Collective Organization of Complex Networks: Emergent Scales, Vulnerability, and Targeting of Desired Dynamics

PhD Thesis

Ricardo Gutiérrez Díez
MSc in Electrical Engineering
MSc in Physics

2013
Collective Organization of Complex Networks: Emergent Scales, Vulnerability, and Targeting of Desired Dynamics

Ricardo Gutiérrez Díez
MSc in Electrical Engineering
MSc in Physics

Advisor
Stefano Boccaletti
PhD in Physics

2013
A mi familia con cariño
Resumen

Cuando una colectividad de sistemas dinámicos acoplados mediante una estructura irregular de interacciones evoluciona, se observan dinámicas de gran complejidad y fenómenos emergentes imposibles de predecir a partir de las propiedades de los sistemas individuales. El objetivo principal de esta tesis es precisamente avanzar en nuestra comprensión de la relación existente entre la topología de interacciones y las dinámicas colectivas que una red compleja es capaz de mantener. Siendo este un tema amplio que se puede abordar desde distintos puntos de vista, en esta tesis se han estudiado tres problemas importantes dentro del mismo que están relacionados entre sí.

Por un lado, en numerosos sistemas naturales y artificiales que se pueden describir mediante una red compleja la topología no es estática, sino que depende de la dinámica que se desarrolla en la red: un ejemplo son las redes de neuronas del cerebro. En estas redes adaptativas la propia topología emerge como consecuencia de una autoorganización del sistema. Para conocer mejor cómo puedenemerger espontáneamente las propiedades comúnmente observadas en redes reales, hemos estudiado el comportamiento de sistemas que evolucionan según reglas adaptativas locales con base empírica. Nuestros resultados numéricos y analíticos muestran que la autoorganización del sistema da lugar a dos de las propiedades más universales de las redes complejas: a escala mesoscópica, la aparición de una estructura de comunidades, y, a escala macroscópica, la existencia de una ley de potencias en la distribución de las interacciones en la red. El hecho de que estas propiedades aparezcan en dos modelos con leyes de evolución cuantitativamente distintas que siguen unos mismos principios adaptativos sugiere que estamos ante un fenómeno que puede ser muy general, y estar en el origen de estas propiedades en sistemas reales.

En segundo lugar, proponemos una medida que permite clasificar los elementos de una red compleja en función de su relevancia para el mantenimiento de dinámicas colectivas. En concreto, estudiamos la vulnerabilidad de los distintos elementos de una red frente a perturbaciones o grandes fluctuaciones, entendida como una medida del impacto que estos acontecimientos externos tienen en la interrupción de una dinámica colectiva. Los resultados que se obtienen indican que la vulnerabilidad dinámica es sobre todo
dependiente de propiedades locales, por tanto nuestras conclusiones abarcan diferentes topologías, y muestran la existencia de una dependencia no trivial entre la vulnerabilidad y la conectividad de los elementos de una red.

Finalmente, proponemos una estrategia de imposición de una dinámica objetivo genérica en una red dada e investigamos su validez en redes con diversas topologías que mantienen regímenes dinámicos turbulentos. Se obtiene como resultado que las redes heterogéneas (y la amplia mayoría de las redes reales estudiadas lo son) son las más adecuadas para nuestra estrategia de \textit{targeting} de dinámicas deseadas, siendo la estrategia muy efectiva incluso en caso de disponer de un conocimiento muy imperfecto de la topología de la red.

Aparte de la relevancia teórica para la comprensión de fenómenos colectivos en sistemas complejos, los métodos y resultados propuestos podrían dar lugar a aplicaciones en sistemas experimentales y tecnológicos, como por ejemplo los sistemas neuronales \textit{in vitro}, el sistema nervioso central (en el estudio de actividades síncronas de carácter patológico), las redes eléctricas o los sistemas de comunicaciones.
Abstract

The time evolution of an ensemble of dynamical systems coupled through an irregular interaction scheme gives rise to dynamics of great of complexity and emergent phenomena that cannot be predicted from the properties of the individual systems. The main objective of this thesis is precisely to increase our understanding of the interplay between the interaction topology and the collective dynamics that a complex network can support. This is a very broad subject, so in this thesis we will limit ourselves to the study of three relevant problems that have strong connections among them.

First, it is a well-known fact that in many natural and manmade systems that can be represented as complex networks the topology is not static; rather, it depends on the dynamics taking place on the network (as it happens, for instance, in the neuronal networks in the brain). In these adaptive networks the topology itself emerges from the self-organization in the system. To better understand how the properties that are commonly observed in real networks spontaneously emerge, we have studied the behavior of systems that evolve according to local adaptive rules that are empirically motivated. Our numerical and analytical results show that self-organization brings about two of the most universally found properties in complex networks: at the mesoscopic scale, the appearance of a community structure, and, at the macroscopic scale, the existence of a power law in the weight distribution of the network interactions. The fact that these properties show up in two models with quantitatively different mechanisms that follow the same general adaptive principles suggests that our results may be generalized to other systems as well, and they may be behind the origin of these properties in some real systems.

We also propose a new measure that provides a ranking of the elements in a network in terms of their relevance for the maintenance of collective dynamics. Specifically, we study the vulnerability of the elements under perturbations or large fluctuations, interpreted as a measure of the impact these external events have on the disruption of collective motion. Our results suggest that the dynamic vulnerability measure depends largely on local properties (our conclusions thus being valid for different topologies) and they show a non-trivial dependence of the vulnerability on the connectivity of the network elements.
Finally, we propose a strategy for the imposition of generic goal dynamics on a given network, and we explore its performance in networks with different topologies that support turbulent dynamical regimes. It turns out that heterogeneous networks (and most real networks that have been studied belong in this category) are the most suitable for our strategy for the targeting of desired dynamics, the strategy being very effective even when the knowledge on the network topology is far from accurate.

Aside from their theoretical relevance for the understanding of collective phenomena in complex systems, the methods and results here discussed might lead to applications in experimental and technological systems, such as in vitro neuronal systems, the central nervous system (where pathological synchronous activity sometimes occurs), communication systems or power grids.
Preface

Systems composed of a large number of interacting dynamical units appear in very many fields of science and technology. Coupled chemical systems, interacting species, flashing fireflies or chirping crickets, neural networks in the brain, the Internet and the World Wide Web, and infrastructural networks are some well-known examples. They can be modeled as networks or graphs, whose nodes represent the dynamical units, while the links joining them stand for the interactions between them; indeed, the usefulness of this approach to the scientific study of such complex systems has been shown in thousands of publications during the last 15 years. Networks are simple mathematical objects that neatly encode the structure of relations among a large number of interdependent elements. Nevertheless, while the classical objects of study of mathematical graph theory are regular in nature, the systems of interest in modern network theory show a large degree of irregularity and heterogeneity. Hence the frequently used term complex networks.

The structural study of complex networks focuses on characterizing the topology of the irregular coupling schemes. Two of the most relevant tasks in this endeavor are finding the unifying properties that, despite the intrinsic differences, are found in many real networks (which are believed to be at the basis of their functioning), and developing models that reproduce these structural properties. Another major subfield in network theory is the study of the dynamics of complex networks, which addresses issues such as the kind of collective motions that can take place in a large ensemble of dynamical units interacting through a complex coupling scheme. Indeed, in every network of dynamical units there is an inherent dependence of the collective dynamics on the network structure. In such systems a wide spectrum of collective phenomena (that are impossible to predict from the properties of the individual units alone) may originate, ranging from completely turbulent regimes to global order, including phase transitions and clustering.

This thesis presents original theoretical results in the field of complex network dynamics that have appeared in the following publications:

Broadly speaking, the main subject of the thesis is the mutual influence between structure and dynamics in complex networks, while the specific objective that inspired the research work behind it has been to provide an answer to the following problems in the field of complex network dynamics: i) how can the interplay of structure and dynamics lead to the spontaneous emergence of structural features found to be common to a large variety of complex natural and manmade systems, ii) how the importance of different elements in a network for preserving a collective dynamical state is correlated to their topological centrality, iii) how structural features can be exploited for imposing a desired dynamics on a given network.

The thesis is organized as follows. The first two chapters are of an introductory character, as they present the background material needed for the reading of the original results presented later. In Chapter 1, a review of the main concepts and results on the structure of complex networks is given, whereas Chapter 2 focuses on the dynamics of complex networks, starting from a general discussion of elementary dynamical systems theory. While, due to the nature of the original work on which this thesis is based, there is a stronger emphasis on dynamics (as reflected also in the sheer number of pages of each introductory chapter), we start with the chapter on the structure of complex networks, as it is self-contained and presents concepts that are later used in the second chapter. Chapter 3 is based on (Assenza et al., 2011; Gutiérrez et al., 2011a), and shows how some empirically-motivated forms of adaptive synchronization in networks of oscillators lead spontaneously to the emergence of meso- and macroscale features that have been empirically observed in a wide range of real complex systems. Chapter 4 shows how the different elements of such networks can be classified according to their importance in preserving a collective (synchronous) dynamics by first introducing a dynamical definition of vulnerability (Gutiérrez et al., 2011b). Chapter 5 focuses on the targeting of complex dynamical systems, and presents a strategy for imposing desired dynamics compatible with the equations of motion on a given network by taking advantage of its structural properties (Gutiérrez et al., 2012). Finally, in Chapter 6 we summarize the most relevant findings in the form of conclusions.
Acknowledgements

It is a great pleasure to thank everyone who helped me through all the research work and the writing that led to this PhD thesis. First of all, I express my warm gratitude to all those friends and colleagues who helped in all possible ways, both by sharing with me their knowledge and talent for research, and with their encouragement, kindness and sense of humor: Juan Almendral, Daniel Alvariño, Javier Buldú, Inmaculada Leyva, Adrián Navas, David Papo, Irene Sendiña-Nadal, and Massimiliano Zanin, and also to our indefatigable visitor Ricardo Sevilla of Universidad de Guadalajara in Mexico. During the last few months, Adrián, David and Massimiliano had to patiently put up with my complaints about the intolerable hardship a thesis writer must endure and responded with a friendly smile to jokes that probably made no sense. I appreciate that too!

Very little remains in this thesis of the work I started doing with Prof. Fernando Maestú, but I remain very grateful to him for having given me the opportunity to start doing research under his supervision, and for giving me the possibility to search for my own path. I also remain deeply grateful to my friends and colleagues in the Laboratory of Cognitive and Computational Neuroscience at the Center for Biomedical Technology: Sara Aurtenetxe, Ricardo Bajo, Pablo Campo, Alejandro Carboni, Nazareth Castellanos, Pablo Cuesta, María Eugenia López, Laura Lorenzo, Stephan Moratti, Ángel Nevado, Guiomar Niso, Javier García Pacios, Claudia Poch, Ignacio de Ramón, David del Río, Cristina Saugar and Elesa Solesio. I also want to express my deep gratitude to Prof. Ernesto Pereda of Universidad de La Laguna for his friendship and for having introduced me to many topics that later, one way or another, were to become part of my thesis work.

I am extremely grateful to Prof. Eshel Ben-Jacob for giving me the opportunity to spend 3 great months doing research in his group at the School of Physics and Astronomy of Tel Aviv University.

I am very much obliged to Prof. Rosa María Benito for her help, her personal support and her generosity throughout all the process of finishing the thesis and preparing the thesis submission and defense. This would not have been possible without her kindness.
and her efforts. I also thank her, together with Prof. Regino Criado and Prof. Juan Carlos Losada, for their most valuable comments that have helped me improve both the thesis and the presentation. I am indebted to Prof. Ruedi Stoop of ETH Zurich and Prof. Francesco Sorrentino of the University of New Mexico for reading the thesis and presenting their evaluation, as well as their comments and suggestions. And I also would like to thank Prof. David Gómez-Ullate of Universidad Complutense de Madrid for his friendly support and his generosity, as well as his willingness to share his ideas with me.

Prof. Francisco del Pozo, director of the Center for Biomedical Technology of the Technical University of Madrid, where my research activities were carried out, played a very important role in my beginnings as a scientist, put me in contact with my advisor, and also did an excellent job in making the Center a great place for doing research. I owe him my sincere gratitude for all that.

Finally, I can hardly find the words to express my gratitude to Prof. Stefano Boccaletti, my advisor. Not only would this thesis be completely unthinkable without him, but he also taught me so many things, both actively and by his own example, and both as a scientist and as a human being, that it would take pages to explain. Grazie, Stefano!
## Contents

1 Introduction I: The structure of complex networks ........................................ 1
   1.1 Basic concepts ................................................................. 2
   1.2 Structural measures and properties ........................................ 4
      1.2.1 Degree and related measures ....................................... 5
      1.2.2 Centrality measures .................................................. 6
      1.2.3 Vulnerability measures .............................................. 9
      1.2.4 Community structure .............................................. 10
   1.3 The Laplacian matrix: graph spectra and diffusive coupling .................. 13
      1.3.1 Definition and spectral properties ................................ 14
      1.3.2 Diffusive coupling .................................................. 16
   1.4 Ubiquitous topological properties ........................................... 17
      1.4.1 The small-world property ........................................... 18
      1.4.2 Scale-free networks .................................................. 19
      1.4.3 Mesoscale structure ................................................ 20
   1.5 Network models I: Erdős-Rényi random graphs .................................. 21
   1.6 Network models II: Scale-free networks ................................... 23

2 Introduction II: The dynamics of complex networks .................................. 27
   2.1 Dynamical systems .......................................................... 27
   2.2 Lyapunov exponents ....................................................... 40
   2.3 The Rössler system ......................................................... 51
   2.4 Synchronization of dynamical systems ................................... 55
   2.5 The Kuramoto model ....................................................... 61
   2.6 The master stability function ............................................. 68
   2.7 Adaptive networks ........................................................ 72
   2.8 Network control and targeting .......................................... 73

3 Emerging meso- and macroscale structural properties from the synchro-
   nization of adaptive networks .................................................. 79
   3.1 Introduction .............................................................. 79
3.2 Model 1 ................................................................. 82
   3.2.1 Methods .......................................................... 82
   3.2.2 Results ........................................................... 85
3.3 Model 2 ................................................................. 91
   3.3.1 Methods .......................................................... 91
   3.3.2 Results ........................................................... 92
3.4 Discussion ............................................................ 98

4 Vulnerability under finite perturbations in complex networks 101
   4.1 Introduction .......................................................... 101
   4.2 Methods .............................................................. 103
   4.3 Results ............................................................... 105
   4.4 Discussion ........................................................... 110

5 Targeting the dynamics of complex networks 115
   5.1 Introduction .......................................................... 115
   5.2 Methods .............................................................. 116
      5.2.1 MSF approach .................................................. 116
      5.2.2 Targeting procedure ............................................ 118
      5.2.3 Network model .................................................. 119
   5.3 Results ............................................................... 120
   5.4 Discussion ........................................................... 125

6 Conclusions .............................................................. 127

References ............................................................... 131
Chapter 1

Introduction I: The structure of complex networks

The theoretical study of networks has been traditionally considered the domain of graph theory, a branch of discrete mathematics born in the 18th century that has been subject to intense efforts and many developments in the intervening years. Additionally, the empirical study of networks has flourished during the last century in some specialized scientific contexts, especially in the study of social networks. These are the two obvious antecedents of the modern science of complex networks, also known as network theory, that has emerged in the last 15 years. Network theory shares with the mathematical theory of graphs the search for a quantitative and precise understanding of network structure, and it also makes use of some of the exact results of the older theory. However, it possesses a stronger emphasis on empirically validated models and on the introduction of measures and algorithms for the analysis of real networks. Many of the scientists working on network theory are physicists and applied mathematicians, whereas truly interdisciplinary work frequently occurs in the empirical analysis of real world networks and in the numerous applications of network theory in other fields of science and engineering.

The objects of study in network theory are systems whose structures are irregular, complex, and sometimes they even evolve in time. They can also be very large, containing thousands or millions of elements, making probability theory and the methods of statistical mechanics very useful for their study, and making the use of computers frequently indispensable for the numerical treatment of network models or the analysis of real network data. The aim has been to create a theory that captures and explains the regularities frequently observed in real world networks across an impressive range of

\[\text{1 Its origins can be traced to the legendary Swiss mathematician Leonhard Euler, and his celebrated solution of the problem of the Seven Bridges of Königsberg in 1735.}\]
disciplines, and the field has already produced an enormous number of theoretical models and results, tools and algorithms for data analysis, and new empirical findings, in the process changing the way complex systems are studied in many fields. The study of networks of dynamical units has also been extremely important in this development, and indeed there has been a good deal of interaction between the science of networks and dynamical systems theory, as the present thesis testifies.

In this chapter we explain some of the main concepts and results in the field of network theory, with special emphasis on those that are relevant for the work presented in Chapters 3, 4 and 5. For the most part, we follow the notation and conventions of (Boccaletti et al., 2006), which is the main reference on which the chapter is based.

1.1 Basic concepts

A network or graph is a pair \((N, \mathcal{L})\) consisting of two sets such that \(N \neq \emptyset\) and \(\mathcal{L}\) is a set of unordered pairs of elements of \(N\). The elements of \(N\) are the nodes or vertices of the network, and the number of them \(N := |N|\) is the size of the network. Nodes are usually referred to by an integer \(i \in \{1, 2, \ldots, N\}\), but the labeling is completely arbitrary. The elements of \(\mathcal{L} := \{l_1, l_2, \ldots, l_L\}\) are the links or edges, and \(L := |\mathcal{L}|\) is simply referred to as the number of links in the network \((0 \leq L \leq N(N - 1)/2)\). We will say that the link \(l_k = (i, j)\) connects nodes \(i\) and \(j\) or is incident in them. Two nodes connected by a link are adjacent or neighboring nodes, and they are sometimes referred to as first neighbors, especially when the terminology is extended to consider second neighbors (nodes connected via another node), third neighbors, and so on. A representation of the graph can be given in terms of an adjacency matrix \(A\), a symmetric \(N \times N\) binary matrix whose entry \(a_{ij}\) is 1 if \((i, j) \in \mathcal{L}\) and 0 if nodes \(i\) and \(j\) are not connected.

A directed network consists again of two sets \((N, \mathcal{L})\), but this time \(\mathcal{L}\) is a set of ordered pairs of elements of \(N\). Now, \(l_k = (i, j)\) is referred to as a link from \(i\) to \(j\). Other properties extend naturally from the previous case: nodes \(i\) and \(j\) are neighbors if they are connected by \((i, j), (j, i)\) or both, and \(0 \leq L \leq N(N - 1)\). Regarding the adjacency matrix, \(a_{ij}\) is 1 if there is a link from \(i\) to \(j\) (that is, if \((i, j) \in \mathcal{L}\)), and 0 otherwise. As this allows for the possibility that \(a_{ij} \neq a_{ji}\), the adjacency matrix of a directed network is not symmetric in general.

\(^2\)Mathematically, establishing a labeling amounts to selecting one representative graph from the equivalence class of isomorphic graphs.
A weighted network is a 3-tuple \((\mathcal{N}, \mathcal{L}, \mathcal{W})\), where \(\mathcal{W} := \{w_1, w_2, \ldots, w_L\}\) is the set of weights associated with the links in the network, with \(w_i \in \mathbb{R}^+\) for all \(i\). Both the link set \(\mathcal{L}\) and the adjacency matrix \(\mathcal{A}\) are defined according to whether the network is directed (it is a weighted directed network) or undirected (it is simply a weighted network), just as it was explained above for unweighted networks. The weight matrix \(\mathcal{W}\) is a \(N \times N\) matrix whose entry \(w_{ij} \in \mathbb{R}\) is the weight assigned to the link \((i, j) \in \mathcal{L}\) if there is such a link, and \(w_{ij} = 0\) if \(i\) and \(j\) are not connected.

Whatever the type of the networks considered, in this thesis they will always assume we deal with simple graphs: in \(l_k = (i, j)\) for any \(k\) it is always the case that \(i \neq j\) (there are no self-loops), and \(k \neq k' \Rightarrow l_k \neq l_{k'}\) (nor multiple edges). The first condition implies that \(a_{ii} = 0\ \forall \ i\), and the same is true of the main diagonal elements of \(\mathcal{W}\) in weighted networks. Figure 1.1 illustrates the previous definitions with a graphical representation of some networks.\(^3\)

---

\(^3\)The usual way to represent a network is by drawing one circle or ball or dot for each node, and joining two nodes by a line whenever there is a link between them. The spatial distribution of the elements in the plane is totally irrelevant; only the pattern of connections matters.
ing $i \rightarrow i'$ of the nodes in $N'$ is assumed).

A walk from node $i$ to node $j$ is a sequence of adjacent nodes that starts with $i$ and ends with $j$. If we imagine an inhabitant of the network visiting node after node using the links as "bridges", the length of the walk is the number of links he/she has gone through to reach his/her destination. A cycle is a closed walk of at least three nodes with no link repeated. A path is a walk where no node is visited more than once. A walk of minimal length between two nodes is a shortest path or geodesic, of which there may be more than one. The diameter of a network is the largest shortest path length considering all possible pairs of nodes. A network is connected if there is a path connecting every pair of distinct nodes, otherwise it is said to be disconnected. A component of a network is a maximally connected induced subgraph. In models considering statistical ensembles of networks of variable size, a component scaling with $N$ is referred to as a giant component.

Most important for later chapters in this thesis is the physical interpretation of a network in the study of networked dynamical systems. In that context, every node is a dynamical unit characterized by a state that evolves in time, and links are interpreted as couplings or interactions among units. Among the unweighted networks, undirected networks show bidirectional symmetric interactions (as $a_{ij} = a_{ji}$), whereas directed networks allow for unidirectional coupling (for instance, $a_{ij} = 1$ and $a_{ji} = 0$ reflects unidirectional coupling from node $i$ to node $j$); both cases assume a uniform coupling strength. In weighted networks, the link weights $w_{ij}$ represent the coupling strengths, which in general vary from one link to another.

1.2 Structural measures and properties

The term network topology or network structure refers generically to the pattern of connections (i.e. links connecting nodes) of a network as a whole. A large number of measures and indices have been proposed to characterize the topology of networks in the last decade. In this section we present those measures that have been the most relevant in our work, as they are necessary background material for the later chapters. For a comprehensive discussion of these topics, see (Boccaletti et al., 2006) and (Newman, 2010).

---

As in the famous problem on the Seven Bridges of Königsberg solved by Euler.

Clearly, such a walk must be a path.
1.2.1 Degree and related measures

The degree is the simplest and most fundamental topological measure. The degree or connectivity of a node is defined as the number of links incident on it. The degree of node $i$ is denoted by $k_i$, and can be defined in terms of the adjacency matrix as follows:

$$k_i = \sum_{j=1}^{N} a_{ij}. \quad (1.1)$$

The list of the node degrees $\{k_1, k_2, \ldots, k_N\}$ is referred to as the degree sequence. Frequently, a statistical approach to the local (first neighbors) properties of a network is useful. The degree sequence $\{k_1, k_2, \ldots, k_N\}$ is then considered to be an $N$-tuple of independent, identically distributed discrete random variables, whose common probability mass function is called the degree distribution $P(k)$. This approach is adopted whenever ensembles of networks with the same statistical (local) properties, as given by some $P(k)$, are considered or whenever the topology of a network is characterized statistically. In the first case, the predefined distribution $P(k)$ is such that $P(k = k')$ denotes the probability that $k'$ is the degree of a node chosen at random. Whenever “the dice are thrown” $N$ times, we get a possible realization of a degree sequence for a representative of the ensemble of size $N$. In the second case, $P(k)$ is inferred from the data, and $P(k = k')$ denotes the fraction (relative frequency) of nodes whose degree is $k'$ in the network. To study the local properties of a network, a simple graphical representation of $P(k)$ is generally resorted to as a first step. For quantitative information, the moments of the distribution are computed as $\langle k^n \rangle = \sum_{k} k^n P(k)$, and as usual the mean degree $\langle k \rangle$ and the fluctuations $\langle k^2 \rangle$ (sometimes expressed in terms of the variance $\text{var}(k) = \langle k^2 \rangle - \langle k \rangle^2$ or the standard deviation $\text{std}(k) = \sqrt{\text{var}(k)}$) are the most frequently employed.

Networks in which $k_i$ and $k_j$ for any $i$ and $j$ are independent random variables, and therefore do not show any kind of correlation even if $i$ and $j$ are adjacent nodes, are called uncorrelated networks. However, in most networks (including many real networks, as well as network ensembles originating from empirically plausible network generation models) the probability that a node of degree $k$ is adjacent to a node of degree $k'$ depends on the value $k$. To study correlated networks, it is natural to consider the conditional probability distribution $P(k'|k)$. However, in order to understand these correlations in empirical data, it is far easier (and less prone to suffer from finite-size effects) to evaluate the average nearest neighbors degree of a node:

$$k_{nn,i} = \frac{1}{k_i} \sum_{j \in N_i} k_j = \frac{1}{k_i} \sum_{j=1}^{N} a_{ij} k_j. \quad (1.2)$$
By averaging $k_{nn,i}$ over the nodes whose degree equals $k_i = k$, it is possible to calculate the average degree of the nearest neighbors for nodes with degree $k$, $k_{nn}(k)$. If the conditional probability is available, this quantity is computed as $k_{nn}(k) = \sum_{k'} k' P(k'|k)$. A network is said to be assortative (dissortative) if $k_{nn}(k)$ is an increasing (decreasing) function of $k$, which means that nodes are connected to nodes of a similar (dissimilar) connectivity. To quantify the degree of assortativeness of a given network, the slope of $k_{nn}$ vs. $k$ or the Pearson correlation coefficient of the degrees of adjacent nodes are sometimes reported.

In undirected weighted networks, the strength of node $i$ is defined as

$$s_i = \sum_{j=1}^{N} w_{ij},$$

which is somehow a generalization of the degree, as it considers both the number and the weight of the links incident on a node. If the weight of a node picked at random is taken to be a (continuous) random variable, the strength distribution $R(s)$ is the probability density function of the weights of a given network. The motivation and usefulness of this approach can be explained in analogous terms as those employed above for the case of $P(k)$. The $n$-order moments of $R(s)$ can be studied, and measures similar to the assortativity in terms of conditional distributions or the weighted average nearest neighbors degree of node $i$, defined as $k_{Wnn,i} = \frac{1}{s_i} \sum_{j \in N} w_{ij} k_j$, can be used to characterize correlations in a network by combining information both on the number of links and their weight.

In the context of directed networks, the out-degree of node $i$ is the number of outgoing links of a node $k_{i \text{out}} = \sum_j a_{ij}$, whereas the in-degree is the number of incoming links $k_{i \text{in}} = \sum_j a_{ji}$, and the total degree is $k_i = k_{i \text{out}} + k_{i \text{in}}$. The in-degree and out-degree sequences of a directed network are defined as expected, and the same can be said about the two degree distributions, $P(k_{\text{in}})$ and $P(k_{\text{out}})$. The extensions of the above definitions and considerations to this case and to the case of directed weighted networks, where $s_{i \text{in}}$ and $s_{i \text{out}}$ can be defined in the obvious way, present no difficulty.

1.2.2 Centrality measures

The quantification of the importance of a node for the functioning of a network is a key issue in network theory. Several definitions of node centrality exist that focus on different topological aspects. Each of them reflects how important a node is for the functioning of a network under a given set of (sometimes implicit) assumptions, which frequently have
to do with the nature of the processes or interaction taking place among the elements of the network. The reader is referred to (Newman, 2010) and (Borgatti, 2005) for a comprehensive discussion of this issue.

The first measure of centrality that comes to mind is the degree itself, which sometimes in this context is referred to as \textit{degree centrality}. In this definition of centrality, a node is simply considered to be the more central the more connections it has. A similar definition in terms of the node strength in weighted networks is also possible. Both definitions are, of course, based on purely local properties. Generally speaking, however, node $i$ in a network together with its neighbors in $\mathcal{N}_i$ do not form anything close to a disconnected component. Rather, \textit{non-local interactions} (indirect interactions with second neighbors and more distant nodes) play an important part in the network functioning or dynamics. Since the local approach may be quite unable to discern the importance of nodes in some cases (an obvious example is presented in Figure 1.2), non-local measures that quantify the centrality of a node are called for.

![Figure 1.2](image)

\textbf{Figure 1.2: A central node with moderately low degree.} The degree of the node highlighted in red is only 2, but it is clearly seen that this does not reflect its importance as a “bottleneck” in the network. Its removal suffices to disconnect the network into two separate components, and its malfunctioning (in case the network supports a given dynamical process) may severely impair a collective dynamics.

One of the most widely-used centrality measures is focused on the extent to which a node is important for the propagation of interactions or perturbations\footnote{This comment reflects the assumption that we are dealing with networked dynamical systems, as we will always do in later chapters. In the context of transportation networks, for instance, one would refer to the propagation of goods, etc.} along geodesics. Thus, the \textit{betweenness} or \textit{load} of node $i$, $b_i$, is defined as

$$b_i = \sum_{j,k \in \mathcal{N}, j \neq k} \frac{n_{jk}(i)}{n_{jk}},$$  

(1.4)
where \( n_{jk}(i) \) is the number of geodesics connecting \( j \) and \( k \) going through \( i \) and \( n_{jk} \) is the total number of geodesics between \( j \) and \( k \). According to this definition, a node of high centrality is one through which a large part of the interactions in the network propagate, if one is willing to assume that they only (or mainly) travel along geodesics.

Another definition of centrality takes implicitly into account interactions along all possible paths in unweighted, undirected networks \([\text{Bonacich, 1987}]\). It is called the *eigenvector centrality*, and is defined for a given node as the corresponding component of the leading eigenvector of the adjacency matrix: if \( \mathcal{A} \mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \), with \( \mathbf{v}_1 = (v_1^{(1)}, v_1^{(2)}, \ldots, v_1^{(N)}) \), then \( v_1^{(i)} \) is the eigenvector centrality of node \( i \) (the subscript “1” appears because the eigenvalues are assumed to be ordered as \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_N \)).\(^7\) This choice can be motivated by the recursive procedure we now explain \( [\text{Newman, 2010}] \).

Let us start by assuming the centrality is initially \( x_i = 1 \) for all \( i \) (which is tantamount to the assumption of absolute ignorance on the network structure). A better estimation of the centrality is obtained by summing the centralities of the neighbors of each node,

\[
x_i' = \sum_j a_{ij} x_j,
\]

or, in matrix notation, \( \mathbf{x}' = \mathcal{A} \mathbf{x} \), where \( \mathbf{x} = (x_1, x_2, \ldots, x_N)^T \). Clearly, this boils down to a computation of the degree centrality. The procedure can be iterated, thus obtaining an improved centrality measure \( \mathbf{x}'' = \mathcal{A} \mathbf{x}' \) that integrates information both on the number of neighbors of a node and on the number of neighbors those neighbors themselves have. The idea is that a node can be central because its neighbors are central, because it has many neighbors or for both reasons. After \( t \) iterations,

\[
x_i' = \mathcal{A} \cdot \mathcal{A} \cdots \mathcal{A} \mathbf{x} = \mathcal{A}^t \mathbf{x}.
\]

Now, as \( \mathcal{A} \) is a symmetric matrix, its normalized eigenvectors \( \mathbf{v}_i \) constitute an orthonormal basis of \( \mathbb{R}^n \). Hence, it is possible to write \( \mathbf{x} = \sum_i c_i \mathbf{v}_i \) by an appropriate choice of \( c_i \). We then have,

\[
x_i' = \mathcal{A}^t \sum_i c_i \mathbf{v}_i = \sum_i c_i \lambda_i^t \mathbf{v}_i = \lambda_1^t \sum_i c_i \left[ \frac{\lambda_i}{\lambda_1} \right]^t \mathbf{v}_i.
\]

Clearly, as \( \lambda_i / \lambda_1 < 1 \) for \( i \neq 1 \),

\[
\lim_{t \to \infty} x_i' = c_1 \lambda_1^t \mathbf{v}_1,
\]

\(^7\)The fact that there can be no more than one eigenvector corresponding to \( \lambda_1 \) and the fact that \( \lambda_1 \geq 0 \) are both proven as part of the Perron-Fröbenius theorem.
and so the limiting centrality measure is simply proportional to the leading eigenvector of the adjacency matrix. As in practice only relative centrality measures are important, the value taken by the constant of proportionality is irrelevant. Notice that $x^t$ results from repeatedly multiplying $A$ and $x = (1, 1, \ldots, 1)^T$, and so its components must be non-negative. Given $\lim_{t \to \infty} x^t = c_1 \lambda_1^t v_1$, and the fact that eigenvectors are defined up to a constant factor, we shall take the eigenvector centralities $v^{(i)}_1$ to be non-negative also. Besides, if the eigenvectors are taken to be normalized (i.e. if $\|v_1\| = 1$), $v^{(i)}_1 \leq 1$ holds for all $i$.

Moreover, since by definition
\[ v^{(i)}_1 = \lambda_{1}^{-1} \sum_{j} a_{ij} v^{(j)}_1, \] (1.9)
the resulting eigenvector centrality is in agreement with the ansatz used in the iterative procedure (i.e. the centrality of node $i$ is proportional to the sum of the centralities of its neighbors, and therefore combines information on the number of neighbors and their own centrality).

We have introduced the concepts of degree centrality, betweenness centrality and eigenvector centrality, but many other definitions of centrality exists in the literature. For a more detailed discussion, see (Boccaletti et al., 2006) and (Newman, 2010).

### 1.2.3 Vulnerability measures

A notion related to centrality is that of *vulnerability*, which may be loosely defined as the tendency of a network to segregate into disconnected components, “break down” or malfunction whenever a given node is perturbed in one way or another, or removed altogether.\(^8\) Again, different measures exist that provide precise operational definitions.

For instance, the vulnerability of a node has been measured as the increase in the network diameter when a node is removed, and it has been quantified by the number of components into which the network fragments too, in (Albert et al., 2000). Vulnerability has also been measured in terms of a so-called performance function, and its variations upon the occurrence of damages, which include the removal of nodes and links. Among the various possible performance functions, the authors in (Latora and Marchiori, 2005)

---

\(^8\)This definition echoes the fact that vulnerability measures are important in applications, especially in technological and biological networks.
chose the network efficiency, defined as

\[ E = \frac{1}{N(N-1)} \sum_{i,j \in N, i \neq j} \frac{1}{d_{ij}}, \]  

(1.10)

where \( d_{ij} \) is the geodesic length between nodes \( i \) and \( j \). The efficiency quantifies the topological closeness of the nodes in a network, and is sometimes interpreted as the traffic capacity of a network in technological applications.

A formalization of the concept of vulnerability in terms of vulnerability functions that meet certain basic mathematical properties consistent with the intuitive notion of the vulnerability of a graph (somehow related to regularity and to the number of alternative edges existing between nodes) has also been proposed (Criado et al., 2005).

Several other vulnerability measures exist. As a matter of fact, centrality measures can also be interpreted as vulnerability measures in some contexts. For instance, a node of high betweenness (such as the red node in Figure 1.2) is generally a vulnerable piece of the network: its removal or that of some of the links that join it to the rest of the network may have dramatic effects. Moreover, a perturbation of its dynamics (in case we are dealing with a networked dynamical system) can crucially affect the dynamics on the whole network. This line of thought will be followed in Chapter 4, where a dynamical definition of vulnerability is presented and used to identify the nodes that are critical for the preservation of a collective dynamics.

### 1.2.4 Community structure

Given a network \((N, L)\), a community is a subgraph induced by \( N' \subset N \), where \( N' \) is a set of nodes that are tightly connected among them, much more so than they are connected with nodes lying outside the community (see Figure 1.3). This is the qualitative idea behind any kind of community structure detection algorithm or community strength index. Depending on the structural cohesion measure, the formal definitions will differ.

A stringent criterion for cohesion consists in requiring maximal connectivity. According to the corresponding definition, a link must exist between each possible pair of nodes in a subgraph for it to be considered a community. A clique (a maximal complete subgraph) is the name for such a strong type of community. Sometimes the definition is relaxed to \( n \)-cliques, which are maximal subgraphs in which the largest geodesic distance between nodes is \( n \) (a 1-clique is a clique, 2-cliques are subgraphs in which every node is

10
Figure 1.3: A network with a clearly visible community structure. Different colors show the membership relation of nodes in different communities.

reachable from another node through no more than one intermediary, and so on). Another possibility is to relax the definition by decreasing the number of nodes to which each node must be connected. Thus, a $k$-plex is a maximal subgraph containing $n$ nodes each of which is adjacent to $n - k$ or more peers.

