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Electrostatic Plasma Instabilities Excited by a High-Frequency Electric Field

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ABSTRACT

An analysis of the electrostatic plasma instabilities excited by the application of a strong, uniform, alternating electric field is made on the basis of the Vlasov equation. A very general dispersion relation is obtained and discussed. Under the assumption $\omega_o^2 >> \omega_{pi}^2$ (where ω_o is the applied frequency and ω_{pi} the ion plasma frequency) a detailed analysis is given for wavelengths of the order of or large compared with the Debye length. It is found that there are two types of instabilities: resonant (or parametric) and non-resonant. The second is caused by the relative streaming of ions and electrons, generated by the field; it seems to exist only if ω_o is less than the electron plasma frequency ω_{pe}. The instability only appears if the field exceeds a certain threshold, which is found. The resonant instability is caused by parametric interaction of natural modes of oscillation of the plasma; it occurs only if $n\omega_o \approx \omega_{pe}$. The case

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$n = 1$ is studied in detail for strong fields, and simple expressions are given for the growth rate and the range of unstable frequencies around ω_{pe}. For either instability, growing waves exist, for large fields, only if the wave number k is less than a maximum value, k_{max}, which is determined. Finally, a physical picture of the resonant instability is discussed.

I. INTRODUCTION

Recently some interest has been shown in the excitation of longitudinal plasma wave instabilities by the application of strong high-frequency electric fields. DuBois and Goldman[1] and Goldman[2] have used a Green's function analysis to show that when the applied frequency ω_o is around the electron plasma frequency ω_{pe}, the plasma becomes unstable if the field exceeds a certain threshold value; Lee and Su[3] use a hydrodynamic approach to show that a relative drift between the electrons and the ions in the equilibrium state when the field is absent, leads to a reduction in the magnitude of the threshold field.

All of the work mentioned above has been done with moderately strong fields only. Silin[4] and Jackson[5] have considered an arbitrary field intensity. Silin used cold plasma equations and his results are given in terms of Bessel series, hardly illuminating; on the other hand, he noticed that when $n\omega_o \approx \omega_{pe}$ $(n \neq 1)$, instabilities similar to the one just discussed were possible and he pointed out further that the plasma could also be unstable even if such a resonant condition was not satisfied.

Jackson's work is based on the Vlasov equation but, as we shall see in Sec. V, he obtained his dispersion relation under improper approximations.

Our analysis is based on the Vlasov equation. In the next section we obtain an infinite, convergent determinant which, when set equal to zero, gives the dispersion relation. A very general analysis of this equation is made in Sec. III. It is found that for infinitely heavy ions, the applied field does not modify the stability of the electrons.[5] It is also found that, for large fields, growing oscillations exist only for wave numbers, k , less than some maximum value, k_{max} , that decreases as the field increases. Finally, assuming $\omega_o^2 >> \omega_{pi}^2$ (ω_{pi} is the ion plasma frequency) and excluding from our analysis wavelengths small compared with the Debye length we obtain two simple conditions either of which (or both) every root of the dispersion relation has to satisfy.

In Sec. IV we investigate those roots that satisfy either one of the two conditions. One type of root is found to be stable and similar to the usual plasma electron wave which exists when the field vanishes. The other type of root can be unstable, even with a purely imaginary value. We obtain a simple equation for the root; in the limit of zero temperatures it agrees with Silin's off-resonance result[4] ($n\omega_o$ not close to ω_{pe}). For the hot plasma we investigate the weak field limit and find a definite threshold that the field has to exceed for the plasma to be unstable; we also study the very strong field limit and obtain k_{max} as a function of the field. Both results are similar to results obtained in the theory of the two-stream instability; it can be concluded that the present instability is due to the relative streaming of the two species of particles under the action of the applied field. A further, tentative conclusion that has no equivalent in the theory of the two-stream instability, is that growing waves can be excited off-resonance only if $\omega_o < \omega_{pe}$.

In Sec. V we study those roots that satisfy simultaneously the two conditions of Sec. III; for long wavelengths this happens for $n\omega_o \simeq \omega_{pe}$. The instability found is stronger than the one just discussed and is due to parametric interaction of natural modes of the system. We study in detail the case n = 1; the weak field limit of Refs. 1-3 is not considered. We find a simple equation for the root; when the cold plasma limit is taken, it agrees with Silin's equation[4] after his Bessel series are summed. We show that Jackson's result[5] for the root are wrong. Finally, we obtain k_{max} for this root too. In Sec. VI we offer a physical picture for this instability.

II. BASIC EQUATIONS

We consider an infinite, homogeneous plasma to which a uniform, alternating electric field is applied. (For an electromagnetic wave of finite but very long wavelength, our homogeneous assumption is equivalent to the dipole approximation; a further condition for such a field is that, if $\omega_o < \omega_{pe}$, the wavelengths of the oscillations under study are to be small compared with the skin depth of the applied field.) Any instability associated with inhomogeneities of the field is, of course, excluded from this analysis.

We assume that the Vlasov equation describes the time evolution of the distribution functions of ions and electrons; thus we look for unstable waves with growth rates much larger than the collision frequency. Then, the equilibrium distribution function of the α-species, F_α, obeys the equation

$$\frac{\partial F_\alpha}{\partial t} + \frac{q_\alpha}{m_\alpha} \underset{\sim}{E} \sin \omega_o t \cdot \frac{\partial F_\alpha}{\partial \underset{\sim}{v}} = 0 \qquad (1)$$

where $\underset{\sim}{E}$ is the field inside the plasma; α stands for e (electrons) and i (ions).

The solution to Eq. (1) is $F_\alpha = N_\alpha F_{\alpha o}(\underset{\sim}{v} + \omega_o \underset{\sim}{\epsilon}_\alpha \cos\omega_o t)$ where $\underset{\sim}{\epsilon}_\alpha = q_\alpha \underset{\sim}{E}/m_\alpha \omega_o^2$, $N_e q_e + N_i q_i = 0$, and $F_{\alpha o}$ is an arbitrary function.

We study now the stability of the plasma around this equilibrium. Let f_α be the perturbation of the distribution function of the α-species. The linearized equation for f_α is

$$\frac{\partial f_\alpha}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial f_\alpha}{\partial \underset{\sim}{r}} + \frac{q_\alpha}{m_\alpha}\underset{\sim}{E}\sin\omega_o t.\frac{\partial f_\alpha}{\partial \underset{\sim}{v}} - \frac{N_\alpha q_\alpha}{m_\alpha}\frac{\partial\phi}{\partial \underset{\sim}{r}}.\frac{\partial F_{\alpha o}}{\partial \underset{\sim}{v}} = 0 \ , \qquad (2)$$

where the small electrostatic potential ϕ obeys Poisson's equation:

$$\frac{\partial^2\phi}{\partial \underset{\sim}{r}^2} = -4\pi \sum_\alpha q_\alpha \int f_\alpha d\underset{\sim}{v} \quad . \qquad (3)$$

We introduce the transformation

$$t = t,\ \underset{\sim}{\rho}_\alpha = \underset{\sim}{r} + \underset{\sim}{\epsilon}_\alpha \sin\omega_o t\ ,$$

$$\underset{\sim}{u}_\alpha = \underset{\sim}{v} + \omega_o \underset{\sim}{\epsilon}_\alpha \cos\omega_o t$$

into Eq. (2) and define

$$\phi_k = \int \phi\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d\underset{\sim}{r} \quad , \qquad (4)$$

$$f_{\alpha k} = q_\alpha e^{-\underset{\sim}{k}\cdot\underset{\sim}{\epsilon}_\alpha \sin\omega_o t}\int f_\alpha e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d\underset{\sim}{r} \equiv q_\alpha \int f_\alpha e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\rho}_\alpha} d\underset{\sim}{\rho}_\alpha \ . \qquad (5)$$

There results for $f_{\alpha k}$:

$$\frac{\partial f_{\alpha k}}{\partial t} + i\underset{\sim}{k}\cdot\underset{\sim}{u}_\alpha f_{\alpha k} - \frac{\omega_{p\alpha}^2}{k^2} i\underset{\sim}{k}.\frac{\partial F_{\alpha o}}{\partial \underset{\sim}{u}_\alpha}\Bigg[\int f_{\alpha k}\, d\underset{\sim}{u}_\alpha + e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\epsilon}_{\alpha\beta}\sin\omega_o t}\int f_{\beta k}\, d\underset{\sim}{u}_\beta\Bigg] \quad , \qquad (6)$$

where ϕ_k has been eliminated between Eqs. (3) and (4); $\omega_{p\alpha}^2 = 4\pi q_\alpha^2 N_\alpha/m_\alpha$ and $\underset{\sim}{\epsilon}_{\alpha\beta} = \underset{\sim}{\epsilon}_\alpha - \underset{\sim}{\epsilon}_\beta$.

