

An Optimal T-S Model for the Estimation and Identification of Nonlinear Functions

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Abstract— A novel optimal method is developed to improve the identification and estimation of Takagi-Sugeno (T-S) fuzzy model. The idea comes from the fact that the main drawback of T-S model is that it can not be applied when the membership functions are overlapped by pairs. This limits the application of the T-S model because this type of membership function has been widely used in the stability and controller design of fuzzy systems. It is also very popular in industrial control applications. The method presented here can be considered as a generalized version of T-S fuzzy model with optimized performance in approximating nonlinear functions. Various examples are chosen to show the high function approximation accuracy and fast convergence obtained by applying the proposed method in approximating nonlinear systems locally and globally in comparison with the original T-S model.

Index Terms—Nonlinear systems, Fuzzy systems, Takagi-Sugeno fuzzy model, Universal approximators, optimization

1 Introduction

Nonlinear control systems based on the Takagi-Sugeno (T-S) fuzzy model [28], [29] have attracted lots of attention during the last twenty years (e.g., see [1], [2], [6], [9], [10], [11], [14], [16], [18], [19], [20], [21], [24], [25], [31] and [37]). It provides a powerful solution for development of function approximation, systematic techniques to stability analysis and controller design of fuzzy control systems in view of fruitful conventional control theory and techniques.

This model is formed by using a set of fuzzy rules to represent a nonlinear system as a set of local affine models which are connected by fuzzy membership functions [3] and [4].

This fuzzy modelling method presents an alternative technique to represent complex nonlinear systems [7], [30], [35] and [38], and reduces the number of rules in modelling higher order nonlinear systems [9] and [29].

T-S fuzzy models are proved to be universal function approximators as they are able to approximate any smooth nonlinear functions to any degree of accuracy in any convex compact region [7], [12], [19], [30], [35] and [38]. This result provides a theoretical foundation for applying T-S fuzzy models

to represent complex nonlinear systems approximately.

Great attention has been paid to the identification of T-S fuzzy models and several results have been obtained [5], [12], [23], [32] and [36]. They are based upon two kinds of approaches, one is to linearize the original nonlinear system in various operating points when the model of the system is known, and the other is based on the input-output data collected from the original nonlinear system when its model is unknown.

The authors in [5] use a fuzzy clustering method to identify T-S fuzzy models, including identification of the number of fuzzy rules and parameters of fuzzy membership functions, and identification of parameters of local linear models by using a least squares method [27] and [33]. The goal is to minimize the error between T-S fuzzy models and the corresponding original nonlinear systems.

The authors in [12] suggest a method to identify T-S fuzzy models. Their method aims at improving the local and global approximation of T-S model. However, this complicates the approximation in order to obtain both targets. It has been shown that constrained and regularized identification methods may improve interpretability of constituent local models as local linearizations, and locally weighted least squares method may explicitly address the trade-off between the local and global accuracy of T-S

fuzzy models.

In [26] a new method of interval fuzzy model identification was developed. The method combines a fuzzy identification methodology with some ideas from linear programming theory. The idea is then extended to modelling the optimal lower and upper bound functions that define the band which contains all the measurement values. This results in lower and upper fuzzy models or a fuzzy model with a set of lower and upper parameters. This approach can also be used to compress information in the case of large amount of data and in the case of robust system identification. The method can be efficiently used in the case of the approximation of the nonlinear functions family. This present paper is inspired by the idea that by using a proper evaluation of triangular shape membership functions it is possible to emulate the simplicial high-level canonical piecewise linear approximation technique [13] by following a fuzzy logic approach. The paper focuses on the development of an interval L_∞ -norm -norm function approximation methodology problem using the LP technique and the TS fuzzy logic approach. This results in lower and upper fuzzy models or a fuzzy model with lower and upper parameters.

In [22] a constructive method to synthesize a MISO TS fuzzy logic system imposing the requested derivative constraints on the function representing its behaviour is presented. The values of that function and its partial derivatives on the grid points of the input space permit to define a suitable interpolator of the function itself.