Other definitions of a statistical nature have been proposed, which are in principle better suited for real networks and applications than those presented above in that they allow for larger irregularities. They are frequently used in conjunction with graph partitioning algorithms (see, for instance, (Newman and Girvan, 2004; Newman, 2004; Duch and Arenas, 2005; Fortunato, 2010)), since the detection of the underlying community structure of a given network is usually performed by partitioning a network in many different possible ways and computing the resulting community structure measure upon each partition. Eventually, the definitive network community structure is taken to be that which maximizes the considered measure. As an example of such a measure, in (Radicchi et al., 2004) the degree of each node $i$ belonging to the community induced by $\mathcal{N}'$ is split into a term quantifying the connectivity of the node within and outside $\mathcal{N}'$,

$$k_i = \sum_{j \in \mathcal{N}'} a_{ij} + \sum_{j \notin \mathcal{N}'} a_{ij}.$$  \hspace{1cm} (1.11)

Communities are characterized in several ways by the general requirement that the first term in this decomposition be larger than the second. While many more criteria exist in the literature, in the rest of this chapter we will limit ourselves to the widely-used community structure measures that are relevant in our work (specifically, in the work presented in Chapter 3).
Given a partition of a network into subgraphs, the *modularity index* provides a widely-used measure to quantify the adequacy of the partition as a potential community structure [Newman and Girvan, 2004]. The modularity index is the average over all subgraphs of the difference between the actual number of links connecting nodes of the subgraph among them and the number of links that would be expected by a totally random placement of links in the network (in the absence of any form of community structure). Mathematically, let $G_i$ be the subgraph to which node $i$ belongs, and let $C$ be the number of subgraphs (potential communities). The number of edges running between vertices of the same type is

$$\frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \delta(G_i, G_j),$$

(1.12)

where $\delta$ behaves as a Kronecker delta function: $\delta(G_i, G_j)$ equals 1 if $G_i = G_j$, and is zero otherwise. As the number of link ends is $2L$ by definition, the probability that in a network with links established at random there is a link incident on node $j$ is $k_j/2L$. Therefore, in such a network the expected number of links connecting $i$ and $j$ is $k_i k_j/2L$, and the expected number of links between nodes in the same community is

$$\frac{1}{2} \sum_{i,j=1}^{N} \frac{k_i k_j}{2L} \delta(G_i, G_j).$$

(1.13)

From the definition mentioned above, combining expressions (1.12) and (1.13), the modularity index $Q$ is defined as

$$Q = \frac{1}{2L} \sum_{ij} \left( a_{ij} - \frac{k_i k_j}{2L} \right) \delta(G_i, G_j).$$

(1.14)

In many instances in the literature, $Q$ considered as a real-valued function having as domain the set of all possible partitions of a network. The partition that maximizes $Q$ is considered to be the one that best captures the community structure in the network. The search for the optimal partition seems to be an NP-hard problem, as the number of possible partitions grows faster than any power of the system size, but several heuristic search strategies have been proposed that are sometimes inspired by optimization problems in statistical physics, such as the search for ground-state energies in spin glasses or the Potts model. In Chapter 3 we use the extremal optimization of modularity, proposed in [Duch and Arenas, 2005], in one case followed by the fast algorithm of [Newman, 2004]. We then deal with weighted networks, and therefore the $Q$ index must be then adapted in the expected way to such case. The extremal optimization algorithm is based on a divide algorithm based on previous algorithms that optimize a global variable by

---

9In Eq. (1.14) $w_{ij}$ and $s_i = \sum_j w_{ij}$ should be used instead of $a_{ij}$ and $k_i$. 

---

12
improving extremal local variables that involve coevolutionary avalanches, and has been shown to be both efficient and accurate. The fast algorithm is based on a classical agglomerative hierarchical clustering method (starting with one member in each community and joining communities in pairs by choosing the join that results in the highest $Q$) and finally looking for the best cut in the resulting dendrogram.

Once the $C$ communities of the network (where $C$ results from the graph partitioning algorithm itself) are found, we evaluate the modularity of the network partition (i.e., how well separated into communities the network is according to the best partition given by the algorithm) by measuring its modular cohesion:

$$MC := \frac{\sum_{\alpha=1}^{C} \sum_{i \in \alpha} \sum_{j \in \alpha} w_{ij}}{\sum_{\alpha=1}^{C} \sum_{\beta=1}^{C} \sum_{i \in \alpha} \sum_{j \in \beta} w_{ij}} \frac{\sum_{\alpha=1}^{C} \sum_{i \in \alpha} \sum_{j \in \alpha} w_{ij}}{N}$$

(1.15)

where $w_{ij}$ are the weights. Indexes $\alpha$ and $\beta$ run through all communities, while $i$ and $j$ are node labels, as usual. The modular cohesion $MC$ takes values in the range $[0, 1]$, and is equal to 1 when the network is partitioned into non-interacting modules (separate components). One way to put it is that the $MC$ measures the fraction of the total network wiring which is used to connect nodes belonging to the same community.

In this subsection, we have discussed the concept of modular (community) structure, and different ways to measure modularity in networks. But we have not explained why community structure is so important scientifically. The presence of community structure in real networks will be briefly treated in Subsection 1.4.3

1.3 The Laplacian matrix: graph spectra and diffusive coupling

The Laplacian matrix provides another matrix representation of a network, which is complementary to that provided by the adjacency matrix. The fact that a simple rule allows to go back and forth between the adjacency matrix and the Laplacian matrix indeed proves that they contain the same information, but some properties of networks may be more transparent using one representation or the other. In this section, we only consider the case of unweighted, undirected networks.
1.3.1 Definition and spectral properties

The Laplacian matrix or Kirchoff matrix of a network is defined as

\[ \Delta = \mathcal{D} - A, \]  

(1.16)

where \( \mathcal{D} := \text{diag}(k_1, k_2, \ldots, k_N) \). The elements of the matrix are therefore \( \Delta_{ij} = \delta_{ij}k_i - a_{ij} \). The Laplacian matrix has several interesting properties. Two obvious facts are its symmetry, \( \Delta_{ij} = \Delta_{ji} \), and the fact that it is a zero-row sum matrix, since for all \( i \)

\[ \sum_j \Delta_{ij} = k_i - \sum_j a_{ij} = 0. \]  

(1.17)

These and other properties set constraints on the eigenvalue spectrum of a Laplacian matrix, which is simply referred to as the (graph) spectrum. The next two results are very important in this respect. We follow the proofs given in (Newman, 2010).

The eigenvalues of the Laplacian matrix are real and non-negative.

Proof: The Laplacian matrix is symmetric, so it is diagonalizable and its eigenvalues are real. In the following we prove the non-negativity of the eigenvalues.

For a given network of size \( N \) and \( L \) links, let us designate one end of each link to be end 1, and the other to be end 2, in an arbitrary fashion. Let \( \mathcal{B} \) be a \( L \times N \) matrix with element \( B_{ij} = +1 \) in case end 1 of link \( i \) is attached to node \( j \), \( B_{ij} = -1 \) if end 2 of link \( i \) is attached to node \( j \), and \( B_{ij} = 0 \) otherwise.\(^{10}\) Each row of \( \mathcal{B} \) contains exactly one +1 and one −1. Now let us consider the sum \( \sum_k B_{ki}B_{kj} \). If \( i = j \), it becomes \( \sum_k (B_{ki})^2 \), which is equal to \( k_i \). In case \( i \neq j \) the only non-zero terms in the sum are −1, and they occur whenever link \( k \) connects nodes \( i \) and \( j \). As there can be no more than one such term in a simple graph, the value of the sum turns out to be −1 if \( i \) and \( j \) are connected, and zero otherwise. Therefore, \( \sum_k B_{ki}B_{kj} = \Delta_{ij} \), or, in matrix notation,

\[ \Delta = \mathcal{B}^T \mathcal{B}. \]  

(1.18)

This fact alone implies that the Laplacian matrix is positive semi-definite, and therefore the theorem is proved. It can also be proved by direct computation as follows. Let \( \mathbf{v}_i \) be a (normalized) eigenvector of \( \Delta \) with eigenvalue \( \lambda_i \). It follows from the above that

\[ \mathbf{v}_i^T \mathcal{B}^T \mathcal{B} \mathbf{v}_i = \mathbf{v}_i^T \Delta \mathbf{v}_i = \lambda_i \mathbf{v}_i^T \mathbf{v}_i = \lambda_i. \]  

(1.19)

\(^{10}\)This matrix is sometimes called the edge incidence matrix or simply the incidence matrix.
Since this is nothing but the square of the Euclidean norm of the vector $Bv_i$, $\lambda_i \geq 0$. □

The Laplacian matrix must have at least one zero eigenvalue.

**Proof:** Let us apply the Laplacian matrix on a vector $v_1 = \frac{1}{\sqrt{N}}(1, 1, \ldots, 1)$. Since

$$
\sum_j \Delta_{ij} v_1^{(j)} = \frac{1}{\sqrt{N}} \sum_j (\delta_{ij} k_i - a_{ij}) = 0,
$$

$v_1$ is an eigenvector of the Laplacian matrix with associated eigenvalue zero. □

Now let us suppose that a network has $C$ connected components. The Laplacian matrix is in block diagonal form, or at least can be put in block diagonal form simply by a permutation of rows and columns (i.e. by a mere change in the labeling of the nodes),

$$
\Delta = \begin{pmatrix}
\Delta_1 & 0 & \ldots \\
0 & \Delta_2 & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix}.
$$

Here, $\Delta_1, \Delta_2, \ldots, \Delta_C$ are the Laplacian matrices of the components. There are at least as many orthogonal eigenvectors with eigenvalue zero as components in the network, as all vectors $(1, 1, 1, \ldots, 0, 0, 0)$, $(0, 0, 0, \ldots, 1, 1, 1, \ldots, 0, 0, 0)^{11}$ that have ones in positions corresponding to a connected component and zeros elsewhere are indeed eigenvectors with associated eigenvalue zero. As we saw above, this rule extends to the case when the network is connected, so in general a network with $C$ components has at least $C$ eigenvectors with eigenvalue zero. In fact, it can be proved that the multiplicity of the eigenvalue zero always equals the number of components.

The $N$ eigenvalues of the Laplacian matrix are conventionally numbered in ascending order, so that $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_N$. It follows from the above that it must always be the case that $\gamma_1 = 0$. If the network is connected $\gamma_i > 0 \ \forall i > 0$. Otherwise $\gamma_1 = \gamma_2 = \ldots = \gamma_C = 0$ and $\gamma_i > 0 \ \forall i > C$, where $C$ is the number of components. We will make use of these spectral properties later, in Section 2.6 on the master stability function and in Chapters 4 and 5.

---

11The vectors are shown in non-normalized form for ease of notation.
1.3.2 Diffusive coupling

The Laplacian matrix is also very important in the study of the dynamics of complex networks. Here we only intend to give an explanation as to why the Laplacian is the fundamental network representation in several problems in network dynamics, and so the number of dynamical concepts used in the explanations are kept to a minimum. All the relevant concepts will be explained at length in Chapter 2.

Whenever a large ensemble of dynamical units interact, the full system can be represented by a dynamical complex network. This is a very high-dimensional dynamical system in which the interactions among the units follow the coupling scheme established by the network topology. In many cases, the rate of change of a variable that is “exchanged” between coupled units (neighboring nodes) is naturally expected to depend on the difference of the values that variable takes in them. Assuming that the dynamical system in each node is 3-dimensional to be concrete, if $i$ and $j$ are neighboring nodes and they are coupled through the state variable $y$, the rate of change of the state of the $i$-th node $x_i = (x_i, y_i, z_i)$ will depend on $y_j - y_i$. The simplest choice is to make it proportional to this difference, and it is known as diffusive coupling. The resulting time evolution is governed by

$$
\dot{y}_i = f_y(x_i) + \sigma \sum_{j \in N_i} (y_j - y_i),
$$

(1.22)

where $\sigma$ is a coupling strength or coupling constant, and the first summand simply represents the $y$-component of the vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ giving the internal dynamics of a node evaluated at $x_i$. By working out the right-hand side of the previous expression we get

$$
\dot{y}_i = f_y(x_i) + \sigma \sum_{j=1}^{N} a_{ij}(y_j - y_i) = f_y(x_i) + \sigma \sum_{j=1}^{N} a_{ij}y_j - \sigma y_i \sum_{j=1}^{N} a_{ij}
$$

(1.23)

$$
= f_y(x_i) - \sigma \sum_{j=1}^{N} \Delta_{ij}y_j.
$$

The equations of motion in the case of diffusive coupling can be written in a compact form by means of the Laplacian matrix.

---

12 If it is a physical magnitude, this guarantees invariance under changes of origin in the measurement scale.
13 We assume that we are in the presence of a time-continuous dynamical system, but the discussion that follows applies to discrete maps as well.
Moreover, we will see how properties related to the stability of dynamical regimes can be derived from properties of the Laplacian matrix, which justifies its being considered the more useful representation of a network in this context. For the moment let us just mention that the zero-row sum condition guarantees that a network of identical dynamical systems can synchronize, since $x_1 = x_2 = \ldots = x_N$, with all nodes evolving according to the autonomous laws of motion, is a possible solution of the dynamics. This is a property that accounts for the popularity of diffusive coupling in studies of collective dynamics on networks (and even in those devoted to the synchronization of a few dynamical systems with regular coupling topologies), and is already obvious from Eq. (1.22).

To conclude, let us briefly comment on the origin of the term. Several variants of the physical phenomenon known as *diffusion* are considered in statistical and thermal physics. The most typical case refers to the process whereby particles in a fluid spread from higher concentration regions to lower concentration regions. By analogy, we see that in Eq. (1.22) $y$ can be said to flow into $i$ from neighboring nodes that have a higher “concentration” and flow out of $i$ to nodes that have lower “concentration”.

### 1.4 Ubiquitous topological properties

The availability of ever increasing sets of empirical data from all kinds of complex systems, together with the development of powerful data analysis tools intended to characterize the pattern of interconnections found in them, have been instrumental in the development of network theory. Physical systems (such as spin networks, or systems exhibiting percolation phenomena), biological systems (genetic networks, neuronal networks, and networks of biochemical reactions), technological systems (transportation networks, communication networks, the Internet and the World Wide Web) and social networks (networks of acquaintances, collaboration networks) have been studied using the tools of network theory (see (Boccaletti et al., 2006) and (Newman, 2010) for a review).

The conclusions drawn from the literature point to the remarkable fact that there exist a few essential topological properties that are frequently observed both in natural and manmade systems, and therefore seem to be important for the collective behavior of inherently different networks. These topological properties differ greatly from those found in the traditional objects of study of mathematical graph theory, such as regular lattices and random graphs. For this reason, great efforts on the part of theoreticians during the last 15 years have been devoted to the search of network models exhibiting
those properties, and to check their validity with empirical observations from real-world networks, an endeavor that continues today, as our work presented in Chapter 3 demonstrates. In the following we briefly discuss the main properties that have been seen to be common to a large variety of networks from different fields. For a more comprehensive discussion see (Boccaletti et al., 2006) and references therein.

1.4.1 The small-world property

Frequently, the existence of shortcuts communicating distant areas in a large network decreases dramatically the average geodesic distance between randomly chosen nodes. Indeed, the empirical finding that in most real networks there is a surprisingly short path between any two nodes (i.e. geodesic distances turn out to be much smaller than one would intuitively expect) has received much attention in the last decades. A scientific approach to this issue originated in the social sciences, where it was first investigated in the 1960s by Milgram in a series of experiments to estimate the number of steps needed to connect randomly chosen people living in distant geographical areas by a chain of acquaintances.

Following these and other pioneering studies, Watts and Strogatz in a seminal paper introduced the so-called small-world property of networks, and showed its relevance in biological and technological applications (Watts and Strogatz, 1998). Small-world networks are networks that combine a high level of clustering with a small average path length. The authors show that the existence of shortcuts in a network allows for the coexistence of these two apparently incompatible features. Indeed, as illustrated in Figure 1.4, adding a few links between distant regions to a highly clustered regular network may be sufficient to drastically reduce the average geodesic distance without significantly alter the high clustering in the network.

In Figure 1.4 the clustering is measured in terms of the clustering coefficient (Watts and Strogatz, 1998), expressing how likely it is for two neighbors of a given node to be adjacent to each other. But the small-world property is a robust property of networks that can be detected by different measures. For instance, the small world property of networks can also be investigated in an efficiency-based formalism, in which a small-world network is a network showing a high value of (global) efficiency as defined in Eq. (1.10), and also a high value of local efficiency, defined as

$$E_{\text{loc}} = \frac{1}{N} \sum_{i \in \mathcal{N}} E(G_i), \quad (1.24)$$
Figure 1.4: A random rewiring of a regular ring lattice of a fraction \( p \) of the links reduces drastically the average geodesic length, while keeping the clustering almost undiminished. a) Random rewiring procedure. The actual network size considered is \( N = 100 \), and the average degree is \( \langle k \rangle = 10 \). b) Average path length \( L(p) \) and clustering coefficient \( C(p) \) as a function of \( p \), normalized with respect to the case \( p = 0 \). The small-world phenomenon is said to occur in the interval where (for \( p \) already quite small) \( L(p) \) drops abruptly while \( C(p) \) remains high. Adapted by permission from Macmillan Publishers Ltd: Nature, [Watts and Strogatz, 1998], copyright © (1998).

where \( E(G_i) \) is the efficiency of \( G_i \), the subgraph induced by \( \mathcal{N}_i \) (the set of nodes adjacent to \( i \)). That being said, a clear-cut mathematical threshold does not exist in any formalism, and the issue of the small-worldness of networks must generally be addressed in statistical terms.

1.4.2 Scale-free networks

In the traditional models of graph theory, such as random graphs and regular lattices, most nodes are (at least approximately) topologically equivalent. Indeed, whereas the degree follows a delta distribution in regular lattices, in Erdős-Rényi random graphs (see Section 1.3) the degree distribution turns out to be binomial (Poisson if the graph is
sufficiently large), and therefore has a well-defined dispersion which is relatively small for large networks, the ratio of the standard deviation over the mean varying as $1/\sqrt{N}$. The pioneers in the analysis of networks expected to find such distributions in their datasets. However, it was found that a large fraction of the real networks that were analyzed had a power-law scaling in the degree distribution

$$P(k) \propto k^{-\gamma},$$

(1.25)

where typically $1 < \gamma < 3$, the exact value being system dependent. The average degree $\langle k \rangle$ of such distributions is well-defined, but the second moment $\langle k^2 \rangle$ of the distribution diverges if the upper integration limit is unbounded, as for large $k_{\text{max}}$ we have

$$\int_{k_{\text{min}}}^{k_{\text{max}}} dk \, k^2 \, P(k) \sim k_{\text{max}}^{3-\gamma},$$

(1.26)

and this results in a divergent variance $\sigma^2 = \langle k^2 \rangle - \langle k \rangle^2$. Networks with such degree distributions are called scale-free networks, since a power-law is the only type of functional form that remains invariant up to a multiplicative factor under a rescaling of the independent variable (the solution to $f(\alpha x) = \beta f(x)$ for all $\alpha$ and $\beta$ must be a power law).

Power laws play an important role in the statistical mechanics of phase transitions and in fractal geometry, and their importance for characterizing empirical distributions in economics, sociology and other fields, has been known for over a century (for a review, see (Newman, 2005)). In the context of network theory, scale-free networks are characterized by a highly inhomogeneous degree distribution, resulting in the presence of a few hubs with a very high degree, and a large number of scarcely connected elements. This fact has fundamental implications for the propagation of information along a networked dynamical system, as will be shown in Chapters 4 and 5.

1.4.3 Mesoscale structure

The scale-freeness of a network, as a property that depends solely on the degree distribution, is a global property. The small-world property is also global in that the average geodesic distance and the clustering (or the efficiency) ultimately depend on averages over the whole network. Such properties are frequently classified as macroscale properties. On the other hand, topological measures ascribed to particular nodes, such as the degree or the node betweenness, can be considered to be microscale properties. The terminology is analogous to that used in statistical mechanics, where macroscopic thermodynamic prop-
erties are seen to emerge from an extremely large number (on the order of Avogadro’s number) of microscopic dynamical degrees of freedom subject to the laws of classical or quantum mechanics.

Nevertheless, there are also structural properties of interest that lie somewhere in between these two extremes. **Mesoscale** properties belong to a topological realm that is neither at the node level, nor at the whole network level, as they characterize subsets of nodes of a typical size scale. A typical example of mesoscale features, **motifs** are connection patterns (i.e. connected subgraphs induced by a small subset of nodes) that occur in a network with a relative frequency much higher than that found in a modified network with the same \( N \), \( L \) and \( P(k) \) but with links distributed at random.

The most celebrated mesoscale objects of study in network theory are the communities, which were discussed in Section 1.2. In an early paper on the subject, Girvan and Newman in [Girvan and Newman, 2002] reported the presence of community structures in social networks\(^{14}\) and in food webs of marine organisms. Since then, communities have been found in many kinds of real networks (see [Porter et al., 2009] for a good introduction and [Fortunato, 2010] for a comprehensive review). Indeed, the presence of a community structure can be considered to be, together with the small-world property and the power-law degree distribution, the single remaining feature that appears to be common to a large variety of real networks from many different disciplines.

### 1.5 Network models I: Erdös-Rényi random graphs

A **random graph** is a network that has some fixed properties (e.g. a given degree distribution) but aside from that the configuration of the links is totally random. A given random graph is considered as an individual **realization** within the **statistical ensemble** of networks with the same properties. Most of results concerning random graphs are thus of a statistical character. Accordingly, when discussing the value of property \( a \) for a generic random graph model with associated ensemble \( G \), one implicitly refers to the **ensemble average** over all possible realizations \( \langle a \rangle \), defined as

\[
\langle a \rangle = \sum_{g \in G} P(g) a(g),
\]

\(^{14}\)The word **community** itself refers to a social context, where the existence of relatively closed groups of people sharing family ties, friendship, a work environment, etc., is the result of community formation processes facilitated by the fact that the acquaintances of one’s acquaintances are likely to meet one another.
where \( a(g) \) is the value of the property for the network \( g \), and \( P(g) \) is the probability assigned to \( g \) within the ensemble distribution.\(^{15}\) Typically, all realizations being given the same weight, \( P(g) \) is uniformly distributed.

The Erdős-Rényi (ER) random graph is perhaps the simplest and most influential model in network theory, and it marks the beginning of the systematic study of random graphs. It was proposed in 1959 within the context of mathematical graph theory (Erdős and Rényi, 1959). Even though ER graphs do not exhibit many of the properties generically found in real-world networks, they remain of fundamental importance in network theory due to their simplicity, the fact that their properties are exactly known, and because of their suitability to be employed as benchmarks for more realistic models, or null cases in the testing of hypothesis on network heterogeneity. In this section we briefly review the motivation behind the ER random graph, its construction procedure, and some of its properties.

The original aim of the ER random graph model was the probabilistic study of the macroscale structure of graphs and its dependence on an increasing number of (randomly placed) links. In its simplest form, an ER graph is a random graph in which the number of nodes \( N \) and the number links \( L \) is fixed. The construction procedure is very simple: starting with \( N \) disconnected nodes, pairs of randomly selected (not yet connected) nodes are connected until the number of links in the network is \( L \). The resulting graph is just one element in the statistical ensemble of random graphs with size \( N \) and \( L \), denoted by \( G_{N,L}^{\text{ER}} \). There is an alternative ER model, consisting in connecting pairs of disconnected nodes with a probability \( 0 < p < 1 \), so that \( L \) is not fixed, the probability of obtaining a graph with a given \( L \) being \( p^L (1-p)^{N(N-1)/2-L} \). In this model \( G_{N,p}^{\text{ER}} \) is the ensemble of random graphs with fixed \( N \) and \( p \). In this section, we focus on this second variant of the model, which is more suitable for an analytical study of its properties, but in Chapters 4 and 5 the networks used for numerical computation with ER topology will be random realizations of the first construction procedure. Both models are related in a way that is analogous to the relationship between the canonical and grand canonical ensembles of statistical mechanics, and indeed they also coincide in the thermodynamic limit \( N \to \infty \) if the limit is taken for fixed \( \langle k \rangle \), which corresponds to \( 2L/N \) in the first model and to \( p(N-1) \) in the second model.

The macroscale structural properties of ER random graphs are very sensitive to variations in \( p \). Erdős and Rényi proved indeed the existence of the following dramatic changes at the critical probability \( p_c = \frac{1}{N} \) (corresponding to \( \langle k \rangle_c = 1 \) for large network size): if

\(^{15}\)Of course, \( P(g) \geq 0 \) for all \( g \) and \( \sum_{g \in G} P(g) = 1 \).
$p < p_c$ almost surely (with probability tending to 1 as $N \to \infty$) the graph has no component that scales at greater rate than $O(\log N)$, and no component has more than one cycle; if $p = p_c$ then almost surely the largest component has size $O(N^{2/3})$; if $p > p_c$ the graph has a component that scales as $O(N)$ with a number of cycles that also is $O(N)$, and no other component is larger than $O(\log N)$ or has more than one cycle. The transition at $p_c$ has the usual features of a second order phase transition.

The probability that any node $i$ has $k$ links follows a binomial distribution

$$P(k_i = k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}. \quad (1.28)$$

As known from probability theory, for fixed $\langle k \rangle$ and large $N$ this degree distribution is well approximated by a Poisson distribution

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}, \quad (1.29)$$

which accounts for the name Poisson random graphs that ER random graphs are sometimes given. ER graphs are uncorrelated graphs by definition, so $P(k|k') = P(k)$. ER random graphs being the most exhaustively studied models of mathematical graph theory, many other properties can be found in the literature (see [Boccaletti et al., 2006; Newman, 2010] and references therein).

1.6 Network models II: Scale-free networks

Given the frequent observation of highly heterogeneous, power-law degree distributions in empirical network data from a wide range of scientific fields, great efforts have been devoted to the proposal and study of network models able to reproduce the scale-free property. In this section, we will limit ourselves to the discussion of two of the simplest and most widely adopted scale-free network models, but many refinements and modifications have appeared in the literature (see [Boccaletti et al., 2006]).

The first model we consider is simply a random graph with a power-law degree distribution. According to the configuration model (CM) introduced by Bender and Canfield in [Bender and Canfield, 1978], one can construct a generalized random graph for a given $P(k)$ based on a degree sequence $D = \{k_1, k_2, \ldots, k_N\}$ such that the fraction of vertices with degree $k'$ tends to $P(k')$ as $N \to \infty$ (the degree sequence is a random $N$-dimensional vector realization generated with $P(k)$ with the constraint that $\sum_i k_i = 2L$). The model
considers the ensemble $G_{N,D}^{CM}$ of all graphs with $N$ nodes and given degree sequence $D$, each of them having the same probability of occurrence. Pictorially speaking, an element of $G_{N,D}^{CM}$ is generated by assigning $k_i$ semi-links to node $i$, for all possible $i$, and then forming links by a uniformly random connection of semi-links. This generates with equal probability each graph compatible with $D$. Several important analytical results on CM generalized random graphs exist, but in this thesis we will just use this as a method for graph generation, namely, for the generation of CM scale-free (CMSF) networks, in which $D$ follows $P(k) \propto k^{-\gamma}$.

The second model is a model of network growth. There are many examples of real networks where the structure changes by a dynamical evolution of the system, and it is natural to assume that some relationship must exist between the dynamical mechanisms leading to the formation of a network and its main topological properties. Growth processes are simple rules derived from experience that are assumed to reflect the mechanisms underlying the formation of network topologies. We concentrate below on the network growth model proposed by Barabási and Albert in \cite{Barabasi1999}, which was a seminal paper in network theory that gave impetus to the study of scale-free topologies.

The Barabási-Albert (BA) model is a network growth model that is inspired by the fact that in some real networks where new elements are constantly being incorporated, the highly connected nodes acquire new links at higher rates than the low-degree nodes. An element of the ensemble of BA networks with fixed $N$ and $L$, denoted by $G_{N,L}^{BA}$, is constructed as follows. The network starts with $m_0$ isolated nodes, and at each time step $n = 1, 2, \ldots, N - m_0$ a new node $j$ with $m \leq m_0$ links is added to the network. According to the preferential attachment rule, the probability that $j$ establishes a link with an existing node $i$ is linearly proportional to the degree of $i$ at the time of the addition of node $j$,

$$P(j \rightarrow i) = \frac{k_i}{\sum_l k_l}. \quad (1.30)$$

At time $n$ the network will have $N = m_0 + n$ nodes and $L = mn$ links, and therefore the average degree for sufficiently large times is $\langle k \rangle \rightarrow 2m$. The BA model has been treated by mean-field approximations, and exactly by means of rate equation and master equation approaches. In the limit $n \rightarrow \infty$, it has been shown that $P(k) \propto k^{-\gamma}$ with exponent $\gamma = 3$, and for this reason the networks that result from the BA model are

---

16 In fact, each configuration has a degeneracy $\prod_i (k_i!)$, since the $k_i!$ permutations of semi-links for each node $i$ are indistinguishable.

17 The classical example is the World Wide Web, where each node is a website that has links to other websites. It is an obvious fact that the probability of a new site linking an already highly connected site on average is higher than the probability that it connects to a site that has few links pointing to it (and consequently probably has a modest number of visits).
called BA scale-free (BASF) networks. The perfect proportionality in (1.30) has been shown to be essential for obtaining a scale-free topology. See (Boccaletti et al., 2006) and references therein, for extensions of the BA model, a model developed by Price in the context of bibliographic networks that predates the BA model by two decades and presents some similarities with it, and other interesting topics on growth models able to reproduce scale-free topologies. Several generalizations of the preferential attachment mechanism in the context of heterogeneous networks, where nodes are characterized by internal properties, have been proposed in (Santiago, 2007).
Chapter 2

Introduction II: The dynamics of complex networks

In this chapter we focus on the dynamics of complex networks. We first deal with some fundamental concepts from the mathematical theory of dynamical systems and nonlinear dynamics, and with the synchronization of dynamical systems. Once this background material is presented, some aspects of the synchronizability and the controllability of networked dynamical systems are treated, and adaptive networks are briefly discussed. All these subjects depend in one way or another on the interplay between the structure and the dynamics of networks. We also present in some detail the Rössler system (a chaotic oscillator) and the Kuramoto model (a model of coupled phase oscillators), celebrated examples of dynamical systems that will reappear in later chapters, when we present our results on the collective dynamics of complex networks of oscillators.

2.1 Dynamical systems

A dynamical system consists of a set of possible states and a rule for time evolution. The rule for time evolution deterministically specifies how the state of the system, as codified by a set of state variables, changes in time from a given initial condition (i.e. the initial state of the system). The concept is much more familiar than it may first appear. For instance, the two-body problem of celestial mechanics that deals with how a planet moves under the gravitational pull of a more massive star, deals ultimately with a dynamical system (where states are determined by positions and momenta, and the time evolution is given by the integration of the equations of classical mechanics), and indeed its study
by Isaac Newton can be taken as the birth of dynamical systems theory. Indeed, any classical physical system with a finite number of degrees of freedom whose evolution is given by differential equations can be considered as a dynamical system, and also simpler systems, such as discrete-time systems or systems with only a finite number of states, are dynamical systems. Examples abound in physics, chemistry, engineering, biology, economics, etc., and extensive use of dynamical systems theory has been made in those areas during the last 50 years.

More formally, a dynamical system is a triple \((M, \Phi, T)\) where \(M\) is the phase space or state space, a set whose elements are the possible states of the system, and the dynamics is ruled by \(\Phi\), which is a one-parameter semigroup or group of transformations that act on \(M\) with parameter \(t \in T\), the time. For each \(t\), there is a map \(\phi_t : M \rightarrow M\) that specifies how states have evolved from the initial state corresponding to \(t = 0\) to the present time \(t\). If \(\Phi\) is a semigroup, then \(T\) can be \(\mathbb{R}^+\) or \(\mathbb{Z}^+\), depending on whether the system is a time-continuous or a time-discrete dynamical system, respectively. Its elements satisfy

\[
\phi_t \circ \phi_s = \phi_{t+s} \quad \quad \phi_0 = \mathbb{I}.
\]

If \(\Phi\) is a group, \(T\) is \(\mathbb{R}\) or \(\mathbb{Z}\), and the action of the transformations is invertible:

\[
\phi_{-t} \circ \phi_t = \mathbb{I},
\]

where \(\phi_{-t} = (\phi_t)^{-1}\). We refer to the set of phase space points followed by the time evolution of the system \(\{\phi_t x_0 : t \in T\}\) as an orbit or trajectory, starting from the initial condition \(x_0 \in M\). For the purpose of illustration, let us complement these abstract definitions with some examples.

The simplest kind of dynamical system one can think of is a discrete-time system with a finite number of states, such as the one represented pictorially in Figure 2.1. The balls indicate the possible states of the system at any given time \(n\), \(s_n\), and the arrows indicate the update rule \(s_{n+1} = f(s_n)\), where \(f\) corresponds to the one-iteration map \(\phi_1\). Here \(M = \{1, 2, 3, 4\}\), \(T = \mathbb{Z}^+\) (the dynamics cannot possibly be invertible, as the map \(\phi_{-1}\) is not well defined for state 4) and \(f : M \rightarrow M\) is as specified by the arrows in the picture. Of course, \(\phi_1 = f\), \(\phi_2 = f \circ f = f^2\), and \(\phi_n = f^n\) for \(n > 2\). This particular case is relatively simple, as the dynamics leads the system to eventually reach state 4 and remain there. One possible orbit (the one that starts from state 1) is \(1, 2, 4, 4, 4, \ldots\)

\(^{1}\)The modern theory of dynamical systems, however, is associated with the name of Henri Poincaré, for his many contributions, including also his mathematical studies in celestial mechanics, on topics such as the stability of the Solar System and the three-body problem.
A schematic representation of one iteration of a discrete-time dynamical system with a finite number of states. The balls represent the four possible states. The system starts at a given initial state and evolves deterministically according to the arrows.

A less trivial example is that of a discrete map on the real line, as defined by $x_{n+1} = f(x_n)$, where $f : \mathbb{R} \to \mathbb{R}$. In this case, $M = \mathbb{R}$, and $\Phi = \{f^n : n \in T\}$, where $T = \mathbb{Z}$ if the function is invertible and $\mathbb{Z}^+$ otherwise. Here, $f^n$ is the function resulting from composition of $f$ with itself $n$ times for positive $n$, while, in case invertible functions are considered, $f^{-n}$ results analogously from the composition of $f^{-1}$ with itself. Assuming the system is in state $x_0$ at $n = 0$, then it will evolve deterministically $x_1 = f(x_0)$, $x_2 = f(x_1) = f^2(x_0)$, $x_3 = f(x_2) = f^2(x_1) = f^3(x_0)$, and so on. In higher dimensions (i.e. if $M = \mathbb{R}^p$ where $p = 2, 3, \ldots$), or already in one dimension if the map is non-invertible, the system may exhibit very complex, so-called chaotic dynamics even if the time evolution is governed by the iteration of a very simple function.

These examples are the simplest one can find, but this thesis will focus on another category of dynamical systems, which is the most important by far in physics and science in general, namely, that of continuous-time dynamical systems with a continuum of states. A continuous-time smooth dynamical system is a dynamical system such that $M$ is a finite-dimensional smooth manifold and the collection of maps $\phi_t : M \to M$ form a one-parameter family of diffeomorphisms that depend smoothly on the real parameter $t$. In this thesis, the time evolution will always be implicitly given by a system of ordinary differential equations (ODEs), as is frequently the case in dynamical systems theory and in many branches of science since the early days of physics, beginning with Newton’s second law of motion. As in most of the literature written by physicists and in books on the theory of ODEs, instead of using the coordinate-free language of differential geometry, we will assume that a single coordinate chart covers the full manifold, and simply write the mathematical expressions in terms of coordinates. Accordingly, the states on
the $n$-dimensional manifold $M$ will be treated as points in $\mathbb{R}^n$ or a connected open subset thereof (the image set of the coordinate chart), given by the $n$-tuple of state variables $(x_1, x_2, \ldots, x_n)^T$. Moreover, we will always consider invertible systems, so $T = \mathbb{R}$.

