Self-similar motion of laser fusion plasmas. Absorption in an unbounded plasma

A. Barrero and Juan R. Sanmartín

Escuela Técnica Superior de Ingenieros Aeronauticos, Universidad Politécnica de Madrid, Madrid, Spain

(Received 28 July 1976, final manuscript received 21 January 1977)

The one-dimensional motion generated in a cold, infinite, uniform plasma of density \( n_0 \) by the absorption, in a certain plane, of a linear pulse of energy per unit time and area \( \Phi = \Phi_0 \tau \), \( 0 < \tau < \tau_0 \), is considered. The analysis allows for thermal conduction and viscosity of ions and electrons, their energy exchange, and an electron heat flux limiter. The resulting motion is self-similar and governed by a single nondimensional parameter \( \alpha \equiv (n_0 \tau^2/\Phi_0)^{1/2} \). Detailed asymptotic results are obtained for both \( \alpha < 1 \) and \( \alpha > 1 \), the general behavior of the solution for arbitrary \( \alpha \) is discussed. The analysis can be extended to the case of a plasma initially occupying a half-space, and throws light on how to optimize the hydrodynamics of laser fusion plasmas. Known approximate results corresponding to motion of a plasma submitted to constant irradiation \( (\Phi) \) are recovered in the present work under appropriate limiting processes.

I. INTRODUCTION

To achieve break-even conditions in microfusion, a laser pulse must compress a DT pellet to densities well above that of the solid state, \( n_s \), and strongly heat its core.\(^1\) This starts an outward burn wave that both minimizes the energy requirements and increases the burn efficiency.\(^2\)

The cold, dense plasma around the core has a low specific entropy; consequently, the absorption of laser radiation must take place outside the dense pellet, since entropy production is associated with the absorption. This condition is met if the critical plasma density, \( n_{cr} \), at the laser frequency, \( \omega \), is smaller than \( n_s \). Then, the radiation energy is deposited in an expanding corona of hot and rarefied plasma, that results from ablation of the pellet.\(^3\)

Clearly, hydrodynamics must play an essential part in microfusion since it affects the attainment of the following goals: (a) the energy lost in the corona outflow should be held to a minimum; (b) entropy, and therefore entropy production is associated with the absorption. Then, the radiation energy is deposited in an expanding corona of hot and rarefied plasma, that results from ablation of the pellet.\(^3\)

In order to clarify how this behavior depends on the laser design parameters (maximum power flux \( \phi_0 \), pulse duration \( \tau \), and frequency \( \omega \)) we consider the quasi-neutral motion of an initially cold and uniform plasma, allowing for viscosity and heat conduction of both electrons and ions, their energy exchange, and an electron heat flux limiter; other effects, like pressure radiation, nuclear fusion, emission (and re-absorption) of radiation, and non-Maxwellian distribution functions (a point discussed later) are not taken into account. Then, a uniform power flux, linear in time (anomalously), absorbed in a certain plane, produces a self-similar motion. Anisimov\(^4\) noticed the existence of self-similar motion including electron conduction and ion-electron heating and Marshak\(^5\) pointed out self-similar motion of a gas with radiation heat conduction. We consider an unbounded plasma, as a step prior to the more difficult analysis of a half-space plasma, the case of interest for laser fusion plasmas, which we carry out in a forthcoming paper (any qualitative results found there should be valid for spherical geometries, except at the core of the pellet).

In Sec. II we discuss the equations and physics of the problem. Section III introduces both self-similar variables and nondimensional parameters. The resulting system of equations is analyzed in Secs. IV and V (Appendices A and B include some related mathematical details). Finally, the results obtained are summarized and discussed in Sec. VI.

II. STATEMENT OF THE PROBLEM

We consider a fully ionized, single ion-species plasma at rest, with uniform density \( n_0 \). At \( t = 0 \), energy per unit time and unit area \( \phi(t) \) starts being deposited at the plane \( x = 0 \), and the plasma becomes one-dimensional. The macroscopic equations of continuity, momentum, and entropy for species \( j \) (\( e \), electrons, \( i \), ions) are

\[
\frac{Dn_j}{Dt} = -n_j \frac{\partial v_j}{\partial x}, \quad \left( \frac{D}{Dt} \frac{\partial}{\partial x} + v_j \frac{\partial}{\partial x} \right), \quad (1)
\]

\[
m_j \frac{Dv_j}{Dt} = -\frac{\partial}{\partial x} (n_j k T_j) + \frac{4}{3} \frac{\partial}{\partial x} \left( \mu_j \frac{\partial v_j}{\partial x} \right) + q_j n_0 E + R_j, \quad (2)
\]

\[
n_j T_j \frac{D}{Dt} \left( \frac{T_j}{n_j} \right) = \frac{\partial}{\partial x} \left( \chi_j K_j \frac{\partial T_j}{\partial x} \right) + \frac{4}{3} \mu_j \left( \frac{\partial v_j}{\partial x} \right)^2 + H_j, \quad (3)
\]

where \( m, q, n, \mu, \) and \( T \) are particle mass and charge, density, macroscopic velocity, and temperature, respectively, and \( \mu \) and \( K \) are the classical coefficients of viscosity and thermal conduction\(^6\),\(^7\)

\[
\mu_j = \mu_j T_j^{1/2}, \quad K_j = K_j T_j^{1/2}. \quad (4)
\]
Both  \( \bar{\mu} \) and  \( \bar{K} \) depend weakly on  \( T \) and  \( n \) through Coulomb logarithms.