In [15], a new approach to fuzzy modeling using the relevance vector learning mechanism based on a kernel-based Bayesian estimation is introduced. The main concern is to find the best structure of the TS fuzzy model for modeling nonlinear dynamic systems with measurement error. The number of rules and the parameter values of membership functions can be found as optimizing the marginal likelihood of the RVM in the proposed FIS. Because the RVM is not necessary to satisfy Mercer's condition, selection of kernel function is beyond the limit of the positive definite continuous symmetric function of SVM. The relaxed condition of kernel function can satisfy various types of membership functions in fuzzy model. The RVM which was compared with support vector learning mechanism in examples had the small model capacity and described good generalization. Simulated results showed the effectiveness of the proposed FIS for modeling of nonlinear dynamic systems with noise.

In [29], the authors develop an interesting method to identify nonlinear systems using input-output data. They divide the identification process in three steps; premise variables, membership functions and consequent parameters. With respect to membership functions, they apply nonlinear programming technique using the complex method for the minimization of the performance index.

In spite of such important works, prior to 1989, most FLS's were still designed with preselected structures and the adjustment of membership functions (MF's) was carried out by trial and error. Since 1989, a number of techniques for structure and/or parameter identification from I/O data have been suggested in the literature. In [27] Sugeno and Tanaka developed a successive method for identifying T-S model. They combined the least square method, the complex method and an unbiased criterion; while parameter adjustment was adjustment rules and a weighted recursive least squares algorithm. In [33] the authors presented a method for generating fuzzy rules by learning from examples.

In 1991, Wang and Mendel developed a method for generating fuzzy rules by learning from examples [34] and proved that a fuzzy inference system is a universal approximator by the Stone-Weierstrass theorem [32].

In 1995, Wang proposed a new state-space analytical approach to fuzzy identification of nonlinear dynamical systems [33]. In 1996, Langari and Wang proposed achieving structure identification of a T-S fuzzy model by using a combination of fuzzy c-means clustering technique and a fuzzy discretization technique [17].

In [23], Nozaki et al. presented a heuristic method for generating T-S fuzzy rules from numerical data, and then converted the consequent parts of T-S fuzzy rules into linguistic representation.

As we will be demonstrated in this article, the T-S model can not be applied when the membership functions are overlapped by pairs. This limits the usage of the model because as it was shown in the last two decades that the major part of the results obtained in the field of stability and controller synthesis are based on this type of membership functions. Moreover, the method presented here is characterized by the high accuracy obtained in approximating nonlinear systems locally and globally in comparison with the original T-S model.

The rest of the paper is organized as follows. Section 2 presents T-S identification Method. Section 3 introduces restrictions of T-S identification Method. Section 4 demonstrates the proposed approach to improve and generalize the T-S model. Section 5

entails various examples to demonstrate the validity of the proposed approach. These examples show that the proposed approach are less conservative than those based on (standard) T-S model and illustrate the utility of the proposed approach in comparison with T-S model.

2 Identification of T-S Model

An interesting method of identification is presented in [29]. The idea is based on estimating the nonlinear system parameters minimizing a quadratic performance index. The method is based on the identification of functions of the following form:

$$f: \mathfrak{R}^n \rightarrow \mathfrak{R}$$

$$y = f(x_1, x_2, \dots, x_n)$$

The continuous fuzzy dynamic model, proposed by Takagi and Sugeno is described by fuzzy IF-THEN rules where the consequent part is a linear input-output relation.

The main feature of this model is to express the local dynamics of each fuzzy implication by an affine model.

The final fuzzy system is resulted from blending of the affine system models.

Each IF-THEN rule $R^{i_1 \dots i_n}$, for an n^{th} order system can be rewritten as follows:

$$S^{(i_1 \dots i_n)} :$$

if x_1 is $M_1^{i_1}$ and x_2 is $M_2^{i_2}$ and ... x_n is $M_n^{i_n}$ then

$$\hat{y} = p_0^{(i_1 \dots i_n)} + p_1^{(i_1 \dots i_n)} x_1 + p_2^{(i_1 \dots i_n)} x_2 + \dots + p_n^{(i_1 \dots i_n)} x_n$$