Let $f \in C^\infty(M, \mathbb{R}^n)$, where $M \subset \mathbb{R}^n$, be a function that implicitly defines the rule for the time evolution of an $n$-dimensional dynamical system by the initial value problem

$$\dot{x} = f(x) \quad x(0) = x_0,$$

$x_0 \in M$ being a given initial condition.\footnote{We do not write $\dot{x} = f(x, t)$ because we only consider autonomous systems for which the time evolution purely depends on the state of the system. As laws of physics and a vast majority of mathematical models are invariant under time translations, an explicit dependence on time generally is linked to the fact that only a subsystem of a full autonomous system is studied. Also, there is no loss in generality whatsoever in considering first-order systems of ODEs, since, as discussed in any book on differential equations, a higher-order system can always be expressed as a first-order system with the appropriate definition of auxiliary variables.} According to the fundamental existence and uniqueness theorem of ODE theory (sometimes referred to as the Picard-Lindelöf theorem or the Lipschitz-Cauchy theorem), a solution exists and is uniquely defined on some maximal time interval of existence $(-a, a)$. For our purposes, all solutions will always be bounded on $M$ for all $t$ and remain well-defined in $(-\infty, \infty)$. The theorem also shows that solutions in addition depend smoothly on the initial condition (see any book on ODE and/or dynamical systems, such as (Arnold, 2006) or (Perko, 2006)).

Geometrically, the vector field $f$ is such that solutions to (2.3) are curves on $M$ that are tangent to $f$ at each point. Indeed, a solution to the initial value problem (2.3) is nothing but the orbit associated to $x_0$, $\{\phi_t(x_0) : t \in T\}$. This is the one-parameter group of diffeomorphisms previously mentioned in the general context of dynamical systems, and indeed the following conditions hold for the solutions of the ODE system

(i) $\phi_0(x) = x$

(ii) $\phi_s(\phi_t(x)) = \phi_{s+t}(x)$ for all $s, t \in \mathbb{R}$

(iii) $\phi_{-t}(\phi_t(x)) = \phi_t(\phi_{-t}(x)) = x$ for all $t \in \mathbb{R}$.

We thus see that (2.3) implicitly defines a dynamical system. Indeed, $\Phi : \mathbb{R} \times M \to M$, where $(t, x) \mapsto \phi_t(x)$ is the family of solutions for all possible phase-space points, is called the flow of the dynamical system. The fact that phase space points are mapped by the flow into phase space points in a manner reminiscent of the evolution in time of material
particles in a fluid flow accounts for the choice of this terminology (see Figure 2.2).

![Figure 2.2: The flow \( \phi_t \) of a dynamical system. A given initial condition \( x_0 \) is mapped into \( \phi_t(x_0) \) at time \( t \). A neighborhood \( U \) of \( x_0 \), is mapped into \( \phi_t(U) \), which is a neighborhood of \( \phi_t(x_0) \).](image)

In short, the time evolution of the dynamical system is implicitly given by the vector field \( f : M \to M \), whereas the explicit time evolution is formally given by \( \phi_t \), whose general properties are known. Nevertheless, the explicit form of the flow can only be obtained by calculating the concrete integral curves (i.e. the parametrized curves that are tangent to the vector field at every point on \( M \)) which provide the specific time evolution of the system. This is tantamount to solving the system of ODEs. General procedures for finding solutions are available in the case of linear systems, that is, systems governed by a linear system of ODEs (see, for instance, (Perko, 2006)). However, most systems of interest are nonlinear systems, and, except for a small number of cases, for the vast majority of nonlinear dynamical systems there are no strategies of general validity for finding solutions. To make things even more difficult, frequently the orbits defy any form of analytical closed form in terms of elementary functions or series representation of any kind. In those cases, concrete orbits can only by obtained numerically, and yet important knowledge on these systems, their asymptotic behavior and stability properties is often derivable from an analytical treatment by using some of the concepts we will discuss below or several others that are outside the scope of this thesis (for a detailed account see (Perko, 2006; Teschl, 2012)).

A set \( U \subseteq M \) is said to be invariant if, for every \( x \in U \), \( \phi_t(x) \subseteq U \) for all \( t \). If a phase-space point is found such that \( \phi_t(x) = x \) for all \( t \), \( x \) is said to be a fixed point or
equilibrium point \( \{x\} \) being the simplest type of invariant set there is). If, for a given \( x \in M \), there is some \( T > 0 \) such that \( \phi_T(x) = x \), the minimum \( T \) for which this holds is called the period of the closed orbit passing through \( x \). An orbit is called periodic if one point of the orbit is periodic, and hence all points are, as it follows by the composition rule of the flow that \( \phi_{t+T}(x) = \phi_t(x) \). Clearly, a periodic orbit is also an invariant set of the dynamics.\(^3\)

A limit cycle is an isolated periodic orbit. This is one of the most highly studied and frequently encountered invariant sets. In this definition, isolated means that orbits in a neighborhood containing the limit cycle are not closed, and therefore they either spiral towards the limit cycle or away from it. It is not difficult to show that a limit cycle can only exist in nonlinear systems. A stable limit cycle is a limit cycle that attracts all neighbouring orbits, whereas they are repelled from unstable limit cycles. In the frequently encountered case of a two-dimensional dynamical system (where a limit cycle partitions \( M \) into the interior set contained within the limit cycle, and the outside region), a half-stable limit cycle can be defined to be a limit cycle that attracts orbits from one side, and repels orbits from the other, as shown in Figure 2.3. Systems with a stable limit cycle are of very high interest in many branches of science, as they exhibit self-sustained oscillations. An approach to the study of the collective dynamics of coupled self-sustained oscillators is presented in Section 2.5 on the Kuramoto model.

The stability properties of an invariant set are of great importance, as they explain what happens if an orbit starts sufficiently close to the set, but not exactly on it. If, for example, any orbit starting in the vicinity of a fixed point ultimately leaves and goes to other phase-space regions, that fixed point, as a zero Lebesgue measure set, is never going to be observable in experimental measurements or even in numerical simulations, not even asymptotically. The same applies to unstable limit cycles or other unstable sets. We will focus on the stability properties of fixed points, as it is the simplest case, and all concepts can be generalized to more complex sets without difficulty.

A fixed point \( \bar{x} \) is said to be stable if for any given neighborhood \( U(\bar{x}) \) (i.e. any connected open set such that \( \bar{x} \in U(\bar{x}) \)), there exists another neighborhood \( V(\bar{x}) \) contained in it, \( V(\bar{x}) \subseteq U(\bar{x}) \), such that any solution starting in \( V(\bar{x}) \) remains in \( U(\bar{x}) \) for all \( t \geq 0 \). The set is said to be asymptotically stable if it is stable, and if there is a neighborhood

\(^3\)Orbits can indeed be classified as: fixed points (which can be considered to be curves of period zero), regular periodic orbits (curves composed of periodic points with positive period) and non-closed orbits (with no periodic points).
Figure 2.3: A dynamical system defined in the plane \((M = \mathbb{R}^2)\) with a half-stable limit cycle. \(C\) is a half-stable limit cycle with an unstable fixed point in the inside region. Orbits starting in the inside region asymptotically converge to \(C\), whereas orbits starting in the outside region are repelled from it.

The stability of a flow near a fixed point can be inferred from the linearized system around the fixed point. Let the resulting linear equation be written as

\[
\dot{x} = Ax,
\]

which represents the time evolution of linear variations around \(\bar{x}\). Here, \(A\) is an \(n \times n\) matrix that corresponds to the tangent map evaluated at the fixed point \(\bar{x}\). \(A\) can be assumed to be given in Jordan canonical form, as the concepts regarding stability are invariant under linear changes of coordinates,\(^4\) and the decomposition based on Jordan blocks implies that the asymptotic behavior of the system is determined by the real part of the eigenvalues of the matrix.\(^5\) Accordingly, the full (linear) tangent space can be partitioned into a stable linear manifold, which is the subspace spanned by the generalized eigenvectors corresponding to eigenvalues with negative real part, an unstable linear manifold, which is spanned by the generalized eigenvectors corresponding to eigenvalues with positive real part, and a center linear manifold, which is spanned by the generalized eigenvectors corresponding to eigenvalues with zero real part. The center linear manifold is generally not stable under small perturbations, and so, for the sake of simplicity, it is often assumed to be trivial.

\(^4\)In the original coordinate frame, this matrix is the Jacobian matrix of the vector field.

\(^5\)For a nice treatment of systems of linear differential equations, see (Perko, 2006).
we will neglect it and focus on the case where the linear space around a fixed point is
spanned by the stable linear manifold and the unstable linear manifold. A fixed point
is asymptotically stable if and only if all the eigenvalues have negative real part. In any
other case, the orbits in the vicinity of the fixed point approach the fixed point only along
the stable linear manifold.

The previous results can be extended to the nonlinear case. Thus, the stable (unstable)
manifold \( W^s(\bar{x}) (W^u(\bar{x})) \) of a fixed point \( \bar{x} \) in a nonlinear dynamical system is defined
to be the set of all points convergent to \( \bar{x} \) for \( t \to \infty \) (\( t \to -\infty \)):

\[
W^s(\bar{x}) = \{ x \in M : \lim_{t \to \infty} \| \phi_t(x) - \bar{x} \| = 0 \}
\]
\[
W^u(\bar{x}) = \{ x \in M : \lim_{t \to -\infty} \| \phi_t(x) - \bar{x} \| = 0 \}.
\] (2.6)

Both turn out to be manifolds that are invariant under the flow, and it can be shown
that they are tangent to their linear counterparts in a neighborhood of \( \bar{x} \), their tangent
space being moreover of the same dimension.

The limit set of a given phase-space point \( x \in M \) is the set of those points \( y \in M \)
for which a temporal sequence \( \{ t_n \} \) exist such that \( \phi_{t_n}(x) = y \) for \( t_n \to \pm \infty \). This is a
concept of the utmost importance, because one of the fundamental issues in dynamical
systems theory is the long-time behavior of dynamical systems. Limit sets are invariant,
since if \( \phi_{t_n}(x) \to y \), then \( \phi_{t_n+t}(x) = \phi_t(\phi_{t_n}x) \to \phi_t(y) \). On the other hand, it is ob-
vious that all invariant sets are at least limit sets of orbits that start on them by definition.

An invariant set \( A \) is attracting if there is some neighborhood \( U \) of \( A \) such that \( U \)
is positively invariant (\( \phi_t(x) \subseteq U \) for all \( x \in U \) and all \( t > 0 \)) and \( d(\phi_t(x), A) \to 0 \) as \( t \to \infty \) for all \( x \in U \), where the distance between a point \( y \in M \) and a set \( S \subset M \) is
defined to be \( d(y, S) = \inf \{ \| y - y' \| : y' \in S \} \). The basin of attraction of \( A \), \( B(A) \), is
defined as

\[
B(A) = \{ x \in M : \lim_{t \to \infty} d(\phi_t(x), A) = 0 \}. \] (2.7)

Equivalently,

\[
B(A) = \bigcup_{t<0} \phi_t(U). \] (2.8)

Yet a third characterization is to define \( B(A) \) to be the set of points in phase space whose
limit set for \( t \to +\infty \) is contained in \( A \).

\footnote{Such systems in which all eigenvalues have non-zero real part are called hyperbolic systems.}
\footnote{It can also be shown that limit sets are closed sets.}
The definition of an attracting set is not always sufficiently restrictive to be useful.\textsuperscript{8} Fortunately, there is a notion that captures the intuitive idea of an “irreducible” attracting set, namely, that of an attractor, which is an attracting set that is topologically transitive. An invariant set $A$ is said to be topologically transitive if for any $U, V \subseteq A$ there is some $t \in \mathbb{R}$ such that $\phi_t(U) \cap V \neq \emptyset$. In words, an attractor cannot be split into smaller attracting sets. Clearly, a sufficient condition for $A$ to be topologically transitive is that a dense orbit be contained in it.

The identification of attractors and the corresponding basins of attraction is of extraordinary importance in dynamical systems theory, as orbits starting at almost every phase-space point asymptotically tend to one of the attractors available in the case of a large class of dynamical systems.\textsuperscript{9} Aside from fixed points and limit cycles, there exist other kinds of attractors in nonlinear systems that will be very important in later chapters, namely, attractors that exhibit nonperiodic, apparently erratic dynamics. Instead of giving the definitions right away, we presently provide some intuition by means of the numerical treatment of a celebrated example of a dynamical system showing this kind of behavior.

The Lorenz system (Lorenz, 1963) is a 3-dimensional dynamical system whose dynamics is governed by the following system of ODEs

$$
\begin{align*}
\dot{x} &= -\sigma(x - y), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz,
\end{align*}
$$

(2.9)

where $\sigma, r, b > 0$ are some fixed parameters, which in the original model had a physical meaning. Indeed, this system was introduced as a simplified model of fluid convection in the 1960s, but, due to the unexpected complexity of the dynamics it exhibited, lately it ended up playing a key role in the development of nonlinear dynamics as a canonical example of so-called chaotic dynamics (to be defined later). The system is invariant under the transformation $(x, y, z) \rightarrow (-x, -y, z)$, and the $z$ axis is easily seen to be an invariant manifold. For $r \leq 1$, there is only one fixed point in the origin, which is not only linearly stable (as can be seen from the spectrum of the linearized system) but also globally attracting (which can be shown by the use of Lyapunov functions; see, for instance,

\textsuperscript{8}In some cases, only subsets of attracting sets thus defined are really attracting, while the corresponding set may also contain repelling fixed points, unstable manifolds or other non-attracting subsets.

\textsuperscript{9}In the sense of Lebesgue measure.

\textsuperscript{10}Dissipative systems, that is, systems showing phase-space volume contraction along orbits. More on this later in this section.
For $1 < r < \sigma(\sigma + \beta + 3)/(\sigma - \beta - 1)$ two new asymptotically stable fixed points show up, while the origin is no longer stable. For higher values of $r$ the fixed points lose stability and the dynamics turns out to behave in a very erratic manner, with all trajectories encircling the two fixed points in a very irregular way. See the blue curve in Figure 2.4 for a representative orbit of the Lorenz system with parameters $\sigma = 10$, $r = 28$, $b = 8/3$.

Figure 2.4: **Typical orbit of the Lorenz system** ($\sigma = 10$, $r = 28$, $b = 8/3$). The blue curve shows a segment of an orbit started from an given initial condition, while the gray curve in the background shows a much longer orbit after an initial transient is left behind (thus giving a good representation of the underlying Lorentz attractor).

It can be shown by a simple application of Lyapunov function techniques that there is an ellipsoid in phase space that all orbits enter at some point, and never leave again (see the details in (Teschl, 2012)). The ellipsoid is therefore a *trapping region* for the system, and there is a corresponding attracting set contained in it, which is called the *Lorenz attractor*, whose basin of attraction is the full phase space $M = \mathbb{R}^3$. The light gray orbit shown in Figure 2.4 in the background represents a very long trajectory starting from a random initial condition, and recorded only after a long transient has been left behind. As generic orbits approach the attractor asymptotically, its shape can be inferred from such an orbit: it is a butterfly-shaped attractor with two lobes at whose centers lie the two unstable fixed points. The lobes look extremely thin, and indeed it can be shown that the attractor is of Lebesgue measure zero (a proof will be provided at the end of this section).

It is a remarkable fact that the dynamics governed by such a simple set of ODEs as (2.9) seems to be absolutely unpredictable, as orbits starting from nearby initial condi-
tions end up evolving in completely different ways. See Figure 2.5. This phenomenon (for which a precise definition will be given below) is known as sensitive dependence on initial conditions, and the orbits that exhibit such sensitive dependence are known as chaotic orbits.

Figure 2.5: Sensitive dependence on initial conditions of the Lorenz system. The three state variables \((x, y, z)\) are separately shown as a function of time. The blue curve represents an orbit starting from \((3, 5, 25)\), whereas the red curve represents an orbit starting from \((3, 5, 25.5)\). Although the initial conditions are quite similar, it takes relatively few oscillations to make the evolutions of the orbits totally uncorrelated.

As the orbits on the attractor are chaotic, the attractor itself is said to be a chaotic attractor, which is the third large category of attractors frequently found in dissipative systems, together with the much simpler fixed points and limit cycles. A large fraction of the attractors that are chaotic from a dynamics perspective, are, from a geometrical point of view, complicated sets that show Cantor-set-like features, with fine structure at arbitrarily small scales.\(^{11}\) Their dimension can be most properly considered to be non-integer, as they are fractal sets. They are frequently called strange attractors.\(^{12}\) A discussion of the fascinating subject of fractal dimensions and related subjects in the context of non-linear dynamical system is outside the scope of this thesis, but good introductions can be found in \(\text{Eckmann and Ruelle, 1985}\) and \(\text{Ott, 2002}\).

\(^{11}\) The Lorenz attractor is one example of a complicated, strange-looking set, but whether its characterization as a strange attractor is accurate according to the rigorous definitions that exist is apparently in dispute.

\(^{12}\) However, not all chaotic attractors are strange, and there are some strange attractors that do not exhibit chaotic dynamics.
Several definitions of chaos or chaotic dynamics have been proposed. We use the definition that can be found in (Skokos, 2010), which is originally attributed to Devaney. According to it, if there is a set $A \subset M$ such that $\phi_t : A \to A$ (the set is invariant under the flow), we say $\phi_t$ is chaotic on $A$ if: i) it has sensitive dependence on initial conditions, ii) it is topologically transitive, iii) periodic points are dense in $A$. Property ii) has been explained before, and property iii) is clear. However, we need a precise definition for sensitive dependence on initial conditions, which is also the property that is most relevant in physics and in applications.

A flow $\phi_t : A \to A$ is said to exhibit sensitive dependence on initial conditions if some $\delta > 0$ can be found, such that for any $x \in A$ and any neighborhood $U(x)$, there exist $y \in U(x)$ and $t \geq 0$ so that $\|\phi_t(x) - \phi_t(y)\| > \delta$. The implications of the existence of sensitive dependence on initial conditions are of obvious scientific importance, as it makes long term prediction effectively impossible due to the fact that knowledge of the initial conditions based on measurements applied on real systems is always affected by experimental error. A similar reasoning applies to any form of perturbation affecting the dynamics. Surprisingly enough, however, we will see that chaotic systems can synchronize (see Section 2.4), and their dynamics can even be controlled (see Section 2.8) by a judicious use of the very instability of chaotic trajectories.

To end this section, we discuss the evolution of phase-space (hyper)volumes, an issue that will also be treated from a relatively different point of view in Section 2.2 on Lyapunov exponents. We need the following well-known theorem.\footnote{Here we follow the presentation and proof given in (Teschl, 2012).}

Let $\dot{x} = f(x)$ be a dynamical system on $M = \mathbb{R}^n$. Let $A \subset M$ be a bounded (measurable) set, whose volume is $V = \int_A dx$. If the flow of the system is denoted $\phi_t(x)$, the set evolves as $A(t) = \phi_t(A)$, and its volume along the orbit is $V(t) = \int_{A(t)} dx$, then

$$\dot{V}(t) = \int_{A(t)} \nabla \cdot f(x) \, dx. \tag{2.10}$$

Proof: By a change of variables, the volume under the flow can be reexpressed as follows

$$V(t) = \int_{A(t)} dx = \int_A \det(d\phi_t(x)) \, dx. \tag{2.11}$$

We can Taylor-expand in the time variable, $d\phi_t = \mathbb{I} + d\mathbf{f} \cdot t + O(t^2)$, where $\mathbb{I}$ is the identity
matrix, and therefore $V(t) = \int_A (1 + \text{tr}(d\mathbf{f})t + O(t^2)) \, d\mathbf{x}$.\(^{14}\) Hence,

$$
\dot{V}(0) = \lim_{t \to 0} \frac{V(t) - V(0)}{t} = \lim_{t \to 0} \int_A \frac{\text{tr}(d\mathbf{f})t + O(t^2)}{t} \, d\mathbf{x} = \int_A \text{tr}(d\mathbf{f}) \, d\mathbf{x},
$$

where the limit can be taken under the integral sign by Lebesgue’s dominated convergence theorem. The result holds for all $t$, not just $t = 0$, if we replace $A$ with $A(t)$. □

A dynamical system is said to be conservative if phase-space volume is conserved along orbits. The paradigmatic example is that of a $2n$-dimensional Hamiltonian system (e.g. a classical mechanical system with $n$ degrees of freedom, free from any form of dissipation), with coordinates $(q, p) := \{q_1, \ldots, q_n, p_1, \ldots, p_n\}$. The time evolution is governed by Hamilton’s equations

$$
\dot{q}_i = \frac{\partial H(q, p)}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H(q, p)}{\partial q_i},
$$

so $f(q, p) = \{\partial H/\partial p_1, \ldots, \partial H/\partial p_n, -\partial H/\partial q_1, \ldots, -\partial H/\partial q_n\}$. The divergence of this vector field vanishes at every phase space point $(q, p)$, since

$$
\nabla \cdot f = \frac{\partial^2 H(q, p)}{\partial q_1 \partial p_1} + \cdots + \frac{\partial^2 H(q, p)}{\partial q_n \partial p_n} - \frac{\partial^2 H(q, p)}{\partial p_1 \partial q_1} - \cdots - \frac{\partial^2 H(q, p)}{\partial p_n \partial q_n} = 0.
$$

According to (2.10), volumes are preserved under the action of a Hamiltonian flow, a fact frequently referred to as Liouville’s theorem. Physically, this represents the conservation of mechanical energy along the orbits.\(^{15}\)

On the other hand, dissipative systems are those for which phase-space volume contracts under the flow. Physically, they represent systems with internal friction or other forms of energy dissipation. Attractors, of course, can only exist in dissipative systems. Both the Rössler system, which is treated in Section 2.3 and reappears in later chapters, and the Lorenz system are canonical examples of dissipative chaotic systems.

Let us conclude by explicitly proving that the Lorenz system is indeed dissipative (as we said we would do). The divergence of the Lorenz vector field (2.9) is

$$
\nabla \cdot f(\mathbf{x}) = -(1 + \sigma + b),
$$

which is seen to be constant over the phase space. According to (2.10), therefore the

\(^{14}\)We used the well-known fact that $\det(I + \epsilon X) = 1 + \text{tr}(X)\epsilon + O(\epsilon^2)$.

\(^{15}\)For an in-depth discussion, see any sufficiently advanced book on classical mechanics, such as (Arnold, 1989) or (José and Saletan, 1998).
phase-space volume evolves as

$$V(t) = V(0) e^{-(1+\sigma+b)t}.$$  \hspace{1cm} (2.16)

As the volume decreases exponentially along the orbits, the measure of the attractor must be zero.

In this section we have just presented a collection of definitions and some results that will be useful in the rest of this thesis. Needless to say, dynamical systems theory is a huge subject, which not even a textbook can treat in all its facets. A number of texts exist that deal with this subject from very different perspectives, on the one hand, ranging from purely mathematical treatments dealing with rigorous results to more physical approaches in which numerical methods and applications are considered, and on the other, from classical dynamical systems based on ODE theory, analysis and geometry, to nonlinear dynamics and chaos, where measure theory and information concepts are used. Even though the treatment in this section does not follow any single book too closely, those we have consulted the most frequently are (Arnold, 2006), (Perko, 2006), (Teschl, 2012), (Alligood et al., 1996), and (Strogatz, 2001c).

### 2.2 Lyapunov exponents

While the onset of chaos can be frequently understood in terms of the qualitative, geometric theory of smooth dynamical systems (by studying the properties of invariant sets, bifurcations, etc., (Ott, 2002; Alligood et al., 1996; Strogatz, 2001c)), in moderately or highly excited chaotic systems any kind of precise geometrical information on specific orbits becomes largely impractical for the characterization of the dynamics under study. A different approach aimed at a statistical characterization of chaotic dynamics, is usually considered to be more suitable for this task.\(^{16}\) Its main focus is the study of dimensions (the effective number of excited degrees of freedom of a given dynamics), entropy (the production of information) and Lyapunov (characteristic) exponents, which is the topic of this section. Indeed, we will discuss Lyapunov exponents at some length, as they will be of fundamental importance in Chapters 4 and 5. We mainly follow (Skokos, 2010), (Benettin et al., 1980a) and (Benettin et al., 1980b) in our discussion.

\(^{16}\)Sometimes this is known as the ergodic theory of dynamical systems, and is the main topic of some texts on nonlinear dynamics. See (Eckmann and Ruelle, 1985) for a review and (Ott, 2002) for a detailed treatment.
The set of \( n \) Lyapunov exponents of a given \( n \)-dimensional dynamical system, frequently called the Lyapunov spectrum of the system, is a very useful indicator of the sensitivity to small orbit perturbations. Lyapunov exponents are asymptotic measures of the average exponential rate of divergence of infinitesimally nearby initial conditions. They provide a means of characterizing the stretching and contracting characteristics of chaotic attractors and other invariant sets, as we now move on to explain. We will see later on that Lyapunov spectra are also central to the study of the synchronization of chaotic systems.

Let us consider a dynamical system evolving on an \( n \)-dimensional phase-space manifold \( M \) and a connected region \( U \subset M \), such that any orbit starting within it, possibly after going through a transient, settles onto an attractor (\( U \) is thus contained in the basin of attraction). Let us further assume the dynamics of the continuous-time system to be represented by a system of first-order ODEs starting from a given initial condition \( x_0 \)

\[
x = f(x) \quad x(t = 0) = x_0 \in U.
\] (2.17)

An infinitesimal displacement along the trajectory represented by \( y(t) \in T_{x(t)}M \), where \( T_{x(t)}M \) is the tangent (linear) space at the orbit point \( x(t) \), evolves according to the following variational equation

\[
y' = Df(x) \cdot y,
\] (2.18)

\( Df(x) \) being the Jacobian matrix evaluated at \( x(t) \). Since a Taylor expansion of the vector field around \( x \) gives \( f(x') = f(x) + Df(x) \cdot (x' - x) + O(\|x' - x\|^2) \), it is clear that (2.18) can be seen as a linearization around the orbit.

The flow corresponding Eq. (2.17) is denoted by \( \phi_t : M \to M \), and an individual orbit is formally given by

\[
x(t) = \phi_t(x_0).
\] (2.19)

As for Eq. (2.18), its solution is denoted

\[
y(t) = d_x\phi_t y(0),
\] (2.20)

where \( d_x\phi_t : T_xM \to T_{\phi_t(x)}M \) is of course a linear map such that \( d_x\phi_{t+s} = d_{\phi_t(x)}\phi_s \circ d_x\phi_s \).

Equivalently, this can be written as

\[
y(t) = Y(t, x) \cdot y(0),
\] (2.21)

\footnote{For a detailed and rigorous discussion of tangent spaces, tangent bundles and related concepts from differential geometry, see [Lee, 2012].}
where $\mathbf{Y}(t, \mathbf{x})$ is the fundamental matrix of solutions of Eq. \eqref{eq:2.18}. By definition, it satisfies
\[ \dot{\mathbf{Y}}(t, \mathbf{x}) = D\mathbf{f}(\mathbf{x}) \cdot \mathbf{Y}(t, \mathbf{x}) \quad \mathbf{Y}(t = 0, \mathbf{x}) = \mathbb{I}, \] (2.22)
where $\mathbb{I}$ is the identity matrix in $n$ dimensions, and the composition property can thus be written
\[ \mathbf{Y}(t + s, \mathbf{x}) = \mathbf{Y}(t, \phi_s(\mathbf{x})) \cdot \mathbf{Y}(s, \mathbf{x}). \] (2.23)

For ease of notation we will simply denote $\mathbf{Y}(t, \mathbf{x}) \cdot \mathbf{y}$ as $\mathbf{Y}_t \mathbf{y}$, as it is obvious that the fundamental matrix is evaluated at the phase-space point $\mathbf{x}$ corresponding to the tangent space $\mathcal{T}_\mathbf{x}M$ that contains the tangent vector $\mathbf{y}$.

The Lyapunov Characteristic Exponent (LCE) of order 1 for an orbit starting at $\mathbf{x}$ and a tangent (deviation) vector $\mathbf{y} \in \mathcal{T}_\mathbf{x}M$ is defined as
\[ \chi(\mathbf{x}, \mathbf{y}) = \limsup_{t \to \infty} \frac{1}{t} \log \| \mathbf{Y}_t \mathbf{y} \|. \] (2.24)
It should be noticed that the value of the norm of the tangent vector $\| \mathbf{y} \|$ does not affect the value of $\chi(\mathbf{x}, \mathbf{y})$.

Furthermore, as the tangent space at a point $\mathbf{x}$ is an $n$-dimensional vector space (i.e. a vector space of the same dimensionality as the phase-space manifold $M$), and all norms in finite-dimensional vector spaces are known to be equivalent, then by the same argument $\chi(\mathbf{x}, \mathbf{y})$ is independent of the norm used in the definition.

This definition can be extended to higher orders. Let $\{\mathbf{y}_i\}, \ i = 1, 2, \ldots, p \leq n$, be a set of linearly independent vectors in $\mathcal{T}_\mathbf{x}M$, and let $E^p$ be the subspace they generate. The phase-space volume of the $p$-parallelogram having the $p$ vectors $\mathbf{Y}_t \mathbf{y}_i$ as edges, is computed as usual as the norm of the wedge product of these vectors
\[ \text{vol}_p(\mathbf{Y}_t, E^p) = \| \mathbf{Y}_t \mathbf{y}_1 \wedge \mathbf{Y}_t \mathbf{y}_2 \wedge \cdots \wedge \mathbf{Y}_t \mathbf{y}_p \|. \] (2.25)

Then, the LCE of order $p$ of $\mathbf{Y}_t$ with respect to $E^p$ is defined as
\[ \chi(\mathbf{x}, E^p) = \limsup_{t \to \infty} \frac{1}{t} \log \text{vol}_p(\mathbf{Y}_t, E^p) \] (2.26)

---

\[18\] See, for instance, (Perko, 2006).
\[19\] If we consider $\beta \mathbf{y}$ instead of $\mathbf{y}$, where $\beta$ is a nonzero constant, the extra term $\log |\beta|/t$ we get vanishes for $t \to \infty$.
\[20\] The norms $\| \cdot \|$ and $\| \cdot \|$' are said to be equivalent if $\beta_1 \| \mathbf{y} \| \leq \| \mathbf{y} \|' \leq \beta_2 \| \mathbf{y} \|$, for positive real numbers $\beta_1$ and $\beta_2$. 
From the definition of LCEs of order 1, it follows that
\[ \chi(x, a_1 y_1 + a_2 y_2) \leq \max \{ \chi(x, y_1), \chi(x, y_2) \} \tag{2.27} \]
for any nonzero \( a_1, a_2 \in \mathbb{R} \). It follows the set of vectors \( \{ y : \chi(x, y) \leq r \} \) for any \( r \in \mathbb{R} \)
is a vector subspace of \( T_x M \), and the function \( \chi(x, y) \), for any \( y \neq 0 \), takes at most \( n \) different values, which are denoted
\[ \nu_1(x) > \cdots > \nu_s(x) \quad 1 \leq s \leq n. \tag{2.28} \]
This naturally yields the subspaces
\[ L_i(x) = \{ y : \chi(x, y) \leq \nu_i(x) \}, \tag{2.29} \]
where \( T_x M = L_1(x) \supset L_2(x) \supset \cdots \supset L_s(x) \supset L_{s+1}(x) = \{ 0 \} \), and \( \chi(x, y) = \nu_i(x) \) if and only if \( y \in L_i(x) \backslash L_{i+1}(x) \).

A basis of the tangent space \( \{ y_i \}, i = 1, 2, \ldots, n \), is said to be a normal basis if
\[ \sum_{i=1}^{n} \chi(x, y_i) \leq \sum_{i=1}^{n} \chi(x, g_i) \tag{2.30} \]
where \( \{ g_i \}, i = 1, 2, \ldots, n \), is any other basis. Whereas a normal basis fails to be unique, the \( \chi(x, y_i) \) depend only on \( Y_t \) and not on the particular normal basis. They are called the LCEs of \( Y_t \). We choose an ordering of the basis elements such that \( \chi(x, y_1) \geq \chi(x, y_2) \geq \cdots \geq \chi(x, y_n) \), and use the notation \( \chi_i(x) := \chi(x, y_i) \). If the value \( \nu_i(x) \) occurs \( k_i \) times among \( \chi_1(x) \geq \chi_2(x) \geq \cdots \geq \chi_n(x) \), the value \( \nu_i(x) \) is said to occur with multiplicity \( k_i \), and the collection \( (\nu_i(x), k_i), i = 1, \ldots, s \leq n \), is the spectrum of LCEs for an orbit starting at \( x \).

It is clear from Hadamard’s inequality\(^{21}\) that
\[ \sum_{i=1}^{n} \chi(x, y_i) \geq \limsup_{t \to \infty} \frac{1}{t} \log |\det Y_t|. \tag{2.31} \]
Additionally, the matrix \( Y_t \) is said to be Lyapunov regular as \( t \to \infty \) if for each normal basis \( \{ y_i \} \)
\[ \sum_{i=1}^{n} \chi(x, y_i) = \liminf_{t \to \infty} \frac{1}{t} \log |\det Y_t|. \tag{2.32} \]

\(^{21}\)According to this inequality, for a matrix \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \), \( |\det A| \leq \prod_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \).
holds. Together with Eq. (2.31), this implies that \( \lim_{t \to \infty} \frac{1}{t} \log |\det Y_t| \) exists, is finite, and satisfies
\[
\lim_{t \to \infty} \frac{1}{t} \log |\det Y_t| = \sum_{i=1}^{n} \chi(x, y_i) = \sum_{i=1}^{s} k_i \nu_i(x).
\] (2.33)

The following very important theorem is stated without proof:

If \( Y_t \) is Lyapunov regular, then the LCEs of all orders are given by
\[
\chi(x, y) = \lim_{t \to \infty} \frac{1}{t} \log \|Y_t y\| \tag{2.34}
\]

\[
\chi(x, E^p) = \lim_{t \to \infty} \frac{1}{t} \log \text{vol}_p(Y_t, E^p) = \sum_{j=1}^{p} \chi_{i_j}(x), \tag{2.35}
\]

where \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq n \).

If we further treat the phase-space (compact) manifold \( M \) as a measure space with normalized measure \( \mu \) (i.e. \( \mu(M) = 1 \)) such that the flow \( \phi_t \) is a measure preserving diffeomorphism of class \( C^1 \) on \( M \) (where both \( \phi_t \) and its inverse are measurable),\(^{22}\) then by Oseledec’s Multiplicative Ergodic Theorem (Benettin et al., 1980a; Oseledec, 1968), the one-parameter family \( d_x \phi_t \) is Lyapunov regular for almost every \( x \in M \) with respect to \( \mu \). Therefore, the previous theorem holds, and the LCE spectrum of the dynamical system can be obtained from (2.34) and (2.35).

If knowledge on the sequence of subspaces \( L_i(x) \) is available, by taking an initial vector \( y_i \in L_i(x) \setminus L_{i+1}(x) \) one could in principle compute
\[
\nu_i(x) = \lim_{t \to \infty} \frac{1}{t} \log \|Y_t y_i\|, \quad i = 1, 2, \ldots, s. \tag{2.36}
\]
The issue is that apart from \( L_1(x) \), the remaining subspaces \( L_i(x) \), for \( i = 2, 3, \ldots, s \), have positive codimension, and thus zero Lebesgue measure. Thus, any \( y \in \mathcal{T}_x M \) chosen at random or from an imperfect knowledge on the subspaces \( L_i(x) \) inevitably leads to the computation of \( \chi_1(x) \), because in principle \( y \) would belong to \( L_1(x) \) and not \( L_i(x) \), \( i = 2, 3, \ldots, s \).\(^{23}\) In short, a generic choice of \( y \) leads to the computation of \( \chi_1(x) \).

\(^{22}\)A measure preserving flow \( \phi_t \) is simply a flow that preserves the measure: \( \mu(\phi_t^{-1}(A)) = \mu(A) \).

