Metrizability of spaces of holomorphic functions

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Abstract

In this paper we prove that if $U$ is an open subset of a metrizable locally convex space $E$ of infinite dimension, the space $\mathcal{H}(U)$ of all holomorphic functions on $U$, endowed with the Nachbin–Coeuré topology $\mathcal{T}_S$, is not metrizable. Our result can be applied to get that, for all usual topologies, $\mathcal{H}(U)$ is metrizable if and only if $E$ has finite dimension.

1. Introduction

The study of locally convex topologies on $\mathcal{H}(U)$ is a topic of interest for many researchers. It is very natural to ask which properties $\mathcal{H}(U)$ has from the point of view of functional analysis. In particular, some mathematicians have been interested in characterizing the locally convex spaces $E$ such that $\mathcal{H}(U)$ is metrizable for all open subsets $U$ of $E$ and for topologies as the compact open, Nachbin ported and Nachbin–Coeuré topologies. In 1968, Alexander proved the following theorem for Banach spaces with Schauder basis, which was generalized by Chae:

Theorem 1. (See [1, p. 13], [4, Theorem 16.10].) Let $U$ be an open subset of an infinite dimensional Banach space. If $\tau$ is a topology on $\mathcal{H}(U)$ finer than the topology of pointwise convergence, then $(\mathcal{H}(U), \tau)$ is not metrizable.

Although this result can be applied to all usual topologies on $\mathcal{H}(U)$, its proof is only valid when $E$ is a Banach space because Baire and Josefson–Nissenzweig Theorems are used (see [6] and [11]).

In 2007, Ansemil and Ponte have got that, for the Nachbin ported topology, Theorem 1 can be generalized to metrizable locally convex spaces $E$ (see [3]). In this article, we prove an analogous result for the Nachbin–Coeuré topology $\mathcal{T}_S$, which answers a question stated by Mujica in [8, Problem 11.9] thirty years ago.

2. Definitions and previous results

Throughout this paper, the letter $E$ will denote a complex locally convex space, $E'$ will represent the dual space of $E$ and $U$ will be an open subset of $E$. A function $f : U \to \mathbb{C}$ is holomorphic on $U$ if it is continuous and for each $z \in U$ and $w \in E$ the function of one complex variable

$$\lambda \mapsto f(z + \lambda w)$$
is holomorphic on a neighborhood of zero in \( \mathbb{C} \). Let \( \mathcal{H}(U) \) denote the space of all holomorphic functions on \( U \). The compact open topology on \( \mathcal{H}(U) \), \( \tau_0 \), is defined by the seminorms

\[
f \in \mathcal{H}(U) \mapsto \sup_{z \in K} |f(z)|
\]

when \( K \) ranges over the compact subsets of \( U \).

Let us recall the definition of other fundamental topologies on \( \mathcal{H}(U) \). A seminorm \( p \) on \( \mathcal{H}(U) \) is ported by a compact subset \( K \) of \( U \) if for every open neighborhood \( V \) of \( K \) in \( U \) there is a constant \( C > 0 \) such that

\[
p(f) \leq C \cdot \sup_{z \in V} |f(z)|
\]

for all \( f \in \mathcal{H}(U) \). The Nachbin topology \( \tau_{\omega} \) is the locally convex topology on \( \mathcal{H}(U) \) defined by the seminorms ported by the compact subsets of \( U \).

The Nachbin–Coeure topology, denoted by \( \tau_\beta \), is the locally convex topology on \( \mathcal{H}(U) \) defined by the seminorms \( p \) which verify the following property: for each increasing countable open cover of \( U \), \( \{V_n\}_{n=1}^\infty \), there exist \( n_0 \in \mathbb{N} \) and \( C > 0 \) such that

\[
p(f) \leq C \cdot \sup_{z \in V_{n_0}} |f(z)|
\]

for all \( f \in \mathcal{H}(U) \). It is well known that the space \( (\mathcal{H}(U), \tau_\beta) \) is bornological and \( \tau_0 \leq \tau_{\omega} \leq \tau_\beta \) on \( \mathcal{H}(U) \) (see [5, Propositions 3.17, 3.18]). Moreover, \( \tau_{\omega} \) and \( \tau_\beta \) coincide on \( E' \), which can be identified with a subspace of \( \mathcal{H}(U) \) (see [5, Proposition 3.22]).

In our main result (Theorem 4), the following proposition will be used:

**Proposition 2.** (See [3, Proposition 1.]) Let \( E \) be a metrizable locally convex space. If \( (E', \tau_{\omega}) \) is metrizable, then \( E \) is a normed space.