If we try to study the normal modes of Eq. (6) we come upon the same improper integral that appears for the case of zero applied field.[6] To specify the problem properly we consider an initial value problem and take the Laplace transform of Eq. (6); we define for arbitrary n ,

$$f^n_{\alpha k} = \int_0^\infty dt\, f_{\alpha k}\, e^{i(\omega + n\omega_o)t} \quad , \qquad \mathrm{Im}\,\omega > 0 ,$$

$$E^n = \int f^n_{ek}\, d\underset{\sim}{u}_e \quad , \quad I^n = \int f^n_{ik}\, d\underset{\sim}{u}_i \quad ,$$

$$A^n_\alpha = \int \frac{i f_{\alpha k}(t=0)\, d\underset{\sim}{u}_\alpha}{\omega + n\omega_o - \underset{\sim}{k}\cdot\underset{\sim}{u}_\alpha}$$

and obtain by straightforward calculation:

$$D^n_e E^n = -\chi^n_e \sum_p J_{n-p}\, I^p + A^n_e \quad , \tag{7a}$$

$$D^n_i I^n = -\chi^n_i \sum_p J_{p-n}\, E^p + A^n_i \tag{7b}$$

for $\alpha =$ e and i, respectively. In Eqs. (7a,b) above, $x = \underset{\sim}{k}\cdot\underset{\sim}{\epsilon}_{ei}$ is the argument of the Bessel functions and

$$D^n_\alpha \equiv 1 + \chi^n_\alpha = 1 + \frac{\omega^2_{p\alpha}}{k^2} \int \frac{\underset{\sim}{k}\cdot(\partial F_{\alpha o}/\partial \underset{\sim}{u}_\alpha)\, d\underset{\sim}{u}_\alpha}{\omega + n\omega_o - \underset{\sim}{k}\cdot\underset{\sim}{u}_\alpha} \quad ;$$

use has been made of the identity

$$e^{ia\sin b} = \sum_p J_p(a)\, e^{ipb} \quad .$$

We have obtained an infinite system of equations for E^n and I^n (n being an arbitrary integer, positive or negative). If we know the solution to this system we can obtain, by Laplace inversion, the behavior in time of

$\int f_{\alpha k}\, d\underset{\sim}{u}_\alpha$ (or $e^{i\underset{\sim}{k}\cdot\underset{\sim}{\epsilon}_\alpha \sin\omega_o t} \int f_{\alpha k}\, d\underset{\sim}{u}_\alpha$) and ϕ_k. For long times, it suffices to look for the singularities of E^n or I^n. For E^s , for instance, we would have

$$E^s = \frac{\Delta_s^e}{\Delta} \qquad (8)$$

where $\Delta = \det \overline{\Delta}$ and $\overline{\Delta}$ is the matrix of the coefficients of the unknowns E^n, I^n in the system (7a, b); also $\Delta_s^e = \det \overline{\Delta}_s^e$ and the matrix $\overline{\Delta}_s^e$ is obtained from $\overline{\Delta}$ by substituting the set $\{A_e^n, A_i^n\}$ for the column of $\overline{\Delta}$ that contains the coefficients of E^s in the system (7a, b). If $\partial F_{\alpha o}/\partial \underset{\sim}{u}_\alpha$ and $f_{\alpha k}(t=0)$ are analytical functions of $\underset{\sim}{u}_\alpha$, as we shall assume here, χ_α^n and A_α^n are known to be entire functions of the variable ω ;[6] from the convergence results obtained at the end of this section it may then be concluded that so are Δ and Δ_s^e . Therefore the singularities of E^s are just the zeros of Δ. The equation

$$\Delta(\omega) = 0 \qquad (9)$$

is our dispersion relation. Before proceeding to its study we point out that for $\text{Im}\omega \leq 0$, the analytical continuation of $\Delta(\omega)$ is obtained by following the Landau contour[6] in the integral for χ_α^n .

The study of the determinant Δ becomes very simple if we consider the homogeneous system

$$D_e^n E^n = -\chi_e^n \sum_p J_{n-p} I^p , \qquad (10a)$$

$$D_i^n I^n = -\chi_i^n \sum_p J_{p-n} E^p . \qquad (10b)$$

Through elimination of either E^n or I^n we obtain

$$I^n - \frac{\chi_i^n}{D_i^n} \sum_p \sum_m J_{p-n} J_{p-m} \frac{\chi_e^p}{D_e^p} I^m = 0 ,$$

$$E^n - \frac{\chi_e^n}{D_e^n} \sum_p \sum_m J_{n-p} J_{m-p} \frac{\chi_i^p}{D_i^p} E^m = 0 .$$

The respective matrices of coefficients, $\bar{\Delta}_i$ and $\bar{\Delta}_e$, are given below:

$$\bar{\Delta}_i^{nn} = 1 - \frac{\chi_i^n}{D_i^n} \sum_p \frac{J_{p-n}^2 \chi_e^p}{D_e^p} \equiv 1 + d_i^{nn} ,$$

$$\bar{\Delta}_i^{nm} = - \frac{\chi_i^n}{D_i^n} \sum_p \frac{J_{p-n} J_{p-m} \chi_e^p}{D_e^p} \equiv d_i^{nm} \qquad (m \neq n), \tag{11a}$$

$$\bar{\Delta}_e^{nn} = 1 - \frac{\chi_e^n}{D_e^n} \sum_p \frac{J_{n-p}^2 \chi_i^p}{D_i^p} \equiv 1 + d_e^{nn} ,$$

$$\bar{\Delta}_e^{nm} = - \frac{\chi_e^n}{D_e^n} \sum_p \frac{J_{n-p} J_{m-p} \chi_i^p}{D_i^p} \equiv d_e^{nm} \qquad (m \neq n) . \tag{11b}$$

The relation between $\Delta_\alpha \equiv \det \bar{\Delta}_\alpha$ and Δ is

$$\Delta = \Delta_\alpha \prod_{-\infty}^{\infty} D_e^n \prod_{-\infty}^{\infty} D_i^n \quad ; \tag{12}$$

therefore, although $\bar{\Delta}_e$ is quite different from $\bar{\Delta}_i$, $\Delta_e = \Delta_i$. We also point out that $\Delta(\omega)$ is periodic in ω with real period ω_o ; hence, we can assume for any root of (9) that $|\mathrm{Re}\,\omega| \leq \omega_o/2$. We shall assume this throughout our study.

Finally, before discussing the solution to Eq. (9), we comment briefly on the validity of our handling of the infinite system (7a, b). We want to show that the result (8) for E^s is valid. The basic point to make is that for finite $|\omega|$, $\chi_\alpha^n = O(|n|^{-2})$ for large $|n|$; this is true for any physically admissible $F_{\alpha o}$. We can then show as follows that the infinite determinants Δ and Δ_s^e are absolutely convergent.

We consider the region $|\mathrm{Re}\,\omega| \leq \omega_o/2$. Let us first exclude any value ω within this region such that $D_\alpha^n(\omega) = 0$ for some α and n. Then the following conclusions follow easily. The Bessel series of the Neumann type that appear in the elements of $\bar{\Delta}_\alpha$ are absolutely and uniformly convergent since for real x, $|J_n(x)| \leq 1$; (these series are also uniformly convergent as functions of x in a strip around the real axis). Then, $\sum_n |d_\alpha^{nn}|$ and $\sum_{n,m} |d_\alpha^{nm}|$ are also convergent: along a column, i.e., for fixed m, d_α^{nm} goes to zero as $|n|^{-2}$; along a row the same conclusion can be reached by observing that as $|m| \to \infty$

$$J_m(x) = O(e^{-|m|\ln|m|})$$

and then writing

$$d_\alpha^{nm} = -\frac{\chi_\alpha^n}{D_\alpha^n}\left[\sum_{|p|\leq N} \frac{J_{p-n}\, J_{p-m}\, \chi_e^p}{D_e^p} + \sum_{|p|>N} \frac{J_{p-n}\, J_{p-m}\, \chi_e^p}{D_e^p}\right], \qquad |m| - 1 \leq 2N \leq |m|$$

and taking the limit $|m| \to \infty$. As a consequence, Δ_α is uniformly and absolutely convergent.[7] So is, of course, the infinite product $\prod_{-\infty}^{\infty} D_\alpha^n$.