\(^{23}\)In (Skokos, 2010), the author gives the following example. Let \( L_1(x) \) be \( \mathbb{R}^3 \), \( L_2(x) \subset L_1(x) \) a particular plane of \( \mathbb{R}^3 \), and \( L_3(x) \subset L_2(x) \) a line, with corresponding LCEs \( \chi_1(x) > \chi_2(x) > \chi_3(x) \) having multiplicities \( k_1 = k_2 = k_3 = 1 \). It is obvious that in such a case \( \mu(L_2(x)) = \mu(L_3(x)) = 0 \), where \( \mu \) here denotes Lebesgue measure.
with probability one. Moreover, from the perspective of the numerical computation of LCEs (except for a few cases, the only way to compute LCEs of time-continuous chaotic systems), even if in some special cases one could be in possession of perfect knowledge of $L_i(x)$, numerical errors would eventually lead to the computation of $\chi_1(x)$ whatever first-order LCE is meant to be computed.

According to the reasoning above, the numerical computation of LCEs of order 1 of a given dynamical system is only effective for the computation of the maximum LCE $\chi_1(x)$. Fortunately, the computation of the maximum LCE is sometimes sufficient for the characterization of dynamical systems, as it proves to be a tool of extraordinary importance for detecting the presence of chaotic dynamics, both in theoretical studies and in the analysis of experimental time series.\footnote{The experimental case is more complex in that it requires the use of embedding techniques for attractor reconstruction, followed by methods that allow for an estimation of the Jacobian of the dynamical system along an orbit, as the precise equations are generally not known.} Indeed, if $\chi_1(x) > 0$, that means there is at least one direction along the orbit showing sensitive dependence to perturbations, and so the dynamics is chaotic. Otherwise, if $\chi_1(x) = 0$ (we will see below that every dynamical system possesses at least one zero LCE that corresponds to the direction along the orbit itself) there is no chaos. The maximum LCE is also extremely useful in the study of the synchronization of chaotic systems, as will be seen in later sections. However, for a full characterization of the dynamics in terms of divergence rates along the orbits, the full LCE spectrum must be computed.

The LCEs of order $p$ can take at most $n!/p!(n-p)!$ different values. According to (2.35), a randomly chosen $p$-dimensional subspace $E^p \subset T_x M$ (which results from the random selection of $p$ linearly independent vectors $y_i, i = 1, 2, \ldots, p$) leads to the computation of the maximal LCE of order $p$, which is the sum of the $p$ largest LCEs of order 1,

$$\chi(x, E^p) = \sum_{i=1}^{p} \chi_i(x). \quad (2.37)$$

By a similar argument as that used in the first order case, even if knowledge of the subspaces $L_i(x) i = 1, 2, \ldots, s$ allowed for the spanning of a particular subspace $E^p \subset T_x M$ that would yield a LCE of order $p$ smaller than $\sum_{i=1}^{p} \chi_i(x)$, numerical errors would lead to the computation of this maximal value. In conclusion, a random choice of the vectors $y_i$, for $i = 1, 2, \ldots, p \leq n$, generically leads to the computation of $\chi_1(x) + \chi_2(x) + \ldots + \chi_p(x)$. This fact is used in a straightforward manner in the standard procedure for the numerical computation of the full LCE spectrum that we explain next.\footnote{The algorithm, due to Benettin and co-authors, together with the relevant theory, is covered in detail in the original references, [Benettin et al., 1980a,b].}
For the numerical computation of the maximum LCE $\chi_1(x)$, one has to simultaneously integrate numerically\(^\text{26}\) an orbit from $x$, and the linear evolution of a randomly chosen deviation vector $y(0) \in T_xM$ under the tangent map, $y(t) = d_x\phi_t y(0)$, which is simply the solution of the corresponding variational equation (2.18). Then the finite-time maximum LCE is
\[
\chi_1(x, t) = \frac{1}{t} \log \frac{\|y(t)\|}{\|y(0)\|},
\]
and obviously, assuming the mathematical conditions for the existence of Lyapunov exponents hold (more on this later), $\chi_1(x) = \lim_{t \to \infty} \chi_1(x, t)$. In principle, by just using the previous expression (2.38) for a sufficiently long time window, it would be possible to get a good estimate of the maximum LCE. However, this procedure meets with some difficulties if the orbit is chaotic, as the norm will increase exponentially with time, resulting in numerical overflow. Instead, we can define a time interval $\tau$ and realize that the finite time LCE at time $t = k\tau$, for $k \in \mathbb{N}$ equals
\[
\chi_1(x, k\tau) = \frac{1}{k\tau} \log \frac{\|y(k\tau)\|}{\|y(0)\|} = \frac{1}{k\tau} \sum_{i=1}^{k} \log \frac{\|y(i\tau)\|}{\|y(((i-1)\tau))\|},
\]
by obvious cancelations and basic properties of the logarithm. Let us denote $D_0 = \|y(0)\|$. If the vector $y(t)$ is renormalized each $\tau$ time units so that its direction is preserved but it is rescaled to a $D_0$-norm vector, by the linearity of the tangent map, this does not change the value of the quotient $\|y(i\tau)\|/\|y(((i-1)\tau))\|$. By calling $D_i$ the norm of $y(i\tau)$ after $\tau$ time units since its last renormalization, we obtain
\[
\chi_1(x) = \lim_{k \to \infty} \chi_1(x, k\tau) = \lim_{k \to \infty} \sum_{i=1}^{k} \log \frac{D_i}{D_0}.
\]
This way the problems due to the exponential growth of the norm disappear at the cost of some computation time. For the sake of simplicity, frequently the choice $D_0 = 1$ is made.

Similarly, to compute the maximal LCE of order $p$ (which is equal to the sum of the $p$ largest LCEs of order 1), which we denote by $\chi_1^{(p)}(x)$, the time evolution of the orbit starting from $x$ and of $p$ linearly independent deviation vectors $y_1(0), y_2(0), \ldots, y_p(0) \in T_xM$ must be simultaneously monitored. We will always assume that $\{y_1(0), y_2(0), \ldots, y_p(0)\}$

\(^{26}\)By using one of the many existing numerical methods for solving ODEs, such as a Runge-Kutta scheme, or a more advanced variable-step method (for a good discussion, see Press et al., 2007).
is an orthonormal basis of $E^p \subset \mathcal{T}_x M$. As expected, the finite-time LCE is

$$\chi_1^{(p)}(x, t) = \frac{1}{t} \log \frac{\| \mathbf{y}_1(t) \wedge \mathbf{y}_2(t) \wedge \cdots \wedge \mathbf{y}_p(t) \|}{\| \mathbf{y}_1(0) \wedge \mathbf{y}_2(0) \wedge \cdots \wedge \mathbf{y}_p(0) \|}, \quad (2.41)$$

where of course $\| \mathbf{y}_1(0) \wedge \mathbf{y}_2(0) \wedge \cdots \wedge \mathbf{y}_p(0) \| = 1$, and again

$$\sum_{i=1}^{p} \chi_i(x) = \chi_1^{(p)}(x) = \lim_{t \to \infty} \chi_1^{(p)}(x, t). \quad (2.42)$$

As in the case of the maximum LCE of order 1, in the actual computation of the maximum LCE of order $p$ a time interval $\tau$ is fixed and, to avoid numerical overflow, the computation is carried out using the following expression for the finite time LCE

$$\chi_1^{(p)}(x, k\tau) = \frac{1}{k\tau} \sum_{i=1}^{k} \log \frac{\| \mathbf{y}_1(i\tau) \wedge \mathbf{y}_2(i\tau) \wedge \cdots \wedge \mathbf{y}_p(i\tau) \|}{\| \mathbf{y}_1((i-1)\tau) \wedge \mathbf{y}_2((i-1)\tau) \wedge \cdots \wedge \mathbf{y}_p((i-1)\tau) \|}, \quad (2.43)$$

which is mathematically equivalent to Eq. (2.41).

Nevertheless, this case meets with a sort of difficulty that does not come up in the case of the maximal LCE of order 1, namely, the fact that when at least two vectors are involved (as in the computation of $\chi_1^{(2)}(x)$) the angles between them tend to become too small, due to the progressive alignment of all tangent vectors in the direction of the greatest divergence rate, which ultimately makes them completely parallel in finite precision calculations. The simplest way to avoid this is to apply the Gram-Schmidt reorthonormalization procedure every time the volume spanned by $\{ \mathbf{y}_1(i\tau), \mathbf{y}_2(i\tau), \ldots, \mathbf{y}_p(i\tau) \}$, for any $i$, is computed. This well-known linear-algebraic procedure simply consists in going through the set of linearly independent vectors one by one, and making them orthogonal to the subspace spanned by the previously considered vectors, and of unit norm:

$$\begin{align*}
\mathbf{w}_1(i\tau) &= \mathbf{y}_1(i\tau) \\
\mathbf{w}_2(i\tau) &= \mathbf{y}_2(i\tau) - \langle \mathbf{y}_2(i\tau), \hat{\mathbf{y}}_1(i\tau) \rangle \hat{\mathbf{y}}_1(i\tau) \\
& \quad \vdots \\
\mathbf{w}_p(i\tau) &= \mathbf{y}_p(i\tau) - \sum_{j=1}^{p-1} \langle \mathbf{y}_p(i\tau), \hat{\mathbf{y}}_j(i\tau) \rangle \hat{\mathbf{y}}_j(i\tau) \\
& \quad \hat{\mathbf{y}}_p(i\tau) = \mathbf{w}_p(i\tau) / \| \mathbf{w}_p(i\tau) \|,
\end{align*} \quad (2.44)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Then, the integration of system equations and the evolution of the reorthonormalized vectors $\{ \hat{\mathbf{y}}_1(i\tau), \hat{\mathbf{y}}_2(i\tau), \ldots, \hat{\mathbf{y}}_p(i\tau) \}$ continue up to $(i+1)\tau$, where the volume measurement is performed on the time-evolved set.

---

\textsuperscript{27} Any set of $p$ linearly independent vectors can be easily transformed into an orthonormal basis of the linear space of dimension $p$ they generate by Gram-Schmidt orthonormalization. The procedure is explained below.
of vectors, simply denoted \{\mathbf{y}_1((i + 1)\tau), \mathbf{y}_2((i + 1)\tau), \ldots, \mathbf{y}_p((i + 1)\tau)\}, and reorthonormalization is again performed. This way the set of tangent vectors are rescaled to unit volume, and the orthogonality between the vectors is re-imposed by the procedure every \(\tau\) time units. We are interested in the limit for long times

\[
\chi_1^{(p)}(\mathbf{x}) = \lim_{k \to \infty} \frac{1}{k\tau} \sum_{i=1}^{k} \log \frac{\|\mathbf{y}_1(i\tau) \wedge \cdots \wedge \mathbf{y}_p(i\tau)\|}{\|\mathbf{y}_1((i-1)\tau) \wedge \cdots \wedge \mathbf{y}_p((i-1)\tau)\|},
\]

(2.45)

where obviously the volume spanned by the orthonormal set of vectors is one, and thus we finally get

\[
\sum_{i=1}^{p} \chi_i(\mathbf{x}) = \chi_1^{(p)}(\mathbf{x}) = \lim_{k \to \infty} \frac{1}{k\tau} \sum_{i=1}^{k} \log\|\mathbf{y}_1(i\tau) \wedge \cdots \wedge \mathbf{y}_p(i\tau)\|.
\]

(2.46)

Benettin et al. in (Benettin et al., 1980b) further proposed a way to compute the largest \(p\) LCEs of order 1, by computing the evolution of \(p\) (initially orthonormal) deviation vectors \(\mathbf{y}_j j = 1, 2, \ldots, p\), as in the computation of LCEs of order \(p\) we just dealt with. It can be shown that the volume of the \(p\)-parallelogram having as edges the vectors \(\{\mathbf{y}_1(i\tau), \mathbf{y}_2(i\tau), \ldots, \mathbf{y}_p(i\tau)\}\) is equal to the volume spanned by the set of vectors \(\{\mathbf{w}_1(i\tau), \mathbf{w}_2(i\tau), \ldots, \mathbf{w}_p(i\tau)\}\) from (2.44). Since the vectors \(\mathbf{w}_j(i\tau) j = 1, 2, \ldots, p\) are orthogonal to one another, the volume of the parallelogram they form is equal to the product of their norms,

\[
\|\mathbf{y}_1(i\tau) \wedge \cdots \wedge \mathbf{y}_p(i\tau)\| = \prod_{j=1}^{p} \|\mathbf{w}_j(i\tau)\|.
\]

(2.47)

Therefore, from (2.46), considering \(\chi_p(\mathbf{x}) = \chi_1^{(p)}(\mathbf{x}) - \chi_1^{(p-1)}(\mathbf{x})\) we arrive at

\[
\chi_p(\mathbf{x}) = \lim_{k \to \infty} \frac{1}{k\tau} \sum_{i=1}^{k} \log \frac{\prod_{j=1}^{p} \|\mathbf{w}_j(i\tau)\|}{\prod_{j=1}^{p-1} \|\mathbf{w}_j(i\tau)\|} = \lim_{k \to \infty} \frac{1}{k\tau} \sum_{i=1}^{k} \log\|\mathbf{w}_p(i\tau)\|.
\]

(2.48)

This is the fundamental expression for the numerical computation of the Lyapunov exponents. Every time the Gram-Schmidt orthonormalization procedure is applied, the norms \(\|\mathbf{w}_j(i\tau)\|\) are computed and the sums eventually converging to \(\chi_1(\mathbf{x}), \ldots, \chi_p(\mathbf{x})\) in (2.48) are updated with the corresponding new summand.

This procedure easily allows for the computation of the full spectrum of LCEs of order 1 if \(p = n\) is chosen. Since in this thesis, as in the vast majority of the literature on nonlinear dynamics and complex dynamical networks, only the LCEs of order 1 are
considered, we will refer to them simply as the Lyapunov exponents, while the largest Lyapunov exponent will be called the maximum Lyapunov exponent or simply the Lyapunov exponent. The full set of $n$ Lyapunov exponents, in its turn, will be referred to as the Lyapunov spectrum.

Furthermore, the Lyapunov spectrum will always be considered to be that of the dynamical system itself, regardless of the orbit along which the Lyapunov exponents are computed. This can be justified on the grounds that we only study the Lyapunov exponents of chaotic dynamical systems with just a single globally attracting chaotic attractor (i.e. the basin of attraction is the whole of $M$ except for possible subsets of zero Lebesgue measure). As every nonperiodic orbit on the attractor covers it densely, any two different orbits are effectively equivalent in the long run. Indeed, in such connected chaotic domain, the Lyapunov exponents are constant almost everywhere with respect to $\mu$ (i.e. the measure invariant under the flow action that appears in Oseledec’s theorem) if the measure is ergodic, which roughly means that orbits asymptotically run across any neighborhood of the attractor however small.\footnote{More mathematically, an invariant measure is an ergodic measure if every set $A$ invariant under the flow, $\phi^{-1}_t (A) = A$, must be such that either $\mu(A) = 0$ or $\mu(A) = 1$. The notion of ergodicity is in a sense a form of irreducibility or non-separability of an attractor. Ergodic theory is a highly developed mathematical theory on which many monographs have been written. For an introductory discussion from a physicist’s perspective see [Eckmann and Ruelle, 1985].}

This means that in practice we can eliminate the explicit dependence on $x$ from our notation, and simply refer to the Lyapunov exponents of a dynamical system. They are denoted as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

(2.49)

according to the notation found in most of the nonlinear dynamics literature. Frequently $\lambda_{\text{max}}$ will be used instead of $\lambda_1$ whenever only the maximum Lyapunov exponent is considered.

The Lyapunov spectrum gives valuable information for the study of the evolution of phase-space (hyper)volumes, a subject that we discussed at the end of the previous section. In the case of conservative systems (such as Hamiltonian systems), $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 0$, according to the fact that phase-space volume is preserved. In the case of dissipative systems, however, $\lambda_1 + \lambda_2 + \ldots + \lambda_n < 0$, as phase-space volume contracts under the action of the flow. Even if there is chaos and therefore there are positive Lyapunov exponents, contraction must outweight expansion in dissipative systems. In both conservative and dissipative systems, there is always at least one zero Lyapunov exponent corresponding to the direction along the orbit, which can be simply explained by the fact that perturbations...
tions along the flow (which physically correspond to starting the system at $x(\delta t)$ instead of at $x(0)$) neither grow nor decay exponentially in time, but rather remain stationary ($\phi_t(x(0)) = x(t)$, whereas $\phi_t(x(\delta t)) = x(t + \delta t)$). As we said, the system shows sensitive dependence on initial conditions if and only if $\lambda_1 > 0$. If moreover $\lambda_2 > 0$ the system is sometimes said to be hyperchaotic, which means there are two or more directions of positive divergence rate of the deviation vectors along orbits.

To conclude this section, we briefly comment on the existence of Lyapunov exponents for a given dynamical system, an issue that is only rarely addressed in the literature. Oseledec’s theorem states that for an invariant measure $\mu$, Lyapunov exponents exist for orbits starting at $\mu$-almost every point. If the dynamical system is conservative, Liouville’s theorem guarantees that it preserves a measure that is equivalent to the Lebesgue measure, and therefore Oseledec’s theorem implies that Lyapunov exponents exist for every point $x$ in $M$ except for a set of Lebesgue measure zero. However, in the case of dissipative systems there are typically basins of attraction whose points contract toward the corresponding attractors under the action of the flow, and every invariant measure with support inside the basins must therefore be zero off the attractor. Therefore the measure is supported on a set of zero Lebesgue measure, and Oseledec’s theorem does not say anything about Lyapunov exponents along orbits starting from a point $x$ in the basin that is not on the attractor itself. As any kind of trajectory studied numerically by a random choice of an initial condition (or by a deliberate choice with finite precision) converges to the attractor asymptotically but cannot be on the (zero-measure) attractor itself, the theorem does not say anything on the existence of the Lyapunov exponents associated with those (generic) orbits. A deeper treatment of this question leads to the notion of natural measures and other concepts that are outside the scope of this thesis. Indeed, we will take for granted the existence of the Lyapunov exponents for those systems we will be studying, which can be justified on ample numerical evidence obtained by us, and by many others before, consistent with the fact that the finite-time Lyapunov exponents converge to the same set of values regardless of the initial condition. However, the problem of the existence of the Lyapunov exponents of dissipative systems seems to be open, and simple dissipative systems of low-dimensionality have been proposed where the Lyapunov exponents fail to exist for orbits starting at almost every point in phase space. See (Ott and Yorke, 2008) for some examples and a good discussion of this issue.

Detailed proofs and in-depth discussions of the topics covered in this section can be found in (Benettin et al., 1980a; Oseledec, 1968). For a pedagogical introduction to Lyapunov exponents see (Skokos, 2010).
2.3 The Rössler system

The well-known Poincaré-Bendixson theorem (see, for instance, Perko, 2006) precludes the existence of chaotic attractors in continuous-time dynamical systems defined on phase-space manifolds of dimension less than three. Accordingly, in one or two dimensional manifolds, such as the line, the circle, the plane, the sphere, the torus, etc., only fixed points and periodic or quasiperiodic attractors are possible.

In the 1970s, Otto Rössler, while searching for chaotic systems that were as simple as possible and chaotic behavior in far-from-equilibrium chemical kinetics, came up with several simple chaotic systems, including three-dimensional systems and four-dimensional systems showing hyperchaos. We will be dealing with the first one he proposed (Rössler, 1976), which, due to its simplicity and robustness among other properties, has since then become a prototype system for the study of chaotic dynamics. Rössler proposed it as a system presenting some similarities with the Lorenz system described in Section 2.1 and yet simpler. According to common practice, we simply call this system the Rössler system. Much of our discussion will be based also on the more recent sources (Gaspard, 2005; Letellier and Rossler, 2006).

More specifically, Rössler was inspired by the reinjection principle of relaxation-type systems, which frequently present a Z-shaped manifold on which motion proceeds slowly until a certain edge is reached, and then the trajectory rapidly jumps to the other branch. In three-dimensional flows, the reinjection principle can lead to chaotic behavior for example if motion spirals out on the slow branch. See Figure 2.6 for an illustration.

![Figure 2.6: Illustration of the reinjection principle. Relaxation behavior in dimension three between the two branches of a Z-shaped slow manifold.](image-url)
The Rössler system is defined by the following system of ODEs

\[
\begin{align*}
\dot{x} & = -y - z \\
\dot{y} & = x + ay \\
\dot{z} & = b + z(x - c),
\end{align*}
\]  

(2.50)

where \(a\), \(b\) and \(c\) are parameters of the system. Rössler focused originally on the case where \(a = 0.2\), \(b = 0.2\), and \(c = 5.7\), but other parameter values have been used since then. The Rössler system presents the minimal ingredients for chaos in that the phase space has the minimal dimension (three, a characteristic also of the Lorenz system), its nonlinearity is based on just a single quadratic term, and, as will be shown below, the chaotic dynamics evolves around one single lobe (unlike the two-lobe Lorenz system seen in Figure 2.4). For a large set of parameter values, it presents one stable chaotic attractor, known as the Rössler attractor.\(^{29}\)

The linear terms in the first two equations lead to oscillations in the \((x, y)\) plane, as setting \(z = 0\), the dynamics in the plane is governed by

\[
\begin{align*}
\dot{x} & = -y \\
\dot{y} & = x + ay.
\end{align*}
\]  

(2.51)

The eigenvalues of the Jacobian of this 2-dimensional system are \((a \pm \sqrt{a^2 - 4})/2\). When \(0 < a < 2\),\(^{30}\) the eigenvalues are thus complex and have a positive real part, and the origin is thus an unstable focus (i.e. in the linear approximation orbits spiral away from the origin). The oscillations are amplified, as the system behaves like a linear oscillator with negative damping, thus resulting in a spiraling-out of trajectories. The third equation in (2.50), in its turn, which contains the nonlinear term, induces the reinjection back to the spiraling motion. As long as \(x < c\), the orbit is kept close to the \((x, y)\) plane, but as the spiraling motion makes \(x\) exceed \(c\), \(z\) increases, which itself diminishes the increase of \(x\). For a better understanding we need to move on to an analysis of the full nonlinear system.

The system has two fixed points, denoted as \(F_{\pm}\), whose coordinates are

\[
F_{\pm} = \left( \frac{c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \pm \sqrt{c^2 - 4ab}}{2a} \right).
\]  

(2.52)

\(^{29}\)Incidentally, this attractor is also the typical example of what is known as a phase coherent chaotic attractor, a very relevant property for the study of the (phase) synchronization of chaos.\(^{\text{Rosenblum et al., 1996; Boccaletti et al., 2002.}}\)

\(^{30}\)This always holds in studies where the Rössler system features prominently.
Figure 2.7: Rössler system fixed points and their associated (stable and unstable) manifolds. A schematic representation of the positions of $F_+$ and $F_-$, and their stable and unstable linear manifolds is shown.

$F_-$ is located in the middle of the attractor and is a saddle-focus with a 2-dimensional unstable manifold (which is an unstable spiral in the $(x, y)$ plane to a good approximation), and therefore it has a central role in the spiralling-out occurring on the main disk of the attractor. When the nonlinearity $z(x - c)$ becomes important, the trajectory visits the neighborhood of $F_+$, which is another saddle focus whose unidimensional unstable manifold sends the trajectory along the unidimensional stable manifold of $F_-$ running through the center of the disk on the $(x, y)$ plane. The cycle then starts anew. For a pictorial representation of the fixed point locations and their associated stable and unstable manifolds, see Figure 2.7.

A study of the limit sets of the dynamics as a function of the parameter values (i.e. a bifurcation analysis, [Alligood et al., 1996; Strogatz, 2001c; Crawford, 1991]) shows both periodic and chaotic regimes. We will focus on the chaotic regimes, where the Lyapunov spectrum is such that $\lambda_1 > 0$, $\lambda_2 = 0$, and $\lambda_3 < 0$. A so-called paper-sheet model, with one regular band and one Möbius band can be shown to encode all topological properties of the unstable periodic orbits embedded within the attractor (see a schematic representation in Figure 2.8). The topology of the attractor indeed can be said to be that of a stretched and folded ribbon. His structure can be inferred from a simple application of the Poincaré section method (see [Perko, 2006; Teschl, 2012] for an introduction).

By inspecting the Lyapunov spectrum of the system, we find that in all cases $|\lambda_3| > \lambda_1$, as it must always be the case for a dissipative system. Indeed, $|\lambda_3|$ is of one or two orders of magnitude above $\lambda_1$, and that explains why the expansion rate of the stretching process
Figure 2.8: **Schematic representation of the paper-sheet model.** The “normal band” and the “Möbius band” are indicated. Adapted from [Letellier and Rossler, 2006].

occurs at a much slower rate than reduction rate of the folding process, and why the normal band and the Möbius band in the attractor become very thin.

Figure 2.9: **Typical orbit and Lyapunov spectrum of the Rössler system** ($a = b = 0.2$, **and** $c = 7.0$). The green curve shows an orbit started from a given initial condition, while the gray background figure shows a much longer orbit (thus giving a good representation of the Rössler attractor). The Lyapunov spectrum appears on the top right corner.

To conclude, we show a generic orbit of the Rössler system in a chaotic regime in Figure 2.9, with parameters $a = b = 0.2$ and $c = 7.0$, as in Chapters 4 and 5.
2.4 Synchronization of dynamical systems

Synchronization is generally understood as a form of agreement or correlation in time of two or more dynamical processes due to some form of physical coupling or forcing. The everyday meaning of the word is clear, and even in most scientific contexts (as in the synchronization of clocks in relativity theory, or the synchronization of biological rhythms) it seems to be out of dispute. There is much more to it than it may seem, however. Indeed, the synchronization of dynamical systems is by now a vast subject that encompasses a rich variety of nontrivial phenomena including different forms of synchronization involving systems of great dynamical complexity. We will deal with the small subset of these phenomena that is needed for understanding our work as explained in the later chapters of this thesis. For a broader view on these topics, see, for instance, (Boccaletti et al., 2002) and (Pikovsky et al., 2003).

Christian Huygens pioneered the study of synchronization as a physical phenomenon when in the 17th century he studied the motion of pendulum clocks hanging from a common beam.\footnote{A colorful account of this story can be found in (Pikovsky et al., 2003).} In this physical setting, the mechanical vibrations in the beam provide a form of mutual coupling, which in some cases induces synchronization. Whereas clocks are generally said to be synchronized if they tick at the same rate, according to some definitions of synchronization we will see that the ticking need not be simultaneous: there can be delays (phase shifts) as long as they stay constant in time. The synchronization of a collection of interacting pendulum clocks provides a nice illustration that permits to introduce some basic concepts, so we will start by considering this case.

Mathematically, a pendulum clock is a phase oscillator,\footnote{Moreover, if the small angle approximation, $\sin \theta \approx \theta$, is justified, they are harmonic oscillators for all purposes.} as it can be described as a two-dimensional dynamical system whose state is given by a phase and an angular velocity, $\{\theta, \dot{\theta}\}$. The phase-space manifold is thus a cylinder

$$M = S^1 \times \mathbb{R},$$

(2.53)

where $S^1$ is the unit circle, on which points can of course be univocally identified by the phase coordinate $\theta \in [0, 2\pi)$. The need for two state variables (i.e. not just a phase, but also an angular velocity) is related to the fact that second order ODEs are required to describe the physical process, as expressed in Newton’s equations, or any equivalent formulation of classical mechanics.\footnote{The physical problem is very well-known, and can be easily solved by energy conservation reasoning.} When isolated, a conservative pendulum is moreover...
a periodic system, and therefore

\[ \phi_{t+T}(\theta(0), \dot{\theta}(0)) = \phi_t(\theta(0), \dot{\theta}(0)) \]  

(2.54)

holds for all initial conditions and for all time. Here, \( T \) denotes the period of oscillation of the clock.

Let us consider the case where there are \( N \) pendulum clocks hanging from a beam, with phases \( \{\theta_i\}, \ i = 1, 2, \ldots, N \). The evolution of an individual clock is no longer autonomous as it depends on its own state, but also on those of the other clocks. We are therefore dealing with a case of coupled oscillators. Now, let us imagine that the following dynamical regime is observed:

\[ \theta_1(t) = \theta_2(t) = \cdots = \theta_N(t). \]  

(2.55)

If the phases oscillate in unison simply due to the fact that the systems are identical and are started from the same state, that may not be considered synchronization, as it does not comply with the requirement that the coherent dynamics be brought about by some form of physical interaction. A more accurate way to put it would be to say that, even for vanishing coupling between the clocks (as one would get by using a perfectly rigid beam), synchronized motion is possible (as the equations of the full \( 2N \)-dimensional dynamical system formed by the clocks are compatible with Eq. (2.55)) but it is not stable, since a slight perturbation irreversibly deviates the joint evolution of the clocks from synchronization. We will formalize these concepts shortly.

The synchronization of phase oscillators is a much broader and deeper subject than the previous example may seem to imply, both in its theoretical scope and its potential for applications in different scientific fields. Oscillating electric circuits, mechanical resorts, vibrating molecules, fireflies, firing neurons, Josephson junctions and hundreds of other natural and manmade systems can be characterized as periodic phase oscillators, and in many situations and applications their collective behavior in the form of synchronized evolution is a very relevant issue.\(^{34}\) A common feature of these systems is that their phase portraits exhibit asymptotically stable limit cycles in the case of dissipative nonlinear systems, or a superposition of (marginally) stable periodic orbits in the case of the harmonic oscillator and other conservative systems. Sometimes, some types of chaotic oscillators where some sort of periodicity can be established (the so-called phase coherent chaotic oscillators, such as the Rössler system from the previous section) are also treated.

\(^{34}\)For a non-technical account, see [Strogatz, 2004].
as phase oscillators, by considering the chaotic amplitude variation as a form of noise or fluctuations. In many cases, there is only one relevant degree of freedom (one canonical coordinate), a phase $\theta$ that indicates where along the cycle the state of the system is located at each time.\textsuperscript{35} In such cases, the dynamical system is one-dimensional, $M = S^1$, and the dynamics is governed by a vector field such that

$$\dot{\theta} = f(\theta) \quad f : S^1 \to \mathbb{R}. \quad (2.56)$$

If, on the other hand, second-order time derivatives are needed to write the laws of motion (as in classical mechanical systems, such as the simple pendulum we just discussed) the system is two-dimensional, $M = S^1 \times \mathbb{R}$. The vector field is then $f : S^1 \times \mathbb{R} \to \mathbb{R}^2$, and its components ($f_v, f_\theta$) are real valued functions defined as

$$\dot{\theta} = v \quad f_\theta(\theta, v) = v, \quad (2.57)$$

which results trivially from the usual trick to turn higher-order systems of ODEs into first-order systems, and

$$\dot{v} = f_v(\theta, v) \quad f_v : S^1 \times \mathbb{R} \to \mathbb{R}. \quad (2.58)$$

whose form is of course system-dependent.

When two one-dimensional phase oscillators interact, they form a 2-dimensional system, and already in this simplest of scenarios including coupled oscillators, new dynamical possibilities emerge. The equations of motion, assuming the more general case of \textit{bidirectional coupling},\textsuperscript{36} are:

\begin{align*}
\dot{\theta}_1 &= f_1(\theta_1, \theta_2) \\
\dot{\theta}_2 &= f_2(\theta_1, \theta_2).
\end{align*}

The system is said to be in a synchronous regime whenever the phases advance coherently

$$\theta_2(t) - \theta_1(t) = \Delta \theta \iff \dot{\theta}_1(t) = \dot{\theta}_2(t), \quad (2.60)$$

which generally (at least if synchronization is stable, and not an artefact due to the choice of initial conditions) happens after some transient if the system is started from arbitrary

\textsuperscript{35}In the experimental study of systems whose equations of motion are not known in detail, the phase can, at least in principle, be extracted by a suitable change of coordinates. See (Chavez et al., 2006) and (Kralemann et al., 2008) for a discussion of the phase extraction from experimental time series of oscillating systems.

\textsuperscript{36}In the case of \textit{unidirectional coupling}, one oscillator (the \textit{master}) evolves autonomously and affects the evolution of the other (the \textit{slave}). If system 1 is the master, the equations of motion are $\dot{\theta}_1 = f(\theta_1)$, $\dot{\theta}_2 = f_2(\theta_1, \theta_2)$. 

57
initial conditions. Any phase lag $\Delta \theta \in [0, 2\pi)$ is in principle possible, but the established terminology highlights two highly symmetric cases that occur in some simplified cases. Thus, if $\Delta \theta = 0$ the systems are in phase, whereas if $\Delta \theta = \pi$ they are in antiphase.

The equations for the case of a system comprising two second-order phase oscillators are only a bit more cumbersome notationally, but the conditions for synchronization are the same. An interesting case is that of $N$ coupled phase oscillators. The coupling in this case can have the topology of a chain, a lattice or an all-to-all coupling scheme, but it can also be much more irregular as in complex network configuration schemes, where the links provide the information on the dynamical dependencies. In all these cases, synchronization occurs if and only if

$$\dot{\theta}_1(t) = \dot{\theta}_2(t) = \ldots = \dot{\theta}_N(t).$$

(2.61)

This case also allows for the existence of partially synchronized states, where only a fraction of the $N$ oscillators evolve synchronously. The most important point, however, is not whether a synchronous solution such as that in Eq. (2.61) is compatible with the equations of motion, but whether it is asymptotically stable, or at least (marginally) stable (and this applies to the two-oscillator case as well, as in fact to any form of synchronization of dynamical systems). If the synchronous state is not recovered even after a small perturbation $\delta \theta_i \ll 1$ affects one oscillator, the possibility of synchronization is of limited physical interest. A stronger form of synchronization is globally stable synchronization, when the relation (2.61) is achieved from any initial condition. At least in principle, a complete analysis of the stability of synchronization can be carried out using the tools described in Section 2.1, as the synchronous states form a manifold embedded on the phase space of the full dynamical system comprising all the coupled oscillators.

Sometimes, noise due to fluctuations or unpredictable interactions with the environment is considered, which in the case of two oscillators leads to the following equations of motion

$$\dot{\theta}_1 = f_1(\theta_1, \theta_2) + \xi_1(t),$$

$$\dot{\theta}_2 = f_2(\theta_1, \theta_2) + \xi_2(t),$$

(2.62)

where $\xi_i(t)$ for $i = 1, 2$ are additive random terms. The resulting equations are stochastic equations reminiscent of the Langevin equation. The purpose of adding a random term in this way is frequently that of pushing the focus of theoretical studies closer to the realities found in experimental dynamical systems, where systems without appreciable noise or fluctuations due to parameter variations are the exception, and see whether the phenomena observed are robust in the presence of noise. As the exact conditions for
synchronization considered above are clearly unattainable in this case, several weaker conditions are considered, such as (focusing again on the case of just two coupled oscillators for the sake of simplicity)

$$\lim_{t \to \infty} |\theta_1(t) - \theta_2(t)| < C,$$

where $C$ is some constant. This somehow implies that the oscillators run synchronously on average, as any consistent difference in angular frequencies would asymptotically lead to an unbounded relative phase $\theta_1(t) - \theta_2(t)$. Conditions such as this one are also important in the study of phase synchronization in chaotic and noisy oscillators (see, for instance, [Rosenblum et al.] and [Parlitz et al., 1996]).

Our discussion so far has been limited to the case of coupled phase oscillators, which is the classical setting for the study of synchronization. Several forms of synchronization, however, can also occur in other types of dynamical systems. Remarkably, in the past two decades the concept of synchronization developed in such a way as to include chaotic dynamics [Pecora and Carroll, 1990]. The fact that chaotic systems can synchronize may sound counterintuitive; indeed, chaotic systems seem to intrinsically defy any form of synchronization, as even two identical systems starting from slightly different initial conditions separate exponentially in time. Nevertheless, there is by now ample numerical and experimental evidence showing that synchronization actually takes places among chaotic oscillators, and the concept itself has been expanded to cover a large variety of synchronization types. In this thesis, we will only be interested in the simplest form of synchronization of (general, possibly chaotic) identical dynamical systems, sometimes called complete synchronization or identical synchronization. More general forms of synchronization occurring also among non-identical systems that we will not discuss (generalized synchronization [Rulkov et al., 1995], phase synchronization [Rosenblum et al.], lag synchronization [Rosenblum et al., 1997], etc.) have been identified and studied. For a review of this fascinating topic, see [Boccaletti et al., 2002].