We have assumed a collision-dominated plasma; that is, at each time  \( t \) the mean free path  \( \lambda \) and the time between collisions, for each species, are assumed to be much less than the length of plasma disturbed,  \( x(t) \) and  \( t \), respectively; the conditions implied by this hypothesis will be discussed in Sec. III. Nonetheless, a flux limiter\textsuperscript{15,16} is included in the electron heat conduction to approximately deal with situations where these conditions are not entirely met (viscous effects, for electrons, are much less important than thermal conduction even though viscous and thermal diffusivities are comparable, because, as will be shown,  \( v_e \approx v_i \) and therefore,  \( m_e v_e^2 \ll kT_e \). On the other hand, since  \( \lambda_e \ll T_e \) and  \( x_e \), the solution of the system (6), (7), and (3), with Eqs. (8) and conditions (10)–(12), while the pulse is on, is self-similar (as shown in the next section) as long as the Coulomb logarithms in Eqs. (4), (5), and (9) can be approximated by constants.

### III. SELF-SIMILAR VARIABLES AND EQUATIONS

Let
\[
\varphi(t) = \varphi_0 (t/\tau)^p \quad 0 < t < \tau
\]
and define
\[
\xi = x/[\omega(t/\tau)]^q, \quad (n_e, t) = n_0 \varphi(\xi),
\]

Introducing (13) and (14) into the equations of the last section, the powers of  \( t \) are found to drop out when
\[
p = 1, \quad q = \frac{1}{2}, \quad r = \frac{1}{5}, \quad s = \frac{3}{5}.
\]

Now, since  \( w, v_0, \) and  \( T_0 \) may be chosen arbitrarily, we set
\[
\frac{v_0}{w} = 3, \quad \frac{K_0}{\bar{K} E} = 1, \quad \frac{R_0}{\bar{R} E} = 1, \quad \frac{\varphi_0 \omega t}{\bar{\mu} \omega w},
\]
in order to simplify the continuity equation in the usual way, and take into account the fact that both energy deposition and electron heat conduction lie at the root of the phenomenon studied. Then, defining
\[
\alpha = 3kT_0/m_1 \varphi_0 w,
\]
we can compute all nondimensional coefficients in the equations in terms of  \( \alpha \) and  \( A_1 \) (the ion mass number) from the known values of  \( \bar{K} \),  \( \bar{R} \), and  \( \bar{E} \).

We thus arrive at the following system:

\[
\frac{d}{d\xi} \left[ \frac{\bar{K} E}{k T_0} \frac{d}{d\xi} \right] = \frac{\alpha}{\bar{K} E} \frac{d}{d\xi} \left[ \bar{K} E \theta_a + \varphi(\xi) \right]
\]

\[
\left( \frac{0.039}{\bar{A} \bar{E} T_0^{5/2}} + \frac{0.00049}{\bar{A} \bar{E} T_0^{7/2}} \right) \frac{d}{d\xi}, \quad (16a)
\]

\[
\frac{d}{d\xi} \left[ \frac{\bar{A} \bar{E} T_0^{5/2}}{k T_0} \frac{d}{d\xi} \right] = \frac{\alpha}{\bar{K} E} \frac{d}{d\xi} \left[ \bar{K} E \theta_a + \varphi(\xi) \right]
\]

\[
\left( \frac{0.039}{\bar{A} \bar{E} T_0^{5/2}} + \frac{0.00049}{\bar{A} \bar{E} T_0^{7/2}} \right) \frac{d}{d\xi}, \quad (16b)
\]
\[ \alpha = \frac{9k}{4m_i} \left( \frac{\beta \nu_0}{\nu} \right) \left( \epsilon_{\text{r}} \right)^{12} \sim 12.9 \left( \frac{\beta \nu_0}{\nu} \right) \left( \epsilon_{\text{r}} \right)^{12}, \]  

(23a)

\[ \nu_0 = \frac{4}{3 \beta \nu_0} \left( \frac{\nu}{\epsilon_{\text{r}}} \right) \approx 1.73 \times 10^7 \left( \frac{\nu}{\epsilon_{\text{r}}} \right) \text{ cm/sec}, \]  

(23b)

\[ T_0 = \left( \frac{\beta \nu_0}{\epsilon_{\text{r}} \kappa_1 \kappa_2} \right)^{1/4} \approx 1.05 \left( \frac{\beta \nu_0}{\epsilon_{\text{r}}} \right)^{1/4} \text{ keV}. \]  

(23c)

The energy deposited in the plasma per unit area is

\[ \int_0^x \phi_0(t/\tau) dt \approx 50 \text{ kJ/cm}^2. \]  

For a sphere of radius 500 \( \mu \), that energy per area leads to a total energy

\[ E \approx 1.57 \text{ kJ}. \]  

The self-similar variables make a discussion of the approximations used in Sec. II easy. First, Eq. (17) shows that

\[ \chi_{\epsilon} \approx 1 \text{ if } \left( \frac{m_i}{m} \right)^{1/2} \ll \frac{1}{n}, \]  

(24)

From the results in the following sections it may be shown that (24) is only violated for \( \alpha \) comparable to \( m_i/m \) or less; thus, \( \chi_{\epsilon} \approx 1 \), if \( \alpha \gg m_i/m \). A similar condition results from the requirement \( \lambda \ll x_f \), since \( \lambda \sim 1/T \) \( (m_i/kT)^{1/2} \), we obtain, for electrons,

\[ \left( \frac{m_i}{m} \right)^{1/2} \ll \frac{1}{n}. \]  

Likewise, the time between electron collisions is much smaller than \( t \) if

\[ \alpha \ll \frac{1}{n}, \]  

the corresponding inequalities for ions are found to be less restrictive.