Therefore, Δ is absolutely and uniformly convergent. Finally, this is also true for Δ_s^e if the set $\{|A_\alpha^n|\}$ is bounded.[7]

Let us assume now that, say, D_e^r vanishes for some ω within the region $|\mathrm{Re}\,\omega| \leq \omega_o/2$. We rewrite Δ in the form

$$\Delta = \Delta_e \prod_{-\infty}^{\infty} D_e^n \prod_{-\infty}^{\infty} D_i^n = \Delta_e^* \prod_{\substack{-\infty \\ n \neq r}}^{\infty} D_e^n \prod_{-\infty}^{\infty} D_i^n$$

where $\Delta_e^* = \det \bar{\Delta}_e^*$ and $\bar{\Delta}_e^*$ and $\bar{\Delta}_e$ differ only in the rth row: the rth row of $\bar{\Delta}_e^*$ is D_e^r times the rth row of $\bar{\Delta}_e$. Then, all the conclusions obtained above remain true if we consider Δ_e^* instead of Δ_e. (We note here that although Δ_i seems to have terms involving the factor $(D_e^r)^{-p}$ with p an arbitrary, positive integer this cannot be the case since $\Delta_e = \Delta_i$ and only the values $p = 0, 1$ appear in Δ_e. This fact about Δ_i is hidden in the complicated structure of the determinant; it can be established from Eq. (12) and the structure of $\bar{\Delta}$ as observed in Eqs. (10a, b). Equation (12) was established by considering the way by which $\bar{\Delta}_i$ was obtained from $\bar{\Delta}$.)

Similar considerations could be made if $D_i^r = 0$.

III. THE DISPERSION RELATION

In this section we discuss in general terms the solution of Eq. (9). We first consider some limiting values of the parameters appearing in $\Delta(\omega)$ for which this function becomes very simple. Then we discuss how to obtain approximate solutions of the dispersion relation by using the fact that the ratio $m_e/m_i = \mu$ is a very small number.

First consider the limit $E = 0$. Since $J_n(0) = 0$ for $n \neq 0$, Δ_α is then diagonal. Our equation becomes

$$\Delta = \prod_{-\infty}^{\infty} D_e^n D_i^n \left(1 - \frac{\chi_e^n \chi_i^n}{D_e^n D_i^n}\right) = \prod_{-\infty}^{\infty} (1 + \chi_e^n + \chi_i^n) = 0 .$$

Δ vanishes if $1 + \chi_e^o + \chi_i^o = 0$; this is the usual dispersion relation as it should be since there is no external field. Δ vanishes also if

$$1 + \chi_e^n + \chi_i^n = 0 \quad , \qquad n \neq 0 . \tag{13}$$

For a given root of $1 + \chi_e^o + \chi_i^o = 0$, (13) gives a corresponding infinite set of roots. However, the imaginary part is the same for the whole set and the real parts differ only by an arbitrary integer multiple of ω_o. This infinite set has no physical relevance here and is due to the introduction of the infinite set of Laplace transforms $f_{\alpha k}^n$ from the single function $f_{\alpha k}$.

Let us take next the limit $|x| \rightarrow \infty$. Since the series for d_α^{nn} and d_α^{nm} are uniformly convergent we can take the limit inside the series. Therefore, d_α^{nn} and d_α^{nm} go to zero and $\Delta_\alpha \rightarrow 1$. Thus we obtain

$$\Delta = \prod_{-\infty}^{\infty} D_e^n \prod_{-\infty}^{\infty} D_i^n = 0 . \tag{14}$$

Electron and ion oscillations become decoupled; if $F_{\alpha o}$ is Maxwellian there are no unstable roots of the dispersion relation in this limit. Since $x = \underset{\sim}{k} \cdot \underset{\sim}{\epsilon}_{ei}$ this result can be made clear by observing that as the electrons go back and forth under the action of the oscillating applied field, they cross so many wavelengths of the perturbing wave that any effect is averaged out. It should be understood, however, that there are always wavelengths so large that, no matter how large E is, the assumption $|x| >> 1$ cannot be considered valid. Therefore, a more meaningful conclusion from (14) is that as the field increases there is a decrease in the maximum value of k, k_{max}, for which instabilities can occur. This result is very similar to one existing in the theory of the two-stream instability with no applied field.[8] This will become more apparent in Sec. IV.

Finally, let $m_i \rightarrow \infty$; then $\omega_{pi}^2 \rightarrow 0$, $D_i^n \rightarrow 1$, and $\Delta_\alpha \rightarrow 1$. Therefore,

$$\Delta = \prod_{-\infty}^{\infty} D_e^n = 0 \; .$$

In this limit the electron stability is not affected by the applied field. This result is made clear by observing that when $m_i \rightarrow \infty$, the ions need not be considered, and in a frame of reference oscillating with the electrons there is no field force.

For arbitrary x and finite m_i , Eq. (9) has to be solved approximately; it is convenient to take advantage of the small value of μ . Let us write Δ_α in a form often used for infinite determinants:

$$\Delta_\alpha = 1 + \sum_n d_\alpha^{nn} + \sum_{n<m} \begin{vmatrix} d_\alpha^{nn} & d_\alpha^{nm} \\ d_\alpha^{mn} & d_\alpha^{mm} \end{vmatrix} \tag{15}$$

plus terms involving products of three or more of the $(d_\alpha^{nn}, d_\alpha^{nm})$ elements. Our purpose is to single out those terms of this expansion which are not small compared with one.

In order to do this we make the following assumption: $\omega_o^2 >> \omega_{pi}^2$; this is, of course, the case when $\omega_o^2/\omega_{pe}^2 \geq O(1)$ since $\omega_{pi}^2/\omega_{pe}^2 = Z\mu << 1$; $(q_i \equiv Ze)$. It seems also the case of most interest for an electromagnetic wave field.

Next we consider those wavelengths such that $k^2/k_i^2 \leq O(1)$; ($k_\alpha^2 \equiv \omega_{p\alpha}^2/v_\alpha^2 = 4\pi q_\alpha^2 N_\alpha/\kappa T_\alpha$, $v_\alpha^2 = \kappa T_\alpha/m_\alpha$). This seems to be a rather weak restriction since for $k^2 >> k_i^2$ collective oscillations are hardly expected, much less instabilities.

Under these two assumptions it is possible to conclude that $|\chi_i^n| = O(\omega_{pi}^2/\omega_o^2) < O(1)$ for $n \neq 0$. In effect, we have then $\omega_o^2 >> k^2 v_i^2$ and this conclusion about $\chi_i^n (n \neq 0)$ can be easily verified by an integration by parts of the integral defining χ_α^n in Sec. II. (We should point out that for $\mathrm{Im}\,\omega > 0$, i.e., for stable roots, we need to assume also $\mathrm{Im}\,\omega << \omega_o$; this is because the analytical continuation of χ_i^n for $\mathrm{Im}\,\omega \geq 0$ involves a term that may not be small when $\mathrm{Im}\,\omega$ is large; see Eq. (16) below, where the exponential term is not small if $\mathrm{Im}\,\omega$ is comparable to ω_o. This condition does not affect the unstable roots and, on the other hand, is satisfied by the stable root that we find.)

For mathematical definiteness we assume now that $F_{\alpha o}(\underset{\sim}{u}_\alpha)$ is a Maxwellian distribution; then we can write explicitly the known asymptotic expansion[9]:

$$\chi_i^n \simeq \frac{-\omega_{pi}^2}{(n\omega_o + \omega)^2 - 3k^2 v_i^2}\left[1 + O\left(\frac{k^4 v_i^4}{n^4 \omega_o^4}\right)\right] + \sigma i\,(\pi/2)^{1/2}\,\frac{k_i^2}{k^2}\,\frac{n\omega_o + \omega}{k v_i}\,\exp\left[\frac{-(n\omega_o + \omega)^2}{2k^2 v_i^2}\right], \qquad n \neq 0 \tag{16}$$

where $\sigma = 1$ if $\mathrm{Im}\,\omega = 0$ and $\sigma = 0, 2$ for $\mathrm{Im}\,\omega$ positive or negative, respectively. We can see explicitly in (16) that the conditions stated above imply $|\chi_i^n| = O(\omega_{pi}^2/\omega_o^2) < O(1)$ for $n \neq 0$.

