

HIERARCHICAL BOUNDARY ELEMENTS

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Abstract—In this paper we show the possibility of applying adaptive procedures as an alternative to the well-known philosophy of standard Boundary Elements. The three characteristic steps of adaptive procedures, i.e. hierarchical shape functions families, indicator criteria, and a *posteriori* estimation, can be defined in order to govern an automatic refinement and stopping of the solution process. A computer program to treat potential problems, called QUEIMADA, has been developed to show the capabilities of the new idea.

1. INTRODUCTION

The last decade has seen the spectacular increment of research and applications of Boundary Element (BE) techniques. Although some authors [1] consider it as a part of Finite Element Method (FEM), in principle its development was almost independent [2–4, 6–9], and the FEM philosophy was incorporated later [5]. Since then, there has been an explosion of books [10–17] to teach the subject, as well as a series of International Conferences [18–22], specially dedicated to BE. The research has maintained a similar pace; our group, for instance, has explored several possibilities [25–35].

After all that effort, the method seems to be well founded, although some problems remain to be solved. Among them, the most important seems to be the evaluation of the influence matrices involving singular integrals, which takes a substantial part of the computer time in all BE codes.

In order to fix ideas, let us concentrate on applications to potential theory, as shown elsewhere [25]. The basic collocation equation for two-dimensional and a Laplacian problem is

$$c\phi(P) + \int_{\partial\Omega} \phi(Q) \frac{\partial\phi^*(P, Q)}{\partial\mathbf{n}(Q)} ds(Q) = \int_{\partial\Omega} \frac{\partial\phi(Q)}{\partial\mathbf{n}(Q)} \phi^*(P, Q) ds(Q), \quad (1)$$

where $\partial\Omega$ is the boundary of the domain, ϕ is the potential, P the collocation point, Q a running dummy point, \mathbf{n} the normal vector at Q , ϕ^* the fundamental solution of the Laplacian equation

$$\phi^* = \ln \frac{1}{PQ}, \quad (2)$$

and c depends on the local geometry. For smooth boundaries,

$$c \begin{cases} 2\Pi & \text{if } P \in \Omega \\ \Pi & \text{if } P \in \partial\Omega \\ 0 & \text{if } P \in \bar{\Omega}. \end{cases} \quad (3)$$

If there is a conical vertex at P , c is the corresponding solid angle.

The standard direct BEM, as described in Ref. [25], proceeds interpolating in an isoparametric fashion the boundary as well as the potential and fluxes defined along it

$$\begin{aligned} \mathbf{x} &= \mathbf{N} \mathbf{x}^e, \\ \phi &= \mathbf{N} \phi^e, \\ \frac{\partial\phi}{\partial\mathbf{n}} &= \mathbf{N} \frac{\partial\phi^e}{\partial\mathbf{n}}, \end{aligned} \quad (4)$$

where \mathbf{N} collects the desired interpolation shape functions. A detailed treatment, as well as a computer program in Fortran, for the linear case, can be seen in Ref. [25]. A Basic version, with some improved features, prepared for the IBM-PC, is included in Ref. [17]. Introducing the previous interpolatory conditions in eqn (1), it is possible to build a system like

$$\mathbf{A} \phi = \mathbf{B} \mathbf{q},$$

where \mathbf{A} and \mathbf{B} are influence matrices containing elements of the type

$$c_i \delta_{ij} + \int_{\partial\Omega_j} N_i \frac{\partial\phi_{ij}^*}{\partial n_j} ds_j \quad (6)$$

and

$$\int_{\partial\Omega_j} N_i \phi_{ij}^* ds, \quad (7)$$

respectively, where ϕ and \mathbf{q} are vectors containing the values of the potential and the fluxes along the boundary. After input of boundary conditions, eqn (5) is written in standard form

$$\mathbf{K} \mathbf{X} = \mathbf{F},$$

where \mathbf{X} contains the boundary unknowns.

The most attractive feature of the method is, of course, the reduction in the dimensions that have

to be discretized, but the payed price is, on the one hand, the nonsymmetry of **A** and **B**, and in the other, the fact that both of them are completely filled, due to the global nature of the fundamental solution [eqn (2)]. In addition, the integrals to be evaluated are singular, and that means an extra effort in computing.