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the state vectors of two identical (i.e. defined by exactly the same equations of motion\footnote{Including parameter values in case there are free parameters in the definition of the common vector field $f$.}) dynamical systems, each of them starting from a given initial state (in general $\mathbf{x}_1(0) \neq \mathbf{x}_2(0)$). In isolation, $\dot{\mathbf{x}}_1 = f(\mathbf{x}_1)$ and $\dot{\mathbf{x}}_2 = f(\mathbf{x}_2)$. If the systems interact, then the time evolution of each system depends also on the state of the other system. Assuming again the most general case of bidirectional coupling, the equations of motion are

$$\begin{align*}
\dot{\mathbf{x}}_1 &= f_1(\mathbf{x}_1, \mathbf{x}_2) \\
\dot{\mathbf{x}}_2 &= f_2(\mathbf{x}_2, \mathbf{x}_1),
\end{align*}$$

(2.64)
where $f_1$ and $f_2$ are the modified vector fields, consisting typically of the autonomous evolution $f$ and one additive coupling term. Synchronization is said to occur if and only if

$$\lim_{t \to \infty} \|x_2(t) - x_1(t)\| = 0. \quad (2.65)$$

As explained in the case of phase oscillators, for synchronization to be scientifically relevant, the synchronized solution $x_1(t) = x_2(t)$ i) must be compatible with the equations of motion (2.64), and ii) it must have some form of stability. Indeed, if synchronization is unstable, any unsynchronized initial condition, however close to synchronization does not lead to a synchronous evolution.

To discuss synchronization in terms of phase-space orbits, it is necessary to introduce the notion of synchronization manifold or synchronization hyperplane $M$, which is the hyperplane $x_1 = x_2$ where the synchronization condition is met.\footnote{When dealing with 1-dimensional systems, $M$ is just the line $x_1 = x_2$ in the $(x_1, x_2)$-plane.} If there is a phase-space open set $U$ around the bounded area of $M$ where the synchronous dynamics takes place (so to speak, the direct sum of the identical attractors of the individual systems) such that any initial condition leads asymptotically to synchronization according to Eq. (2.65), synchronization is said to be \textit{stable}. Equivalently, synchronization is stable if and only if $M$ is asymptotically stable. If $U$ is the full phase space, synchronization is said to be \textit{globally stable}. Of course, what we just said extends naturally to the case of ensembles of $N$ dynamical systems for $N > 2$, where synchronization is expressed by extending the condition (2.65) to all possible pairs of (sub)systems.\footnote{In this case, $M$ is defined to be the hyperplane $x_1 = x_2 = \cdots = x_N$.}

To assess the stability of completely synchronized states of identical coupled systems, it is convenient to follow the norm of the time evolution of the synchronization error $e(t) := x_2(t) - x_1(t)$. We limit ourselves to the case of two oscillators, as it makes the explanation simpler, but the discussion can be easily extended to larger systems. The time evolution of the $e(t)$ is given by

$$\dot{e}(t) = f_2(x_2(t), x_1(t)) - f_1(x_1(t), x_2(t)). \quad (2.66)$$

For synchronization to be possible $M$ must be an invariant set of the dynamics. Assuming that for $x_1 = x_2 =: x_S$ the coupling terms vanish (as in the case of diffusive coupling explained in Subsection 1.3.2), and therefore $M$ is indeed an invariant set, one gets the autonomous evolution on $M$ given by $x_S = f(x_S)$ simultaneously for both systems. The linear stability of $M$ corresponding to a first order expansion $x_2 = x_S + \delta e$, where by
definition $x_1 = x_S$, can be studied by means of the following equation

$$\delta e = Df(x_S)\delta e,$$  \hspace{1cm} (2.67)

where $Df(x_S)$ is the Jacobian of the vector field $f$ evaluated on $\mathcal{M}$.

When the orbit $x_S$ on $\mathcal{M}$ is constant (a fixed point) or periodic (e.g. a limit cycle) the stability problem can be tackled by studying the spectrum of $Df$ or the Floquet multipliers (Perko, 2006), respectively. We will be mainly interested in the case that synchronization takes place on a chaotic attractor embedded within $\mathcal{M}$. The Lyapunov exponents based on the linear kernel (2.67), as average measurements of expansion or contraction rates along orbits, can serve for this purpose. Indeed, these transversal Lyapunov exponents express whether the error shrinks or grows in directions that are transverse to $\mathcal{M}$, and therefore the negativity of all of them provides a necessary condition for the stability of the synchronized state.\footnote{No sufficient condition can be derived from this fact, as precisely the transversal Lyapunov exponents (being Lyapunov exponents of a given dynamical system) are temporal averages over the whole chaotic attractor.}

In Section 2.6 we will further continue the study of the stability of synchronization of networks of oscillators (i.e. ensembles of interacting oscillators whose coupling follows an arbitrary, generally irregular, topology) in terms of the transversal Lyapunov exponents, and see the way it relates to topological properties.

To conclude, we mention that other criteria for the stability of synchronized motion (including sufficient conditions as well) can be given in terms of Lyapunov functions and other methods. See (Boccaletti et al., 2002) for a comprehensive review.

### 2.5 The Kuramoto model

Synchronization phenomena in large populations of interacting dynamical units with periodic dynamics have been devoted intense research efforts in the last decades, as they are important at a fundamental level, and have potential interest in the study of a large number of physical, biological and social systems, and in technical applications (Kuramoto, 1984, 1975; Strogatz, 2001a; Acebrón et al., 2005). Large populations of interacting oscillating systems are taken to form an ensemble of weakly coupled nonlinear oscillators, the autonomous dynamics of each of them being that of dynamical system with a globally attracting limit cycle. As the oscillators are coupled weakly, their individual phase space is, in principle, not greatly distorted by the coupling, each oscillator in the ensemble being
described by just one degree of freedom, namely, the corresponding phase. A simplifying, useful approach consists in treating each unit as a phase oscillator.

A Kuramoto-type model consists of an ensemble of \( N \) phase oscillators running at arbitrary intrinsic frequencies and coupled through a sine function. The evolution of the phase of the \( i \)-th oscillator \( \theta_i(t) \) is governed by the equation

\[
\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} w_{ij} \sin (\theta_j - \theta_i),
\]

where the natural frequencies \( \omega_i \) are assigned to the oscillators following a probability density \( g(\omega) \), and \( w_{ij} \geq 0 \) (\( w_{ij} \) is zero if oscillators \( i \) and \( j \) are not coupled, otherwise it represents the interaction strength). While the oscillators in isolation would run at their own frequencies, the coupling tends to synchronize each oscillator to all the others if the coupling strength is sufficiently high. By changing the reference frame to a rotating frame, \( \theta_i \rightarrow \theta_i - \Omega t \), where \( \Omega = \int d\omega \omega g(\omega) \), Eq. (2.68) can be easily transformed into an equivalent system of phase oscillators with zero mean frequency. This rotating frame is always implied in the analysis, as it simplifies the discussion with no lack of generality.

System (2.68) is a representative model of coupled phase oscillators whose synchronization properties have been very well studied in a large body of literature, since it is simple enough to be mathematically tractable, yet sufficiently rich and flexible to display a large variety of possible dynamics (Acebrón et al., 2005; Strogatz, 2001b). It is also very general in that the coupling matrix entries \( w_{ij} \) are simply assumed to be nonnegative real numbers. Therefore it encompasses different topologies and interaction schemes, in general both directional (the matrix need not be symmetric) and weighted (the matrix need not be binary).

In the rest of this section, we focus on the original Kuramoto model (Kuramoto, 1975, 1984), which is one of the simplest cases contained in Eq. (2.68), but a very important one, as it provides great insight into collective dynamics. In this model, there is an all-to-all homogeneous coupling, such that \( w_{ij} = K/N \forall i, j \), with \( K \geq 0 \):

\[
\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin (\theta_j - \theta_i).
\]

The phenomenology of synchronization in such models as simply observed in numerical simulations is as follows. When the coupling is weak, the oscillators evolve incoherently. Above a given coupling threshold, however, collective synchronization emerges, in what
seems to be the temporal analog of a second-order phase transition. See this evolution in Figure 2.10 in terms of the modulus of the order parameter (to be defined in the next paragraph), which measures the degree of synchronization in the system, and also by means of an instantaneous snapshot that illustrates the phase distribution in the ensemble. Intuitively, one expects that synchronization should increase with the coupling strength, and that the higher the number of oscillators that become synchronized, the more coherent are the summands in the coupling term in Eq. (2.69), and thus the stronger is the force acting upon not-yet-synchronized oscillators onto the collective dynamics. It is the purpose of the rest of this section to derive the analytical results that lend support to these expectations. In our exposition, we mainly follow (Acebrón et al., 2005; Strogatz, 2001b).

![Figure 2.10: Order parameter and instantaneous phase distribution of a Kuramoto model comprising $N = 100$ oscillators.](image)

The complex-valued order parameter is defined as follows

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j},$$

(2.70)

where $r \in [0, 1]$ measures the phase coherence of the ensemble and $\psi \in [-\pi, \pi)$ gives the
average phase.\footnote{In Chapter 3 we will repeatedly use $r$ to quantify the global order in an ensemble of phase oscillators.} Multiplying both sides of Eq. (2.70) by $e^{-i\theta_i}$

$$re^{i(\psi - \theta_i)} = \frac{1}{N} \sum_{j=1}^{N} e^{i(\theta_j - \theta_i)}, \quad (2.71)$$

and equating the imaginary parts, Eq. (2.69) can be rewritten

$$\dot{\theta}_i = \omega_i + Kr \sin (\psi - \theta_i). \quad (2.72)$$

It now becomes clear that each oscillator can be thought of as coupled to the common average phase $\psi$ with coupling strength $Kr$. Also, the existence of a positive feedback loop between coupling strength and coherence is apparent: as $r$ grows, the effective coupling $Kr$ increases, and this, in its turn, allows to lock so far unsynchronized oscillators onto the coherent rhythm of oscillation.

Let us now focus on the thermodynamic limit, $N \to \infty$. We begin by noticing that, formally, the order parameter in Eq. (2.70) can be expressed this way

$$re^{i\psi} = \int_{-\pi}^{\pi} d\theta \left( \frac{1}{N} \sum_{j=1}^{N} \delta(\theta - \theta_j) \right) e^{i\theta}. \quad (2.73)$$

Accordingly, in the thermodynamic limit, where the phases of the oscillators running at a given natural frequency $\omega$ are distributed with a time-dependent probability density $\rho_t(\theta; \omega)$, the order parameter is

$$re^{i\psi} = \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} d\theta d\omega \, g(\omega) \, \rho_t(\theta; \omega) \, e^{i\theta}. \quad (2.74)$$

Of course, along with this equation, a normalization condition must be satisfied such that for any $\omega$ and any time

$$\int_{-\pi}^{\pi} d\theta \, \rho_t(\theta; \omega) = 1. \quad (2.75)$$

When $K \to 0$ in Eq. (2.72), (i.e. in the fully unsynchronized limit), the oscillators practically rotate at their natural frequencies $\theta_i(t) \approx \omega_i t + \theta_i(0)$. Setting $\rho_t(\theta; \omega) = \delta(\theta - \omega t)$ in Eq. (2.74), for $t \to \infty$ one obtains $r \to 0$ by the Riemann-Lebesgue lemma. On the other hand, in the strong coupling limit, $K \to \infty$, the oscillators are locked to the mean phase, $\theta_i \approx \psi$. In this limiting case, setting $\rho_t(\theta; \omega) = \delta(\theta - \psi)$, one obtains
full synchronization, \( r = 1 \). Based on the numerical evidence from the finite-size case (see Figure 2.10), the existence of a critical coupling \( K_c \) above which the fully unsynchronized dynamics is replaced by some form of partial order is assumed (we will see in the following that this assumption is fully justified). One thus expects that for intermediate couplings, \( K_c < K < \infty \), a fraction of the oscillators are phase locked and the rest evolve out of synchrony, thus giving rise to a state of partial synchronization, \( 0 < r < 1 \). In the mean-field \( N \to \infty \) case, synchronization in general is said to occur if and only if \( r > 0 \) as \( t \to \infty \).

Since according to Eq. (2.72), the \( i \)-th oscillator moves with drift velocity \( \nu_i = \omega_i + Kr \sin (\psi - \theta_i) \), a continuity equation for the oscillator density can be written

\[
\partial_t \rho_i(\theta; \omega) + \partial_\theta \left\{ \left[ \omega + Kr \sin (\psi - \theta) \right] \rho_i(\theta; \omega) \right\} = 0.
\] (2.76)

This equation must be solved together with Eq. (2.74) (and the normalization condition in Eq. (2.75)) from a given initial condition. Now, the limiting cases heuristically treated above can be analyzed more rigorously, so let us go back to them. The stationary solution corresponding to the fully unsynchronized case \( \rho_i(\theta; \omega) = 1/(2\pi) \) (i.e. the phases are uniformly distributed over \([-\pi, \pi]\) for all \( \omega \)) when set into Eq. (2.74) yields \( r = 0 \) for all \( g(\omega) \), and therefore trivially satisfies Eq. (2.76). This is frequently called the incoherent solution. The strong coupling limit \( \rho_i(\theta; \omega) = \delta(\theta - \psi) \), on the other hand, when set into Eq. (2.74) yields \( r = 1 \) due to the normalization condition (2.75). This is also seen to satisfy the continuity equation (2.76). This is the global synchronization solution.

For an intermediate coupling regime, in the natural laboratory frame of reference \( (\theta \to \theta - \Omega t) \) an oscillator evolving with velocity \( \nu = \omega + Kr \sin (\psi - \theta) \) cannot synchronize if \( |\omega| > Kr \), and the stationary density \( \rho(\theta; \omega) \) must then obey \( \nu \rho(\theta; \omega) = A \), where \( A \) is a constant independent of \( \theta \). On the other hand, if the oscillator can synchronize, it becomes locked at an angle \( \theta \) such that \( Kr \sin (\theta - \psi) = \omega \) with \(-\pi/2 \leq (\theta - \psi) \leq \pi/2\). The corresponding stationary density is thus

\[
\rho(\theta; \omega) = \begin{cases} 
\delta \left[ \theta - \psi - \sin^{-1} \left( \omega/Kr \right) \right] H(\cos \theta) & |\omega| < Kr \\
A/|\omega + Kr \sin (\psi - \theta)| & |\omega| > Kr,
\end{cases}
\] (2.77)

where \( H(x) \) is the Heaviside step function. According to the well known rule \( \delta(f(\theta)) = \)

\[\text{Of course, the Dirac delta function is indeed a distribution or generalized function and not a regular function, and the partial derivatives are therefore not applicable without further ado. The argument, however, can be carried out by invoking the limiting case of a sequence of increasingly peaked functions (as commonly done when treating with Delta functions), or simply by noticing that in equation (2.76) \( \theta = \psi \) and \( \rho_i(\theta; \omega) \) is stationary and does not change with \( \theta \).} \]
\[ \frac{\delta(\theta - \theta_0)}{|f'(\theta_0)|} \], where \( \theta_0 \) is assumed to be the only root of \( f(\theta) \), taking \( f(x) = \omega + Kr \sin(\psi - \theta) \), the stationary density for \(|\omega| < Kr\) can be written as \( \rho(\theta; \omega) = \sqrt{K^2r^2 - \omega^2} \delta \left[ \omega + Kr \sin (\psi - \theta) \right] H(\cos \theta) \). Additionally, the normalization condition (2.75) imposes that, for a given \( \omega \), \( A = \sqrt{\omega^2 - K^2r^2}/(2\pi) \). Plugging the stationary density for a partially synchronized state into (2.74), one obtains

\[
\begin{align*}
\int_{\pi/2}^{\pi/2} \int_{-\infty}^{\infty} d\theta d\omega g(\omega) \delta \left[ \theta - \psi - \sin^{-1} \left( \frac{\omega}{Kr} \right) \right] e^{i(\theta - \psi)} \\
+ \int_{-\pi}^{\pi} \int_{|\omega| > Kr} d\theta d\omega g(\omega) \frac{A}{|\omega + Kr \sin (\psi - \theta)|} e^{i(\theta - \psi)}.
\end{align*}
\] (2.78)

To proceed with the derivation, from now on we assume that the frequency distribution is symmetric around the mean, \( g(\omega) = g(-\omega) \).

Under this assumption, then the second summand in (2.78) vanishes due to the symmetry relation \( \rho(\theta + \pi; -\omega) = \rho(\theta; \omega) \). The order parameter is thus

\[
r = \int_{|\omega| < Kr} d\omega g(\omega) \cos \left[ \sin^{-1} \left( \frac{\omega}{Kr} \right) \right] = Kr \int_{-\pi/2}^{\pi/2} d\theta g(Kr \sin \theta) \cos^2 \theta.
\] (2.79)

Aside from the trivial \( r = 0 \) solution that corresponds to the \( \rho(\theta; \omega) = (2\pi)^{-1} \) case, there are solutions corresponding to the partial synchronization state \((0 < r < 1)\) that satisfy

\[
1 = K \int_{-\pi/2}^{\pi/2} d\theta g(Kr \sin \theta) \cos^2 \theta.
\] (2.80)

Indeed, the critical coupling \( K = K_c \), which is the lower end of the partial synchronization interval, can be obtained by simply setting \( r = 0 \) in Eq. (2.80),

\[
K_c = \frac{2}{\pi g(0)}.
\] (2.81)

Additionally, by expanding the right-hand side of (2.80) in powers of \( Kr \) around \( K_c \) (see the details in (Acebrón et al., 2005; Kuramoto, 1975)) the following scaling is found

\[
r \sim \sqrt{\frac{-16(K - K_c)}{\pi K_c^2 g''(0)}}.
\] (2.82)

The order parameter bifurcates for \( K > K_c \) supercritically if \( g''(0) < 0 \) and subcritically.

---

43 Normal distributions, uniform distributions, and Lorentzian distributions, among others, possess this form of symmetry. The reader should note that calculations are done in the rotating frame of reference, and therefore \( \int d\omega \omega g(\omega) = 0 \).
for $K < K_c$ if $g''(0) > 0$. Kuramoto in (Kuramoto, 1975) used the Lorentzian frequency
distribution

$$g(\omega) = \frac{\gamma/\pi}{\gamma^2 + \omega^2}, \quad (2.83)$$

which permits the analytical evaluation of the integral in (2.80), which gives $r = \sqrt{1 - (K_c/K)}$
for $K > K_c = 2\gamma$.

So far we have followed Kuramoto’s original reasoning, which leads to stationary state
solutions corresponding to incoherent and partially synchronized states, but we have not
discussed the stability properties of these solutions. The linear stability of the incoherent
solution can be studied by plugging $\rho_t(\theta; \omega) = 1/(2\pi) + \exp (\lambda t) \mu(\theta, \omega)$ into Eq. (2.76),
which gives

$$\lambda \exp (\lambda t) \mu(\theta, \omega) + \omega \exp (\lambda t) \partial_\theta \mu(\theta, \omega) - Kr \cos(\psi - \theta) [1/(2\pi) + \exp (\lambda t) \mu(\theta, \omega)] + Kr \sin(\psi - \theta) \partial_\theta \mu(\theta, \omega) = 0. \quad (2.84)$$

The linear stability is guaranteed if $\text{Re} \lambda < 0$, whereas if there is an admissible $\lambda$
with positive real part incoherence is unstable. Inserting $r$ according to Eq. (2.74), and keeping
only the terms linear in $\mu(\theta, \omega)$,

$$\lambda \mu(\theta, \omega) + \omega \partial_\theta \mu(\theta, \omega) - \frac{K}{2\pi} \text{Re} e^{-i\theta} \int_{-\pi}^\pi \int_{-\infty}^{+\infty} d\omega' d\omega' g(\omega') e^{i\omega'} \mu(\theta', \omega') = 0, \quad (2.85)$$

while the normalization condition is in this case equivalent to

$$\int_{-\pi}^\pi d\theta \mu(\theta, \omega) = 0. \quad (2.86)$$

Since $\mu$ is periodic in $\theta$, the function can be expanded in a Fourier series $\mu = \sum_{n=-\infty}^{\infty} b_n(\omega)e^{in\theta}$. Inserting this expansion in Eq. (2.85) we obtain for a given $n$

$$(\lambda + in\omega) b_n(\omega) = \frac{K}{2} (\delta_{n,1} + \delta_{n,-1}) \int_{-\infty}^{+\infty} d\omega' g(\omega') b_n(\omega'), \quad (2.87)$$

where $b_{-n} = b_n^*$ has been used.\footnote{The asterisk denotes complex conjugation.} Viewing the right-hand side of (2.87) as the action of
a linear operator on the function $b_n(\omega)$, then the $\lambda_n(\omega) = -in\omega$, for $n = \pm1, \pm2, \ldots$, belong to the continuous spectrum of the operator. When $g(\omega)$ is unimodal (i.e. it
has even symmetry and is nonincreasing for $\omega > 0$) Strogatz et al. in (Strogatz et al.,
1992) have used this fact to show that the incoherent solution is neutrally stable for
$K < K_c = 2/[\pi g(0)]$, the continuous spectrum lying on the imaginary axis, whereas for
$K > K_c$ there is a positive eigenvalue.

The typical dependence of $r$ on $K$ as obtained in numerical simulations, of which Figure 2.10 provides an example, shows that for $K < K_c$ the oscillators starting from any generic initial condition act as if they were uncoupled, whereas for $K > K_c$ this incoherent state becomes unstable and there is a nucleation of a small cluster of oscillators which gives a positive $r(t)$ for $t \to \infty$. The asymptotic value of $r$ grows exponentially as $K$ is increased until it reaches its maximum value $\lim_{t \to \infty} r(t) = 1$, corresponding to full synchronization. While Kuramoto was able to give analytical expressions for $K_c$ and the asymptotic dependence $r(K)$ and Strogatz and collaborators solved the linear stability problem for incoherent states many years later (and Crawford and others made some extensions, see (Strogatz, 2001b) and references therein), however, the partial synchronization branch where asymptotically $r(K) > 0$ has not been shown to be linearly stable, and the problems of global stability or convergence results on the finite-size system as $N \to \infty$ have not been satisfactorily addressed (Strogatz, 2001b).

Despite these difficulties in the analytical treatment of the model, by simply describing the way a large ensemble of oscillators running at different natural frequencies spontaneously lock to a common rhythm, the Kuramoto model has established itself as a fundamental piece in the study of collective dynamics in complex systems. Biological systems (such as pacemaker cells in the heart or in the suprachiasmatic nucleus of the brain, congregations of flashing fireflies or chirping crickets) and technological systems (such as laser arrays or superconducting Josephson junctions) feature prominently in its applications. Furthermore, from a theoretical point of view the model provides important clues as to the setting and robustness of collective motion in dynamical systems.

Many interesting aspects of the Kuramoto model are discussed pedagogically in (Strogatz, 2001b). For a comprehensive treatment and additional rigorous results, see (Acebrón et al., 2005).

### 2.6 The master stability function

The master stability function (MSF) approach provides an elegant and powerful way to study the linear stability of synchronization in ensembles of identical dynamical systems. While it was initially proposed for the study of arrays of coupled oscillators (Pecora and Carroll, 1998; Fink et al., 2000), the method was soon to be adopted in the context of...
networks of oscillators with arbitrary topologies (Barahona and Pecora, 2002), eventually becoming the standard approach to the study of the synchronization of complex networks.

Let there be a network of \( N \) identical \( m \)-dimensional chaotic oscillators. Topologically, these are the nodes of the network, while their coupling provides the links. The autonomous (isolated) dynamics of each of them is governed by \( \dot{x}_i = f(x_i) \), where \( x_i \in \mathbb{R}^m \) is the state vector of the \( i \)-th oscillator. Let the nodes be diffusively coupled, and let the coupling scheme be that of a connected network with a topology represented by a binary symmetric \( N \times N \) adjacency matrix \( \mathcal{A} = (a_{ij}) \) (or, equivalently, by its corresponding Laplacian matrix \( \Delta \)). Moreover, let the coupling function \( h : \mathbb{R}^m \rightarrow \mathbb{R}^m \) be the same for all pairs of connected nodes. The dynamics of each node is then governed by

\[
\dot{x}_i = f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} [h(x_j) - h(x_i)] = f(x_i) - \sigma \sum_{j=1}^{N} \Delta_{ij} h(x_j), \tag{2.88}
\]

where \( \sigma \) is the coupling constant, and the reasoning in Eq. (1.23) has been followed. As \( \sum_j \Delta_{ij} = 0 \), the synchronization manifold \( \mathcal{M} \) is an invariant set of the dynamics (i.e. there exist orbits satisfying \( x_1(t) = x_2(t) = \cdots = x_N(t) =: x_S(t) \forall t \)). Synchronization, \( \dot{x}_S = f(x_S) \) in every node, is therefore possible. Is it stable? To answer this question we resort to the MSF approach.

We start by condensing into a single equation the network dynamics as given by the \( N \) equations of the form (2.88). Let us define \( x = (x_1, x_2, \cdots, x_N)^T \), \( f(x) = [f(x_1), f(x_2), \cdots, f(x_N)]^T \), and \( h(x) = [h(x_1), h(x_2), \cdots, h(x_N)]^T \). The equation we are looking for can be simply written as follows

\[
\dot{x} = f(x) - \sigma \Delta \otimes h(x), \tag{2.89}
\]

where \( \otimes \) is the Kronecker product. Linearization around \( \mathcal{M} \) yields the following variational equation

\[
\dot{\xi} = [I \otimes Df(x_S) - \sigma \Delta \otimes Dh(x_S)] \xi, \tag{2.90}
\]

where \( I \) is the \( N \times N \) unit matrix. The first term is block diagonal, with \( m \times m \) blocks, whereas the second term can be put in block diagonal form by diagonalizing \( \Delta \) (which, as seen in Section 1.3, is diagonalizable, has \( N \) real, non-negative eigenvalues, \( \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N \), and orthogonal eigenvectors). Diagonalization leaves us with \( N \) block diagonalized variational equations

\[
\dot{\xi}_k = [Df(x_S) - \sigma \gamma_k Dh(x_S)] \xi_k, \tag{2.91}
\]
where \( k = 1, 2, \ldots, N \) denote different \( m \)-dimensional eigenmodes. Since the network is assumed to be connected, \( \gamma_1 = 0 \) corresponds to the eigenmode along \( \mathcal{M} \), whereas the eigenvectors associated to \( \gamma_i > 0 \) for \( i = 2, 3, \ldots, N \) form a basis of the orthogonal complement to \( \mathcal{M} \).

The simplification is enormous: regardless of topological details, we can characterize the linear stability of \( \mathcal{M} \) for a given dynamics \( f(x) \) and output function \( h(x) \), by just considering the network spectrum \( \gamma_2, \gamma_3, \ldots, \gamma_N \). But we can do even better than this simply by parametrizing the variational equations (2.91). Indeed, the resulting one-parameter variational equation with parameter \( \nu \),

\[
\dot{\zeta} = \left[ Df(x_S) - \nu Dh(x_S) \right] \zeta,
\]

(2.92)
can be used to study the stability of synchronization corresponding to a given dynamics \( f(x) \) and coupling function \( h(x) \), and, in some sense, to do so for all topologies and all values of \( \sigma \) once and for all. Let us see how this can be done.

By computing the maximum (conditional) Lyapunov exponent based on the kernel (2.92) as a function of \( \nu \), \( \lambda_{\text{max}}(\nu) \), we can learn about the stability of synchronization in our system. If \( \lambda_{\text{max}}(\nu') < 0 \) we have a necessary condition for variations around \( \mathcal{M} \) to damp out along a generic eigenmode corresponding to \( \gamma = \nu'/\sigma \). Thus, for some \( f(x) \) and \( h(x) \) we have a single MSF curve \( \lambda_{\text{max}}(\nu) \) and, given a topology, by just computing the spectrum we can know whether synchronization is stable or not (and if not, which eigenmodes carry the unstability) for a given \( \sigma \), or which \( \sigma \) values allow for stable synchronization. The latter possibility, however, will not always be available; when synchronization can be stabilized in a network by a judicious choice of \( \sigma \) we say that the network is synchronizable.

Whether \( \lambda_{\text{max}}(\nu = 0) \) is positive or zero depends on whether the dynamics \( \dot{x} = f(x) \) is chaotic or periodic, respectively; as we will focus on the chaotic case, for us \( \lambda_{\text{max}}(0) > 0 \). For \( \nu > 0 \) three qualitative behaviors are possible in the vicinity of the origin that can be associated with three different classes of coupled dynamical systems, according to the classification first proposed in (Boccaletti et al., 2006): I) \( \lambda_{\text{max}}(\nu) \) is a function that remains positive for \( \nu > 0 \), II) \( \lambda_{\text{max}}(\nu) \) is a function that intercepts the abcissa at some \( \nu_c > 0 \) and then remains negative, III) \( \lambda_{\text{max}}(\nu) \) is a function that becomes negative, reaches at least one minimum and rapidly rises again and becomes positive, and there-

\[45\text{Of course, } \lambda_{\text{max}}(0) \text{ is simply the maximum Lyapunov exponent of a single (uncoupled) system whose dynamics is given by } f(x).\]
fore has two zero crossings at $\nu_1$ and $\nu_2$ ($0 < \nu_1 < \nu_2$). Of course other possibilities may exist, depending also on the $\nu$ range considered, but these are the most frequently encountered cases. A sketch illustrating the three MSF classes is presented in Figure 2.11.

![Figure 2.11: MSF classes of networked chaotic systems for a given dynamical system and coupling function.](image)

Curves corresponding to a class I (blue), a class II (red) and a class III (green) chaotic system are presented.

For class I systems, $\mathcal{M}$ is transversally unstable whatever the topology and the coupling strength $\sigma$. For class II systems, the topology and $\sigma$ must be such that $\sigma \lambda_2 > \nu_c$ holds for synchronization to be stable. Such systems always (for any topology) admit (stable) synchronization for large enough $\sigma$, while the effect of the connection topology is simply a rescaling (imposed by $\lambda_2$) of the threshold for synchronizability. Class III systems, in their turn, can only be synchronized whenever $\lambda_N/\lambda_2 < \nu_2/\nu_1$. If this condition is satisfied, the system is synchronizable for any $\sigma$ such that $\nu_1 < \sigma \gamma_i < \nu_2$ for all $i$ (equivalently, it must be the case that $\sigma \gamma_2 > \nu_1$ and $\sigma \gamma_N < \nu_2$). Thus, for class III systems, the ability of a given topology to synchronize is associated with the ratio $\lambda_N/\lambda_2$, as the more packed the eigenvalues of $\Delta$ are, the higher is the probability of having all conditional Lyapunov exponents within the stability range for some $\sigma$.

Here, we have only concerned ourselves with the basics of the classical MSF approach. For an insightful account of the MSF rationale together with a systematic study of the MSF behaviors of several well-known dynamical systems see (Huang et al., 2009). Many of the variants that have appeared in the literature have been reviewed in (Boccaletti et al., 2006). Some of those variants that are relevant in the study of network control and related subjects are explained in Section 2.8 and in Chapter 5 where we present a new
formulation of the MSF approach in the context of the targeting of network dynamics that has been recently proposed (Gutiérrez et al., 2012). The related problem of synchronizing a (slowly) time-varying complex network has been addressed by an adaptive mechanism in (Sorrentino and Ott, 2008).

2.7 Adaptive networks

So far in this chapter we have been discussing dynamics on networks (or simply individual dynamical systems). But, already in Section 1.6, we saw an example of a growth model, which is also a network dynamical process. Such processes, however, should be more properly considered as examples of dynamics of networks, as in those cases evolution takes place in the space of topologies (or, equivalently, the space of $N \times N$ adjacency/weight matrices) (Gross and Blasius, 2008). In the first case, each node is a dynamical system and the state of the network is given by the collective state of all nodes, while the topology is static; in the second case, the state of the network is the topology itself, which is time-dependent, and no endogenous node dynamics is considered.

These days attention is being increasingly devoted to adaptive networks, i.e. networks that combine topological evolution with internal dynamics in the network nodes. As in the case of dynamics on networks, the topology reflects the coupling scheme and therefore influences the time evolution of the node states. Now, however, the coupling topology is not static, indeed it is coupled to the node dynamics. In adaptive networks, the dynamical state $\mathbf{x}(t) \equiv (x_1(t), x_2(t), \ldots, x_N(t))^T \in \mathbb{R}^{Nm}$ (where $N$ is the network size and $m$ the dimensionality of the dynamics in each node), which of course depends on the coupling scheme, in its turn reshapes the network topology $\mathcal{W}(t) = (w_{ij}(t)) \in \mathbb{R}^{N \times N}$ (assuming the general case of weighted, directed networks), thus closing the loop of mutual influence between structure and dynamics. The full network is thus an $(Nm + L)$-dimensional dynamical system ($L$ being the number of links) where the evolution of the state of the $i$-th node, $x_i(t) \in \mathbb{R}^m$, depends on both the states of its neighbors $\{x_j(t)\}$ and, through the coupling scheme, on the link weights connecting them to it $\{w_{ij}(t)\}$, where $j$ takes all possible values in the set of neighbors $\mathcal{N}_i$. In most of the cases of interest the topology update is uniquely based on local rules, i.e. the evolution of $w_{ij}(t)$ depends explicitly on $x_i(t)$ and $x_j(t)$, but not on the state of other nodes.

The fact that in many interesting self-organization phenomena found in nature there actually is a reshaping of the topology that depends upon the dynamics of the interacting nodes.
units is a strong incentive for studying collective dynamics in adaptive network. Examples can be found in the central nervous system, where neuronal plasticity is known to reshape the pattern of synaptic connections by potentiating synapses of highly cooperative neurons and depressing those where a pre-synaptic action potential does not elicit a post-synaptic response, or in manmade transportation and communication networks, where traffic congestions eventually lead to the creation of new links, to name but two examples from totally unrelated fields. Additionally, several theoretical studies dealing with adaptive networks have consistently shown that some remarkable dynamical phenomena frequently occur in this context, including the formation of complex topologies, robust self-organization, spontaneous emergence of well differentiated classes of nodes and time-scale competition, among others. For a review of important recent results on adaptive networks, see (Gross and Blasius, 2008).

2.8 Network control and targeting

As explained in Section 2.1, sensitive dependence on initial conditions is the hallmark of chaos. It is also the reason why chaotic dynamics has been traditionally considered undesirable in experiments or applications: given that initial conditions are known with finite precision, the presence of chaos implies that the system under study is in practice intrinsically unpredictable (even under the best imaginable experimental conditions). It turns out, however, that for the very same reasons, chaotic systems are flexible sources of a wide range of dynamics. In fact, the intrinsic instability of chaotic orbits opens up the possibility of exploiting this richness by a judicious application of perturbations that slightly modify the state of the system. In short, chaotic systems are controllable.

Sensitive dependence on initial conditions aside, chaotic systems are characterized by two other relevant properties: i) infinitely many unstable periodic orbits are embedded in the chaotic attractor, and ii) a system ergodically visits the neighborhoods of the phase points in each of these unstable periodic orbits. The original idea of chaos control is to apply small perturbations to the system when it is moving through a small neighborhood of a periodic orbit with the aim of stabilizing the orbit (see (Boccaletti et al., 2000) and references therein). One thus takes advantage of the fact that a tiny perturbation can give rise to a large response in the long run. In contrast, nonchaotic dynamics generally need perturbations of a magnitude comparable to the range of the unperturbed evolution to obtain a similar result.
But perturbations can also be applied simply in order to reach one particular orbit along which the system could be naturally evolving had it started at the corresponding initial condition. Indeed, targeting methods propose strategies for an intelligent application of perturbations that are able to lead the system to some goal dynamics, i.e., some desired orbit that is compatible with the laws of motion, and is not unstable (Boccaletti et al., 2000). The main difference with respect to the control problem is that the stability properties of the goal dynamics make its observation perfectly possible: the system could autonomously evolve according to such dynamics if only there were ways to control the initial conditions to any desired level of precision, which in general is not possible.

It is not our purpose to discuss the traditional methods for the targeting and control of low-dimensional individual chaotic systems, but some typical examples of these methods will be very briefly reviewed in Chapter 5. A solid introduction is provided in (Ott, 2002), while a comprehensive review can be found in (Boccaletti et al., 2000).