The conclusion just found for $|\chi_i^n|$, $(n \neq 0)$ allows us to establish a very general and important result. This is that there are no roots of the dispersion relation satisfying simultaneously the conditions $|\chi_i^o(\omega)| < O(1)$ and $|D_\alpha^n(\omega)| \geq O(1)$ (for all n). Therefore any ω such that $\Delta(\omega) = 0$, will satisfy at least one of these two conditions: $|D_e^n(\omega)| << 1$ for some n ; $|\chi_i^o(\omega)| \geq O(1)$, i.e., $|\omega/\omega_{pi}| \leq O(1)$.

It is very simple to prove the result just given, once we have established that $|\chi_i^n| < O(1)$. In effect, if $|\chi_i^o| < O(1)$, $D_i^n \simeq 1$ for all n . Moreover, from Eqs. (11a, b) we can notice that all d_α^{nn} , d_α^{nm} involve some χ_i^n factor; therefore, $\Delta_\alpha \simeq 1$. Hence $\Delta \simeq \prod_{-\infty}^{\infty} D_e^n$ and this cannot vanish if $|D_e^n| \geq O(1)$, for all n .

When D_e^n is very small (for some n), or χ_i^o is not very small, some elements in the matrices Δ_e, Δ_i can be O(1), as can be concluded from a simple observation of (11a, b). Since neither D_e^n nor D_i^o can vanish exactly because some terms in (11a, b) would go to infinity, $\Delta = 0$ is equivalent to $\Delta_\alpha = 0$, $\alpha = e, i$. If by using (15) we can enumerate in a simple way the terms of this expansion which are not small compared to one, we shall have found a closed equation for the roots of $\Delta(\omega) = 0$.

We shall make a detailed study of the dispersion relation in the next two sections. In the next section we shall assume that ω is such that conditions $|\chi_i^o| \geq O(1)$, and $|D_e^n| < O(1)$ (say for n = r) are not satisfied simultaneously. For the second condition we shall find that the oscillations have large ω and are stable; the first condition corresponds in some cases to unstable waves. There are purely imaginary roots, but $|\omega|$ is always small (compared with ω_{pe}). This instability is of the "bunching" or two-stream type;[8] we shall call it non-parametric or non-resonant, to distinquish it here from the one we shall indicate now.

Those values of ω such that $|\chi_i^o| \geq O(1)$ and $|D_e^r| < O(1)$ simultaneously, are discussed in Sec. V. The corresponding instabilities will be found to be much stronger; they shall be called parametric or resonant, because they are similar to the parametric instabilities found in many other physical problems. For $k \ll k_e$, they correspond to instabilities studied in Refs. 1-5 for several limiting conditions. We also point out, finally, that strictly speaking $|\chi_i^o| \geq O(1)$ is an overstatement in the condition for the resonant instability: it will be seen in Sec. V that, although for most cases $\omega^2/\omega_{pi}^2 \leq O(1)$ at resonance, sometimes ω can be as large as $\omega_{pe}^{2/3}\,\omega_{pi}^{1/3}$; then $|\chi_i^o| < O(1)$. It would be more exact to write $|\chi_i^o| \gg |\chi_i^n|$ $(n \neq 0)$ or that an enhancement in χ_i^o [and $(D_e^r)^{-1}$] occurs. Off-resonance, the strict condition $|\chi_i^o| \geq O(1)$ can be used.

IV. THE NON-PARAMETRIC INSTABILITY

Let us consider first the case $|\chi_i^o| < O(1)$, $D_e^r \simeq 0$. We now observe the following crucial point: All the elements of $\bar{\Delta}_i$ contain the factor $(D_e^r)^{-1}$; hence, expansion (15) for Δ_i would appear to be quite complicated. On the other hand, only the <u>r</u>th row of $\bar{\Delta}_e$ contains the factor $(D_e^r)^{-1}$; moreover, as seen in (15), the non-diagonal elements of this row appear in Δ_e multiplying always some elements off the <u>r</u>th row, elements which are small.

Therefore, to order unity, $\Delta_e = 0$ adopts an extremely simple form:

$$\Delta_e \simeq 1 + d_e^{rr} = 1 - \frac{\chi_e^r}{D_e^r} \sum_p J_{r-p}^2 \frac{\chi_i^p}{D_i^p} = 0 \,. \tag{17}$$

Consider now the case $\chi_i^o \geq O(1)$, $|D_e^n| \geq 1$ for all n. We observe that χ_i^o appears in all rows of $\bar{\Delta}_e$ but only in the zeroth row of $\bar{\Delta}_i$. Following an argument similar to the one just used to obtain Eq. (17), there results a very simple expression for Δ_i ; the dispersion relation becomes

$$\Delta_i \simeq 1 + d_i^{oo} = 1 - \frac{\chi_i^o}{D_i^o} \sum_p J_p^2 \frac{\chi_e^p}{D_e^p} = o \,. \tag{18}$$

When $|\chi_i^o| \geq O(1)$ and $D_e^r \simeq 0$, as in the next section, the expansion for either Δ_e or Δ_i is much more complicated.

Now we study Eq. (17). (We remember here that since we took $|\mathrm{Re}\,\omega| \leq \omega_o/2$, r is not really arbitrary but can be determined after Eq. (17) has been solved; for $D_e^n \simeq 0$, $n \neq r$, we shall have the same roots as for $n = r$ except for an arbitrary multiple of ω_o.) Equation (17) can be written as

$$1 + \chi_e^r - \chi_e^r \sum_p J_{r-p}^2 \frac{\chi_i^p}{D_i^p} = 0 \,.$$

We can substitute (-1) for χ_e^r in the last term since this term is small. *Using the dominant term of expansion (16) (for $n = 0$, too), we obtain*

$$1 + \chi_e^r \simeq \sum_p \frac{J_{r-p}^2 \, \omega_{pi}^2}{(p\,\omega_o + \omega)^2} \,. \tag{19}$$

For $k << k_e$ we obtain immediately

$$(R_e\omega + r\omega_o)^2 = \omega_{pe}^2 \left\{ 1 + \frac{3k^2}{k_e^2} + \sum_p \frac{J_{r-p}^2 \; \omega_{pi}^2}{[(p-r)\omega_o + \omega_{pe}]^2} \right\} \quad ,$$

$$\mathrm{Im}\,\omega = \gamma_L(\mathrm{Re}\,\omega + r\omega_o) < 0 \ ,$$

where the function γ_L is the known expression for the Landau damping; m is to be chosen so that $|\omega_{pe} - r\omega_o| \leq \omega_o/2$. The requirement $|\chi_i^o| < O(1)$ is violated when $r\,\omega_o \approx \omega_{pe}$ as seen in the equations just given; therefore, $r\,\omega_o \approx \omega_{pe}$ is the resonance condition. (This is true, however, only for $k << k_e$, i.e., when $|\mathrm{Re}\,\omega + r\omega_o| \simeq \omega_{pe}$ and then for $r = 1$ we would have the parametric resonance that was studied in Refs. 1-5 and for $r \neq 1$ we would have the parametric resonances indicated by Silin.[4] For instance, for some k around k_e, $\mathrm{Re}\,\omega + r\omega_o \approx 2\omega_{pe}$; for this k, resonances are to be expected for $r\omega_o \approx 2\omega_{pe}$, but instabilities, if any, are expected to be weak.) The resonances or parametric effects will be studied, as already indicated, in the next section.