In spite of the previous difficulties, the method is worthwhile and its performances are excellent as shown in the abovementioned references. In particular, its behavior under an *h*-convergence type test can be seen in Ref. [49].

2. NATURAL ELEMENTS. HIERARCHY OF INTERPOLATING FUNCTIONS

The introduction of the isoparametric idea was, no doubt, a major step in the BEM evolution. Given a single set of points, everything is defined: the geometry, the collocation points, and the locally based interpolation functions. The procedure can be very efficiently automatized although some details are worth noting. First the enormous rigidity of the approach: a complicated geometry enforces a very detailed mesh, even if the boundary conditions are simple. Of course, the reciprocal, i.e. simple geometry with complicated boundary conditions is also possible. And the mesh refinement always increases the computing burden. While more effective computing procedures are developed, it is interesting to adopt a different approach.

Evidently, the current FEM tendencies push us to choose locally based interpolating functions but, why to do it when the weighting functions are globally based?

What is necessary is to identify in each problem, where the discontinuities arise and to adjust then our elements (and their corresponding shape functions) to the particular needs of the problem under study.

That is very simple in two-dimensional potential problems, the discontinuities are due to sharp corners or sudden changes in nature (or value) of the boundary conditions, and they define the "natural" elements of the problem. Inside them it is possible then to apply a global approach with all the advantages of progressive refinement inherent to the classical Ritz approach.

In fact, what we are facing is the idea of Adaptive Elements as described for FEM in Refs. [43-47].

Needless to say, the problem now is how to define the geometry and how to choose the collocation points when the initial linear approach has been exceeded, but the freedom introduced will pay the change by reducing the computational effort to those parts where it is really needed.

That is the idea that we presented in Refs. [42] and [48]. In order to assimilate the adaptive philosophy, it is necessary first of all to define a hi-

erarchy of functions that, in our case, is the well known Peano's family [47-50].

$$N_P = \frac{1}{P!} [\xi^P - b], \quad -1 \leq \xi \leq \Delta; \quad P \geq 2;$$

$$b = \begin{cases} 1 & \text{if } P \text{ is odd} \\ \xi & \text{if } P \text{ is even} \end{cases} \quad (9)$$

plus the linear ones

$$N_0 = \frac{1 - \xi}{2}, \quad N_1 = \frac{1 + \xi}{2}. \quad (10)$$

The introduction of a new element after a certain ϕ_n, q_n approach

$$\begin{aligned} \phi_n &= a_1 \phi_1 + a_2 \phi_2 + \dots + a_n \phi_n, \\ q_n &= b_1 q_1 + b_2 q_2 + \dots + b_n q_n \end{aligned} \quad (11)$$

produces the nesting of the successive influence matrices. For instance, if

$$\hat{A} = \begin{bmatrix} \int q_{11}^* J_1 & \int q_{12}^* J_2 & \dots & \int q_{1n}^* J_n \\ \int q_{21}^* J_1 & \int q_{22}^* J_2 & \dots & \int q_{2n}^* J_n \\ \int q_{n1}^* J_1 & \int q_{n2}^* J_2 & \dots & \int q_{nn}^* J_n \end{bmatrix}, \quad (12)$$

then

$$\hat{A} = \begin{bmatrix} \hat{A} & \int q_{j,n+1}^* J_{n+1} \\ \int q_{n+1,j}^* J_j & \int q_{n+1,n+1}^* J_{n+1} \end{bmatrix} \begin{matrix} N \\ 1 \end{matrix} \quad (13)$$

To produce the last row it has been necessary to define a new collocation point. Currently we are choosing them at the center and quarters of the activated elements.

Surely, the hierarchy induced in both **A** and **B** is integrally transmitted to the final system, producing nested matrices, better conditioned than usual and prone to an iterative solution. (Although, to say the truth, and probably due to the small size of the matrices we are working with, we have not found any advantage in using iterative in place of direct-elimination solvers).

3. REFINEMENT AND STOPPING

As was said in the Abstract, in order to get an adaptive procedure it is necessary to establish an indicator of the places needing future refinements and an estimator giving an idea of the level of precision reached with the preexisting solution.