We will also need some results about bounding and limited sets. A subset \( A \) of \( E \) is said to be bounding if \( \sup_{z \in A} |f(z)| < \infty \) for all \( f \in \mathcal{H}(E) \). A set \( B \subset E \) is limited if

\[
\lim_{n \to \infty} \left( \sup_{z \in B} |\varphi_n(z)| \right) = 0
\]

for every sequence \( \{\varphi_n\}_{n=1}^\infty \subset E' \) such that \( \lim_{n \to \infty} \varphi_n(z) = 0 \) for all \( z \in E \).

**Proposition 3.** Let \( E \) be a Banach space.

1. Every bounding set in \( E \) is limited (see [10, Corollary 2.13]).
2. If \( A \) is a limited subset of \( E \), the closed convex balanced hull of \( A \) is also limited (see [10, Remark 4.2(c)]).
3. If \( E \) has infinite dimension, every limited set in \( E \) has empty interior (see [10, Corollary 4.13]).

The third property is a consequence of the Josefson–Nissenzweig Theorem. Indeed, if a limited subset \( A \) of an infinite dimensional Banach space \( E \) has no empty interior, there exist \( z_0 \in A \) and \( r > 0 \) such that \( B_E(z_0, r) \subset A \). Then \( B_E(0, 1) \subset \frac{1}{r}(A - z_0) \) and we obtain that \( B_E(0, 1) \) is also limited. By the Josefson–Nissenzweig Theorem, there is a sequence \( \{\varphi_n\}_{n=1}^\infty \) in \( E' \) such that \( \|\varphi_n\| = 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \varphi_n(z) = 0 \) for all \( z \in E \). Therefore,

\[
1 = \lim_{n \to \infty} \|\varphi_n\| = \lim_{n \to \infty} \left( \sup_{z \in B_E(0, 1)} |\varphi_n(z)| \right) = 0.
\]

This is absurd; so limited subsets in \( E \) have empty interior.

**3. Metrizability of \( \mathcal{H}(U) \)**

**Theorem 4.** Let \( U \) be an open subset of a metrizable locally convex space \( E \). If \( (\mathcal{H}(U), \tau_\beta) \) is metrizable, then \( E \) is a finite dimensional normed space.

**Proof.** The symbol \( \widehat{E} \) will denote the completion of \( E \). If \( r > 0 \), \( B_E(0, r) \) and \( B_{\widehat{E}}(0, r) \) will represent the open balls with center 0 and radius \( r \) in \( E \) and \( \widehat{E} \), respectively. If \( (\mathcal{H}(U), \tau_\beta) \) is metrizable, then the subspace \( (E', \tau_\beta) \) is also metrizable. As the topologies \( \tau_{\omega} \) and \( \tau_\beta \) coincide on \( E' \), by Proposition 2 we obtain that \( E \) is a normed space. Its completion \( \widehat{E} \) is a Banach space.

Let \( \{\mathcal{F}_n\}_{n=1}^\infty \) be a fundamental system of neighborhoods of 0 in \( (\mathcal{H}(U), \tau_\beta) \). For each \( n \in \mathbb{N} \) let

\[
A_n = \{z \in U: |f(z)| \leq 1 \text{ for every } f \in \mathcal{F}_n\}.
\]
We claim that the sets \( A_n \) are limited in \( E \). Indeed, if \( n \in \mathbb{N} \) and \( f \in \mathcal{H}(E) \), then \( f|_U \) is holomorphic on \( U \). Since \( \mathcal{F}_n \) is a neighborhood of zero, there is \( \alpha > 0 \) such that \( \alpha f|_U \in \mathcal{F}_n \). If \( z \in A_n \), then \( |\alpha f(z)| \leq 1 \) and so
\[
\sup_{z \in A_n} |f(z)| \leq \frac{1}{\alpha} < \infty.
\]
This shows that \( A_n \) is a bounding subset of \( E \). Hence, by Proposition 3, \( A_n \) and \( \mathcal{T}(A_n)^E \), the closed convex balanced hull of \( A_n \), are limited in \( E \) for every \( n \in \mathbb{N} \).

If \( z_0 \in U \), the mapping
\[
T : (\mathcal{H}(U), \tau_5) \to (\mathcal{H}(U - z_0), \tau_5)
\]
defined by
\[
Tf(z) = f(z + z_0)
\]
for each \( f \in \mathcal{H}(U) \) and \( z \in U - z_0 \), is a homeomorphism. Therefore, the space \((\mathcal{H}(U), \tau_5)\) is metrizable if and only if \((\mathcal{H}(U - z_0), \tau_5)\) is also metrizable and so we can assume that \( 0 \in U \).