Finally, the quasi-neutrality condition, \( \lambda_0 \sim \left( kT_e/4\pi n e^2 \right)^{1/2} \ll x_f \), written in self-similar variables becomes

\[ \left( \frac{\beta \nu_0}{\nu} \right)^{1/3} \ll \left( \omega_{\text{pl}}/\kappa_1 \right)t/\tau, \]  

(25)

where \( \omega_{\text{pl}} \) is the ion plasma frequency. It is thus clear that charge separation is not self-similar. Using expressions (21)–(23a), we obtain, from (25),

\[ \left( \frac{\beta \nu_0}{\nu} \right)^{1/3} \ll \frac{A_{1/8}^{1/3}}{n^{1/6}} \left( \frac{t}{\tau} \right)^{1/3} \ll 10^5 \frac{t}{\tau}. \]  

The results for \( \theta_{\epsilon}, \beta, \) and \( \epsilon_f \) to be found later show that this inequality is easily satisfied for all reasonable values of \( \alpha, A_{1/8}, \beta, \) and \( \epsilon_f \).

IV. ASYMPTOTIC SOLUTION FOR \( \alpha \ll 1 \)

Expanding all variables in powers of \( \alpha \)

\[ n = n_0 + \alpha n_1 + \cdots, \]  

\[ \phi = \phi_0 + \alpha \phi_1 + \cdots, \]  

\[ \epsilon_f = \epsilon_{f0} + \alpha \epsilon_{f1} + \cdots, \]  

Eqs. (16) yield, to lowest order,

\[ \frac{n}{n_0} = 1, \quad \frac{\phi}{\phi_0} = 0. \]  

(21)
and, assuming 1 ≫ \alpha ≫ m_e/m_i (x_e = 1 except in a narrow neighborhood of \xi_f),

\[\theta_{el} - 2\xi \frac{d\theta_{el}}{d\xi} = \frac{1}{\xi} \left( \phi_{el} \frac{d\theta_{el}}{d\xi} \right)\]  

(26)

with the boundary conditions

\[\theta_{el}(\xi_0) = 0, \quad \frac{d\theta_{el}}{d\xi}|_{\xi=0} = -\frac{1}{\xi} .\]

Equation (26) represents a self-similar thermal wave. It may be shown that there exists a value, \xi_f, such that \theta_{el} = 0 for \xi > \xi_f. Then, \theta_{el}(\xi_f) = 0, \xi_f being given by

\[\int_0^{\xi_f} \theta_{el} d\xi = \frac{1}{6} .\]

this condition results from integrating (26) between 0 and \xi and making the heat flux vanish at \xi_f. Figure 1 shows \theta_{el}(\xi) as given both by a numerical computation and by an approximate, integral method solution.

\[\theta_{el}(\xi) = 0.76(1 - \xi/0.31)^{0.41}, \quad \xi < 0.31;\]

actually, \theta_{el} behaves as \((\xi - \xi_f)^{0.4}\) near \xi_f.

The equations for \bar{n}_2, u_2, and \theta_{i2}, neglecting viscosities and ion conduction, are

\[d\bar{n}_2/d\xi = \xi^{-1} du_2/d\xi,\]

\[u_2 - 4\xi du_2/d\xi = -d\theta_{i2}/d\xi,\]

\[\theta_{i2} - 2\xi d\theta_{i2}/d\xi = 4.3\theta_{el}^{-1/2},\]

(27)

FIG. 2. Dimensionless velocity, \bar{u}_2, vs dimensionless distance, \xi, for \alpha ≪ 1. [The insert represents u_2 vs \xi in the inner layer, \xi = 0 (\alpha/3)].

A. Inner layer

Defining new, stretched variables

\[\hat{\xi} = \xi/\alpha^{1/3}\theta_{el}(0)^{1/2}, \quad \hat{\bar{u}} = -\bar{u}_2/\theta_{el}(0),\]

Eq. (16b) becomes

\[\hat{\bar{u}} - 1 = (4\hat{\xi} - 2^{1/2}) d\hat{\bar{u}}/d\hat{\xi}\]

whose solution is

\[\hat{\bar{u}} = 1 - A \left| 1 - 4\hat{\xi}^2 \right|^{1/3}.\]  

(31)

Boundary condition \hat{\bar{u}}(0) = 0 leads to A = 1; on the other hand, matching (31) for large \hat{\xi} to (30a), we get

\[A = \left[ a\theta_{el}(0)/4 \right]^{1/3} C/\theta_{el}(0).\]

Thus, \hat{\bar{u}} presents a cusp at \hat{\xi} = 1/2 (see Fig. 2).