While an ensemble of networked dynamical units can be treated as just one high-dimensional dynamical system, the possibility of using traditional methods for the control or targeting of chaotic systems in this context meets with increasing difficulties as the dimensionality of the system $Nm$ grows (again, $N$ is the network size and $m$ the dimensionality of the node dynamics). Moreover, such an approach would completely ignore the topology of the network instead of taking advantage of the fact that different nodes play different roles in a heterogeneous complex network. A more promising approach is that of basing the control/targeting strategy on the coupling scheme (the topology of the network), by choosing judiciously where (i.e. on which node) to apply the perturbations in the network, rather than where (i.e. on which state) to apply them on the underlying high-dimensional attractor. In the rest of this section we describe some of the control schemes for dynamical networks that have been proposed in the past few years, while in Chapter 5 we discuss our recent work on the targeting of network dynamics.

The pinning control of scale-free networks has been studied in (Wang and Chen, 2002a). The proposed pinning method is based on applying local linear feedback injections to some nodes of a network, the effect of which are then propagated to the rest of the network. Specifically, the authors study a system of $N$ linearly and diffusively coupled identical $m$-dimensional dynamical systems

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} \Delta_{ij} h x_j, \quad (2.93)$$

74
where $h = (h_{ab}) \in \mathbb{R}^{m \times m}$ is a binary diagonal matrix with $h_{aa} = 1$ for a particular $a$ and $h_{bb} \neq 0$ for $b \neq a$ (two coupled nodes are linked through the variable $a$). The aim is to stabilize the network on a global (possibly unstable) fixed point

$$x_1 = x_2 = \cdots = x_N = \bar{x},$$

(2.94)

where by definition $f(\bar{x}) = 0$. In order to achieve this goal, local linear feedback injections are applied to a small fraction of the nodes, denoted by the indices $i_1, i_2, \ldots, i_l$, whose evolution is given by

$$\dot{x}_{i_k} = f(x_{i_k}) - \sigma \sum_{j=1}^{N} \Delta_{i_kj} h(x_j) - \sigma d h(x_{i_k} - \bar{x}) \quad k = 1, 2, \ldots, l,$$

(2.95)

where $d > 0$ is the feedback gain. Meanwhile, the rest of the nodes simply follow the dynamics given in Eq. (2.93). Linearizing around the fixed point $\bar{x}$ gives

$$\dot{\eta} = [Df(\bar{x}) - \sigma \Gamma h] \eta,$$

(2.96)

where $\eta_i(t) = x_i(t) - \bar{x}$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_N)^T \in \mathbb{R}^{N \times m}$. On the other hand, $B = \Delta + D$, where $D = \text{diag}(d_1, d_2, \ldots, d_N)$, and $d_{ik} = d$ whereas the rest of the diagonal entries are zero. If $\gamma_1, \gamma_2, \ldots, \gamma_N$ are the eigenvalues of matrix $B$ and $\Phi = [\phi_1, \phi_2, \ldots, \phi_N] \in \mathbb{R}^{N \times N}$ is the eigenvector basis, such that $B\phi_k = \gamma_k \phi_k$, by expanding each column $\eta$ on the basis $\Phi$, $\eta = \Phi \nu$ where $\nu \in \mathbb{R}^{N \times m}$, one obtains

$$\dot{\nu} = [Df(\bar{x}) - \sigma \gamma_k h] \nu,$$

(2.97)

where $\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_N)$. Each row $\nu_k$ gives

$$\dot{\nu}_k^T = [Df(\bar{x}) - \sigma \gamma_k h] \nu_k^T.$$

(2.98)

If the kernel $[Df(\bar{x}) - \sigma \gamma_k h]$ is such that the real parts of its eigenvalues are all negative for all $k$, then this implies that there is local exponential stability (see (Wang and Chen, 2002a) for further details). The dependence of the controllability on the set of pinned nodes and the feedback gain $d$ is studied. The authors conclude that the most effective control strategy consists in pinning the nodes with the highest degrees (which is considerably superior to the strategy consisting in a random selection of nodes to be pinned). In (Wang and Chen, 2002b), the authors refine and extend this method, and random networks are considered (in that case the differences are not so large between the random pinning scheme and the degree ranking pinning scheme). Analogous results in a different context are reported in Chapter 5.
The pinning controllability of synchronized dynamics has been studied in (Sorrentino et al., 2007). A system of $N$ identical oscillators that are diffusively coupled is again considered,

$$
\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} \Delta_{ij} h(x_j) - \sigma \sum_{k=1}^{l} \delta(i - i_k) \kappa_i (x_i - s), \tag{2.99}
$$

where $h : \mathbb{R}^m \to \mathbb{R}^m$ is an output function. Here, $s(t)$ is a desired reference evolution, $\dot{s} = f(s)$, and $\kappa_i$ is the control gain of node $i$. The last term on the right-hand side represents the pinning control action, and is present only for those nodes that belong to the set of pinned nodes $\{i_1, i_2, \ldots, i_l\}$. The objective is to drive a network toward a desired synchronous evolution $s(t)$, so that $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ for $t \to \infty$.

Indeed, the study of the pinning controllability amounts to studying the stability of this solution. To do so, the authors consider an extended network of $N + 1$ dynamical systems $y_i$, where $y_i = x_i$ for $i = 1, 2, \ldots, N$ and $y_{N+1} = s$ (i.e. the desired common evolution is treated as the state of an extra virtual node added to the network). Eq. (2.99) can then be rewritten as follows

$$
\dot{y}_i = f(y_i) - \sigma \sum_{j=1}^{N+1} P_{ij} h(y_j), \quad i = 1, 2, \ldots, N + 1. \tag{2.100}
$$

The matrix $P = (P_{ij}) \in \mathbb{R}^{(N+1) \times (N+1)}$ is

$$
P = \begin{pmatrix}
\Delta_{11} + \xi_1 \kappa_1 & \Delta_{12} & \cdots & \Delta_{1N} & -\xi_1 \kappa_1 \\
\Delta_{21} & \Delta_{22} + \xi_2 \kappa_2 & \cdots & \Delta_{2N} & -\xi_2 \kappa_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} + \xi_N \kappa_N & -\xi_N \kappa_N \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \tag{2.101}
$$

with $\xi_i = \sum_{k=1}^{l} \delta(i - i_k)$. $P$ is a nonsymmetric, zero row-sum matrix with nonnegative values along the main diagonal. If the adjacency matrix (and hence $\Delta$) is assumed to be symmetric, $P$ is diagonalizable, as its spectrum contains that of its minor spanning the first $N$ rows and first $N$ columns (a symmetric matrix) plus one zero eigenvalue. Let $\{\gamma_i = \gamma_i^r + i \gamma_i^i\}$ be the eigenvalues of $P$ and assume they are ordered in such a way that $\gamma_1^r \leq \gamma_2^r \leq \cdots \leq \gamma_N^r$. By spectral graph theory, it can be proved that $\gamma_i^r \geq 0$ for all $i$, and $\gamma_1$ is the only zero eigenvalue (Sorrentino et al., 2007). By invoking the MSF rationale, the linear stability of the synchronized evolution $y_1(t) = y_2(t) = \cdots = y_N(t) = y_{N+1}(t)$ can be studied from Eq. (2.100). For given $f$ and $h$, this can be evaluated as the stability of $N + 1 m$-dimensional blocks for each $\nu_i = \sigma \gamma_i$ for $i = 1, 2, \ldots, N + 1$, as explained in Section 2.6. When $P$ has a real spectrum, the pinning controllability can be evaluated.
from $R^{N+1} = \gamma_{N+1} / \gamma_2$, assuming the system is type III according to the classification based on MSF properties (see Section 2.6).\textsuperscript{46} Numerical results included in (Sorrentino et al., 2007) show that for either too high or too low $\kappa$, where $\kappa_1 = \cdots = \kappa_p = \kappa$, the pinning controllability is reduced, with a maximum around a given intermediate value. Using a larger number of controllers proves to be an effective strategy in all cases considered.

Complex network control is a thriving field that originated only a few years ago and is currently expanding at a rapid rate. The two methods described above constitute two attempts at a general strategy for pinning control of networks of oscillators. Instead of briefly reviewing a relatively high number of contributions found in the literature, we have chosen to explain at some length two of the most significant ones, which are the closest in spirit to our approach to network targeting that is presented in Chapter 5. But before we finish this section, let us briefly dwell on a recent contribution to the study of the controllability of real complex networks that are only imperfectly known, both for its scientific significance and in order to give a flavor of the great diversity of methods employed (which are by no means restricted to MSF approaches).

A strategy for studying the controllability of directed, weighted networks whose node dynamics is unknown and whose connection weights may also be unknown (and time dependent) has been recently presented in (Liu et al., 2011). Whereas the underlying model is in some respects rather stringent (e.g. it assumes linearity and scalar node states), the fact that it allows to give precise, general rules for the controllability of networks based on rigorous results from control theory and mathematical graph theory, assuming only knowledge of the adjacency matrix, makes them potentially useful in a wide range of contexts, including experimental studies based on imperfectly known networks with unknown dynamics (the most frequently encountered case in the experimental study of networks). The authors give analytical conditions for identifying the minimal set of nodes that, if forced to follow a corresponding set of time evolutions, are enough for a full control of the network dynamics. They focus on the controllability of linear systems\textsuperscript{47} of the type

$$\dot{x}(t) = Ax(t) + Bu(t),$$

(2.102)

where $x(t) = (x_1(t), \ldots, x_N(t))^T$ represents the (scalar) states of the nodes. $A$ is an $N \times N$ matrix representing the wiring in the network (a nonsymmetric weight matrix, in

\textsuperscript{46}If the spectrum is complex, both $R^{N+1} = \gamma_{N+1}^r / \gamma_2^r$ and $M^{N+1} = \max_j \gamma_j^l$ should be employed. See (Pecora and Carroll, 1998; Sorrentino et al., 2007).

\textsuperscript{47}According to the authors, this assumption is apparently not as stringent as it may sound, since “because structural controllability implies controllability of a continuum of linearized systems, our results can also provide a sufficient condition for controllability for most non-linear systems.” (Liu et al., 2011)
the general case), and $B$ is an $N \times M$ input matrix (where $M \leq N$) that identifies the nodes acted upon by an external controller. The input vector $\mathbf{u}(t) = (u_1(t), \ldots, u_M(t))^T$ gives the signals used in forcing the $M$ driver nodes. The system (2.102) is considered to be controllable if it can be driven from any initial state to a desired target dynamics in finite time. By Kalman’s controllability rank condition (see (Liu et al., 2011) and references therein), this is possible if and only if the controllability matrix

$$C = (B, AB, A^2B, \ldots, A^{N-1}B)$$  \hspace{1cm} (2.103)

is a full-rank $N \times NM$ matrix (i.e. $\text{rank}(C) = N$).

The authors’ proposal is to consider the system $(A, B)$ to be structurally controllable if it is possible to choose the non-zero weights in $A$ and $B$ in such a way that the system satisfies Kalman’s rank condition. This concept provides a means to overcome the incomplete or approximate knowledge of $A$ if at least the adjacency matrix is known. Indeed, a structurally controllable system is controllable for all weight combinations except for a set of zero measure cases where the parameters satisfy some exact constraints. Whereas Kalman’s controllability condition for large networks is computationally prohibitive, important graph-theoretical results that give sufficient conditions for controllability can be proven (see the Supplementary Information in (Liu et al., 2011) for the full argument).

The minimum number of driver nodes needed for the network control is shown to be determined by the maximum set of links that do not share start or end nodes with any other link. Furthermore, network control is shown to be possible if and only if each node that does not have an incoming link from the previously defined set is controlled and there are directed paths from the input signals to all nodes that do have such an incoming link.

Furthermore, by employing methods from statistical mechanics the authors show that the number of driver nodes needed to control the network is to a large extent determined simply by the degree distribution $P(k_{\text{in}}, k_{\text{out}})$. Somewhat in contradiction to the results from other approaches to network controllability, they also find that the denser and more homogeneous a network is, the fewer nodes are needed to control the network, whereas sparse inhomogeneous nodes are the hardest to control, and that the driver nodes tend to avoid the high-degree nodes. Moreover, most links are found to be non-critical, in that they have a role in some control schemes, but the network can be controlled in their absence. See (Liu et al., 2011) for a detailed account of the method and of the results obtained in the analyses of some empirical networks.
Chapter 3

Emerging meso- and macroscale structural properties from the synchronization of adaptive networks

Synchronization is a collective phenomenon occurring in systems of interacting units that is ubiquitous in nature, society and technology (for a discussion of synchronization in dynamical systems see Section 2.4 while examples of synchronization phenomena in different fields appear in (Pikovsky et al., 2003; Strogatz, 2004)). Recent studies have enlightened the important role played by the interaction topology on the emergence of synchronized states in complex networks. However, most of these studies disregard the fact that real world systems change their interaction patterns in time. In this chapter, we analyze synchronization features in adaptive networks, where structural and dynamical features co-evolve (see Section 2.7). The feedback of the node dynamics on the interaction pattern is governed by the competition of two empirically motivated mechanisms: homophily (the reinforcement of interactions between correlated units in the graph) and homeostasis (the conservation of the input strength received by each unit). The competition between these two adaptive principles leads to the emergence of key meso- and macro-scale structural properties observed in real-world networks, such as modular and scale-free structures, together with a striking enhancement of local synchronization in systems with no global order. The work we present here has been published in (Assenza et al., 2011) and (Gutiérrez et al., 2011a).

3.1 Introduction

The interest of scientists in understanding the relationship between the emergence of synchronous behavior, and the physical mechanisms governing the transfer and processing of
information in a system of networking units, motivated the first studies of synchronization in oscillator networks with complex topologies (Watts and Strogatz, 1998; Strogatz, 2001a). Since then, synchronization phenomena and complex networks have been extensively studied hand in hand, highlighting the crucial role of the network topology in the emergence and stability of synchronous motion (Boccaletti et al., 2006; Moreno and Pacheco, 2004; Arenas et al., 2006; Nishikawa and Motter, 2006; Motter et al., 2005b; Chavez et al., 2005; Zhou et al., 2006; Arenas et al., 2008). For instance, it has been found that synchronization can emerge more easily in networks with highly heterogeneous degree distributions, due to the presence of nodes sharing a large number of connections (the hubs) that may act as pacemakers for the rest of the oscillators (Moreno and Pacheco, 2004; Gómez-Gardeñes et al., 2007, 2011). On the other hand, the presence of hubs reduces considerably the stability of fully synchronized states (Nishikawa et al., 2003). In addition to this, since densely interconnected sets of oscillators synchronize more easily than those with sparse connections (Moreno et al., 2004; Lodato et al., 2007), the analysis of synchronization has also been used as a tool to detect the presence of modules at different topological scales (Fortunato, 2010; Arenas et al., 2006; Boccaletti et al., 2007).

All the works mentioned above consider synchronization on static complex networks. However, fixed interaction patterns turn out to be inadequate for the description of many real-world networks, which are intrinsically time-varying (Holme, 2003; Valencia et al., 2008; Tang et al., 2010; Stehlé et al., 2010). More importantly, the stationary hypothesis must be abandoned when modeling the situations where the very same network topology emerges as a result of the dynamical interaction between its constituents. For these reasons, the interest has moved to adaptive networks, i.e. to graphs where the topology co-evolves with the dynamical process taking place on top of them, thus creating a feedback loop between structure and dynamics. See Section 2.7 for a very brief introduction and (Gross and Blasius, 2008) for a review of adaptive networks.

Models considering the effects of synchronization on the structure of a network have been recently proposed (Ott and Yorke, 2008; Zhou and Kurths, 2006; Aoki and Aoyagi, 2009; Fujiwara et al., 2011), in which the ties between units (or groups of them) strengthen as their dynamical states become more similar. Indeed, such an interplay between structure and dynamics is a rather general principle that is frequently observed in sociology and neuroscience, for example. However, in order to describe the structure and dynamics of real systems, such as social or neural ones, where the available resources (time, synapses, etc.) to establish connections are limited, we need to add a further constraint to the reinforcing mechanism acting on each single link. Thus, it is reasonable to introduce a competition mechanism by which the enhancement of some connection from
a node is counter-balanced by the weakening of other connections of the same node to
the network.

As explained in Section 1.4, the analysis of real networked systems has revealed that,
despite the intrinsic differences, their structure is characterized by two unifying prop-
erties: i) a power law (scale-free) scaling in the network connectivity (Barabási and Albert,
1999), and ii) the presence of modules (community structures) at a mesoscopic scale (For-
tunato, 2010). While no model exists yet able to reproduce at once these two features,
their interplay is believed to be the basis of the system general functioning and perfor-

We here introduce two simple models of an adaptive network of phase oscillators, in
which the coupling is directed and time-varying, and the temporal evolution of the link
weights is coupled to the dynamics of the oscillators. The oscillator dynamics is regu-
lated by the Kuramoto model (see Section 2.5) which is a paradigmatic framework for the
study of synchronization processes in many systems (Manrubia et al., 2004; Osipov et al.,
2007). The mechanism through which the dynamics reshapes the network structure is
governed by the competition between the two principles that have been mentioned before:
(a) the connections between synchronized units are enhanced, (b) the available resources
per unit to sustain interactions with the rest of the ensemble are limited. Principle (a)
is known to be relevant in neuronal plasticity and in spreading in sociology, under the
terms of Hebbian learning (Hebb, 1949) and homophily (McPherson et al., 2001), respec-
tively.1 The limitation in the associative capacity established by principle (b) is known
to play a relevant role in neuroscience under the term homeostasis (Turrigiano and Nel-
son, 2004),2 while in social systems it is related to the Dunbar’s number (Dunbar, 1992).
Neuroscience is, indeed, the field in which more efforts have been put into elucidating the
role of adaptive mechanisms, and various studies on neuronal plasticity (based on spe-
cific forms of competitive adaptation underlying memory and learning abilities) reported
emergent properties. See, for instance, the experimental and numerical evidence of an en-
hancement of synchronization of neurons originally spiking at different rates in (Nowotny
et al., 2003), and the emergence of synchronized clusters of different sizes allowing for
the storing of information in (Seliger et al., 2002). Furthermore, a similar interplay be-
tween these two adaptive mechanisms in networks of spiking integrate-and-fire neurons

---

1The homophily principle is, for instance, at the core of the internal mechanisms governing the transfer
and processing of information at the level of the individual and, at higher scales, it plays a leading role
in the development of cultural consensus (Axelrod, 1997) in society and cognitive tasks in the brain
(Uhlhaas et al., 2009).

2In neural systems, competition appears in combination with Hebbian learning in the development
has been shown to lead to a very promising approach to data clustering based on the self-organization that results from adaptation [Landis et al., 2010].

Numerical and analytical results allow to conclude that the competitive adaptive mechanism provided by the combination of homophily and homeostasis leads to the simultaneous emergence of a meso-scale of strongly modular communities and a scale-free distribution in the connection weights, as well as a striking enhancement of local synchronization in the absence of global dynamic order. There are two reasons for using two different models with mathematically different adaptive laws (though both following the same general principles of homophily and homeostasis): on the one hand, whereas Model 1 was the first to be proposed and gave very clear numerical results, we found in Model 2 a related model that moreover could be treated analytically; on the other hand, the fact that two different models lead to qualitatively similar results lends support to the possibility that our results are robust to changes in the precise equations. For the second reason, we also use different definitions in some parameters, measures and initial conditions when studying one or the other model. Similar adaptive mechanisms following those general principles might lead to the emergence of such ubiquitous features in quite a general context.

3.2 Model 1

In this section we give detailed information on the model and results that have appeared in [Assenza et al., 2011].

3.2.1 Methods

We consider a weighted and directed network of $N$ coupled phase-oscillators, where the phase of the $i$-th unit ($i = 1, \ldots, N$) is denoted by $\theta_i(t)$, and evolves in time according to

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} w_{ij} \sin(\theta_j - \theta_i). \quad (3.1)$$

Here, $\omega_i$ stands for the natural frequency of $i$, $\lambda$ is the coupling constant and $w_{ij} := w_{ij}(t)$ are non-negative quantities representing the strength at time $t$ of the links pointing from node $j$ to node $i$. The specific case in which the weights $w_{ij}$ are the same for all pairs of
nodes $i$ and $j$, and do not depend on time, is that of the Kuramoto model (see Section 2.5).

In our model, the weights of the interactions $w_{ij}(t)$ in Eq. (3.1) co-evolve with the system dynamics as

$$
\dot{w}_{ij}(t) = w_{ij}(t) \left[ s_i \cdot p_{ij}^T(t) - \sum_{l=1}^{N} w_{il}(t) \cdot p_{il}^T(t) \right] (3.2)
$$

where $s_i$ is the total incoming strength of node $i$, $s_i = \sum_{j=1}^{N} w_{ij}$, and $p_{ij}^T(t)$ is the degree of local synchronization between oscillators $i$ and $j$ averaged over the time interval $[t-T, t]$ [Gómez-Gardeñes et al., 2007, 2011]

$$
p_{ij}^T(t) = \left| \frac{1}{T} \int_{t-T}^{t} e^{i[\theta_j(t')-\theta_i(t')]} \, dt' \right|. (3.3)
$$

In the above equations, $T$ is a control parameter that quantifies the amount of memory used by each oscillator in the updating process, and the quantities $p_{ij}^T(t)$ take values in $[0, 1]$, with $p_{ij}^T(t) = 1$ meaning that oscillators $i$ and $j$ have been perfectly synchronized along the last $T$ time units.

The adaptive scheme in Eq. (3.2) has the form of the replicator equation of evolutionary dynamics [Hofbauer and Sigmund, 1998] and it implements the main characteristics of both homophily and homeostasis. The nodes $j$ that during the last $T$ time units have been highly synchronized with the target node $i$ ($p_{ij}^T > s_i^{-1} \sum_l w_{il} p_{il}^T$), will enhance their connection strength, according to homophily. On the other hand, homeostasis implies that the weights of the remaining inputs (those with $p_{ij}^T < s_i^{-1} \sum_l w_{il} p_{il}^T$) must be depressed to keep constant the total incoming strength of oscillator $i$, which is initially set as $s_i = 1$ for all $i$. Indeed, homeostasis can be seen to arise from Eq. (3.2) by the fact that $\dot{s}_i = \sum_{j=1}^{N} \dot{w}_{ij} = 0$. As a consequence, all the links pointing to the same oscillator compete for the available resources.

The model has been implemented on weighted and directed random regular graphs, i.e. graphs where all the nodes have the same number of incoming and outgoing links, $k_{in} = k_{out} = k$, and the connections are purely random. The size of the graph is $N = 300$. For each simulation, we initially assign to each link of the network a constant weight $w_{ij}(t_0) = 1/k$. This ensures that the constraint $\sum_j w_{ij}(t_0) = 1$ ($i = 1, ..., N$) is satisfied. We have checked that the results are almost independent of the connectivity $k$, provided that its value is sufficiently high (the same qualitative results emerge for any $k \geq 20$). Hence, in all results presented here we fixed the initial connectivity at the value of $k = 20$. 

83
Eqs. (3.1) and (3.2) are numerically integrated by a 4th order Runge-Kutta algorithm, with double precision and 0.02 integration time step. We consider a uniform distribution of natural frequencies $g(\omega)$ in the interval $[-1/2, 1/2]$, and also a uniform distribution of initial phases $\theta_i(t_0)$ in the range $[-\pi, \pi]$. The structural changes that occur from time $t - 1$ to $t$ are quantified by the quantity $\Delta(t) \equiv \sqrt{\sum_{i,j} [w_{ij}(t) - w_{ij}(t - 1)]^2}$. The system is considered to be in a stationary state when the condition $\Delta < 10^{-6}$ holds.

The original Kuramoto model assumes static and all-to-all interactions of equal strength, but it has been numerically shown that the modulus of Kuramoto’s order parameter $r(t) = \frac{1}{N} \left| \sum_{i=1}^{N} e^{i\theta_i(t)} \right|$ displays a second order phase transition also when a non-trivial interaction pattern is considered, with the critical coupling depending on the topological properties of the network (Boccaletti et al., 2006). Apart from the global synchronization parameter $r$, it is also interesting to consider the average degree of local (phase) synchronization between connected pairs of nodes, which can be quantified by the local order parameter introduced in (Gómez-Gardeñes et al., 2007, 2011), slightly modified to take into account the weights of the links:

$$r_{\text{link}} = \frac{1}{N} \sum_i \sum_{j \in \Gamma_i} \left| \lim_{\Delta t \to \infty} \frac{w_{ij}}{\Delta t} \int_{t_s}^{t_s + \Delta t} e^{i[\theta_i(t) - \theta_j(t)]} dt \right|$$  \hspace{1cm} (3.4)

where $\Gamma_i$ is the set of all the nodes pointing to $i$, and $t_s$ is the time corresponding to the onset of the stationary regime. Notice that, in spite of the resemblance with the definition of $p_{ij}^T(t)$ in equation (3.3), $r_{\text{link}}$ directly depends on this quantity only when the limit $T \to \infty$ is considered. In particular, in this case $r_{\text{link}}$ turns out to be proportional to the weighted average of all the $p_{ij}^T$. In our simulations we took $\Delta t = 500$ time units after checking that (for the distribution of internal frequencies $\{\omega_i\}$ used in this work) a larger value of $\Delta t$ does not change the values obtained for $r_{\text{link}}$. Global and local order in the corresponding oscillator dynamics is thus characterized by the time-average of $r(t)$ and $r_{\text{link}}$ respectively. The values reported in the figures for each value of $\lambda$ and $T$ are calculated as averages over 100 independent network realizations. Time is expressed in steps of the Runge-Kutta algorithm.

As for the structural properties of the emerging networks in the stationary state, we focus on both their macroscopic and mesoscopic description. Macroscopically, we analyze the distribution of the link weights, $P(w)$ (i.e. the probability of finding a link with weight $w$). On the other hand, for the mesoscopic description, we focus on the modular structure of the networks. In particular, we apply the extremal optimization of modularity algorithm proposed by Duch and Arenas in (Duch and Arenas, 2005), which gives
us a partition of the network into non-overlapping communities or modules, and then we evaluate the modularity of the partition by means of the $MC$ measure (see Section 1.2).

### 3.2.2 Results

In the top panels of Figure 3.1 we report the typical time-evolution of the order parameter $r$ for different values of the two control parameters $\lambda$ and $T$. For $t < 0$, we integrated numerically equation Eq. (3.1) on a homogeneous network in which all the nodes have $k$ neighbors, and the weights $w_{ij}$ do not change in time and are fixed to $1/k$ for all links. Then, for $t \geq 0$ we considered the full dynamics of the adaptive model by switching on the weight evolution governed by Eq. (3.2). When the weights co-evolve with the oscillator dynamics, a clear enhancement of synchronization is observed for any values of $\lambda$ and $T$. However, in some cases, as in the panels corresponding to $\lambda = 0.50$ and $0.65$, the order parameter $r$ exhibits quasi-periodic oscillations in time.

The different phases of the oscillator dynamics in the $\lambda-T$ plane are characterized by the contour-plots in Figure (3.1). The diagrams report the values of the degree of global, $r$ (left), and local, $r_{link}$ (right), synchrony obtained in the stationary state of the dynamics. By looking at the contour-plot of the global order parameter, $r$, we can identify three different dynamical regimes, whose boundaries are highlighted by dashed curves. In particular, going from small to large values of $\lambda$ we move from an incoherent phase (phase I) to a totally synchronized region (phase III), passing by a partially ordered phase (phase II). This phase transition can be understood by referring to the classical Kuramoto model. Denoting, indeed, the natural frequency distribution by $g(\omega)$, the critical coupling of the Kuramoto model is $\lambda_c = 2/\pi g(0)$, which turns out to be approximately 0.64 in our case [Kuramoto, 1975]. On the other hand, the contour-plot of $r_{link}$ describes the local degree of synchronization within the network. When the system is ordered at a global scale, both partially or totally, it also attains a perfect local synchronization. Conversely, different areas in the region of global incoherence correspond to different local regimes. We will now discuss in detail each of the three phases.

We start from phase III which, as shown in the contour plots of Figure (3.1), appears for relatively large values of the coupling strength $\lambda$. In this phase the dynamics of the system ends up in a perfectly synchronized state ($r \simeq 1$), and the resulting network is very similar to the initial network, i.e. a regular structure in which all the links share roughly the same value of the weights. This result indicates that, for values of $\lambda$ close to the critical point of the non-adaptive network, the adaptive system simply needs
Figure 3.1: **Time evolution of the global order parameter and phase diagrams.** In the top panels the time evolution of the Kuramoto order parameter $r$ is reported for $T = 60, 100, 130$ and $\lambda = 0.50, 0.65$. The dynamics of the oscillators is initially (for $t < 0$) implemented on a fixed network, while for $t \geq 0$ the weights co-evolve with the dynamics. We observe that the network behavior leads to an enhancement of the order parameter for each value of $\lambda$ and $T$. We also notice the quasi-periodic behavior of $r(t)$ for the top panels corresponding to $\lambda = 0.50$. The two contour-plots in the bottom panels report, respectively, the average value of $r$ (left) and $r_{\text{link}}$ (right) in the stationary state of the system as a function of $\lambda$ and $T$. Three different dynamical regimes, whose boundaries are highlighted by dashed lines, clearly emerge from the contour-plot of $r$ and $r_{\text{link}}$. From left to right, they correspond respectively to an incoherent (phase I), a partially ordered (phase II) and a totally synchronized region (phase III). In particular, the oscillatory evolution of $r(t)$ shown in the top panels corresponds to phase II.

a weak reorganization of the weights to achieve full synchrony. However, this perfect synchronization is hardly ever observed in real scenarios. Therefore, in the remainder of this subsection we will rather focus on describing the emergent behavior in the dynamical regimes where a local synchronization appears while a perfect global dynamical order is absent, i.e. phases I and II of the parameter space.
We first turn our attention to phase II. Here, the system exhibits partial global order, with a value of $r(t)$ oscillating in time, together with an almost perfect local degree of synchronization. In Figure 3.2 (A) we report a typical case in which the order parameter $r(t)$ behaves harmonically. A careful analysis of the network structure points out that, as an effect of the adaptive dynamics, a large fraction of the links have been suppressed, i.e. they are left with a weight which is practically zero. This behavior is very different from what we have observed found in phase III, where all the links survive, slightly modifying their original weights. Here the network splits into two clusters, practically two separate components, which are clearly visualized by plotting the weight matrix, after an opportune relabeling of the nodes. Each of the two components displays a large degree of internal synchrony. In fact, if we evaluate $r(t)$ separately for each of the two clusters, we find in both cases a value of $r$ which is close to 1 and constant in time. Hence, the clusters can be regarded as two almost non-interacting oscillators with different natural frequencies $\omega_1, \omega_2$. The interference between such frequencies produces the harmonic behavior of $r(t)$ with frequency $\Omega_{1,2} = \omega_1 - \omega_2$. In Figure 3.2 (B) we also report a case in which the network splits into three components, each one exhibiting an almost perfect degree of internal synchronization. Again, the modules can be regarded as three independent oscillators with natural frequencies $\omega_1, \omega_2, \omega_3$ and, as expected, the order parameter $r(t)$ of the whole network oscillates periodically with leading frequencies $\Omega_{1,2} = \omega_1 - \omega_2, \Omega_{1,3} = \omega_1 - \omega_3$ and $\Omega_{2,3} = \omega_2 - \omega_3$. The spontaneous break up of the initial network into separate components is the typical situation we have found for all values of $\lambda$ and $T$ in phase II. However, the partitions can be far more complex than the two cases described above, and multiple components of different sizes can coexist in the asymptotic network state. Moreover, we have also observed that an additional community structure may appear inside one of the components of the network (as we will later show in Figure 3.4) pointing out a highly nontrivial modular structure.

To better characterize the modularity of the emerging networks, we use the $MC$ measure of the optimal partition obtained from the graph partitioning method. As shown in Figure 3.3, $MC$ takes its maximal value, $MC = 1$, in phase II, indicating a partition of the network into separated frequency components. The value of $MC$ remains rather large also in phase I, where the network still displays a modular structure, while some weak links connecting different modules appear, thus making the overall network connected. The appearance of different connected modules in phase I reveals the emergence of meso-scale features from the dynamical reorganization of the weights. Modular structures frequently appear in large social and neural systems. In particular, in the brain of mammals, they efficiently describe the organization of the cortico-cortical pathways into anatomical-functional modules (Varela et al., 2001; Bullmore and Sporns, 2009).
Figure 3.2: **Emergence of cluster structures in the partially ordered phase II.**

(A) For $\lambda = 0.5$ and $T = 150$, the network splits into two components of similar size, while the global order parameter $r$ oscillates harmonically in time. However, if $r(t)$ is evaluated separately for the two clusters, a stationary value close to 1 is found in both cases. (B) For $\lambda = 0.5$ and $T = 180$, the network splits into three components and the global order parameter displays a periodic behavior with three frequencies. Again, each component has an almost perfect degree of internal synchronization. The plots on the left show the weight matrices after a suitable relabeling of the nodes.

In a social context, modular patterns are identified with densely interconnected groups of individuals sharing similar opinions and interests.

In phase I of our model, a modular structure emerges spontaneously together with other key features empirically found in large scale cortical networks (Valencia et al., 2009). Namely, for large values of $T$ the internal synchrony of each module is remarkably large, while the global order parameter is very close to zero, as indicated by $r_{link}$ and $r$ shown in Figure 3.1. Even more strikingly, the network structure obtained in phase
Figure 3.3: **Modularity cohesion.** The contour-plot reports the value of the modular cohesion $MC$ as a function of $\lambda$ and $T$. The large values of $MC$ correspond to the high modular character of the network in phase II, and to the presence of a modular structure also in phase I.

I displays scale-free architectures for the local connectivity patterns. In particular, in Figure 3.4 we report the distribution of link weights, $P(w)$, obtained at $T = 100$ for different values of $\lambda$. For low values of $\lambda$ (corresponding to phase I and the beginning of phase II), our model produces a hierarchical distribution of weights at all scales, which can be fitted by a power-law distribution, $P(w) = w^{-\alpha}$, with an exponent $\alpha$ ranging in $[0.85, 1.20]$. These scale-free architectures coexist (see the network snapshots in the right part of Figure 3.4 corresponding to $\lambda = 0.1$ and 0.2) with a highly modular architecture in which networks are composed of several communities of different sizes. The modules are connected by small weight links, while internal links have strong weights. As $\lambda$ increases (see $\lambda = 0.3$ and 0.4), the intra-modules links increase their weight, as can be observed from the increase in the peak of $P(w)$ at large values of $w$. The enhancement of intra-module ties occurs at the expense of the weakening of the weights of inter-module links, and the eventual break up of the network into several components with independent dynamical behaviors. However, as can be observed from the network snapshots, each of the components may contain several modules of smaller sizes, thus leading to networks with hierarchical modular behavior. Finally, when phase III approaches (see $\lambda = 0.5$ and 0.6), the topology of the network turns to be more compact, and link weights tend to be distributed in a rather homogeneous fashion (there is a comparatively weak reorganization of the weights from the initial condition).
Figure 3.4: Distribution of the link weights. In the left panels we report the distribution of the weights, $P(w)$, for the case $T = 100$, and for different values of $\lambda$ (increasing from $\lambda = 0.1$ to $\lambda = 0.6$). Notice that the regime of small $\lambda$ ($\lambda = 0.1$, $\lambda = 0.2$ and $\lambda = 0.3$) displays a power law distribution of weights, $P(w) = w^{-\alpha}$. In the right panels we show different snapshots of the network structure for the corresponding values of $\lambda$. Note that as $\lambda$ increases from $\lambda = 0.1$ to $\lambda = 0.4$ the modules increase in size and tend to be less overlapping until the network breaks up into several disconnected components. For higher $\lambda$ values the components start coalescing up to forming a rather homogeneous network.
3.3 Model 2

In this section we give detailed information on the model and results that have appeared in (Gutiérrez et al., 2011a).