Now we use an adaptation of [7, p. 184] for open subsets made in [3, Theorem 4]. Let \( r > 0 \) such that \( B_E(0, 2r) \subset U \). If \( \hat{z} \in B_E(0, r) \), there is a point \( z_1 \in B_E(0, r) \) such that
\[
\|\hat{z} - z_1\| < \frac{r}{4}.
\]
Therefore,
\[
\hat{z} - z_1 \in B_E\left(0, \frac{r}{4}\right)
\]
and there is \( z_2 \in B_E(0, \frac{r}{4}) \) such that
\[
\|\hat{z} - z_1 - z_2\| < \frac{r}{4^2}.
\]
If we repeat this argument, for each \( n \in \mathbb{N} \) we can find a point \( z_n \in B_E(0, \frac{r}{4^{n-1}}) \) such that
\[
\left\|\hat{z} - \sum_{k=1}^{n} z_k\right\| < \frac{r}{4^n}.
\]
Hence \( \hat{z} = \sum_{k=1}^{\infty} z_k \).

Let \( w_n = 2^n z_n \in E \) for each \( n \in \mathbb{N} \). The sequence \( \{w_n\}_{n=1}^{\infty} \) converges to zero:
\[
\|w_n\| = 2^n \|z_n\| < 2^n \cdot \frac{r}{4^n-1} = \frac{2r}{2^n-1} \xrightarrow{n \to \infty} 0.
\]
Moreover,
\[
w_n \in B_E\left(0, \frac{2r}{2^n-1}\right) \subset B_E(0, 2r) \subset U
\]
for all \( n \). Therefore,
\[
K = \{w_n : n \in \mathbb{N}\} \cup \{0\}
\]
is a compact subset of \( U \) and
\[
\left\{ f \in \mathcal{H}(U) : \sup_{z \in K} |f(z)| \leq 1 \right\}
\]
is a neighborhood of zero in \((\mathcal{H}(U), \tau_5)\). Since \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) is a fundamental system of neighborhoods of zero in \((\mathcal{H}(U), \tau_5)\), there is \( n_1 \in \mathbb{N} \) such that
\[
\mathcal{F}_{n_1} \subset \left\{ f \in \mathcal{H}(U) : \sup_{z \in K} |f(z)| \leq 1 \right\}.
\]
Let \( w \in K \). If \( f \in \mathcal{F}_{n_1} \), then
\[
|f(w)| \leq \sup_{z \in K} |f(z)| \leq 1.
\]
Hence \( w \in A_{n_1} \) and so \( K \subset A_{n_1} \).
For every \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} \frac{1}{2^k} w_k \) is a convex linear combination of elements of \( K \), which implies that
\[
\sum_{k=1}^{n} \frac{1}{2^k} w_k \in \Gamma(K) \subseteq \Gamma(A_{n_1})^{\hat{E}}.
\]
As the last set is closed in \( \hat{E} \), we have
\[
\hat{z} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2^k} w_k \in \Gamma(A_{n_1})^{\hat{E}}.
\]
Thus, we have proved that
\[
B_{\hat{E}}(0, r) = \bigcup_{n=1}^{\infty} \Gamma(A_{n_1})^{\hat{E}}.
\]
Since \( B_{\hat{E}}(0, r) \) is an open subset of the Banach space \( \hat{E} \), there exists \( n_2 \in \mathbb{N} \) such that \( \Gamma(A_{n_2})^{\hat{E}} \) has no empty interior in \( \hat{E} \).
As we have seen that \( \Gamma(A_{n_2})^{\hat{E}} \) is a limited set, by Proposition 3 \( \hat{E} \) has finite dimension and so \( E \) is also a finite dimensional space. \( \square \)

**Theorem 5.** Let \( U \) be an open subset of a metrizable locally convex space \( E \). Let \( \tau \) be a locally convex topology on \( \mathcal{H}(U) \) such that \( \tau_0 \leq \tau \leq \tau_5 \). Then \( (\mathcal{H}(U), \tau) \) is a metrizable space if and only if \( E \) has finite dimension.

**Proof.** As \( E \) is a metrizable space, \( \tau_5 \) is the bornological topology associated with \( \tau_0 \) (see [5, Example 3.20(a)]). If \( E \) has finite dimension, it is well known that \( (\mathcal{H}(U), \tau_0) \) is a metrizable space. Hence \( \tau_0 \) is bornological and so \( \tau_5 = \tau_0 \). This implies that \( \tau = \tau_0 \) and, therefore, \( (\mathcal{H}(U), \tau) \) is metrizable.