The \xi = 1/6 plane is, therefore, a weak discontinuity
surface.\(^{31}\) Such surfaces move relative to the fluid with the speed of propagation of small disturbances, \(c\), which, due to heat conduction, is given here by \(c^2 = m T_e / \rho_0\), \(\rho_0\) being the position of the discontinuity surface.

\[4(\xi - u)^2 = 4 \xi^2 = \alpha \theta_0(0),\]

or

\[(dx/dt - v)^2 = k T_e / m_1,\]

\(\chi_d\) being the position of the discontinuity surface.

Obviously, there must exist a very thin, viscous sub-layer, centered at \(\xi = \frac{1}{3}\); a detailed analysis of this sub-layer is given in Appendix A.

We notice that \(\eta_2\), as given by (29b) and (31), is no longer singular at \(\xi = 0\). On the other hand, to satisfy condition \(d\theta_{12} / d\xi = 0\) at \(\xi = 0\), heat conduction should be retained in the ion energy equation, within a very thin, thermal, boundary layer around \(\xi = 0\).

\section*{V. ASYMPTOTIC SOLUTION FOR \(\alpha \gg 1\)}

Condition \(\alpha \ll 1\) implies either \(n_0\), and thus, heat capacity per unit volume, small, for given \(\phi_0 / T\), or \(\phi_0 / T\) large for given \(n_0\); either way \(T_e\) must grow very fast with time. Heat conduction is then dominant, leading to a rapid equalization of both temperature and pressure; consequently, the plasma is unable to begin moving, and convection is negligible, as found in last section. In the opposite limit \(\alpha \gg 1\), on the contrary, convective energy flow must be dominant and give rise (neglecting viscosity) to a shock bounding the disturbed plasma.\(^{19}\) However, since convection must vanish at \(\xi = 0\), there must exist a region where heat conduction is important, lying between the origin and the (isentropic) region where convection is dominant.

\subsection*{A. Isentropic region}

The flow behind the shock is isentropic, ion and electron temperatures being equal and viscosity and conduction negligible. The jump conditions across the shock, which may be directly obtained from system (19), are

\[\bar{\eta}_f(\xi_f - u_f) = \xi_f,\]

\[4\bar{\eta}_f u_f (\xi_f - u_f) - 2 \alpha \bar{\eta}_f \theta_f = 0,\]

\[4\bar{\eta}_f (\theta_f + 2 \epsilon_3 / 3 \alpha)(\xi_f - u_f) - 8 \bar{\eta}_f \mu \theta_f / 3 = 0,\]

where the subscript \(f\) labels the conditions just behind the shock; then,

\[\bar{\eta}_f = 4, \quad u_f = 3 \xi_f / 4, \quad \theta_f = 3 \xi_f^2 / 3 \alpha.\]

Defining the normalized variables

\[\eta = \xi / \xi_f, \quad \nu = \bar{\eta} / \bar{\eta}_f, \quad y = u / u_f, \quad z_f = \theta / \theta_f,\]

Eq. (20c) becomes

\[3 \nu (\varepsilon_f + 2 \xi_f^2) d\eta = 3 \xi_f^2 / 3 \alpha;\]

since \(\eta, \nu, y, \) and \(z_f\) are of order unity behind the shock, \(\xi_f = \gamma a^{1/2}\), where \(\gamma\) is an unknown constant of order of unity, determining the shock position.

Then, system (16) becomes

\[dy / d\eta = - 3 \nu \frac{dy}{d\eta}, \quad \frac{d^2 y}{d\eta^2} = - 3 \nu \frac{d^2 y}{d\eta^2}, \quad (32a)\]

\[\nu y + \nu (3y - 4 \eta) \frac{dy}{d\eta} = \frac{1}{2} \frac{d}{d\eta} \left[\nu (\varepsilon_f + z_f) + O(\alpha^{3/2})\right], \quad (32b)\]

\[\nu \left[ z_f (1 + \frac{d y}{d \eta})^2 + \frac{1}{2} (3y - 4 \eta) \frac{dz}{d\eta}^2 \right] = \pi \left[ 75 \alpha^{1/2} \xi_f^2 \right] \frac{z_f - 3 \xi_f}{z_f^2} + O(\alpha^{3/2}), \quad (32c)\]

the upper sign corresponding to electrons. The viscosity and conduction terms are \(O(\alpha^{3/2})\), as indicated, and may be neglected to lowest order. Similarly, we get \(\chi_d = 1 \uparrow\) order \(\alpha^{3/2}\). On the other hand, the energy exchange term is dominant, so that \(x_y ^2 \approx z_f = z_f\). Thus, Eq. (32b) will read

\[\nu y + \nu (3y - 4 \eta) d y / d \eta = - \frac{d (\nu \chi)}{d \eta},\]

while adding the (32c) equations for ions and electrons, we obtain

\[2 \alpha (1 + \frac{d y}{d \eta}) + (3y - 4 \eta) d z / d \eta = 0. \quad (32e)\]

Equations (32a), (32d), and (32e) must be solved subject to boundary conditions

\[\phi(1) = \phi(1) = z(1).\]

Having neglected second-order derivatives, that system will clearly not be uniformly valid in the whole interval \(0 < \eta < 1\).