We have concluded that conditions $|\chi_i^o| << 1$, $D_e^r \simeq 0$ give rise to stable roots. Consider now Eq. (18); consider k such that $\omega_o >> kv_e$. Then the expansion (16) can be used also for χ_e^n, $n \neq 0$; we obtain

$$1 + \chi_i^o \left[1 - \sum_{p \neq 0} \frac{J_p^2 \, \omega_{pe}^2}{\omega_{pe}^2 + 3k^2 v_e^2 - p^2 \omega_o^2} - J_o^2 \, \frac{\chi_e^o}{D_e^o} \right] = 0 \ . \qquad (20)$$

Since for $\chi_i^o \geq O(1)$ there results that $\omega^2/\omega_{pi}^2 \leq O(1)$, we have dropped terms of order ω^2/ω_o^2. This equation can be rewritten as:

$$1 + \chi_i^o \delta + \chi_e^o + (\delta - J_o^2)\, \chi_e^o \chi_i^o = 0 \qquad (21)$$

where

$$\delta = 1 - \sum_{p \neq 0} \frac{J_p^2\, \omega_{pe}^2}{\omega_{pe}^2 + 3k^2 v_e^2 - p^2 \omega_o^2} .$$

When $\delta = J_o^2(x)$ or $J_o^2(x) = 0$, Eq. (21) has no unstable roots since it adopts the forms $[1 + \chi_e^o + \chi_i^o \delta = 0]$ and $[(1 + \chi_e^o)(1 + \delta\chi_i^o) = 0]$. Using an argument similar to one given in Ref. 8 it is also possible to show, that there are no purely imaginary, unstable roots, when δ and $\delta - J_o^2$ are both positive. As will be seen below, this is not the case when $\delta - J_o^2 < 0$. Although we could not prove that there are not unstable roots in general when δ and $\delta - J_o^2$ are positive, we suspect that this is the case. (For instance, it will be shown below that for weak fields, when only terms in E^2 are retained, there can be instabilities for $\delta - J_o^2 < 0$ but none otherwise.) We conclude tentatively, then, that the system can be unstable off resonance, only when $\omega_o < \omega_{pe}$. In effect, since $J_o^2 \leq 1$, the system would not be unstable if $\delta > 1$. From the definition of δ, and since $\omega_o^2 >> k^2 v_e^2$, $\delta > 1$ if $\omega_o > \omega_{pe}$ and our conclusion follows. (We assumed throughout that $k^2 << \omega_o^2/v_e^2$. The plasma is expected to be more stable the smaller the wavelength but care has to be taken if $k < k_i$ but $k >> k_e = \omega_{pe}/v_e$, possible in a non-isothermal plasma; we exclude such case here.)

Condition $\omega_o < \omega_{pe}$ seems similar to one found in the theory of the two-stream instability.[8] If u is the relative streaming velocity of the two species, it is found there that only if $ku < \omega_{pe}$ can the plasma be unstable.

We referred to this condition when Eq. (14) was discussed; there it was understood as imposing an upper limit to k . Since ku has dimensions of frequency, it can also be thought of an imposing a frequency limit; if this limit is exceeded, the electrons move too fast for any "bunching" to occur. (The "bunching" of particles or density modulation in space is the physical process by which waves are excited when two groups of particles flow relatively to each other; see Sec. VI or Ref. 8 for an explanation of the "bunching" process.)

We point out, finally, that for roots such that $|\omega/kv_e| < O(1)$, condition $\delta < 1$ (or $\omega_o < \omega_{pe}$) is easily seen to be necessary for growing waves to exist. In effect, then, $\chi_e^o \simeq k_e^2/k^2$ and Eq. (21) can be written as

$$1 + \frac{k^2 \delta + k_e^2 (\delta - J_o^2)}{k_e^2 + k^2} \chi_i^o = 0$$

and this is the dispersion relation for a field-free, one-species, stable plasma, if $\delta > 1$.

Now, the calculation of the roots of (20) can be greatly simplified by using the identity (see Eq. (A-1) in the Appendix):

$$\sum_p \frac{J_p^2(x)\, \nu^2}{\nu^2 - p^2} = \frac{\pi \nu}{\sin \pi \nu} J_\nu(x)\, J_{-\nu}(x) \quad .$$

Equation (20) becomes [ν being here $(\omega_{pe}^2 + 3k^2 v_e^2)^{1/2}/\omega_o$]:

$$1 + \chi_i^o \left[1 - \frac{\omega_{pe}^2}{\omega_{pe}^2 + 3k^2 v_e^2} \frac{\pi \nu}{\sin \pi \nu} J_\nu J_{-\nu} - J_o^2 \left(\frac{\chi_e^o}{D_e^o} - \frac{\omega_{pe}^2}{\omega_{pe}^2 + 3k^2 v_e^2} \right) \right] = 0. \tag{22}$$

Since $k^2 << k_e^2$ follows from $\omega_o < \omega_{pe}$, $\omega_o >> kv_e$, the factor multiplying J_o^2 is quite small when $|\omega/kv_e| \leq O(1)$ and the second term inside the bracket can be further simplified to $(\pi\nu/\sin\pi\nu)\, J_\nu J_{-\nu}$, $\nu = \omega_{pe}/\omega_o$. Finally, we obtain

$$1 + \chi_i^o \left[1 - \frac{\pi\nu}{\sin\pi\nu} J_\nu J_{-\nu}\right] = 0 , \quad \nu = \frac{\omega_{pe}}{\omega_o} . \qquad (23)$$

If the cold plasma limit is taken, $\chi_i^o = -\omega_{pi}^2/\omega^2$ and (23) becomes identical to Eq. 3.36 of Ref. 4. When v_i is finite and we look for purely imaginary roots of (23), this equation becomes

$$1 + k_i^2/k^2 \left[1 - \pi^{1/2} \frac{\gamma}{kv_i} e^{\gamma^2/k^2v_i^2}\left(1 - \operatorname{erf}\frac{\gamma}{kv_i}\right)\right]$$

$$\times \left[1 - \frac{\pi\nu}{\sin\pi\nu} J_\nu J_{-\nu}\right] = 0 , \qquad \omega = i\gamma .$$

The function inside the first bracket[9] has a very simple behavior, decaying smoothly from 1 to 0 as γ grows from zero to infinity.

Next, we proceed to determine whether there is a threshold which the field has to exceed for the onset of instabilities. We take the limit $|x| << 1$ in Eq. (20):

$$1 + \chi_i^o \left[1 - \frac{x^2}{2} \frac{\omega_{pe}^2}{\omega_{pe}^2 + 3k_e^2v_e^2 - \omega_o^2} - \left(1 - \frac{x^2}{2}\right)\frac{\chi_e^o}{D_e^o}\right] = 0.$$

We look for purely imaginary roots and assume that the field just exceeds the threshold, so that $|\omega/kv_\alpha| << 1$. Then we have ($\omega = i\gamma$)

$$1 + \frac{k_i^2}{k^2}\left[1 - \pi^{1/2}\frac{\gamma}{kv_i}\right]\left[\frac{k^2}{k_e^2 + k^2} - \frac{x^2}{2}\left(\frac{k_e^2}{k_e^2 + 3k^2 - k_e^2\,\omega_o^2/\omega_{pe}^2} - \frac{k_e^2}{k^2 + k_e^2}\right)\right] = 0 .$$

The conditions for instability are then found to be:

$$v_E \equiv \omega_o |\underset{\sim}{\epsilon}_{ei}| = \left|\frac{\underset{\sim}{E}}{\omega_o}(q_e/m_e - q_i/m_i)\right| > v_e\, 2^{1/2}(1 + T_i/T_e)^{1/2}\left(1 - \frac{\omega_o^2}{\omega_{pe}^2}\right)^{1/2} , \quad (24a)$$

$$\omega_o < \omega_{pe} \quad ; \quad (24b)$$

(24b) agrees with our former result on the frequency limitation; (24a) establishes a threshold that the field intensity has to exceed for the onset of instabilities. We point out that for the cold plasma limit the threshold vanishes; if the usual macroscopic, warm plasma equations had been used, so that we would have taken $\chi_i^o \approx -\omega_{pi}^2/(\omega^2 - 3k^2 v_i^2)$ instead of $\chi_i^o \simeq k_i^2/k^2\,(1 + i\,\pi^{1/2}\omega/kv_i)$, the factor $(1 + T_i/T_e)^{1/2}$ in (24a) would be changed into $(1 + 3T_i/T_e)^{1/2}$.[10]

Finally, we consider a given k and take the limit of very large fields; we indicated in Sec. III that the plasma would become ultimately stable for the given wavelength . We take the limit $|x| \gg 1$ and again expand χ_i^o for small $\omega = i\gamma$; using the asymptotic expansion for $J_\nu, J_{-\nu}$, (22) becomes, after some rearrangement:

$$1 + \frac{k_i^2}{k^2}\left[1 - \pi^{1/2}\frac{\gamma}{kv_i}\right]\left[1 - \frac{k_e^2\,\nu\,(\cos\pi\nu - \sin 2x)}{(k_e^2 + 3k^2)\,|x|\,\sin\pi\nu}\right] = 0 \quad ;$$

we have dropped terms of order of k^2/k_e^2. The plasma is unstable if

$$k < \frac{\omega_{pe}}{v_E}\,\frac{k_i^2}{k_i^2 + k^2}\,\frac{\sin 2x + \cos\pi\nu}{\sin\pi\nu} \equiv k_{max} , \qquad \omega_o \ll \omega_{pe} ; \quad (25)$$

(we remember that the case $\nu \simeq n$ is excluded from this analysis). As a final conclusion we point out that (24a) and (25) are very similar to the results of the two-stream instability; it is self-evident that the streaming v_E, generated by the field causes the present instability.

