 \leq i \leq t$, and leaving unchanged the nonterminals. Let $\mathcal{T} = \{P'_{n+1}\}$ be a team in Γ'.

It is easy to prove that if $w_0 \Longrightarrow_{\Gamma}^{*} w$, then $\bar{w}_0 Z \Longrightarrow_{\Gamma'}^{*} \bar{w} Z'$ and if $\bar{w}_0 Z \Longrightarrow_{\Gamma'}^{*} X_1 X_2 \ldots X_m Z'$, $X_i \in N'$, $i = 1, \ldots, m$, then $w_0 \Longrightarrow_{\Gamma}^{*} X_1' \ldots X_m'$, where

$$X_j' = \begin{cases} X_j, \text{ if } X_j \in N, \\ a_i, \text{ if } X_j = A_i \in N", \\ \lambda, \text{ if } X_j = \Lambda. \end{cases}$$

Note that, if the team $\mathcal{T}$ plays in Γ' (it actually plays successfully), then there exists a sentential form $\bar{u}Z'$ with $\bar{u} \in (N" \cup \{\Lambda\})^*$ such that $\bar{w}_0 Z \Longrightarrow^{*} \bar{u} Z'$ in Γ'. Therefore, it follows that $w_0 \Longrightarrow_{\Gamma}^{*} u$, $u \in T^*$. Consequently, $L(\Gamma) \neq \emptyset$ holds.

Conversely, assume that $L(\Gamma) \neq \emptyset$. Hence, there is a $w \in T^*$ such that $w_0 \Longrightarrow_{\Gamma}^{*} w$. Consequently, we obtain that $\bar{w}_0 Z \Longrightarrow_{\Gamma'}^{*} \bar{w} Z'$. Note that $\bar{w} \in (N" \cup \{\Lambda\})^*$ and therefore the team $\mathcal{T}$ can be applied to $\bar{w}$. Thus, $\mathcal{T}$ plays successfully in Γ' as the derivation successfully ends in the next step after $\mathcal{T}$ is disabled. Consequently, $\mathcal{T}$ plays successfully in Γ' if and only if $L(\Gamma) \neq \emptyset$. □

Is the converse true as well? In other words, is the decidability status of the problem "Does a team play successfully?" the same as that of the emptiness problem for languages generated by teams in CD grammar systems? We give below a proof for a weaker variant of this statement, namely the word "successfully" is removed.

Proposition 8. *If the emptiness problem for languages generated by teams in CD grammar systems is decidable, then the problem "Does a team play?" is decidable.*

Proof. Let $\Gamma = (N, T, w_0, P_1, \ldots, P_n)$ be a CD grammar system and let $\mathcal{T} = \{P_{i_1}, \ldots, P_{i_r}\}$ be a fixed team in Γ. In order to solve the problem if $\mathcal{T}$ plays in Γ, we consider the CD grammar system Γ',

$$\Gamma' = (N', T', w_0' Z, P_1', \ldots, P_n', P_{n+1}', P_{n+2}'),$$

where $N' = N \cup \{Z, Z', Y\}$ and T' is a disjoint copy of N, $T' = \{X' \mid X \in N\}$. The components of Γ' are:

$$P_i' = \{X \to \varphi(\alpha) \mid X \to \alpha \in P_i\} \cup \{X \to X' \mid X \in dom(P_i)\} \cup \{Z \to Z, Z \to Z'\},$$

for all $1 \leq i \leq n$, where φ is the morphism that erases all symbols from T and preserves symbols from N and

$$P_{n+1}' = \{X \to X' \mid X \in N\} \cup \{Z' \to Z', Z' \to Y\}, \qquad P_{n+2}' = \{Y \to \lambda\}.$$

Note that: $L(\Gamma') = \{\psi(u) \mid w_0 \Longrightarrow_{\Gamma}^{*} u\}$, where, $\psi(X) = \begin{cases} \lambda, \text{ if } X \in T, \\ X', \text{ if } X \in N. \end{cases}$ Now, for any nonempty subset D, $D \subseteq dom(P_{i_1}) \cap \cdots \cap dom(P_{i_r})$ consider the set $D' = \{X' \mid X \in D\}$ and the regular language $R_{\mathcal{T}}$:

$$R_{\mathcal{T}} = (D')^* - (\bigcup_{i=0}^{r-1} (D')^i) = \{v \in (D')^* \ : \mid v \mid \geq r\}.$$

Now, it is easy to notice that the team $\mathcal{T}$ plays if and only if

$$L_{\mathcal{T}} = (L(\Gamma') \cap R_{\mathcal{T}}) \neq \emptyset.$$

It is an easy exercise to show that $L_{\mathcal{T}}$ can be generated by teams in a CD grammar system. Thus, if we can decide whether or not $L_{\mathcal{T}}$ is empty, then we can decide whether or not $\mathcal{T}$ plays. □

Although the emptiness for $ET0L$ is decidable we do not know the decidability status of the emptiness problem for languages generated by teams in CD grammar systems.

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