A first integral of the system can be obtained by combining Eqs. (32a, e),

\[\nu^{1/2} (4 \eta - 3y) z_f / z = 1;\]

this is the usual adiabatic integral of self-similar isentropic flow.\(^{28}\) Furthermore, Eq. (32d) may be rewritten as

\[\frac{d y}{d \eta} = - \frac{y (4 \eta - 3 \eta) + 2 \varepsilon_f}{5 \xi_f - (4 \eta - 3 \eta)^2}.\]

This equation, together with Eq. (32a), may now be easily solved defining

\[Y = y / \eta, \quad Z = z / \eta^2,\]

this leads to a \((Y, Z)\) phase-space equation

\[d Z = 2 Z (15 - 15 Y + (4 - 3 Y) Y - 3 (Y - 1) (4 - 3 Y)] / d Y = 4 - 3 Y \]

\[2 (2 + 5 Y) + 3 Y (Y - 1) (4 - 3 Y)]^2, \quad (32e)\]

together with

\[\eta d Y / d \eta = - Y (4 - 3 Y) + 2 Z / 5 \xi_f - (4 - 3 \eta)^2.\]

It may be shown analytically, that \(Y\) increases, starting from unity, as \(\eta\) decreases and that for \(Y < 4 / 3, Z < (4 - 3 \eta)^{1/2}\); then, defining \(\eta \bar{y} \) by \(\eta \bar{y} = 4 / 3\), we obtain

\[Z \approx B_1 (\eta - \bar{y})^{3/2}, \quad Y \approx \frac{1}{4} \eta - \bar{y} (\eta - \bar{y}), \quad \eta \approx B_2 (\eta - \bar{y})^{3/2} \]

Numerical results for \(Y, Z\) versus \(\eta\) are given in Fig. 4; we also find \(\eta \approx 0.82, B_1 \approx 1.70, B_2 \approx 0.78.\)

This solution corresponds to the classical problem of a gas at rest, compressed by a plane piston moving with a velocity \(U = t^n\). For certain values of \(n\), the density
Fig. 4. Normalized dimensionless density, \( \nu \), velocity, \( y \), and ion and electron temperatures, \( z \), vs normalized dimensionless distance, \( \eta \); isentropic region, \( \alpha \gg 1 \).

B. Conduction region

The isentropic solution ceases to be valid in the neighborhood of \( \eta \). Moreover, in a region containing the origin, conduction must be taken into account. To determine the order of magnitude of the variables in this region we first notice that the last boundary condition in (18), may be rewritten

\[
(3/8)^{1/2} \gamma^3 \alpha^2 \mu_2 \alpha^2 \frac{d \sigma}{d \eta} \bigg|_{\eta=0} = -\frac{3}{2} ;
\]

thus, \( \sigma = O(\alpha^{3/2}) \), and it is easy to show that \( \sigma = \sigma(\alpha^{3/2}) \), \( \chi = 1 \). On the other hand, \( y = O(1) \), since \( y(\eta) \) is of the order of unity. Finally, Eq. (20a) gives \( \int \frac{1}{4} \nu d \eta = 1/4 \), while the isentropic solution is such that \( \int \frac{1}{4} \nu d \eta = 1/4 \); thus, \( \nu = 0 \) must be much less than unity to the left of \( \eta \).

Then, from (20b) we get \( \int \frac{1}{4} 5 \nu y d \eta = \nu(0) x(0) \), implying \( \nu = O(\alpha^{3/2}) \).

Now defining \( \delta = \alpha^{3/2} \nu \), \( \hat{\nu} = \alpha^{3/2} z \), system (16) to lowest order in \( \alpha \) becomes

\[
\begin{align*}
\frac{d \delta}{d \eta} &= \frac{3 \hat{\nu}}{\eta - 3 \nu} d \eta, \\
\frac{d (\hat{\nu} \delta)}{d \eta} &= 0, \\
2 \hat{\nu} \delta \left( 1 + \frac{d y}{d \eta} \right) - \nu (4 \eta - 3 \nu) \frac{d \delta}{d \eta} &= \left( \frac{3}{8} \right)^{1/2} \gamma^3 \frac{d}{d \eta} \left( \frac{2 \nu \eta}{2} \right) ;
\end{align*}
\]

Eq. (33c) results from adding (16c, d). We have neglected viscosities and ion conduction. System (33) must be solved subject to five boundary conditions, since \( \gamma \) is unknown: three conditions for matching with the isentropic solution, and

\[
y(0) = 0, \quad \left( \frac{3}{8} \right)^{1/2} \gamma^3 \frac{d \sigma}{d \eta} \bigg|_{\eta=0} = -\frac{1}{2} .
\]

Integrating (33b) and matching to the isentropic solution, we obtain

\[
\hat{\nu} = B_1 H_2 \approx 1.32 .
\]

Then, using (33a) and (34), Eq. (33c) may be integrated once, giving

\[
1.32 (2 \eta + 3 \nu) = (3/8)^{1/2} (\nu^2 / 4) \hat{\nu} d \eta / d \eta + 1/3 \nu^2 .
\]

Also, Eq. (33a) becomes

\[
3 \hat{\nu} d \delta / d \eta = - (4 \eta - 3 \nu) d \delta / d \eta .
\]