V. THE PARAMETRIC INSTABILITY

In this section we consider the case $|D_e^n| < O(1)$ (say, for n = r) and $|\chi_i^o| \geq O(1)$; (see last paragraph of Sec. III). The enhancement of χ_i^o implies a small $|\omega|$; therefore $D_e^r(r\omega_o + \omega) \simeq D_e^r(r\omega_o)$. If $k \ll k_e$, we see that the resonant condition $r\omega_o \simeq \omega_{pe}$ is obtained, since for $k \ll k_e$, $D_e(\omega_{pe}) \simeq 0$. Moreover, D_e is then nearly even in its argument so that $D_e^{-r} \simeq 0$ too. Thus, χ_i^o, $(D_e^r)^{-1}$, and $(D_e^{-r})^{-1}$ contribute dominant terms in (15); the results (17) and (18) for Δ_e and Δ_i cannot be used now. It has been found that the subsequent analysis is simpler when made for $\bar{\Delta}_i$; hence consider expansion (15) for Δ_i:

$$\Delta_i = 1 - \sum_n \frac{\chi_i^n}{D_i^n} \sum_p \frac{J_{p-n}^2 \chi_e^p}{D_e^p} + \sum_{n<m} \begin{vmatrix} \dfrac{-\chi_i^n}{D_i^n} \sum\limits_p \dfrac{J_{p-n}^2 \chi_e^p}{D_e^p} & \dfrac{-\chi_i^n}{D_i^n} \sum\limits_p \dfrac{J_{p-n} J_{p-m} \chi_e^p}{D_e^p} \\ \dfrac{-\chi_i^m}{D_i^m} \sum\limits_q \dfrac{J_{q-m} J_{q-n} \chi_e^q}{D_e^q} & \dfrac{-\chi_i^m}{D_i^m} \sum\limits_q \dfrac{J_{q-m}^2 \chi_e^q}{D_e^q} \end{vmatrix} \tag{26}$$

plus terms involving products of three or more d_i^{nn}, d_i^{nm}. To close this expansion, the ansatz $|\chi_i^n / D_e^{\pm r}| < O(1)$ $(n \neq 0)$ is made. We also assume that $|x| \geq O(1)$; to include the case $|x| \ll 1$ would complicate the result too much. Moreover, it has not been determined whether our ansatz and condition $|x| \ll 1$ are consistent with each other. It is easier to treat separately the case $|x| \ll 1$ since then it is possible to close the expansion (26) without using the ansatz $|\chi_i^n / D_e^{\pm r}| < O(1)$; however, this limit has been correctly treated already[1,2] and the threshold for instability determined. (It would be of interest, nevertheless, to obtain a closed expansion for Δ_i valid for both $|x| < O(1)$ and $|x| \geq O(1)$, as we did for the non-resonant case.) We also point out that, if for $x \geq O(1)$ there are roots for which the ansatz is invalid, these roots can be lost when the ansatz is used to close the expansion (26). In Ref. 5, Jackson indicated that, besides the known parametric instability of Refs. 1 and 2, there is another with very small $|\omega|$; it would seem that it is lost here through our ansatz. Jackson's results are doubtful, however, since his treatment is based on an improper approximation as shown later in this section.

With condition $\chi_i^n / D_e^{\pm r} < O(1)$ (for all $n \neq 0$), all elements d_i^{nn}, d_i^{nm}, $n \neq 0$, are very small. At first sight it would seem, therefore, that Δ_i reduces again to (18):

$$\Delta_i \simeq 1 - \frac{\chi_i^o}{D_i^o} \sum_p \frac{J_p^2 \chi_e^2}{D_e^p} = 1 - \frac{\chi_i^o}{D_i^o} \left[\sum_{p \neq \pm r} \frac{J_p^2 \chi_e^p}{D_e^p} + J_r^2 \frac{\chi_e^r}{D_e^r} + J_{-r}^2 \frac{\chi_e^{-r}}{D_e^{-r}} \right] .$$

The last two terms in the bracket are much larger than the first one; however, they may cancel each other because $J_r^2 = J_{-r}^2$ and possibly $|\chi_e^r/D_e^r + \chi_e^{-r}/D_e^{-r}| \ll |\chi_e^{\pm r}/D_e^{\pm r}|$. In principle, therefore, the whole bracket should be retained. The elements d_i^{om}, $m \neq 0$ contain these very large terms

$$-\frac{\chi_i^o}{D_i^o}\left[\frac{J_r J_{r-m}\chi_e^r}{D_e^r} + \frac{J_{-r} J_{-r-m}\chi_e^{-r}}{D_e^{-r}}\right] ,$$

but here there is no near cancellation as for d_i^{oo}. Although these terms are multiplied by small numbers, (the elements of $\bar{\Delta}_i$ off the zeroth row), the largest such products should be retained in principle. Since we are neglecting terms of (26) which are known to be small compared with some of those retained, the dominant terms involving d_i^{om} are contained in the third term of (26). We reach the conclusion that with our ansatz for $\chi_i^n/D_e^{\pm r}$ $(n \neq 0)$, Δ_i is given, to dominant order, as follows:

$$\Delta_i \simeq 1 - \frac{\chi_i^o}{D_i^o}\left[\sum_{p\neq\pm r} \frac{J_p^2\chi_e^p}{D_e^p} + J_r^2\left(\frac{\chi_e^r}{D_e^r} + \frac{\chi_e^{-r}}{D_e^{-r}}\right)\right]$$

$$+ \frac{\chi_i^o}{D_i^o}\sum_{m\neq 0} \frac{\chi_i^m}{D_i^m} J_r^2 \frac{\chi_e^r}{D_e^r}\frac{\chi_e^{-r}}{D_e^{-r}} (J_{m+r} + J_{m-r})^2 . \qquad (27)$$

Since $\omega_o^2 \gg k^2 v_e^2$, we can use (16) for both electrons and ions, for $n \neq 0$. Our dispersion relation becomes

$$D_i^o \Delta_i \simeq 1 + \chi_i^o \left[1 - \frac{J_o^2 \chi_e^o}{D_e^o} - \sum_{\substack{p \neq \pm r \\ p \neq 0}} \frac{J_p^2 \omega_{pe}^2}{\omega_{pe}^2 + 3k^2 v_e^2 - p^2 \omega_o^2} \right.$$

$$- J_r^2 \frac{2\omega_{pe}^2 (\omega_{pe}^2 + 3k^2 v_e^2 - r^2 \omega_o^2 - \omega^2)}{(\omega_{pe}^2 + 3k^2 v_e^2 - r^2 \omega_o^2 - \omega^2)^2 - 4r^2 \omega_o^2 \omega^2}$$

$$\left. - J_r^2 \frac{\omega_{pe}^4 \omega_{pi}^2 / \omega_o^2}{(\omega_{pe}^2 + 3k^2 v_e^2 - r^2 \omega_o^2 - \omega^2)^2 - 4r^2 \omega_o^2 \omega^2} \sum_{m \neq 0} \frac{(J_{m+r} + J_{m-r})^2}{m^2} \right] = 0. \tag{28}$$

We shall call $\lambda = \omega_{pe}^2 + 3k^2 v_e^2 - r^2 \omega_o^2$. Since $\omega_o^2 >> k^2 v_e^2$, $D_e^{\pm r}$ will be very small when $\omega_{pe}^2 \approx r^2 \omega_o^2$. Therefore, $\omega_{pe}^2 >> 3k^2 v_e^2$, i.e., $k_e^2 >> k^2$; we can drop the term $3k^2 v_e^2$ in the denominator of the first series inside the bracket of Eq. (28). Also, since $\lambda << \omega_o^2$, the denominator of the last two terms inside the bracket can be approximated as

$$(\lambda - \omega^2)^2 - 4r^2 \omega_o^2 \omega^2 \simeq \lambda^2 - 4r^2 \omega_o^2 \omega^2 \quad .$$

Then Eq. (28) is rewritten in the form

$$1 + \chi_i^o \left[1 - J_o^2 \frac{\chi_e^o}{D_e^o} - \sum_{\substack{p \neq \pm r \\ p \neq 0}} \frac{J_p^2 \omega_{pe}^2 / \omega_o^2}{\omega_{pe}^2 / \omega_o^2 - p^2} - J_r^2 \frac{2\omega_{pe}^2 (\lambda - \omega^2)}{\lambda^2 - 4r^2 \omega_o^2 \omega^2} \right.$$

$$\left. - J_r^2 \frac{\omega_{pe}^4 \omega_{pi}^2}{\omega_o^2 (\lambda^2 - 4r^2 \omega_o^2 \omega^2)} \sum_{m \neq 0} \left(\frac{J_{m+r} + J_{m-r}}{m} \right)^2 \right] = 0 \quad . \tag{29}$$

We note here that the ansatz $\chi_i^n / D_e^{\pm r} < O(1)$ is equivalent to the requirement $\omega_{pi}^2 << \lambda \pm 2r \omega_o \omega$.