For FEM the development of a *a posteriori* error estimates by Babuska, Szabo, Rheinbolt, etc. has

allowed the splendid flourishing of adaptive methods.

Unfortunately, in our case that mathematical approach is still lacking, and we have been forced to work under physical or intuitive reasons.

About the indicator, we have followed closely the method proposed by Peano *et al.* [50]. That is, over the initial system,

$$\mathbf{K}_i \delta_i = \mathbf{P}_i, \quad (14)$$

new degrees of freedom (one per element and with the hierarchy consecutive to the previously incorporated) are assembled.

$$\begin{bmatrix} \mathbf{K}_i & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} \end{bmatrix} \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix} = \begin{bmatrix} \mathbf{P}_i \\ \mathbf{P}_j \end{bmatrix}. \quad (15)$$

Note that, due to the BEM properties,

$$\mathbf{K}_{ij}^T \neq \mathbf{K}_{ji}. \quad (16)$$

Taking the second row box,

$$\mathbf{K}_{jj} \delta_j = \mathbf{P}_j - \mathbf{K}_{ji} \delta_i. \quad (17)$$

Assuming for δ_i the values obtained after having solved eqn (14), eqn (17) will give a first approach to δ_j . This is why, after some energy reasons, Peano *et al.* propose to make the indication through

$$Q_j = \frac{\sum_{k \neq j} K_{jk} \delta_k - P_j}{\sqrt{K_{jj}}}. \quad (18)$$

If Q_j is less than a prescribed tolerance, the equation j is not assembled.

In our case it is very easy to have the \mathbf{K}_{ij} prepared during the previous step without much additional effort. On the contrary, the computation of \mathbf{K}_{ji} would be very onerous. This is why we have decided to use eqn (18) but through the previously computed \mathbf{K}_{ij} . In that way, every refinement needs only the computation of the diagonal of \mathbf{K}_{jj} and that of \mathbf{P}_j .

Once the active new degrees of freedom are selected, we compute the remaining integrals and start the iteration process with an estimate based on eqn (18).

We have tried also other criteria, see Ref. [48], but Peano's seem to work well enough.

So the computer runs until some limit is reached. This limit is usually provided by an *a posteriori* estimation. As said above, the lack of a rigorous estimator, has moved us to check an "equilibrium-type" condition.

It is well known that, due to the very nature of the problem, it is necessary for the flux to be globally

equilibrated, i.e.

$$\int_{\partial\Omega} \frac{\partial\phi}{\partial n} ds = 0. \quad (19)$$

That is precisely the condition we have choose to check. For every refinement, the residual flux is computed and only when it reaches a certain precision level, the process is stopped.

In some occasions we have found even useful to enforce condition (19) as the last step of the solving procedure. There are various methods, see Refs. [51] or [52], the simplest one being the direct change of one of the computed equations by eqn (19).

4. IMPLEMENTATION AND EXAMPLES

The previous ideas have been implemented in a computer program called QUEIMADA. Due to the simplicity of the problem, it has been possible to do it in an IBM-PC microcomputer, according to the plan developed in Fig. 1 that follows closely the philosophy presented in Ref. [50].

The preprocessor module contains (1) input of geometric and boundary conditions, specifying vertex nodes and indicating whether the evolution is linear or curve. (2) Interpolation using preselected paths (analytical, Peano's interpolation, etc.). (3) Printing and plotting of data. (4) Automatic coding of vertexes, see Ref. [25], and elimination of sharp-Dirichlet type corners, see Ref. [17]. (5) Saving of results.

The primary solver allows for a first approach using a linear interpolation (N_0 and N_1 , shape functions) for the unknowns. The essential parts are: (1) Reading of data and precomputed arrays (related to numerical integrations). (2) Computation of influence coefficients and matrices. (3) Solving the system of equations. (4) Recording of results as well as influence coefficients needed by the next step.

The self-adaptive solver proceeds along the above developed lines and with the following general pattern: (1) Read of data and precomputed arrays. (2) Indicator computations. (3) Activation of new degrees of freedom. (4) Assembling of new influence coefficients. (5) Solving the system of equations. (6) Estimation of the accuracy and decision about a new refinement. In Fig. 2 we present a simple example described for instance in Ref. [40].