The last two equations are to be solved with boundary conditions

\[
y(0) = 0, \quad y(\eta) = \frac{4}{3} \eta , \quad \hat{\nu}(\eta) = 0 .
\]

In the neighborhood of \( \eta \), the solution is found to behave in the form

\[
\hat{\nu} \approx D_1 (\eta - \eta) \eta^{1/2} , \quad \gamma \approx \frac{2}{3} \eta - D_2 (\eta - \eta) \eta^{3/2} .
\]

Then, \( \hat{\nu} \eta^{1/2} d \delta / d \eta \rightarrow 0 \) as \( \eta \rightarrow \eta \), and from Eq. (35)

\[
y = 0.33 .
\]

Numerical results for \( \nu \), \( y \), and \( \hat{\nu} \) versus \( \eta \) are given in Fig. 5.

It is important to notice that expression \( \nu (4 \eta - 3 \nu) \) must increase monotonically with \( \eta \), for all \( \eta \) [see Eq. (19a)], and that the solution in this section violates this condition at \( \eta \):

\[
\nu (4 \eta - 3 \nu) \rightarrow (\eta - \eta) \eta^{1/2} \rightarrow 0 \text{ as } \eta \rightarrow \eta^* ,
\]

\[
\nu (4 \eta - 3 \nu) \rightarrow O(\alpha^{3/2}) (\eta - \eta) \eta^{3/2} \eta^{-3/2} \text{ as } \eta \rightarrow \eta^* ,
\]

and therefore, \( \nu (4 \eta - 3 \nu) \) (mass flow relative to the surface \( \eta \)) presents a negative jump at \( \eta^* \), of order \( \alpha^{3/2} \). A detailed analysis of the solution in the neighborhood of \( \eta^* \), carried out in Appendix B, shows that \( \nu (4 \eta - 3 \nu) \) is continuous at this point, and that \( \nu \) and \( \nu \) present a minimum (\( \approx 0.77 \alpha^{3/2} \)) and a maximum (\( \approx 1.72 \alpha^{3/2} \)), respectively, at \( \eta \approx 0.032 \alpha^{1/2} \). Naturally, the solution given in Figs. 4 and 5, will not be modified by this correction, except in a narrow layer around \( \eta = \eta^* \).

VI. DISCUSSION OF RESULTS

We have studied the one-dimensional motion generated in a cold, unbounded plasma of density \( n_0 \), when a pulse of energy per unit time and area, \( \phi = \phi_0 / \tau \), is anomalously absorbed in a given plane. The analysis, includ-
ing heat conduction, viscosity, ion-electron energy exchange, and an electron flux limiter, shows that the motion is self-similar and governed by a single parameter $\alpha \propto n_0 T_0/\phi_0$. For all reasonable values of $n_0$, $\tau$, and $\phi_0$, the motion is quasi-neutral, and for $\alpha \gg m_e/m_i$ the plasma is collision dominated (no flux limiter).

For $\alpha \ll 1$, a thermal wave carries the energy, convection and ion temperature being negligible. The order of magnitude of the self-similar variables defined in Sec. III is

$$\epsilon = O(1), \quad \theta_1 = O(1), \quad \Delta \xi = O(\alpha^{1/3});$$

Both $\tilde{n}$ and $u$ present a maximum for $\xi = O(\alpha^{1/2})$.

For $\alpha \gg 1$, ion and electron temperatures are practically equal to each other. Close to the origin, there is a region where conduction and thermal energy convection are comparable; a thin layer of cold and very dense plasma separates it from an isentropic region further ahead, where thermal and kinetic energies are of the same order, and which is bounded by a shock from the undisturbed plasma. The order of magnitude of the self-similar variables is

Conduction region:

$$\tilde{n} = O(\alpha^{-2/3}), \quad \theta_j = O(\alpha^{-2/3}), \quad \Delta \xi = O(\alpha^{1/3});$$

Isentropic region:

$$\tilde{n} = O(1), \quad \theta_j = O(\alpha^{-1/3}), \quad \Delta \xi = O(\alpha^{1/3});$$

Intermediate layer:

$$\tilde{n} = O(\alpha^{2/3}), \quad \theta_j = O(\alpha^{-2/3}), \quad \Delta \xi = O(\alpha^{-1/3});$$

where $\Delta \xi$ is the thickness of the region considered; $u = O(\alpha^{1/3})$ everywhere. There is also a shock precursor of thickness $\Delta \xi = O(\alpha^{-1/3})$.

Qualitative information on the solution for $\alpha = O(1)$ may be obtained from the results for $\alpha \ll 1$ and $\alpha \gg 1$.

For $\alpha$ large and decreasing, the density maximum and temperature minimum become less sharp, while $\Delta \xi$, for all zones, approaches $O(1)$. Thus for $\alpha = O(1)$, a shock will stand in the middle of a thermal wave; in other words, the precursor thickness and the distance of the shock to the origin are comparable and of order of unity. Then, convection and conduction of energy are comparable everywhere; moreover, $\theta_{e1}$, $\tilde{n}$, and $u$ are of order unity, and $\theta_{e1} \neq \theta_{i1}$. As $\alpha$ decreases further the shock collapses toward the origin until, at a certain $\alpha$, becomes a weak discontinuity surface, where $\tilde{n}$ and $u$ present a maximum ($\alpha \ll 1$ case).