The first series inside the bracket can be summed for arbitrary r (see Appendix). However, from now on we shall consider the case $r = 1$. It corresponds to the case studied in Refs. 1 and 2 for $|x| << 1$. The case $r \neq 1$ was treated by Silin[4] within the cold plasma approximation. All the subsequent calculations can be extended simply to include an arbitrary r ; however, the instabilities that would be found, would undoubtedly be much weaker. (For instance, to obtain the threshold for the instabilities to set in, the weak field limit has to be taken and, since the terms involving the resonant factors appear multiplied by J_r^2 in Eq. (29) and for $|x| << 1$ $J_r^2 = O(x^{2r})$, the threshold would appear to increase rapidly as r increases.)

For $r = 1$, Eq. (29) can be rewritten as

$$1 + \chi_i^o \left[1 - J_o^2 \left(\frac{\chi_e^o}{D_e^o} - 1 \right) - \left(\frac{2J_o J_1}{x} - J_1^2/2 \right) - J_1^2 \frac{2\omega_{pe}^2 (\lambda - \omega^2)}{\lambda^2 - 4\omega_o^2 \omega^2} \right.$$

$$\left. - J_1^2 \frac{\omega_{pe}^4 \omega_{pi}^2}{\omega_o^2 (\lambda^2 - 4\omega_o^2 \omega^2)} \frac{4}{x^2} (1 - J_o^2) \right] = 0 . \qquad (30)$$

We have used the equality

$$\sum_{p \neq \pm 1} \frac{J_p^2 \nu^2}{\nu^2 - p^2} = \frac{2J_o J_1}{x} - \frac{J_1^2}{2} + O(1 - \nu)$$

[see Appendix, Eq. (A-2)] and the fact that $\nu \equiv \omega_{pe}/\omega_o \approx 1$; the recurrence relation $J_{m+1} + J_{m-1} = 2m J_m/x$, and the identity $\sum_p J_p^2 = 1$ have been also used.

The factor $(\chi_e^o/D_e^o - 1) = -1/D_e^o$ is always very small; it is of order of ω^2/ω_{pe}^2 and k^2/k_e^2 for $\omega >> kv_e$ and $\omega << kv_e$, respectively; it will be dropped. Moreover, since we are only retaining the dominant terms, we can substitute ω_{pe}^2 for ω_o^2 (except, of course, when its difference is divided by another small number). Multiplying (30) by $(\lambda^2 - 4\omega_o^2\omega^2)/4\omega_{pe}^4$ and rearranging terms, we obtain, finally,

$$\left(\frac{\lambda}{2\omega_{pe}^2}\right)^2 - \frac{\omega^2}{\omega_{pe}^2} = \chi_i^o\left[\frac{\omega^2}{\omega_{pe}^2}\left(1 - \frac{2J_oJ_1}{x}\right) + J_1^2\left(\frac{\lambda}{2\omega_{pe}^2} + \frac{\omega_{pi}^2}{\omega_{pe}^2}\,\frac{1 - J_o^2}{x^2}\right)\right].$$

By using tables of $\chi_i^o(\omega/kv_i)$,[9] this equation can be solved for arbitrary ω/kv_i. Let us consider here the limit $\omega >> kv_i$; we can write $\chi_i^o \simeq -\omega_{pi}^2/(\omega^2 - 3k^2v_i^2)$ and our equation becomes:

$$\frac{\omega^4}{\omega_{pe}^4} - \frac{\omega^2}{\omega_{pe}^2}\left[\left(\frac{\lambda}{2\omega_{pe}^2}\right)^2 + \frac{\omega_{pi}^2}{\omega_{pe}^2}\left(1 - \frac{2J_oJ_1}{x}\right) + \frac{3k^2\omega_{pi}^2}{k_i^2\omega_{pe}^2}\right] - \frac{\omega_{pi}^2}{\omega_{pe}^2}\,J_1^2\left(\frac{\lambda}{2\omega_{pe}^2} + \frac{\omega_{pi}^2}{\omega_{pe}^2}\,\frac{1 - J_o^2}{x^2}\right) = 0 \quad , \tag{31}$$

or

$$2\,\frac{\omega^2}{\omega_{pe}^2} = \left(\frac{\lambda}{2\omega_{pe}^2}\right)^2 + \frac{\omega_{pi}^2}{\omega_{pe}^2}\left(1 - \frac{2J_oJ_1}{x}\right) + \frac{3k^2\omega_{pi}^2}{k_i^2\omega_{pe}^2} \pm\left\{\left[\left(\frac{\lambda}{2\omega_{pe}}\right)^2 + \frac{\omega_{pi}^2}{\omega_{pe}^2}\left(1 - \frac{2J_oJ_1}{x}\right) + \frac{3k^2\omega_{pi}^2}{k_i^2\omega_{pe}^2}\right]^2 + \frac{4\omega_{pi}^2J_1^2}{\omega_{pe}^2} \times\left(\frac{\lambda}{2\omega_{pe}^2} + \frac{\omega_{pi}^2}{\omega_{pe}^2}\,\frac{1 - J_o^2}{x^2}\right)\right\}^{1/2} \quad . \tag{32}$$

We can compare this result with those of Refs. 4 and 5, for the same limit $\omega >> kv_i$. If Eq. (28) of Ref. 5 is rewritten with our symbols and dominant terms are retained, it is found that it is identical to our Eq. (31) except for

Jackson's use of $(J_o^2 + J_1^2)/2$ for both $J_o J_1/x$ and $(1 - J_o^2)/x^2$ in our equation. The source of Jackson's error, we think, can be traced to the way he simplified Eqs. (10a,b) [Eq. (20) of his paper]. He neglected χ_α^n for both $\alpha = e$ and $\alpha = i$, when $|n| > 1$; hence, he takes as small $\chi_e^{\pm 2}$, $\chi_e^{\pm 3}$, ... , which are of the order of $\omega_{pe}^2/2^2\omega_o^2 \simeq 1/4$, $\omega_{pe}^2/3^2\omega_o^2 \simeq 1/9$, ... This amounts to neglecting all terms with $|n| > 1$ in a series $\sum_n c_n$, where $c_n \sim 1/n^2$.

If we take the limit $v_e = v_i = 0$ (cold plasma), we can compare our results with Eq. (3.30) of Ref. 4. Silin gives his results in terms of two Bessel series; they are the same series we summed for Eq. (30). After the summation is made, his equation is identical to ours. (Take $n = 1$ in Silin's equation.)

Now, from Eq. (32) the following conditions are found to be required if the plasma is to be stable:

$$(\lambda/2\omega_{pe}^2)^2 + \frac{\omega_{pi}^2}{\omega_{pe}^2}\left(1 - \frac{2J_o J_1}{x} + \frac{3k^2}{k_i^2}\right) \geq 0 \quad ,$$

$$\left[(\lambda/2\omega_{pe}^2)^2 + \frac{\omega_{pi}^2}{\omega_{pe}^2}\left(1 - \frac{2J_o J_1}{x} + \frac{3k^2}{k_i^2}\right)\right]^2 \geq - \frac{4\omega_{pi}^2 J_1^2}{\omega_{pe}^2}\left(\frac{\lambda}{2\omega_{pe}^2} + \frac{\omega_{pi}^2}{\omega_{pe}^2}\frac{1 - J_o^2}{x^2}\right) \geq 0.$$

The first condition is always satisfied since $1 - 2J_o J_1/x \geq 0$. The second can be rewritten as:

$$\lambda \equiv \omega_{pe}^2 + 3k^2 v_e^2 - \omega_o^2 \leq - 2\omega_{pi}^2 \frac{1 - J_o^2}{x^2} \quad , \tag{33a}$$

$$\lambda \geq - \frac{\omega_{pe}^4}{2\omega_{pi}^2 J_1^2}\left[\left(\frac{\lambda}{2\omega_{pe}^2}\right)^2 + \frac{\omega_{pi}^2}{\omega_{pe}^2}\left(1 - \frac{2J_o J_1}{x}\right)\right]^2 - 2\omega_{pi}^2 \frac{1 - J_o^2}{x^2} \quad . \tag{33b}$$

These inequalities and Eq. (32) give the growth rate and the range of unstable frequencies for a given x , i.e., E and k . The minus sign should be taken in front of the root in Eq. (32). By assigning an arbitrary order of magnitude to the ratio λ/ω_{pi}^2 it is possible to simplify the equation further; (see Ref. 4). In all cases the results for ω are consistent with the ansatz $|\chi_i^n/D_e^{\pm r}| < O(1)$, $n \neq 0$, if $|x| = O(1)$. The maximum growth rate found is known to be of the order of $\omega_{pe}^{2/3}\ \omega_{pi}^{1/3}$, much larger than the off-resonance growth rate.