The same was solved in a previous paper [17] where we employed 32 linear elements. In comparison, here the natural elements are only five and the final refinement has only 10 dof.

If the equilibrium condition is taken as an estimation of the accuracy, it is possible to draw the curve $[\log E - \log (1/N)]$ where E is the error and N the number of dof. It is seen that the global rate of convergence is 3.6.

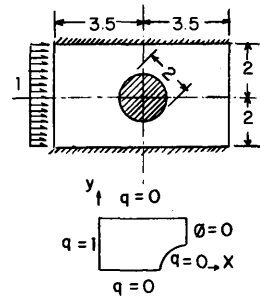
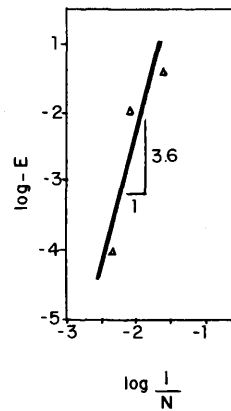
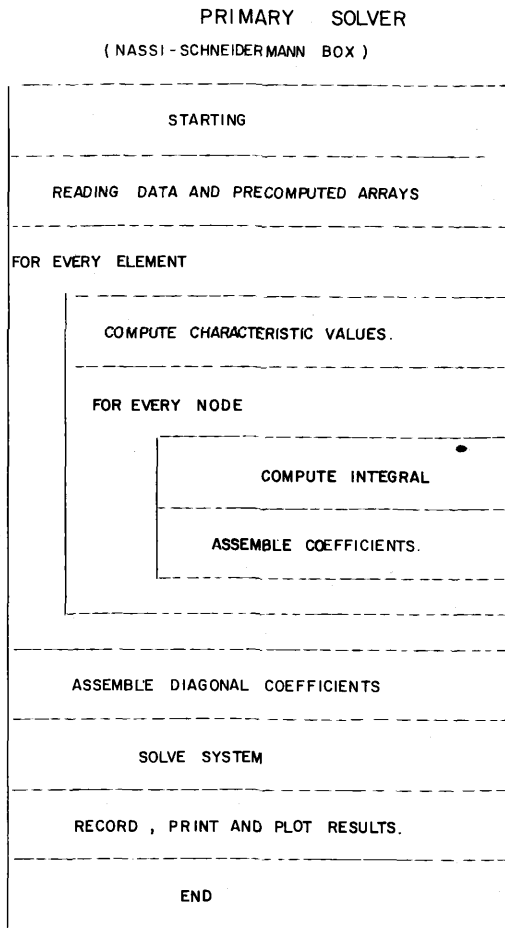


Fig. 2. Convergence of a simple case.

In Fig. 3, a more interesting case is discussed. The subject is the Saint-Venant torsion of an elliptical shaft that is reduced to a mixed potential problem using symmetry. A solution with linear elements was presented in Ref. [24] where we used 30 elements. In comparison, here the natural elements are only 3, although due to the bad linear starting provoked by the condition $\phi = 0$ we have repeated the problem with four elements subdividing the curved side by 2. The last situation is seen in parts A1 and B1, for the unknown potential and fluxes, respectively, while A2 and B2 present the three-element discretization.

As can be seen the first case needs only six dof to produce an almost perfect solution, while the

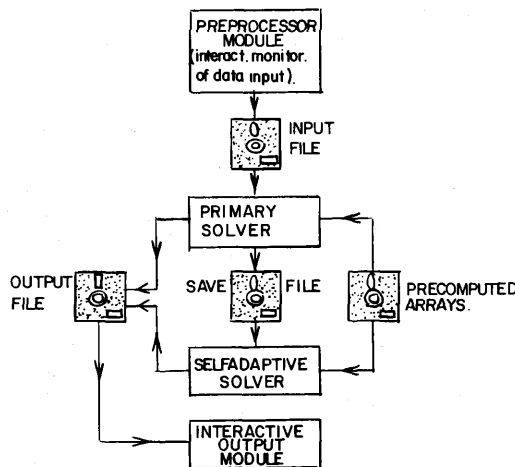
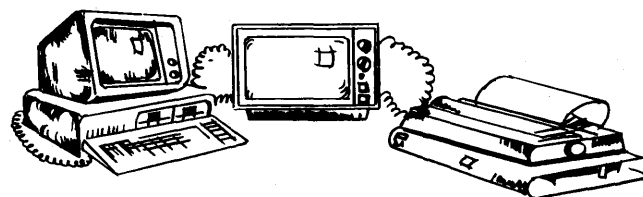


Fig. 1. Interactive environment for program QUEIMADA.