The dimensional variables $x_j, v, T_j$ ($j = e, i$), at $t = \tau$, depend on $\alpha$ in the way

$$x_j/\tau = (n_0 \tau)^{1/3} \xi^{1/3} \propto (\phi_0 \tau)^{1/3} \xi^{1/3} \propto \alpha^{-7/12},$$

$$v = (n_0 \tau)^{1/3} \theta_j \propto (\phi_0 \tau)^{1/3} \theta_j \propto \alpha^{-7/12},$$

$$T_j = (n_0 \tau)^{1/3} \xi^{1/3} \propto (\phi_0 \tau)^{1/3} \xi^{1/3} \propto \alpha^{-7/12}.$$}

These expressions are schematically represented in Figs. 6 and 7; for large $\alpha$, $T_j$ presents different behavior depending on the value of $\alpha$.

We next point out some qualitative conclusions of interest for laser fusion plasmas:

(a) Both velocity (and therefore kinetic energy) and ion temperature, as function of $\alpha$, present a maximum for a value of $\alpha$ of order unity.

(b) Both production and flux of entropy depend critically on $\alpha$ since, first, $T_j \propto \tau$ for $\alpha \ll 1$ and $T_j \propto \tau^{-1/3}$ for $\alpha < O(1)$, and second, for $\alpha > 1$ the motion of the plasma in the front is isentropic, while for $\alpha < O(1)$ heat conduction is important everywhere.

We notice here that these conclusions will remain valid for spherical geometries and for nonlinear pulses (if appropriate values of $df/dt$ are used instead of $\phi_0/\tau$).

Finally, some approximate results by previous authors may be recovered as certain limits of our solution:

(1) For given $\phi_0$ and $t \propto \tau^{-\infty}$ ($\phi \propto \phi_0$, $\phi/t \propto 0$), $\alpha \propto \infty$, and therefore

$$x_j/\tau \propto (\phi_0 \tau)^{1/3}, \quad T_j \propto (\phi_0 \tau)^{2/3}.$$}

This behavior has been previously obtained in Refs. 6 and 7, for constant irradiation $\phi = \phi_0$, and long times (neglecting any transient) so that $\phi/t \propto 0$.

![FIG. 6. Schematic dependence of length of plasma disturbed, velocity and temperatures as functions of $n_0$, $\tau$, and $\alpha$.](image-url)
FIG. 7. Schematic dependence of length of plasma disturbed, velocity and temperatures as functions of \( \phi_0, \tau, \) and \( \alpha. \)

(2) For given \( \phi_0 \) and \( t \approx 0 \) (\( \phi \approx \phi_0, \phi/t \approx 0 \)), \( \alpha \approx 0, \) and therefore,

\[
x_f \sim (\phi_0^7/\eta_0)_{1/6}, \quad T_e \sim (\phi_0^7/\eta_0)^{2/3}.
\]

This behavior has been previously obtained in Refs. 8 and 9, for constant irradiation \( \phi = \phi_0 \), and short times (neglecting motion) so that \( \phi_0/t \rightarrow \infty \). One may also consider the instantaneous deposition of energy per unit area \( W = \phi_0 \tau; \) this leads to

\[
x_f \sim (\dot{W}^3/\eta_0)^{1/6}, \quad T_e \sim (\dot{W}^3/\eta_0)^{3/7}.
\]

ACKNOWLEDGMENTS

This work was conducted at the Universidad Politécnica de Madrid, Spain, in partial fulfillment of the requirements for the doctoral degree of one of the authors (A.B.).

This research was performed under the auspices of the Junta de Energía Nuclear of Spain.

APPENDIX A

To study the viscous sublayer around \( \xi = 1/2 \), for \( \alpha \ll 1 \), we define

\[
x^* = (\xi - 1/2)(\alpha \theta_1(0)/\sigma)^{1/2}, \quad u^* = 1 - \tilde{u},
\]

where

\[
\sigma = 0.039(\theta_1(0)\alpha)^{3/2}/A_1 + 0.00049(\theta_1(0)\alpha)^{3/2}/A_1.
\]

Then, the momentum equation becomes

\[
\frac{d^2u^*}{d\xi^2} + 8 \left[ \theta_1^* (0) \theta_0^{(0)} (\xi^* - u^*) \right] \frac{du^*}{d\xi^*} - a^* = 0;
\]

it is easy to show that, \( a^* \approx 1/2 \) has a maximum value of 1.76, for \( A_1 = 2.5, \) at \( \alpha = 0.039. \)

For \( \xi^* \approx \pm \infty, \) we find

\[
u^* = a^* (\pm \xi^*)^{1/3}.
\]

Matching to the solution of the inner layer for \( \xi = -1/2 \)

\[
s^* = (16\alpha/\alpha \theta_1(0))^{1/16}.
\]

Equation (A1) together with the boundary conditions (A2), (A3) may now be solved numerically for any given \( \alpha. \)

APPENDIX B

To study the cold, highly dense, narrow layer around \( \eta, \) for \( \alpha > 1 \), we define

\[
z = \tilde{z}, \quad J = \nu\eta (\tilde{z} - 3), \quad \tilde{n} = \mu \eta .
\]