We shall now find how k_{max} decreases as E increases, for large values of E ; k_{max} being the maximum value of k for which instabilities can appear. As in Sec. IV, we shall take $|x| \gg 1$ and assume that we are so close to k_{max} that Im ω will be vanishingly small; assuming that ω is purely imaginary, we shall neglect ω^p for $p > 1$. Then, from Eq. (30), we obtain:

$$1 + \chi_i^o \left[1 - \frac{2}{\pi |x|} \sin^2(x - \pi/4) \frac{2\omega_{pe}^2}{\lambda} - \frac{2}{\pi |x|} \sin^2(x - \pi/4) \times \frac{4\omega_{pe}^2 \omega_{pi}^2}{\lambda^2 x^2} \right] = 0 \ . \tag{34}$$

Writing $\chi_i^o \simeq k_i^2/k^2 [1 + i(\pi/2)^{1/2} \omega/kv_i]$ and $\omega = i\gamma$ we have

$$1 + k_i^2/k^2 [1 - (\pi/2)^{1/2} \gamma/kv_i] \left[1 - \frac{4 \sin^2(x - \pi/4)}{\pi |x|} \frac{\omega_{pe}^2}{\lambda} \right] = 0 \ .$$

We have neglected the last term inside the bracket of (34) because it has to be small compared with the second, if our ansatz $|\chi_i^n/D_e^{\pm 1}| < O(1)$, i.e., $\omega_{pi}^2 \ll \lambda \pm 2\omega_o\omega$, is to be satisfied.

For γ to be positive, it is necessary that

$$\lambda > 0 \ ,$$

$$k < \frac{k_i^2}{k^2 + k_i^2} \frac{4 \sin^2 (x - \pi/4)}{\pi \,|\, \underset{\sim}{\epsilon}_{ei} \,|} \frac{\omega_{pe}^2}{\lambda} \approx \frac{4\omega_{pe}}{\pi v_E} \frac{\omega_{pe}^2}{\lambda} \quad ;$$

this k_{max} is much larger than the one found for the non-resonant case, because of the factor ω_{pe}^2/λ .

VI. CONCLUSIONS

We have found two types of instabilities. In Sec. IV we concluded from the observation of Eqs. (24a) and (25) that the nonparametric one was simply a two-stream instability. The parametric instability is caused by the parametric interaction of the applied field with natural modes of oscillation of the plasma, as is apparent from our mathematical analysis.

From another point of view, it seems interesting to present a physical picture of this instability that allows us to visualize the instability mechanism. The parametric instability may be considered, we think, as a resonant enhancement of the two-stream instability that is present off-resonance.

Consider the two-stream instability in the usual case of no field. In the equilibrium state, the species stream relative to each other. If a density perturbation appears, a perturbing electrostatic potential wave will result. As the particles move over the crests and troughs of the wave, they gain and lose speed alternately. As seen from a simple consideration of the continuity equation, this velocity modulation can produce a density modulation (in space); thus, the equilibrium streaming makes the particles form "bunches" alternately placed in space. If they are properly placed with respect to the initial potential, the peaks and valleys of the potential will be enhanced and the instability will set in.

In our problem, the streaming velocity, generated by the field, is itself (time) modulated. Unless this modulation is a very definite one, the results will not be essentially different from the usual two-stream results. If, however, this modulation is well timed, we might say "in phase," we should expect a substantial enhancement of the bunching of the particles and therefore a large instability. From the results of Secs. IV and V we may state now that (for $k << k_e$) $\omega_{pe} \approx n\omega_o$ is the requirement for the modulation of the streaming to be "in phase." When such conditions is not satisfied, we can write Eqs. (24a) and (25) as $v_E > v_e$ and $k < \omega_{pe}/v_E$ (except for numerical factors) where v_E is an average of the streaming velocity; these results are similar to those of the usual two-stream instability. However, as $\omega_o \rightarrow \omega_{pe}/n$, we see in (24a) and (25) that the threshold decays substantially and k_{max} becomes very large. The results of Refs. 1,2 and Sec. V would be obtained here if some terms now important had not been neglected.

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APPENDIX

The identity

$$\sum_{p} \frac{J_p^2 \nu^2}{\nu^2 - p^2} = \frac{\pi\nu}{\sin \pi\nu} J_\nu J_{-\nu} \qquad \text{(A-1)}$$

was given without proof by Silin[4]; the proof will be sketched here for completeness. Define $C(\nu)$,

$$C(\nu) = \sum_{p} \frac{J_p^2 \nu^2}{\nu^2 - p^2} - \frac{\pi\nu}{\sin \pi\nu} J_\nu J_{-\nu} \quad ;$$

it can be noticed that $C(\nu)$ has no singularities at $\nu = \pm p$. Moreover, ν, ν^2 $\sin \pi\nu$, J_ν, and $J_{-\nu}$ are entire functions of ν ; therefore, if $C(\nu)$ is bounded as $|\nu| \to \infty$, it will be a constant from Liouville's theorem. Since $C(0) = 0$, we will have $C(\nu) \equiv 0$. (To determine whether $C(\nu)$ is bounded we remember that $\pi\nu/\sin \pi\nu = \Gamma(1+\nu)\,\Gamma(1-\nu)$ so that the second term in (A-1) can be written as

$$\sum_{p,\,q>0} (-x^2/4)^{p+q} \frac{\Gamma(1+\nu)\,\Gamma(1-\nu)}{\Gamma(q+1+\nu)\,\Gamma(p+1-\nu)} \quad ,$$

the argument of the Bessel functions in (A-1) being x.)

Using (A-1), it is now possible to prove that for $\nu \simeq 1$,

$$\sum_{p \neq \pm 1} \frac{J_p^2 \nu^2}{\nu^2 - p^2} = \frac{2 J_o J_1}{x} - \frac{J_1^2}{2} + O(1-\nu). \qquad \text{(A-2)}$$

Let $\nu = r + \varepsilon$, $|\varepsilon| \ll 1$; from (A-1)

$$\sum_{p \neq \pm r} \frac{J_p^2 \nu^2}{\nu^2 - p^2} = \frac{\pi\nu J_\nu J_{-\nu}}{\sin \pi\nu} - \frac{2 J_r^2 \nu^2}{\nu^2 - r^2}$$

$$= \frac{\pi\nu(-1)^r}{\sin\pi\varepsilon}\left[J_r J_{-r} + \varepsilon \frac{\partial}{\partial\nu} J_\nu J_{-\nu}\Big|_{\nu=r} + O(\varepsilon^2)\right] - \frac{2J_r^2\nu^2}{\varepsilon(2\nu-\varepsilon)}$$

$$= -\frac{J_r^2}{2} + \nu r!\, J_r (2/x)^r \sum_{m=0}^{r-1} \frac{J_m (x/2)^m}{m!\,(r-m)!} \quad .$$

For $r = 1$, (A-2) follows. (We have made use of the relations

$$Yr = 1/\pi\left[\frac{\partial J_\nu}{\partial\nu} - (-1)^r \frac{\partial J_{-\nu}}{\partial\nu}\right]_{\nu=r} \quad ,$$

$$\frac{\partial J_\nu}{\partial\nu}\Big|_{\nu=r} = \frac{\pi}{2} Yr + \frac{r!}{2}(x/2)^{-r} \sum_{m=0}^{r-1} \frac{J_m (x/2)^m}{m!\,(r-m)!} \quad .)$$