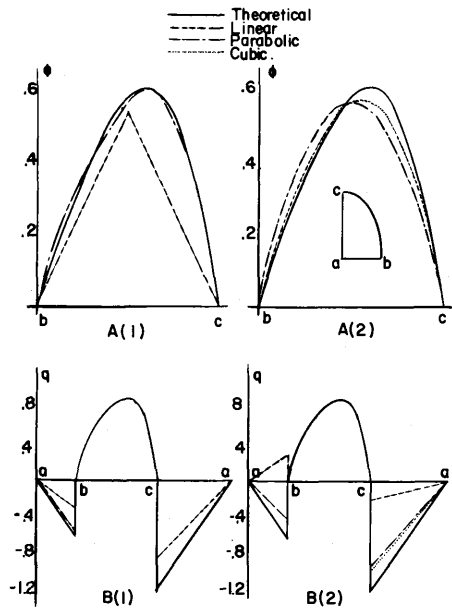


Fig. 3. Behavior of an ellipse under Saint-Venant torsion. (A1) Evolution of the potential along the curved side, when it is modeled with two elements. (A2) Ditto with one element. (B1) Evolution of the fluxes along the straight sides when the curved side is modeled with two elements. (B2) Ditto with one element.

three-element discretization behaves poorer. A dramatic improvement in accuracy can be obtained in the last case by using the technique suggested at the end of point 3.

Finally, Fig. 4 presents the convergence results for both situations. In (A) we have measured the quadratic error of the potential along the curved sides, while in (B) we have taken the residual flux estimator. As can be seen, the rate of convergence is 4 in the first case, 2 when we use only one element

to discretize the curved side, and 3.6 when subdividing it in two elements.

5. FUTURE PROJECTIONS AND RESEARCH DIRECTION

Paraphrasing Lachat and Watson [5], we think that the adaptive approach can be called "the third generation" of boundary integral equations programs, and it can have the same importance on future research as the similar concepts had on FE.

There are obvious directions to study. The first of course is the extension to elasticity problems. It is straightforward and we present some results in this Symposium. Fracture and plasticity can be seen now under a new light.

From a mathematical viewpoint, the method is a challenge to found reliable estimators, and indicators better founded than those presented in this paper.

Finally the importance of the collocation point on the final accuracy needs to be studied in order to optimize the presented algorithm.

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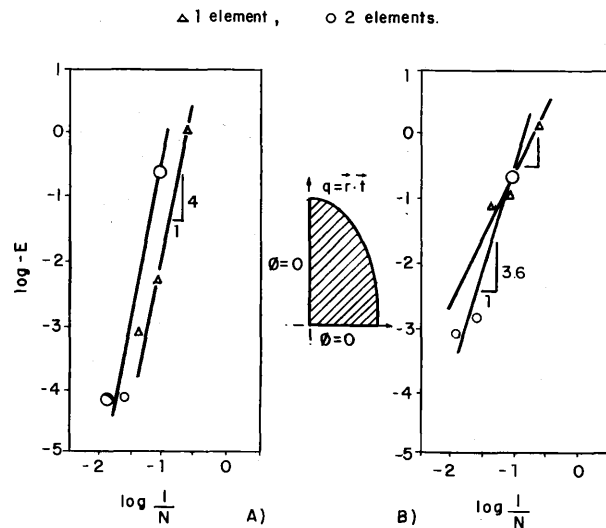


Fig. 4. Convergence results for the torsion problem. (A) When measuring the quadratic error of the potential along the curved sides. (B) When measuring the residual flux error.

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