Now we show the opposite implication. Since \( \tau_0 \leq \tau \leq \tau_5 \), \( \tau_5 \) is also the bornological topology associated with \( \tau \). If \( (\mathcal{H}(U), \tau) \) is metrizable, then \( \tau \) is bornological and so \( \tau = \tau_5 \). Hence, by Theorem 4, \( E \) is a finite dimensional space. \( \square \)

Theorem 5 can be applied to all usual topologies on \( \mathcal{H}(U) \), among them, \( \tau_{oo}, \tau_0 \) and \( \beta \). The definition of \( \tau_{oo} \) is based on the topology of uniform convergence on bounded sets, while \( \beta \) is the strong topology when \( \mathcal{H}(U) \) is considered as a dual space (see [5, Definitions 3.29 and 3.39]).

Our last proposition will show that the only hypothesis on Theorems 4 and 5 (\( E \) is metrizable) cannot be suppressed. We recall that a topological space \( X \) is said to be hemicompact if there is a fundamental sequence of compact subsets of \( X \). A locally convex space \( E \) is a DFC space if there exists a Fréchet space \( F \) such that \( E = (F', \tau_0) \).

Infinite dimensional DFC spaces are not metrizable. Indeed, if \( F \) is a Fréchet space and \( (F', \tau_0) \) is metrizable, then there is a fundamental sequence \( \{K_n\}_{n=1}^{\infty} \) of compact subsets of \( F \). Hence we have
\[
F = \bigcup_{n=1}^{\infty} K_n.
\]
Since \( F \) is a Baire space, there is \( n \in \mathbb{N} \) such that \( K_n \) has no empty interior and, therefore, \( F \) is a finite dimensional space.

**Proposition 6.** Let \( F \) be a separable Fréchet space and let \( U \) be an open subset of \( E = (F', \tau_0) \). Then \( \tau_0 = \tau_5 \) on \( \mathcal{H}(U) \) and \( (\mathcal{H}(U), \tau_5) \) is a Fréchet space.

**Proof.** Using the Banach–Dieudonné Theorem, it is possible to prove that \( (F', \tau_0) \) is a \( k \)-space and so the open subset \( U \) is a \( k \)-space as well (see [9, Theorem 7.6]). Hence \( (C(U), \tau_0) \), the space of all continuous functions on \( U \) with the compact open topology, is complete and \( (\mathcal{H}(U), \tau_0) \) is also complete because it is closed in \( (C(U), \tau_0) \).

Now we use [9, Theorem 7.4]. Let \( \{V_m\}_{m=1}^{\infty} \) be a fundamental system of neighborhoods of \( 0 \) in \( F \). Then the polar sets \( \{V_m^\circ\}_{m=1}^{\infty} \) form a fundamental sequence of compact subsets of \( (F', \tau_0) \). Since \( F \) is separable, there is a countable dense subset \( D \) in \( F \) and then the topology \( \sigma(F', D) \) on \( F' \) is defined by a metric \( \rho \). Moreover \( \sigma(F', D) \) coincides with \( \tau_0 \) on the compact subsets of \( U \).

Consider the sets
\[
L_{m,n} = \left\{ x \in V_m^\circ \cap U : \rho(x, V_m^\circ \backslash U) \geq \frac{1}{n} \right\},
\]
where \( m \) and \( n \) are any natural numbers. Each \( L_{m,n} \) is a compact subset of \( U \) because it is closed in \( V_m^\circ \). In the proof of [9, Theorem 7.4], Mujica asserts that \( \{L_{m,n} = L_{m,n+1} \}_{n=1}^{\infty} \) is a fundamental system of compact subsets of \( U \). As he has recognized in a private communication, it is not clear whether this is true. However, it is possible to prove that \( \{L_{m,n} : m, n \in \mathbb{N} \} \) is a
fundamental sequence of compact subsets of $U$ and thus $U$ is hemicompact. Hence the compact open topology on $\mathcal{H}(U)$ is metrizable and $(\mathcal{H}(U), \tau_0)$ is a Fréchet space.

As $U$ is a $k$-space, $\tau_3$ is the bornological topology associated with $\tau_0$ (see [2, Theorem 1]). Since $\tau_0$ is metrizable, it follows that $\tau_0$ is bornological and then we have $\tau_0 = \tau_3$. □

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References