3.3.1 Methods

The model again consists of a Kuramoto type ensemble of \( N \) phase oscillators, where node \( i \) is characterized by the phase \( \theta_i \), and interacts with \( k \) randomly selected neighboring units following Eq. (3.1). The natural frequencies \( \omega_i \) are randomly assigned following a uniform distribution, this time in \([-\pi, \pi]\). Again, the initial phases are randomly selected in the interval \([-\pi, \pi]\), and the weights are set to \( 1/k \). In this model, the adaptive evolution of the weights \( w_{ij} \) is governed by

\[
\dot{w}_{ij} = \tilde{p}_{ij}^T - \left( \sum_{k \in \mathcal{N}_i} \tilde{p}_{ik}^T \right) w_{ij},
\]

where the quantity \( \tilde{p}_{ij}^T \) denotes the average phase correlation between oscillators \( i \) and \( j \) over a characteristic memory time \( T \), and is defined as

\[
\tilde{p}_{ij}^T(t) = \frac{1}{T} \left| \int_{-\infty}^{t} e^{-\frac{1}{T} (t-t')} e^{i[\theta_i(t')-\theta_j(t')]} dt' \right|,
\]

which differs from the definition of \( p_{ij}^T \) in Eq. (3.3) in that now the memory effects exponentially decay in time as we move into the past. It follows from (3.5) that the normalization condition \( \sum_{j \in \mathcal{N}_i} w_{ij} = 1 \) holds at all times, i.e. the sum of the weights of all incoming connections at each node is conserved. Notice that the first and the second term in the right hand side of Eq. (3.5) account respectively for homophily and homeostasis in the adaptation mechanism.

Global synchronization can again be quantified through the time dependent Kuramoto order parameter \( r(t) \). For a generic choice of \( \lambda \) and \( T \), the system evolves into a stationary state after some time \( t_s \), where the weights \( w_{ij} \) take well defined values, with only marginal fluctuations in time. Then, we can define a time averaged order parameter \( r \) as

\[
r = \lim_{\Delta t \to \infty} \frac{1}{\Delta t N} \int_{t_s}^{t_s+\Delta t} \left| \sum_{i=1}^{N} e^{i\theta_i(t')} \right| dt',
\]

\( \Delta t \) being a suitably long time interval, to be later specified. Furthermore, we denote by
the time average of the pairwise synchronization between connected units,

\[ r_{ij} = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \left| \int_{t_s}^{t_s+\Delta t} e^{i[\theta_i(t')-\theta_j(t')]} \, dt' \right|, \tag{3.8} \]

and define a local synchronization measure (Gómez-Gardeñes et al., 2007) (a different one from that employed in the study of Model 1) as the ensemble weighted average of \( r_{ij} \)

\[ \tilde{r}_{\text{link}} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \in N_i} w_{ij} r_{ij}. \tag{3.9} \]

We once again consider a system with \( N = 300 \) and \( k = 20 \) and explore the structural and dynamical features as functions of \( \lambda \) and \( T \). First, the system is integrated without adaptation (i.e. at fixed values of \( w_{ij} = 1/k \)) during 200 time units. Then, at an instant defined to be \( t = 0 \), the adaptation mechanism is activated, i.e. Eq. (3.5) is incorporated, and the network dynamics is integrated for another \( t_s = 2,000 \) time units (in all studied cases, such a period is at least one order of magnitude larger than that needed to reach the stationary state). Time \( t_s \) also marks the beginning of the period of monitoring the global and local synchronization features in the network, which, in our simulations, is performed along an interval of \( \Delta t = 1,000 \) time units. All reported values result from a further ensemble average over 30 independent integrations of the system, each one starting from a different initial condition for \( \{\omega_i\}, \{\theta_i\} \), and adjacency matrix. The numerical integration scheme is a 4th order Runge-Kutta algorithm, with double precision and 0.02 integration time step.

### 3.3.2 Results

Figure 3.5 (a) shows \( r \) as a function of \( \lambda \) and \( T \). At relatively large values of \( T \), \( r \) depends almost exclusively on \( \lambda \), featuring a quasi-linear dependence up to \( \lambda \approx 4 \), and then rising up steeply into its saturation plateau \( r \sim 1 \). Again, this phase transition can be understood by referring to the classical Kuramoto model (see Section 2.5). Denoting, indeed, the natural frequency distribution by \( g(\omega) \), the critical coupling of the Kuramoto model is \( \lambda_c = 2/\pi g(0) \), which turns out to be 4 in our case (Kuramoto, 1975). We expect that, with increasing \( \lambda \), a higher and higher synchronization level sets in the network, causing the \( \tilde{p}_{ij}^c \) to approach the value of one. Thus, from Eq. (3.5), the \( w_{ij} \) will remain close to their initial value \( 1/k \). At high enough \( \lambda \), therefore, the system is almost indistinguishable from a modified Kuramoto model in which \( N(N-1)/2 - Nk \) links have been pruned.
By monitoring \( r(t) \) for \( T = 15 \) and several \( \lambda \) values (see Fig. 3.5 (c)) one observes that \( r(t) \) is already large at the pre-adaptive stage of the dynamics for \( \lambda \) close to \( \lambda_c \). Both Figure 3.5 (a) and (c) show that the adaptive mechanism has the effect of generically enhancing global synchronization in the network to a remarkable extent already for coupling strengths below the critical value.

Figure 3.5: Global and local synchronization indicators \( r \) (a) and \( r_{\text{link}} \) (b) in the \((T, \lambda)\) parameter space. The color coding is reported in the right columns. (c) \( r(t) \) for \( T = 15 \) and \( \lambda \in \{0.05, 1.50, 2.50, 3.80, 4.20\} \) (specific values indicated by the arrows) for the nonadaptive \((t < 0)\) and the adaptive \((t \geq 0)\) evolution of the network.

Figure 3.5 (b) shows \( r_{\text{link}} \) as a function of \( \lambda \) and \( T \). For a given \( T \), the local synchronization is a nonlinear, concave downwards function of \( \lambda \), with, again, a sudden rise at \( \lambda \approx \lambda_c \), that has the necessary amplitude to make \( r_{\text{link}} \) identically equal to one within our numerical accuracy. The growth of \( r_{\text{link}} \) with \( \lambda \) is, however, much faster than that of \( r \), which delimits (for \( \lambda < 1 \) and \( T > 1 \)) a wide region where the ratio \( \bar{r}_{\text{link}}/r \) is always higher than 3 (and, for the highest \( T \) and lowest \( \lambda \) values it can be even greater than 6), which is the hallmark of the emergence of modular (cluster) synchronization. This
corresponds to Phase II in the previous model.³

Further insight on such a modular structure can be obtained from Figure 3.6 where we report (Fig. 3.6(a)) the $MC$ of the network in the parameter space, defined as in Eq. (1.15). The communities are identified by the optimization procedure proposed by Duch and Arenas described in (Duch and Arenas, 2005), this time followed by Newman’s fast algorithm described in (Newman, 2004) (see Section 1.5). The striking result is that our network appears to be very well separated into components in the entire region where $r$ is relatively low, as happened in the case of Model 1.

Figure 3.6: Emergence of mesoscale structures. (a) $MC$ in the $(T, \lambda)$ parameter space. The color coding is reported in the right column. (b) Distribution of the effective frequencies $\omega_{\text{eff}}$, as a function of $\lambda$, for $T = 15$. The grey code (see column at the right of the panel) indicates the number of oscillators in the network displaying each specific value of $\omega_{\text{eff}}$.

Figure 3.6 (b) shows the distribution of the effective frequencies $\omega_{\text{eff}}$ (i.e. the average frequencies of the oscillators in the stationary regime) vs. $\lambda$, for a typical realization using $T = 15$. Starting from the uniform distribution at $\lambda = 0$ (where $\omega_{\text{eff}}$ are all equal to the natural frequencies), increasing the coupling has the effect of producing a rich scenario of meso-scale patterns, forming different numbers of distinct synchronized communities at certain intervals of the $\lambda$ axis. Eventually, at $\lambda \geq \lambda_c$, all oscillators coalesce into a single synchronous module.

This change in the modularity with increasing coupling strength is also reflected in the associated weight distribution $P(w)$ (i.e. the topology resulting from competitive adaptation), as demonstrated in Figure 3.7 for $T = 15$. For $\lambda = 0.05$, there is no pre-

³Indeed, a similar partition of the parameter space could be adopted in terms of the global and local synchronization properties in this model.
ferred scale and the bulk of the distribution seems to follow closely a power-law scaling, $P(w) \propto w^{-\alpha}$ with $\alpha = 1.215 \pm 0.030$. In other words, adaptation produces here a complete redistribution of the weights, leading eventually to a highly heterogeneous topology.

Figure 3.7: Emergence of macroscale structures. (Left panels) Weight distribution $P(w)$ for $T = 15$ and $\lambda \in \{0.05, 2.00, 3.00, 4.20\}$. The red line in the upper plot ($\lambda = 0.05$) is the analytically obtained distribution of Eq. (3.16) calculated at $\lambda = 0$. (Right panels) Visual sketch of the corresponding network topologies, obtained by applying attractive forces between nodes proportional to the corresponding link weights.

As $\lambda$ increases, a local maximum appears in the distribution, and the network segregates into more and more distinct modules. The formation of three communities at $\lambda = 2$, and of two communities at $\lambda = 3$, have visible dynamical counterparts in Fig. 3.6 (b). At large coupling strengths ($\lambda > \lambda_c$), the maximum is shifted toward $1/k = 0.05$

\footnote{A power law fit was performed based on the nonlinear least-squares Newton-Gauss algorithm.}
(i.e. the initial condition), and becomes more prominent. This is a direct consequence of the fact that for \( \lambda > \lambda_c \), the network is able to synchronize even without adaptation (see Fig. 3.5(c)), and therefore the redistribution of weights is negligible.

The weight distribution can be derived analytically in the limit \( \lambda = 0 \). It turns out, indeed, that this limit is useful for the estimation of the weight distribution not only for strictly vanishing \( \lambda \), but also in the limit \( \lambda \to 0 \), as one could expect from heuristic continuity arguments. In such a case, the phases are uncoupled and evolve as \( \theta_i = \omega_i t + \theta_i^0 \). Plugging this evolution into equation (3.6), one obtains

\[
\tilde{p}^{(0)}_{ij}(0) = \frac{1}{T} \left| \int_{-\infty}^{t} \exp \left( -\frac{t - t'}{T} \right) \exp \left( i[\omega_i t' - \omega_j t'] \right) dt' \right| = \frac{1}{\sqrt{1 + [(\omega_i - \omega_j) T]^2}} = L(\omega_i - \omega_j),
\]

where the superscript in \( \tilde{p}^{(0)}_{ij} \) highlights that we are in the uncoupled limit, and where we used the equality \( |\exp \left( i[\theta_i^0 - \theta_j^0] \right)| = 1 \). Let us define \( L(\Delta) := \left[ 1 + T^2 \Delta^2 \right]^{-1/2} \). This function takes values in \([0, 1]\), and, as \( L(\Delta) = L(-\Delta) \), it has no unique inverse.

As in the uncoupled limit each node forms a closed subsystem with its neighbors, we can focus on the subsystem centered around a generic node \( i \). Now, \( \Delta_j = \omega_j - \omega_i \) are independent identically distributed (iid) random variables with conditioned probability distribution function (pdf) \( p_{\Delta_j}(\Delta_j | \omega_i) \) uniform in \([ -\pi - \omega_i, \pi - \omega_i ] \). All \( L(\omega_i - \omega_j) \) such that \( j \in \mathcal{N}_i \) are also iid random variables. We denote them by \( \hat{L}_j \), and designate indistinctly as \( L(\Delta_j) \) or \( L_j \) the values they take. The pdf of \( \hat{L}_j \) conditioned on \( \omega_i \) is:

\[
p_{L_j}(L_j | \omega_i) = \sum_{\Delta_j \in (L_j^{-1})} \frac{1}{d\Delta_j} p_{\Delta_j}(\Delta_j | \omega_i) = \frac{\#\{\Delta | L(\Delta) = L_j\} \cap [-\pi - \omega_i, \pi - \omega_i]}{2\pi |L_j^2 T \sqrt{1 - L_j^2}|} \times \begin{cases} 
0 & \text{if } L_j \geq 1 \\
2 & \text{if } L(\pi - |\omega_i|) < L_j < 1 \\
1 & \text{if } L(\pi + |\omega_i|) < L_j < L(\pi - |\omega_i|) \\
0 & \text{if } L_j \leq L(\pi + |\omega_i|)
\end{cases}
\]

(3.11)

The expectation value of \( L_j \) conditioned on \( \omega_i \) is given by

\[
\langle L_j(\omega_i) \rangle = \int_{-\infty}^{\infty} L_j p_{L_j}(L_j | \omega_i) dL_j = \frac{1}{2\pi} \int_{-\pi - \omega_i}^{\pi - \omega_i} L(\Delta_j) d\Delta_j
\]

\[
= \frac{1}{2\pi T} \left[ \arcsinh(T(-\pi - \omega_i)) + \arcsinh(T(\pi + \omega_i)) \right],
\]

(3.12)
and the corresponding variance is

\[
\text{var}[L_j(\omega_i)] = \int_{-\infty}^{\infty} L_j^2 p_{L_j}(L_j \mid \omega_i) dL_j - \langle L_j(\omega_i) \rangle^2 = \frac{1}{2\pi} \int_{-\omega_i}^{\pi-\omega_i} (L(\Delta_j))^2 d\Delta_j - \langle L_j(\omega_i) \rangle^2
\]

\[
= \frac{1}{2\pi T} [\text{atan}(T(-\pi - \omega_i)) + \text{atan}(T(\pi + \omega_i))] - \langle L_j(\omega_i) \rangle^2. \tag{3.13}
\]

By Eq. (3.5), the weights have an asymptotic stable value given by \( w_{ij}^{(0)} = \frac{\tilde{p}_{ij}^{(0)}}{\sum_{k \in \mathcal{N}_j \setminus \{j\}} \tilde{p}_{ik}^{(0)}} = \frac{L_i}{L_j + \tilde{G}} \). Let us then define the random variable \( \tilde{G} := \sum_{k \in \mathcal{N}_j \setminus \{j\}} \tilde{L}_k \). As both the expectation value and the variance of \( \tilde{L}_j \) are finite and independent of \( j \), we invoke the central limit theorem for the sum in \( \tilde{G} \) and assert that it converges (in the limit of large \( k \)) to a Gaussian random variable with \( \langle G \rangle = k \langle L_j(\omega_i) \rangle \), and \text{var}[G] = k \text{var}[L_j(\omega_i)].

The pdf of \( \tilde{w}_{ij} := \tilde{L}_j/(\tilde{L}_j + \tilde{G}) \) conditioned on \( \omega_i \) can be obtained by introducing the joint pdf for the independent variables \( \tilde{L}_j \) and \( \tilde{G} \): \( p_{L_j,G}(L_j, G \mid \omega_i) = p_{L_j}(L_j \mid \omega_i) p_G(G \mid \omega_i) \). As \( P(\tilde{w}_{ij} < w_{ij} + \Delta w_{ij}) - P(\tilde{w}_{ij}) = \int_{\Omega} p_{L_j,G}(L_j, G \mid \omega_i) dL_j dG \), by definition, where \( \Omega = \{(L, G) \mid w_{ij} \leq \frac{L_j}{L_j + \tilde{G}} \leq w_{ij} + \Delta w_{ij}\} \), and this set can be re-expressed as

\[
\left\{(L, G) \mid \left( \frac{1}{w_{ij}} - 1 \right)L \leq G \leq \left( \frac{1}{w_{ij} + \Delta w_{ij}} - 1 \right)L \right\}, \tag{3.14}
\]

such a parametrization allows one to rewrite

\[
P(\tilde{w}_{ij} < w_{ij} + \Delta w_{ij}) - P(\tilde{w}_{ij}) = \int_{0}^{1} dL_j \int_{\frac{1}{w_{ij} + \Delta w_{ij}} - 1}^{\frac{1}{w_{ij}} - 1} \frac{L_j}{w_{ij}} p_{L_j}(L_j \mid \omega_i) p_G \left( \left( \frac{1}{w_{ij}} - 1 \right)L_j \mid \omega_i \right). \tag{3.15}
\]

As, by definition, \( p_{\tilde{w}_{ij}}(w_{ij} \mid \omega_i) = \lim_{\Delta w_{ij} \to 0^+} \frac{P(\tilde{w}_{ij} < w_{ij} + \Delta w_{ij}) - P(\tilde{w}_{ij})}{\Delta w_{ij}} \), then one has

\[
p_{\tilde{w}_{ij}}(w_{ij}) = \int_{-\infty}^{\infty} p_{\tilde{w}_{ij}}(w_{ij} \mid \omega_i) p_{\tilde{w}}(\omega_i) d\omega_i
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega_i \int_{0}^{1} dL_j \frac{L_j}{w_{ij}^2} p_{L_j}(L_j \mid \omega_i) p_G \left( \left( \frac{1}{w_{ij}} - 1 \right)L_j \mid \omega_i \right), \tag{3.16}
\]

which provides the analytic expression for the weight distribution in a generic closed subsystem, and thus in the whole network, at \( \lambda = 0 \). Figure 3.7 shows \( p_{\tilde{w}_{ij}}(w_{ij}) \) (see the red continuous line) superimposed to the numerically obtained weight distribution \( P(w) \) for \( \lambda = 0.05 \). The fit is strikingly good, even though the central limit approximation for \( k = 20 \) is only mildly justified. Notice that the analytical curve in the uncoupled
regime reproduces almost perfectly the numerical results for $\lambda = 0.05$, thus suggesting that equation (3.16) is a good approximation to the weight distribution in the regime of small $\lambda$.

### 3.4 Discussion

The emergence of highly modular structures together with scale-free interaction patterns in our adaptive network models reproduces, respectively at the mesoscopic and macroscopic level, two universal properties of real networks (see Section 1.4). These structural features underlie a dynamical regime where the system displays a large degree of local synchronization within the modules in the absence of global dynamical order. Our findings are thus in agreement with both dynamical and structural features observed in real neural systems [Eguíluz et al., 2005; Meunier et al., 2010] and networks of social relations [Scott, 2000]. In such systems, on one hand, local synchronization and consensus coexist with a lack of global order while, on the other hand, modularity and scale-free interaction patterns are core features of their backbone. In particular, regarding the scaling of edge weights, recent quantitative studies on the wiring of fibers in large cortical brain networks have reported power-law distributions for axon fiber densities [Hagmann et al., 2007, 2008] in agreement with those displayed by our adaptive network model. Concerning social systems, the recent development of large scale online social networks and the burst of data about social communications through mobile networks is allowing to monitor the degree of friendship between connected users through the analysis of the load of communication between connected users. Recent research in this direction points out that the load of information in this one-to-one communication channels follows a power-law distribution [Onnela et al., 2007b; a, Sousa et al., 2010] corroborating again the scale-free patterns of the interaction weights. Moreover, we have checked that other structural features such as the strength distribution of nodes, or the reciprocity of links, are also in agreement with those found experimentally.

Summing up, in this chapter we have introduced and analyzed two simple models of an adaptive network of oscillators, where the evolution of the topology is regulated by the synchronization dynamics through the competing mechanisms of homophily and homeostasis. The adaptive nature of the interactions produces a better synchronization both at the global and local scale with respect to the non-adaptive case. At the same time, the link weights evolve towards non-trivial stationary states. The models present three main phases as a function of its control parameters. In the first phase, an enhancement
of global and local synchronization is achieved by a coordinated fine-tuning of the link
weights. In the second phase, partial global order is attained together with an almost
perfect degree of local synchronization. In practice, the network spontaneously splits
into a number of modules, each formed by perfectly synchronized oscillators. Moreover,
the link weights follow a scale-free distribution, while that for node strengths is homo-
geneous with a fast (exponentially) decaying tail (not shown here, see (Assenza et al.,
2011)). From a dynamical perspective, such a structure supports a high degree of local
synchronization, although no global order is achieved, in agreement with observations in
real systems. Our results are consistent with many observed properties of the relation-
ships between structure and dynamics during the formation of synchronized clusters (Li
et al., 2008), indicating how network adaptation can actually be the mechanism at work
in many real complex systems.

The fact that numerical and analytical results based on two different models of adap-
tive synchronization governed by the same competition between the two underlying prin-
ciples of homophily and homeostasis present so many similarities raises the possibility
that such general principles lead to the spontaneous emergence of such ubiquitous meso-
and macroscale properties in a rather general context.
Chapter 4

Vulnerability under finite perturbations in complex networks

Now that we have seen how network properties can emerge from adaptation, we move on to deal with the issue of how a networked dynamical system responds to perturbations applied on its elements. In this chapter, a measure to quantify vulnerability under perturbations (attacks, failures, large fluctuations) in ensembles (networks) of coupled dynamical systems is proposed. Rather than addressing the issue of how the network properties change upon removal of elements of the graph (the strategy followed by most of the existing methods for studying the vulnerability of a network based on its topology), here a dynamical definition of vulnerability is introduced, referring to the robustness of a collective dynamical state to perturbing events occurring over a fixed topology. In particular, we study how the collective (synchronized) dynamics of a network of chaotic units is disrupted under the action of a finite-size perturbation on one of its nodes. Illustrative examples are provided for three systems of identical chaotic oscillators coupled according to three distinct well-known network topologies. A quantitative comparison between the obtained vulnerability rankings and the classical connectivity/centrality rankings is made that yields conclusive results. Possible applications of the proposed strategy and conclusions are also discussed in the last section. The chapter is based on the original research work published in (Gutiérrez et al., 2011b).

4.1 Introduction

Given the high level of heterogeneity in the degree distribution of real networks, a central issue in the analysis of such systems is the assessment of the centrality of the nodes, and
of the network robustness and stability. In this sense, one of the main aims is to properly rank each one of the networking units in terms of the response of the whole system under attacks or dysfunctions of any type that may affect that specific element. In particular, a very important concept that has been used to assess the stability and robustness of the global behavior (or performance) of networks under the action of external perturbations (as failures or malicious attacks) is that of vulnerability.

According to the discussion in Section 1.2, many different approaches have been proposed to define a measure for network vulnerability, relating it to, for instance, decreased cohesion and network fragmentation under random failures and attacks (Albert et al., 2000; Cohen et al., 2000, 2001; Holme et al., 2002) or variations of the network efficiency after topological damages or improvements (Latora and Marchiori, 2005). A formalization of the concept in terms of vulnerability functions that meet certain basic mathematical properties consistent with the intuitive notion of the vulnerability of a graph (somehow related to regularity and to the number of alternative edges existing between nodes) has also been proposed (Criado et al., 2005). These early efforts were essentially devoted to the study of how certain properties of a generic graph are affected by changes in the topology, such as the accidental (random) or intentional removal of elements of the network.

In this chapter, we introduce an alternative approach to the definition of network vulnerability, that connects it to the way the network dynamics abandons a collective (synchronized) state under the action of a perturbation acting on one of the nodes of the graph. Thus, the graph topology is assumed to be constant over the time scales that are relevant for the propagation of the perturbation, and we deal with the vulnerability of a given collective state (the synchronous evolution of the network), making this approach substantially different from the studies previously referred to. The strategy, albeit close in spirit to the studies on the linear stability of synchronized states in chaotic systems, differs considerably from those studies in that it provides a ranking of the nodes in a network in terms of the vulnerability of the collective state under finite-size perturbations applied on them. Relevant applications can be found, indeed, in technological or infrastructural networks, where a practical issue is often to design the better protection strategy for each one of the units to avoid the spreading over the system of an occasional breakdown or intentional large damage. Furthermore, we will show that the approach actually suggests the definition of a finite-time ranking of the units, thus offering a way to distinguish the most vulnerable network nodes in all cases in which the goal is to repair or restore the network dynamics over a finite time.
4.2 Methods

In order to illustrate our method, we refer to three different networking systems, each one made up of $N = 500$ nodes and $L = 8N = 4000$ links. Topologically, the three networks are: 1) an ER random graph, 2) a BASF network, and 3) a CMSF network. All the relevant models are explained in Sections 1.5 and 1.6. In particular, we choose a CMSF with a degree distribution following the scaling law $P(k) \propto k^{-3}$, so as to make the network connectivity comparable to the case of BASF. Dynamically, node $i$ is assigned a vector state $x_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, with internal evolution following the Rössler system of Section 2.3 with $a = b = 0.2$, $c = 7.0$ (in one of its chaotic regimes).

Furthermore, we consider each node as diffusively coupled with its nearest neighbors in the graph, so that the network evolution equations read as

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} \Delta_{ij} H(x_j). \quad (4.1)$$

The connected nodes are linearly coupled through their $x$ variable by the output function $H(x) = x$. The system under study turns out to be class III according to the classification in (Boccaletti et al., 2006) (see Section 2.6). For all three networks $\sigma$ was chosen so that i) $\sigma \gamma_2 = (\nu_1 - 0.0002)$; and ii) $\sigma \gamma_N < \nu_2$ (where $\nu_1 = 0.1395$ and $\nu_2 = 4.4780$ are, to our best numerical evidence, the first and second zeroes of the MSF, respectively). The result of this choice is that all the networks are considered in a dynamical regime in which the manifold $\mathcal{M}$ is slightly linearly unstable along the eigenmode corresponding to the second smallest eigenvalue of the Laplacian matrix (with an associated maximum Lyapunov exponent which amounts, according to our computations, to $\lambda_{\text{max}}(0.1393) = 1.36 \cdot 10^{-4}$), whereas it is linearly stable along all other eigenmodes. This allows for an unbiased comparison between the three considered topologies.

The MSF describes the local linear stability properties of the synchronized dynamics and, as so, it describes the evolution of infinitesimal perturbations affecting the dynamics on $\mathcal{M}$. Since our aim is, instead, to study how the systems diverge from synchronization under the action of finite perturbations applied on individual nodes, and to relate this divergence to the topological features of the perturbed nodes, it is evident that a different strategy has to be followed.

To this purpose, the first step is to evolve a single Rössler oscillator from an arbitrary initial condition for a given time well beyond its initial transient. The final state of such
an evolution is then taken as the initial condition $x_{\text{ini}}$ for the synchronous evolution of the full networked system ($x_1(t = 0) = x_2(t = 0) = \ldots = x_N(t = 0) = x_{\text{ini}}$). An identical copy of the system is also started, with a finite perturbation applied on an arbitrary node $j$. The initial conditions of this second system are thus $\tilde{x}_j(t = 0) = x_{\text{ini}} + v_{\text{pert}}$, and $\tilde{x}_i(t = 0) = x_{\text{ini}}$ for $i \neq j$. The chosen perturbation is of finite size of a scale comparable to that of the attractor, namely, it is a 4.0-norm vector (approximately 17% of the radius of the Rössler attractor) with the norm equally divided into the three components $v_{\text{pert}} = (4, 4, 4)/\sqrt{3}$. It is the comparison of the network states resulting from the simultaneous integration of both systems that allows one to monitor how the perturbed system dynamics deviates in time from that of the unperturbed, synchronized system.

To quantify this deviation we use the global divergence rate (DR), a sort of Lyapunov exponent for finite perturbations and bounded time intervals. The DR for a perturbation applied on node $j$ as a function of time is denoted as $\text{DR}_j(t)$. The DR is defined as the cumulative time average of the natural logarithm of the Euclidean distance between the perturbed system state and the unperturbed (synchronized) system state (in $3N$-phase space, $d(\tilde{x}, x) \equiv (\sum_{i=1}^{3N}[(\tilde{x}_i - x_i)^2 + (\tilde{y}_i - y_i)^2 + (\tilde{z}_i - z_i)^2])^{1/2}$) divided by the norm of the initial perturbation,

$$\text{DR}_j(t) = \frac{1}{t} \int_0^t \log \left( \frac{d(\tilde{x}(t'), x(t'))}{|v_{\text{pert}}|} \right) dt'$$

Interested as we are in the time evolution under perturbations of individual oscillators, we also consider the local divergence rate (dr). By $\text{dr}_{ij}(t)$ we denote the local divergence rate corresponding to the deviation from synchronization of the dynamics of node $i$ for a perturbation applied on node $j$. This latter quantity is calculated as the cumulative time average of the natural logarithm of the Euclidean distance (this time, in 3-space) between the specific $i$-th node state in the perturbed and the unperturbed system, again divided by the norm of the initial perturbation.

$$\text{dr}_{ij}(t) = \frac{1}{t} \int_0^t \log \left( \frac{d(\tilde{r}_i(t'), r_i(t'))}{|v_{\text{pert}}|} \right) dt'$$

In the following, we report numerical results obtained with a classical 4th order Runge-Kutta integration algorithm, with double precision and 0.01 integration time step. Furthermore, all values of the local and global divergence rates shown in the Figures refer to a further ensemble average over 50 independent integrations of the system, each one corresponding to a different choice of the initial state $x_{\text{ini}}$, on top of which the perturbation is applied.
4.3 Results

We start by showing in Figure 4.1 the main topological and synchronizability features of the chosen ER, BASF and CMSF networks.

Figure 4.1: Topology and synchronizability (following the MSF approach) of the considered networks. (Left column) ER random graph: subgraph containing 80 randomly chosen nodes (top panel) with color depending on the degree according to color bar, degree sequence and least-squares fit curve (middle panel), localization of the eigenvalues of the Laplacian matrix superimposed on MSF curve (bottom panel), and eigenvalues around the first zero of the MSF (bottom panel, inset). (Middle column) Same for BASF network. (Right column) Same for CMSF network.

Precisely, the top panels contain a sketch of a representative subgraph of the network, with nodes colored according to their degree following the color code of the bar on the right. The middle panels contain plots showing a histogram of the degree sequence, together with a least-squares fit curve (providing an estimate of the underlying $P(k)$). The bottom panels, in their turn, show the corresponding distribution of the eigenvalues of $\Delta$ (red crosses) superimposed on the MSF curve (which is equal for the three cases, as it is independent of the topology). The spectra of the scale-free networks span a larger portion of the positive semi-axis than does the spectrum of the ER random graph, which turns out to be the easiest to synchronize of all three networks in spite of its homogene-
ity, a seemingly paradoxical fact that has been reported and previously explained in the literature (Nishikawa et al., 2003; Motter et al., 2005a). Another plot zooming on the area in the proximity of the first zero $\nu_1$ is shown in the inset, where one can see how the slightly linearly unstable regime is obtained by our choice of the coupling strength $\sigma$.

Figure 4.2 reports the DRs as a function of time for the ER, BASF, and CMSF network. Each curve corresponds to $\text{DR}_j(t)$ for a perturbation applied on each possible node $j$, and the colors of the curves are indicative of the degree of the perturbed node $k_j$ (to avoid cluttering the Figure, we report the evolution of DR for perturbations applied on only 100 randomly selected nodes). Initially, there is a sort of damping or dissipation of the perturbation up to reaching a minimum in the curve, followed by a steady increase that eventually approaches an asymptotic value. The same qualitative evolution of DR holds all three networks, as well as for different subsets of nodes spanning the entire degree range. By a close inspection of Figure 4.2 it becomes clear that, in all cases, the DRs corresponding to perturbations applied on top of the most isolated nodes (those having the lowest $k$) undergo the lightest damping and diverge rapidly from synchronization. Therefore, it is appropriate to refer to these nodes as the most vulnerable to a finite perturbation. On the other hand, as we inspect perturbations upon more and more connected nodes, the DR goes through a heavier damping and the divergence is slower. However, such a behavior happens to reverse at some point (at least for the BASF and CMSF networks, the ER having only nodes of low or intermediate connectivity), so that the least vulnerable nodes are not the hubs as one could have expected from the above observations.

Figure 4.2: Divergence rates of the perturbed (global) dynamics. Divergence rates of the perturbed (global) dynamics from the synchronization manifold, for perturbations applied on 100 randomly selected nodes. A) ER network, B) BASF network, C) CMSF network. Curves are colored according to the degree of the node upon which the perturbation is applied (see the color bar on the right).
While the DR curves in Figure 4.2 suggest a definition of a time dependent vulnerability, as the ranking of nodes that follows (at a given time $t$) the corresponding distribution of $DR_j(t)$, a remarkable result is that the curves appear to show, in fact, relatively few crossings between them. Precisely, the node ranking is almost conserved along the entire time epoch, a fact that can be used to simplify the operational relationship between vulnerability of a node under finite perturbations and its degree (or other centrality measures), by assuming the minimum of the $DR_j(t)$ as a reliable measure of vulnerability. The idea is that the more negative the minimum is the larger the damping and the more “inertia” the system shows in escaping from synchronization.

Figure 4.3: Minimum of the divergence rate as a function of connectivity/centrality. A) Minimum DR vs $k$. A simplifying partition of the nodes into three sets according to their connectivity (ISOL: isolated nodes; MEDIUM: nodes of intermediate connectivity; HUBS) is sketched for the discussion of results. B) Minimum DR vs eigenvector centrality.

Guided by the above discussion, we show in Figure 4.3 the minimum of each $DR_j(t)$ for all three networks as a function of $k_j$ (Figure 4.3 A) and as a function of the eigenvector centrality (Figure 4.3 B; see Section 1.2). Based on the qualitative time evolution of the global divergence rates and its dependence on the connectivity/centrality as seen in the Figure, we have grouped the nodes into three classes that roughly correspond to the isolated nodes (ISOL in the Figure), the nodes of intermediate connectivity (MEDIUM) and the hubs (HUBS). The transition between the isolated nodes and the nodes of intermediate connectivity has been further adjusted so as to correspond approximately to the point on the connectivity/centrality axis at which the stabilizing effect of the rest of the network becomes strong enough to cause a visible damping of the initial perturbation in the local divergence rate of the perturbed node $dr_{jj}(t)$ (such a phenomenon will be apparent in Figure 4.4).
As we anticipated, isolated nodes are the most vulnerable: a perturbation applied on them rapidly takes the system away from $\mathcal{M}$ (see the region labeled as ISOL in Figure 4.3 A). As we perturb nodes that are more and more connected (those inside the region labeled MEDIUM in Figure 4.3 A) the escape from synchronization becomes slower: these nodes are less vulnerable to the perturbation, the speed at which the system desynchronizes is smaller, and, from the point of view of control of dynamical systems, a restoring or protecting action in technical applications could wait a bit more than in the previous case. In the case of the scale-free networks there are also nodes whose centrality/connectivity is still much higher (those inside the region labeled HUBS in Figure 4.3 A). As we perturb nodes of higher and higher degrees, we reach a point at which the trend is reversed into a situation where centrality and vulnerability to a perturbation are positively correlated quantities. This trend continues up to the most connected hubs as seen in the Figure.

The mechanism that underlies such a nontrivial dependence of the vulnerability on the centrality must be related to the way the perturbations are propagated over the network. The local divergence rates for individual nodes, $\text{dr}_{ij}(t)$ (where $i$ is the node under study and $j$ is the node initially perturbed) are then used to elucidate the situation. Figure 4.4 shows $\text{dr}_{ij}(t)$ for every individual node $i$, for perturbations applied on a few nodes which are either clearly isolated (left column), intermediately connected around the the region of lowest vulnerability shown in Figure 4.3 (middle column), or undisputed hubs of the network (right column); the degree of the perturbed node is shown on each plot. In each case, the thick black line corresponds to $\text{dr}_{jj}(t)$ (the node under study is the node initially perturbed). The remaining curves corresponding to $\text{dr}_{ij}(t)$ for $i \neq j$ are represented as dotted lines whenever $i$ is a first neighbor of $j$ or as continuous lines whenever it is not.

Our results show that perturbations applied on very isolated nodes (those that are the most vulnerable) have some peculiar properties that become less distinct as we approach the boundary between isolation and intermediate connectivity. Indeed, the relatively light global damping (visible in Figures 4.2 and 4.3) is seen to be due mainly to the fact that the perturbed node hardly suffers from any damping itself. As for the propagation of the perturbation, it is very heterogeneous, first affecting only the first neighbors in a way that seems to be inversely related to their degree (the most isolated are the fastest in abandoning the synchronized state), but eventually reaching the rest of the network. This second stage of the propagation occurs, instead, somewhat accordingly to the node degree: hubs respond generally faster than intermediate or isolated nodes. As the region of intermediate connectivity is approached, the damping starts to be more and more prominent and the propagation of the perturbation more homogeneous.
Figure 4.4: **Propagation of perturbations over the network.** Local divergence rates $\text{dr}_{ij}(t)$ vs time for the ER network (upper plots), the BASF network (middle plots) and the CMSF network (lower plots). The degree of the specific node $j$ on which the perturbation is applied is reported on each plot. In each case, the divergence rate of the perturbed node, $\text{dr}_{jj}(t)$, is shown in black, while the other ($i \neq j$) node degrees follow the color bar in the second plot of the first row. Dotted lines correspond to the first neighbors of the perturbed node, continuous lines to the rest of the nodes.