Then, system (19), to lowest order but retaining electron conduction, becomes

\[
\begin{align*}
4\tilde{P} &= \hat{z} \frac{dJ}{d\eta}, \\
3 \frac{d^2\tilde{J}}{d\eta^2} &= 10J - \frac{d}{d\eta} \left( \frac{\dot{J}^2}{\tilde{P}} + \eta J \right), \\
\frac{d}{d\eta} \left( \frac{\ddot{J}^2}{\tilde{P}} \right) &= \frac{\gamma J^3}{8} \left( \frac{\dot{J}^2}{\tilde{P}} \right) - \frac{\gamma J^3}{2} \frac{d}{d\eta} \frac{\dot{J}^2}{\tilde{P}}.
\end{align*}
\]

In the isentropic solution (\( \eta > \tilde{\eta} \)),

\[
J = f(\hat{z} + \hat{\eta} - \tilde{\eta})^{11/13},
\]

which is \( O(\alpha^{-5/7}) \) for \( \eta = \tilde{\eta} = O(\alpha^{-39/70}). \) Defining a point \( \eta_1 \) and a new variable \( \hat{\eta} \)

\[
\eta_1 = \tilde{\eta} + b\alpha^{39/70}, \quad \hat{\eta} = \alpha^{3}(\eta - \eta_1) \quad (\nu > 39/70)
\]

the isentropic solution, near \( \eta_1, \) behaves as

\[
\begin{align*}
\hat{z} &= B_1(\eta - \tilde{\eta})^{13/15} \approx \alpha^{39/70}B_2(\eta - \tilde{\eta})^{13/15}(1 + \alpha^{-117/440}(30/13) + \cdots), \\
J &= \alpha^{-3/7}J_1 + \cdots, \\
\hat{P} &= B_1B_2 + \cdots,
\end{align*}
\]

where we have set \( \nu = 39/28 \) to retain heat conduction in (B1) to lowest order.

We next expand the variables in the neighborhood of \( \eta_1 \)

\[
\begin{align*}
\hat{z} &= \alpha^{39/70}B_1B_2^{13/15} + \alpha^{-27/35}B_2 + \cdots, \\
J &= \alpha^{-3/7}J_1 + \cdots, \\
\hat{P} &= B_1B_2 + \cdots,
\end{align*}
\]

so that (B1) becomes

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\[ 4B_z = b^{9/13} d J_1 / d\hat{n}, \quad d\hat{P} / d\hat{n} = 0, \]

\[ \left( \frac{3}{8} \right)^{5/12} \frac{\gamma^3}{2} B_1^{15/28} B_z^{15/28} d^2 S_1 / d\hat{n}^2 = 4B_1 B_2 - \frac{52}{3} B_2 b^{10/13} dS_1 / d\hat{n}; \]

then

\[ J_1 = 4B_2 \hat{n} / b^{5/13} + F, \]

\[ \hat{P} = B_1 B_2, \]

\[ \hat{n}_1 = G + H \exp \left[ -\frac{52}{3} \left( \frac{1}{8} \right)^{3/2} \frac{2 \hat{n}}{3} \right] - \frac{3B_2 \hat{n}}{13B_1^{10/13}}, \]

where \( F, G, H \) are constants. Clearly, the expansion (B2) ceases to be valid for \( \hat{n} = 0 \) (linear).

Defining

\[ \hat{n} = \alpha^{-9/70} \frac{2}{B}, \]

Eq. (B1c), to lowest order, becomes

\[ \left( \frac{3}{8} \right)^{5/12} \frac{\gamma^3}{7} d^2 S_1 / d\hat{n}^2 + \frac{52}{3} B_2 b^{10/13} d\hat{n}^2 / d\hat{n}_1 = 0. \]

Integrating once, we get

\[ 3\left( \frac{3}{8} \right)^{5/2} \gamma \frac{S_1}{2} d\hat{n} / d\hat{n}_1 + 104B_2 b^{10/13} \left( \hat{n} - B_1 b^{5/13} \right) = 0 \]

whose solution is

\[ \hat{n} = \left( \frac{3}{8} \right)^{5/2} \frac{52}{3} B_2 b^{10/13} \left( \frac{2}{5} - B_1 b^{5/13} \right) \]

\[ + B_2 b^{10/13} \frac{52}{5} \ln \left( \frac{2}{5} - B_1 b^{5/13} \right). \]

Then, for \( \hat{n} \to -\infty \)

\[ \hat{n} \approx \alpha^{-9/70} \left( \frac{2}{3} \right) \left( \frac{260}{3} B_2 b^{10/13} \right)^{3/5} \left( -\hat{n} \right)^{3/5}, \]

\[ J \approx \alpha^{-3/5} 2B_2 b^{10/13}, \quad \hat{P} = B_1 B_2. \]

Matching to the solution of the conduction region gives

\[ b = \left( 15D_2 B_1 / 26D_1 \right)^{11/10} \approx 0.032, \]

\[ \varepsilon_{\text{mix}} \approx \alpha^{-9/70} B_0 b^{3/13} \approx 0.77 \alpha^{-2/70}, \]

\[ \nu_{\text{max}} \approx \alpha^{-9/70} B_0 b^{3/13} \approx 1.72 \alpha^{2/70}. \]

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