When the perturbation is applied on nodes of intermediate to high connectivity, the generic behavior clearly corresponds to a more homogeneous divergence of the full network from $\mathcal{M}$. The way the individual nodes escape from synchronization as a function of connectivity is similar to that described in the previous paragraph. The reason why the hubs are more vulnerable than the nodes of intermediate connectivity, as previously observed, appears to be that for the hubs the damping turns out to be smaller, and the propagation over the network facilitated by a high connectivity makes the divergence from synchronization not only more coherent but also faster.

These results are robust for nodes of similar degree regardless of the specific topol-
ogy of the considered network, suggesting that (at least for the networks under study) the differences in vulnerability under finite perturbations are largely (but not trivially) dependent on local (first-neighbors) properties. Moreover, the same qualitative features were found for other realizations of the three topologies of size $N = 500$ and $N = 2,000$, and also, in the CMSF case, for networks with a degree distribution $P(k) \propto k^{-\gamma}$ with $\gamma = 2.5$ and 3.5.

4.4 Discussion

In this chapter we propose a novel method to study the vulnerability under perturbations (attacks, failures, large fluctuations) in large ensembles (networks) of coupled dynamical systems. Our method differs significantly from those of the classical studies on vulnerability in networks in that we do not address the issue of how the network properties change upon removal of elements of the graph. Rather, we consider a dynamical definition of vulnerability, namely, the vulnerability of a collective dynamical state to perturbing events occurring over a fixed topology. Specifically, we study how the collective (synchronized) dynamics is disrupted depending on the topological properties of the node in the network on which a perturbation acts. Moreover, we put the method to work by measuring the node vulnerability of three systems of identical chaotic oscillators coupled according to three distinct well-known topologies. We find conclusive results regarding the relationship between vulnerability under perturbations and node connectivity/centrality that seem to be robust and generally valid for different topologies.

The method consists in monitoring simultaneously both the original system and a copy of it subjected to a relatively large perturbation (i.e. a large additive term in the initial conditions of one of the networking units), and measuring the divergence rate between both systems, both globally and at the node level (if one is to study the propagation of the perturbation over the network in some detail). In some technological, physical, chemical or biological experimental settings where the systems are sufficiently controllable, a similar strategy could be followed with the same system used successively in two separate experimental runs, one for each initial condition.

Our numerical results highlight that there is a clear (yet non-trivial) dependence of the vulnerability on the node degree and centrality. This dependence turns out to be highly robust and largely dependent on local properties, different topologies yielding essentially equal qualitative features. We have studied separately the action of a
perturbation on isolated nodes, nodes of intermediate connectivity/centrality, and hubs. According to our results, a perturbation pushes the system out of synchronization most rapidly when applied on the most isolated nodes, and becomes less destructive as the perturbed node approaches the region of intermediate connectivity/centrality, showing a negative correlation between vulnerability and connectivity/centrality. After a certain value of connectivity/centrality is reached (where the vulnerability reaches a minimum), vulnerability and connectivity/centrality start to correlate positively, and the hubs of the network turn out to be as vulnerable under perturbations as some of the relatively isolated nodes.

When inspected at the node level, the divergence rates show that the propagation of the perturbation from the initially perturbed node to the rest of the network is very different depending on the connectivity/centrality of the perturbed node and of the other nodes. Some of the most conspicuous features are: a tendency for the most isolated nodes to stay away from the synchronization manifold since right after the perturbation, while recruiting more and more neighbors in a heterogeneous manner starting by the most isolated ones; and a tendency for the rest of the nodes to undergo some kind of damping after being perturbed and then diverge from synchronization with the rest of the network in a more homogenous way. Also, generally speaking, the first neighbors of the perturbed node seem to abandon the synchronized state with rates inversely proportional to their degree, whereas the rest of the nodes seem to do so at rates roughly proportional to their connectivity/centrality.

These results can perhaps be interpreted as the interplay of two opposite “forces” or factors. On the one hand, there is the stabilizing influence of the other nodes in the network, which pull the dynamics of the perturbed node onto the synchronization manifold and seems to be responsible for the damping of the perturbations. On the other hand, there is the fast propagation of information over such inter-connected networks, which helps the perturbation to reach all the nodes relatively rapidly (at least for the topologies considered, for which the geodesic distances are of necessity quite short). The first factor by itself would result in a monotonic decreasing dependence of the vulnerability on the connectivity/centrality; the second factor by itself would result in a monotonic increasing dependence of the vulnerability on the connectivity/centrality. The fact that we find a non-trivial dependence that is decreasing up to a minimum value and then increasing suggests that both factors (and probably others) are present to an extent, and become more or less prominent at different regions of the connectivity/centrality axis. Actually, results found at the local level (as shown in Figure 4.4) seem to agree pretty well with this explanation: isolated nodes are hardly subject to any damping but they propagate
the perturbation relatively heterogeneously, nodes of intermediate connectivity are subject to a heavier damping and propagate the perturbation more homogeneously, whereas hubs behave qualitatively as nodes of intermediate connectivity, but with a still more efficient propagation over the whole network (which is assumed to make the damping lighter because the whole network is more rapidly taken away from synchronization). For the purpose of illustration, Figure 4.5 shows a simplifying, idealized version of this speculative explanation superimposed on the global results (shown in Figure 4.3 A). Further work along these lines is in progress to assess the full validity of the present interpretation as well as the generality of the results.

Figure 4.5: Illustration of a plausible interpretation of the results. The relationship between vulnerability and connectivity is assumed to result from the interplay between two opposing factors: 1) the more isolated a node is, the more free it is to remain perturbed, whereas the more connected the node, the heavier the damping it is subject to (magenta line); 2) the more isolated a node is, the weaker the propagation to other nodes, whereas the more connected the node, the better it is at propagating the perturbation throughout the network (orange line). The combined effect is represented by the black line, which is assumed to capture the main qualitative features of the numerical results.

The relationship between the effect of large perturbations on a network of synchronized oscillators and the connectivity of the perturbed oscillator has been previously studied in the context of Kuramoto oscillators coupled following a scale-free topology (Moreno and Pacheco, 2004). Although the results are not strictly comparable with those reported here—in our work there is an irreversible disruption of the (unstable) synchronized dy-
amics whereas Moreno and Pacheco work on a system where (asymptotically stable) synchronized states are spontaneously reached—, they find an interesting inverse proportionality between the vulnerability of the synchronized dynamics under perturbations (as measured by the resynchronization time) and the degree of the perturbed node. There is an obvious analogy between this finding and the damping of the divergence rates that we attribute to the stabilizing influence of the rest of the network on the node that has been perturbed, which, in our interpretation, should become more and more important as the perturbed nodes are more connected/central. The fact that the trend we see for small degrees/centralities in Figure 4.3 is reversed at some point is the effect of the propagation of the perturbation over the network, the second competing force. Nevertheless, the absence of any trace of an analogous effect in the results reported in (Moreno and Pacheco, 2004) is not surprising, as only the first force is relevant in that context. Therefore these previous results are in good agreement with those reported here, and we believe they somehow lend support to our interpretation.

To summarize, the approach we introduce to the study of the vulnerability under finite perturbations in complex networks may be useful to unveil which nodes in a network are the most vulnerable to large damages or attacks, and thus those that are in more need of protection or rapid restoring action, when the collective dynamics is desirable, or those to be subject to an intentional attack if the build up of collective dynamics is to be prevented. For instance, the ranking of the nodes in terms of our measure of vulnerability could be of interest in the study of simulated networking systems and also experimentally in systems created for testing complex communication protocols (a perturbation could be a failure in one of the subsystems), neuronal cultures (an electric pulse applied on one neuron), etc. Moreover, we have tested the method numerically with three systems that are representative idealizations of many cases of interest. The results reported in this chapter show a very definite dependence of the vulnerability on the connectivity/centrality of the perturbed node, which turns out to be relatively independent of the detailed coupling topology. This makes them potentially fit for extrapolation to a greater variety of systems. The protection of infrastructural networks, such as power grids, and the localization of the best spot for an intentional attack (electric current pulse or magnetic stimulation) meant to prevent or reduce undesired highly synchronized behavior in the central nervous system (e.g. Parkinson’s disease, epilepsy, and other pathological rhythmic activities) are two relevant cases where our results may be applied.
Chapter 5

Targeting the dynamics of complex networks

This is the last chapter of this thesis, which is devoted to the targeting of the dynamics of complex networks. Once we have seen the main properties that emerge in complex networks, and we have studied the vulnerability of the different elements composing a network, we move on to study how to externally influence a network dynamics. We specifically report on a generic procedure to steer (target) a network dynamics towards a given, desired evolution. The problem is here tackled through a MSF approach (see Section 2.6) assessing the stability of the aimed dynamics, and through a selection of the nodes to be targeted. We show that the degree of a node is a crucial element in this selection process, and that the targeting mechanism is most effective in heterogeneous scale-free architectures. This makes the proposed approach applicable to the large majority of natural and manmade networked systems. The work we will be describing has been published in (Gutiérrez et al., 2012).

5.1 Introduction

For a given coupling scheme, an issue of the utmost importance is how to make the network abandon its current time evolution (as defined by its equations of motion and initial condition) and approach a goal dynamics. Traditionally, this has been the subject of the theory of chaos control and targeting of dynamical systems, whose methods have laid the basis for a judicious manipulation of a nonlinear dynamics, cleverly directing it towards a desired one. The idea behind control (Ott et al., 1990) is that of stabilizing one of the infinite set of unstable orbits embedded in chaotic attractors by the application
of small time dependent perturbations. The targeting procedure (Shinbrot et al., 1990, 1992; Kostelich et al., 1993; Boccaletti et al., 1997), instead, seeks to steer the dynamics of the system in the shortest possible time towards another trajectory fully compatible with the equations of the system, but originating from a different initial condition.

As we explained in Section 2.8, there is, therefore, an important difference between the two processes. While control aims to stabilize an otherwise unreachable trajectory, targeting allows to pursue a goal dynamics out of the set of those achievable by the natural evolution of the system from the attractor basin, and, as such, it can be interpreted as forcing the system to forget its initial condition.

While applications of both procedures to low dimensional chaotic (and spatially extended (Grigoriev et al., 1997; Sinha and Gupte, 1998; Parekh et al., 1998)) systems have been one of the fields of major interest within the last 20 years, only recently has the issue of network controllability been approached (Wang and Chen, 2002a; Sorrentino et al., 2007; Liu et al., 2011; Li et al., 2004). Imagine a network of $N$ $m$-dimensional units, such that a generic trajectory meanders in a $m \times N$-dimensional phase space. When the goal is having the network realize a desired evolution $g(t) \in \mathbb{R}^{m \times N}$, then tertium non datur: either i) one has to set the network to the initial condition producing $g(t)$, or ii) one has to target a generic initial condition towards $g(t)$. Clearly, the first case is fully impracticable: the extremely high dimensionality (linearly growing with the network size) of the phase space renders the selection of the specific initial state almost impossible, especially in those applications where noise or disturbances are present. Here, we address the as yet unexplored issue of targeting the dynamics of a complex network.

## 5.2 Methods

### 5.2.1 MSF approach

We consider a master network (MN) composed of $N$ identical diffusively coupled chaotic units. The MN evolution is ruled by:

$$\dot{x}_i^M = f(x_i^M) - \sigma_1 \sum_{j=1}^{N} \Delta_{ij} h_1(x_j^M)$$

where $x_i^M \in \mathbb{R}^m$ is the vector state of the node $i$ and $h_1(x) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $\sigma_1$ are the intra-network coupling function and coupling strength, respectively. $f$ and $h_1$ are chosen
such that the networked units are class I (see Section 2.6), with a monotonically increasing trend of the MSF associated to the synchronous state $x_1^M = x_2^M = \cdots = x_N^M = x_{sync}$ (Boccaletti et al., 2006), which is, therefore, always unstable, regardless of the choice of $\sigma_1$.

Let now
\[
\dot{x}_i^S = f(x_i^S) - \sigma_1 \sum_{j=1}^N \Delta_{ij} h_1(x_j^S)
\]
be the slave network (SN), obtained as a copy of the MN. As long as both networks remain uncoupled and start from different initial conditions, each one will sustain a different turbulent state, and the SN will never realize the desired goal dynamics: $X^S(t) \equiv (x_1^S(t), \ldots, x_N^S(t)) \neq X^M(t) \equiv (x_1^M(t), \ldots, x_N^M(t)) = g(t)$. At this stage, we implement a pinning strategy to target $g(t)$, that consists in sequentially establishing unidirectional links between nodes in the MN and their copies in the SN. The dynamical evolution of the SN is now described by:
\[
\dot{x}_i^S = f(x_i^S) - \sigma_2 \chi_i h_2(x_i^S - x_i^M) - \sigma_1 \sum_{j=1}^N \Delta_{ij} h_1(x_j^S),
\]
where $h_2 : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the coupling function between MN and SN [with the condition that $h_2(0) = 0$], and $\chi_i$ encodes the pinning procedure, i.e. $\chi_i = 1$ if there is a link from the $i$th node of the MN to the $i$th node of the SN, and 0 otherwise. Furthermore, $\sigma_2$ is the parameter ruling the inter-network link strength (see Figure 5.1).

It follows that the equation for the vector describing the difference between the dynamics of both networks $\delta X = X^S - X^M \equiv (\delta x_1, \ldots, \delta x_N)$ can be written (in terms of its components $\delta x_i$) as
\[
\delta \dot{x}_i = f(x_i^S) - f(x_i^M) - \sigma_1 \sum_{j=1}^N \Delta_{ij} [h_1(x_j^S) - h_1(x_j^M)] - \sigma_2 \chi_i h_2(x_i^S - x_i^M).
\]
A stable fixed point of Eqs. (5.4) at $\delta X = 0$ is a necessary condition for networks (5.1) and (5.3) to display the synchronized state $X^M = X^S$. The synchronization error is asymptotically defined as $\lim_{T \rightarrow \infty} (1/T) \int_0^T \|\delta X(t)\|dt$.

The linear stability of this solution can be assessed rigorously by the analysis of the linearized system for small $\delta X$, which reads:
\[
\delta \dot{x}_i = [Df(x_i^M) - \sigma_2 \chi_i D h_2(0)] \delta x_i - \sigma_1 \sum_{j=1}^N \Delta_{ij} Dh_1(x_j^M) \delta x_j
\]
Figure 5.1: **The targeting procedure.** The MN and SN are initially disconnected. Each targeting step establishes a new unidirectional connection between both networks. At the current step, a subgraph of the MN is connected to its counterpart in the SN (see gray continuous line). One further connection is established via that node (see gray dotted line) that minimizes the associated MSF.

being $Df$, $Dh_1$, and $Dh_2$ the corresponding Jacobian functions, and $X^M = g(t)$ the MN state to be targeted. This equation naturally provide us with a MSF kernel for the stability of the solution corresponding to the identical synchronization between MN and SN. Each of the linear equations (5.5), solved in parallel to the $N$ nonlinear equations for the MN defined in (5.1), corresponds to a set of $m$ conditional Lyapunov exponents at each pinning configuration (PC). Therefore, each peculiar PC that renders negative the largest of all such exponents makes the synchronous state $X^M = X^S$ stable.

### 5.2.2 Targeting procedure

Unidirectional MN-to-SN links are always established between analogous nodes, so it is natural to denote them by the label $i$ of the corresponding node in both networks (2, in the example of Fig. 5.1). To refer to a generic link established at a given step $t$, we use $n_t$ (evidently, $\forall t, n_t \in \mathcal{N} \equiv \{1, 2, \ldots, N\}$, but also $t \in \mathcal{N}$, as there are as many possible steps as nodes in the networks). At the targeting step $t$, we suppose to have already established a PC represented by the sequence of links $\mathcal{T}_t = \{n_1, n_2, \ldots, n_{t-1}\}$, where $n_i \neq n_j$, and we seek to identify the next pair of MN-SN nodes to be pinned, i.e. the specific integer $n_t$. Obviously, at $t = 1$ we start from the scenario where MN and SN are disconnected, and thus $\mathcal{T}_1 = \emptyset$, and the issue is to identify $n_1$, whereas at $t = N$, the only disconnected pair of nodes is eventually coupled by the link $n_N$. 118
As for the sequence search, we use the following procedure. At step \( t \), we go through all possible PC comprising \( T_t \) and one extra link between the not yet connected pairs. Denoting each configuration by \( T_o^t = \{ n_1, n_2, \ldots, n_{t-1}, n^o \} \) (where \( o \in \{ 1, 2, \ldots, N - (t - 1) \} \) and \( n^o \in \mathcal{N} \setminus T_t \)) and calling \( \lambda^t_o \) the maximum Lyapunov exponent (\( \lambda_{\text{max}} \)) associated with the corresponding MSF, we choose \( n_t \) to be the specific \( n^o \) that corresponds to the smallest \( \lambda^t_o \), and include it in \( T_{t+1} \) for the next step. A typical computation of \( \lambda_{\text{max}} \) for a given configuration includes the integration of the full system from a random initial condition across a time window of 50,000 time units (about 7,960 cycles of the Rössler oscillator). As the system is integrated, the \( \lambda_{\text{max}} \) value is updated and saved once every 5 time units. At the end of the entire simulation, we ended up with a sequence of 10,000 \( \lambda_{\text{max}} \) estimations, which is generically observed to converge to an asymptotic value with very small fluctuations. The \( \lambda_{\text{max}} \) values reported in the text correspond to the average taken across the second half (5,000 estimations) of this sequence. Repeated tests with several hundreds of such computations show that the standard deviations (the absolute error in the evaluation of \( \lambda_{\text{max}} \)) are always between \( 10^{-5} \) and \( 10^{-4} \), the largest value ever observed being \( 4.87 \times 10^{-4} \).

5.2.3 Network model

The equations of the network of Rössler oscillators (see Section 2.3), with \( a = b = 0.2 \), \( c = 7.0 \), written for simplicity in their scalar form, are:

\[
\begin{align*}
\dot{x}_i^M &= -y_i^M - z_i^M, \\
\dot{y}_i^M &= x_i^M + 0.2 y_i^M, \\
\dot{z}_i^M &= 0.2 + z_i^M (x_i^M - 7.0) + \sigma_1 \sum_{j \in \mathcal{N}_i} (z_j^M - z_i^M), \\
\dot{x}_i^S &= -y_i^S - z_i^S, \\
\dot{y}_i^S &= x_i^S + 0.2 y_i^S + \sigma_2 \chi_i (y_i^M - y_i^S), \\
\dot{z}_i^S &= 0.2 + z_i^S (x_i^S - 7.0) + \sigma_1 \sum_{j \in \mathcal{N}_i} (z_j^S - z_i^S),
\end{align*}
\]

(5.6)

(5.7)

where, at targeting step \( t \), \( \chi_i = 1 \) if \( i \in T_t \) and 0 otherwise. This corresponds, in our notations, to \( \mathbf{x}^M \equiv (x^M, y^M, z^M), \mathbf{x}^S \equiv (x^S, y^S, z^S), \mathbf{h}_1(\mathbf{x}) = (0, 0, z), \mathbf{h}_2(\mathbf{x}) = (0, y, 0), \mathbf{f} = [-y - z, x + 0.2y, 0.2 + z(x - 7.0)]. \mathcal{N}_i \) is the set of neighbors of node \( i \). All our results refer to the case \( \sigma_1 = \sigma_2 = 1 \).
5.3 Results

The feasibility of targeting a general goal dynamics $g(t)$, compatible with the natural evolution of the graph, is tantamount to demonstrating that a suitable perturbation may have a network leaving its current state, and attaining another one that would be naturally produced by a different initial condition. Therefore, the effect of the perturbation can be understood as removing the sensitivity of the network to the initial condition. A way of engineering such a perturbation is to consider two identical networks (configured in a master-slave scheme), and specify when they identically synchronize [Pecora and Carroll, 1990; Kocarev and Parlitz, 1995], when starting from different initial conditions. Therefore, our approach consists in: i) considering two identical graphs, both producing a turbulent regime (i.e. a generic dynamics not showing any particular global or local order), but starting from different initial conditions [a master network (MN) providing us the specific $g(t)$, and a slave network (SN)], and ii) properly engineering a unidirectional pinning action from MN to SN, as shown in Figure 5.1, which ultimately leads to the synchronization of their dynamics.

We analytically assess the stability of the solution corresponding to the synchronization between the two networks, as unidirectional links are sequentially created from nodes in MN to their images in SN. To do so, we introduce the associated MSF (see the previous section on Methods), giving the maximum Lyapunov exponent $\lambda_{\text{max}}$ for each pinning configuration (PC) between the two networks. Eventually, after a given fraction of nodes has been pinned, we show that $\lambda_{\text{max}}$ becomes negative, indicating that SN attained the goal dynamics $g(t)$. Notice that $\lambda_{\text{max}}$ is an ensemble average over all possible attractor trajectories, and therefore, assessing synchronizability by the negativity of $\lambda_{\text{max}}$ independently of the initial state for a given sequence of pinned nodes corresponds to addressing the targetability of the slave network into one of its possible natural evolutions.

A relevant issue is optimizing the search of the best PC, i.e. that PC requiring the minimal number of pinning actions from the MN to the SN. It is evident that such a problem becomes intractable as the network size grows. This is because one would have to apply the MSF approach based on Eq. (5.5) to all possible $N!$ permuting sequences of nodes susceptible to be pinned, and look for the shortest one that renders $\lambda_{\text{max}}$ negative. Here we pursue a suboptimal solution, provided by the outcome of a greedy algorithm that sequentially selects the best pinning action (i.e. that minimizing $\lambda_{\text{max}}$) at each step.

We proceed as follows. For a given PC, the changes in $\lambda_{\text{max}}$ brought about by temporarily establishing an additional link from the MN to the SN are exhaustively explored.
The best link at decreasing $\lambda_{\text{max}}$ is eventually created. When iterated (see Methods section), the procedure leads to a specific targeting sequence, or node ranking, which we call suboptimal ranking. For very large networks, however, this suboptimal strategy can also be computationally demanding. To overcome this limitation, we extend recent results on criticality of network resilience to perturbations on the highly connected nodes ([Wang and Chen, 2002a,b]). Namely, we show that the node pinning sequence minimizing the MSF very much coincides with the sequence of decreasing degrees for small size networks (and can therefore be replaced to a good approximation by such a sequence), thus allowing for the use of the MSF approach also for very large graphs, and compare different graph structures in terms of their targetability.

As an illustration of both the greedy search procedure and the correlation with the node degree, we consider a network made of $N = 50$ Rössler oscillators ([Rössler, 1976] (see the Methods section, and Section 2.3). It is important to remark that the MSF approach is in principle applicable to any continuous time dynamics, and not necessarily limited to a specific kind of oscillator (like the Rössler case we are here considering), nor to a specific dimension of the phase space containing the attractor of each network unit. As for the network structure, we use the topology resulting from a realization of the CM ([Bender and Canfield, 1978] (see Section 1.6) based on a degree sequence taken from a random generation of $N$ integers uniformly distributed between 5 and 45. This will provide us with a suitable statistics to compare our suboptimal sequence with the degree sequence of the graph. In Figure 5.2 we report $\lambda_{\text{max}}$, starting from disconnected MN and SN, up to the point where they are fully (unidirectionally) coupled, each step following the suboptimal ranking procedure (blue circles). It can be seen that each additional targeting action results in about the same decrease of $\lambda_{\text{max}}$, and that the targeting of 12 nodes is enough to reach a negative $\lambda_{\text{max}}$, indicating the achievement of the goal dynamics $g(t)$.

The degree of the nodes in the suboptimal ranking is shown in the inset of Figure 5.2. There is a striking correlation between such a ranking and the inverse of the degree sequence. Indeed, the values of $\lambda_{\text{max}}$ obtained with a targeting based on the degree ranking are also shown in Figure 5.2 (red continuous line), and are almost identical to those obtained with the suboptimal ranking. This raises the possibility that a targeting procedure based on the degree ranking produces quite similar results in a much shorter computation time ($N$ vs. $N(N + 1)/2$ computations), thus allowing for the application of our MSF approach to networks of considerably greater size. Qualitatively identical results were obtained by varying the initial conditions and configuration model realization, as well as with more familiar topologies (ER random graphs and BASF networks; see
Figure 5.2: **Suboptimal vs. degree ranking.** Targeting procedure applied to a $N = 50$ network of Rössler oscillators. $\lambda_{\text{max}}$ extracted from the MSF as a function of the targeting step, for the suboptimal sequence (blue open circles), and with the node degree sequence (red continuous line). The inset shows the degree $k$ of each pinned node in the suboptimal sequence vs. the targeting step.

Sections 1.5 and 1.6) of $N = 50$. Nevertheless, we remark that the results presented in Figure 5.2, though obtained with a somewhat uncommon topology, are those that best illustrate the correlation between suboptimal and degree ranking (due to their including nodes of almost all possible degrees). Indeed, for the specific network size used, both the BASF and the ER topologies result in a less precise correspondence of degree and suboptimal ranking, due to the strongly reduced degree variability.

The ability to study large-sized networks opens up the way for an exploration of topological ensembles, and a quantification of their propensity to be targeted. On the other hand, it is still an open question how different the results would be if based on a less judicious ranking, such as a random ranking, or, generically, a ranking not aiming at reaching the imposition of the MN dynamics upon SN.

To address both issues, we apply the targeting procedure to a network of $N = 500$ Rössler oscillators, with ER and BASF topologies. In Figure 5.3 (top and middle panels) we show the results of the first 200 targeting steps [sufficient, in all cases, for the attainment of $g(t)$]. Both the degree ranking (the solution proven to greatly approximate the suboptimal ranking) and a random ranking (in which the order of appearance of each node is fully random) are inspected, thus allowing for a comparison between both ends of the spectrum of possible targeting strategies. The results are ensemble averages over
Figure 5.3: **Targetability of different topologies.** (Top and middle panels) Targeting scheme for ER and BASF networks of size $N = 500$. $\lambda_{\text{max}}$ (top panel) and synchronization error (middle panel) as functions of the targeting step, for the degree and random rankings. The vertical dashed lines originating in the middle panel mark the targeting step at which the synchronization error vanishes. The inset of the top panel is a zoom around $\lambda_{\text{max}} = 0$. The horizontal dashed line in the inset of the top panel marks the negative value at which the vertical lines intersect the $\lambda_{\text{max}}$ curves. (Bottom panel) Robustness of the targeting strategy against network uncertainties. The panel reports the $\lambda_{\text{max}}$ curves vs. the targeting step for the BASF network used in the top and middle panels. The different curves refer to the degree ranking (empty circles), the random ranking (full circles with the darkest blue intensity), and a degree ranking based on the knowledge of the degrees of only 80%, 60% and 40% of randomly selected nodes in the network.
40 different graph realizations, and over 40 different ranking realizations for the random ranking case. Aside from $\lambda_{\text{max}}$, we also report the synchronization error, which vanishes for the same targeting step at which $\lambda_{\text{max}}$ gets slightly negative. Notice, indeed, that the negativity of $\lambda_{\text{max}}$ is a necessary, but not sufficient condition for synchronization, as $\lambda_{\text{max}}$ is by definition an ensemble average over the attractor trajectories. In our case, the dynamics $g(t)$ is a turbulent one, and as such it is spanning the entire phase space. As a result, having a negative (yet very small in absolute value) $\lambda_{\text{max}}$ does not prevent that, in some areas of the phase space, expanding events will take place that will locally separate the trajectories of the two networks. This, in its turn is reflected in a not perfectly vanishing value of the time averaged synchronization error, which will instead vanish perfectly when the negative $\lambda_{\text{max}}$ will have a sufficiently high absolute value to warrant contraction in the whole phase space.

While in the top and middle panels of Figure 5.3 it is evident that the degree ranking leads to a much better targeting, as compared to the random scheme, there is an additional feature that deserves the utmost attention: the relative improvement in BASF networks results to be much more evident than that of the ER case, indicating that the dynamics of heterogeneous structures, like the ones encountered in real world networks, are much easier to be manipulated and targeted.

Regarding the application of the method to real networks, one should prove the robustness of the degree sequence criterion in cases where only partial information on the network is available. To this purpose, we have considered the same scale-free topology of the top and middle panels of Figure 5.3 and artificially screened out the information on the degree of a higher and higher percentage of randomly selected nodes in the network. We then took the remaining nodes (whose degree is supposed to be known), and constructed the corresponding degree sequences to be used for the targeting. This process is equivalent to a mixed-strategy, in which part of the information on the network degree is lost in favor of a random selection of nodes. The bottom panel of Figure 5.3 reports $\lambda_{\text{max}}$ vs. the targeting steps for the cases in which the percentage of nodes with available degree is 80%, 60% and 40%. For comparison, the same panel contains also the two curves already shown in the top panel of the same Figure, and referring to the pure degree and random rankings. It is evident that, while the targeting procedure is still effective and robust for a relatively high fraction of nodes with unknown degrees (the curves labeled with 80% and 60% are still very close to that of the pure degree ranking), as the uncertainty is further increased, the $\lambda_{\text{max}}$ curve approaches that of the pure random ranking.
5.4 Discussion

While a relevant part of the research on complex networks concentrated on their response (e.g. vulnerability) to various types of external interference, recently the attention has shifted towards methods for engineering a set of perturbations for a network to feature a particular behavior. In this chapter, we have shown the feasibility of targeting a general goal dynamics $g(t)$ compatible with the natural evolution of a network, and we have given necessary conditions for attaining such a dynamics by just pinning a limited number of nodes in the graph. By further comparing the targetability of different topological structures, we demonstrated that the dynamics exhibited by heterogeneous topologies are easier to be targeted. Our conclusion is, therefore, that the vast majority of real world networks are easily manageable from a targeting point of view with our approach, as most of them (from transportation networks in technology, to networks of acquaintances in social science, to metabolic and genetic networks in biology) display a scale free topology, with some nodes concentrating most of the connections. In fact, such networks can be efficiently targeted even when a large fraction of its nodes are not considered, as they are unknown or their degrees are only known with significant uncertainty.
Chapter 6

Conclusions

In this thesis, we have presented methodological approaches and original results in the field of the dynamics of complex networks. While networked dynamical systems can be in principle studied as sufficiently high dimensional dynamical systems, in practice a consideration of the topology of interactions is not only necessary in applications, but clearly advantageous in most theoretical approaches. In one way or another, all the work presented here has dealt with the interplay between topology and dynamics in networked dynamical systems.

Three main issues have been addressed in this thesis. A) The first is the study of the mechanisms leading to the emergence of structural properties out of adaptive synchronization in networks. We propose two models of self-organizing networks of phase oscillators that follow the two principles of homophily (the interactions between dynamically correlated nodes are strengthened) and homeostasis (the resources for a node to establish interactions are limited), which have been observed in many real systems, notably in social and biological networked systems. B) Once we have seen how the properties of networks emerge from adaptation, we consider the issue of the vulnerability of collective states in a network. Specifically, we propose a novel vulnerability measure that gives a node ranking based on how vulnerable each node of the network is in terms of the speed with which the network abandons a collective synchronous evolution if a large perturbation is applied on it. C) Finally, we also address the issue of the targeting of desired goal dynamics in networks. We propose a strategy that allows to impose general dynamics on a given network efficiently by taking advantage of the topology, and analyze its performance in heterogeneous and homogeneous networks. The following conclusions result from this work (where the letter A, B or C identifies which of these problems that we have studied is being considered):
A.1 In general, the adaptive mechanism in the form of a competition between homophily and homeostasis greatly enhances the synchronization in a network.

A.2 Whenever the coupling strength is not extremely high, such systems are in a state of partial order (the global synchronization is low, but the local synchronization tends to be much higher) characterized by the presence of cluster synchronization (i.e. the appearance of several groups of synchronous oscillators).

A.3 At the mesoscopic scale, a community structure spontaneously emerges from adaptive synchronization. This community structure is the topological counterpart of the cluster synchronization mentioned above.

A.4 At the macroscopic scale, a power law in the weight distribution appears for small couplings. In one of the two models considered, we are able to provide analytical support to this finding, which shows an excellent agreement with the numerical results.

A.5 Whereas several mechanisms have been proposed for the emergence of one or the other property, to our knowledge, our models are the first where both a community structure and scale-freeness emerge simultaneously.

A.6 The fact that two models implementing the same general adaptive mechanisms by means of different concrete equations lead to very similar results raises the possibility that such adaptive principles lead to those ubiquitous properties in very general circumstances.

B.1 We propose a definition of vulnerability that is dynamical in nature, as opposed to the purely topological measures traditionally found in the literature. It quantifies the vulnerability of the nodes in a network in terms of the effect of perturbations applied on them: the more destructive a perturbation turns out to be for the preservation of a collective dynamics, the more vulnerable the node on which the perturbation is applied is considered to be.

B.2 By applying the dynamical vulnerability measure to three different network topologies we conclude that it is largely dependent on local properties. Therefore, a vulnerability ranking of the nodes can be proposed in quite general terms for any topology.

B.3 Moreover, this ranking shows a non-trivial dependence of the vulnerability on the node connectivity that is very consistent across different topologies. Isolated nodes are found to be the most vulnerable ones, followed by the hubs. There is a region in between...
these two ends populated by nodes of intermediate connectivity, which turn out to be the less vulnerable to a perturbation.

**B.4** We offer an interpretation of this dependence in terms the propagation of information on the network, on the one hand, and the damping effect of the neighbors of the perturbed node (and, indirectly, of the rest of the network). Although of course we cannot be certain about its validity, all global and local results are in agreement with this interpretation.

**B.5** The applicability of both the vulnerability measure and of the conclusions derived from our results are very straightforward in the case of real systems that can be described by a networked dynamical system. Systems as diverse as neuronal networks *in vitro*, pathological oscillations in the central nervous systems, power grids or communications systems are obvious examples, but there may be many others.

**C.1** A MSF formulation of the targeting problem in complex networks is proposed, which allows the study of this important question for the first time in the literature.

**C.2** Since an optimal search for the best pinning sequence is extremely costly in terms of computation time, we propose a suboptimal strategy based on a greedy algorithm. This strategy is furthermore seen to be approximately equivalent to the strategy based on the degree sequence, which makes our approach applicable to very large networks (as the computation time turns out to be linear in the network size, instead of being quadratic as in the suboptimal case).

**C.3** The targeting strategy is more effective (in terms of the required number of pinning actions) in heterogeneous networks than in homogeneous networks. Also, the strategy based on the degree sequence is much more effective than an approach based on a random sequence, the difference being much greater in the case of heterogeneous networks too.

**C.4** Since most of the real networked systems that have been studied are heterogenous networks, our strategy seems to have great potential for experimental settings and applications in very many different systems. The auxiliary master network we use in that case may be a numerical simulation or a recording of the same system or a similar one evolving according to a desired dynamics.

**C.5** We consider the case of incompletely known networks with heterogenous topologies. We find that our strategy is robust (i.e. the increase in the required number of pinning
actions is small) even when the knowledge of the network topology is so imperfect that 30 or 40% of the network nodes are neglected (either because their degree is unknown, or because the nodes themselves are). Our method is thus confirmed to be very relevant for targeting desired dynamics upon networks in experimental and technological settings.

All these results and methods show that the interplay between topology and dynamics in complex networks is a worthy subject of inquiry that reveals many important results that are relevant for understanding the emergence of structural properties, and for controlling the appearance and stability of collective behaviors. This topic is indeed at the forefront of the research nowadays carried out in the fields of complex systems and nonlinear dynamics, and many important insights are expected in the coming years. We hope that our contributions and (partial) answers to the problems we have investigated in this field will prove useful in the understanding of these fascinating issues that arise when topology and dynamics are simultaneously considered. Also, our theoretical work having some obvious experimental and practical implications, we hope it will lead to applications and extensions in concrete scientific and technological contexts where the systems under study can be represented as networked dynamical systems.
References


D. Sousa, L. Sarmento, and E. Mendes Rodrigues. Characterization of the twitter @replies network: are user ties social or topical? In *Proceedings of the 2nd international workshop on Search and mining user-generated contents*, SMUC ’10, pages 63–70. ACM, 2010.