Where the fuzzy estimation of the output is:

$$\hat{y} = \frac{\sum_{i_1=1}^{r_1} \dots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(x) [p_0^{(i_1 \dots i_n)} + p_1^{(i_1 \dots i_n)} x_1 + \dots + p_n^{(i_1 \dots i_n)} x_n]}{\sum_{i_1=1}^{r_1} \dots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(x)} \quad (1)$$

Where $M_1^{i_1}$ ($i_1 = 1, 2, \dots, r_1$) are fuzzy sets for x_1 , $M_2^{i_2}$ ($i_2 = 1, 2, \dots, r_2$) are fuzzy sets for x_2 and $M_n^{i_n}$ ($i_n = 1, 2, \dots, r_n$) are fuzzy sets for x_n . $x(t)$ is the

state vector. Therefore the whole fuzzy system consists of r_1, r_2, \dots, r_n rules.

Let m be a set of input/output system samples $\{x_{1k}, x_{2k}, \dots, x_{nk}, y_k\}$. The parameters of the fuzzy system can be calculated by minimizing the following quadratic performance index:

$$J = \sum_{k=1}^m (y_k - \hat{y}_k)^2 = \|Y - XP\|^2 \quad (2)$$

Where

$$Y = [y_1 \ y_2 \ \dots \ y_m]^t$$

$$P = [p_0^{(1 \dots 1)} \ p_1^{(1 \dots 1)} \ p_2^{(1 \dots 1)} \ \dots \ p_n^{(1 \dots 1)} \ \dots \ p_0^{(r_1 \dots r_n)} \ \dots \ p_n^{(r_1 \dots r_n)}]^t$$

$$X =$$

$$\begin{bmatrix} \beta_1^{(1 \dots 1)} & \beta_1^{(1 \dots 1)} x_{11} & \dots & \beta_1^{(1 \dots 1)} x_{n1} & \dots & \beta_1^{(r_1 \dots r_n)} & \dots & \beta_1^{(r_1 \dots r_n)} x_{n1} \\ & & & & & \vdots & & \\ \beta_m^{(1 \dots 1)} & \beta_m^{(1 \dots 1)} x_{1m} & \dots & \beta_m^{(1 \dots 1)} x_{nm} & \dots & \beta_m^{(r_1 \dots r_n)} & \dots & \beta_m^{(r_1 \dots r_n)} x_{nm} \end{bmatrix}$$

and

$$\beta_k^{(i_1 \dots i_n)} = \frac{w^{(i_1 \dots i_n)}(x_k)}{\sum_{i_1=1}^{r_1} \dots \sum_{i_n=1}^{r_n} w^{(i_1 \dots i_n)}(x_k)} \quad (3)$$

If X is a matrix of complete rank, the parameters of the fuzzy system are obtained as follows:

$$J = \|Y - XP\|^2 = (Y - XP)^t (Y - XP)$$

$$\nabla J = X^t (Y - XP) = X^t Y - X^t XP = 0 \quad (4)$$

$$P = (X^t X)^{-1} X^t Y$$

2 Restrictions of T-S identification Method

The method proposed in [29] arises serious problems as it can not be applied in the most common case where the membership functions are those shown in fig. 1.

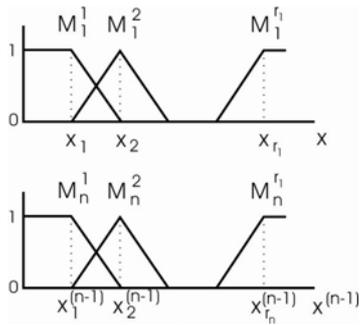


Fig. 1. Membership functions

The membership functions

$$\mu_{i1}(x_i) = \frac{x_{i2} - x_i}{x_{i2} - x_{i1}} \quad \text{and} \quad \mu_{i2}(x_i) = \frac{x_i - x_{i1}}{x_{i2} - x_{i1}}$$

defined in an interval $[x_{i1}, x_{i2}]$ which should verify:

$$\begin{aligned} \mu_{i1}(x_{i1}) &= 1 \\ \mu_{i1}(x_{i2}) &= 0 \\ \mu_{i2}(x_{i1}) &= 0 \\ \mu_{i2}(x_{i2}) &= 1 \\ \mu_{i1}(x_i) + \mu_{i2}(x_i) &= 1 \end{aligned}$$

In this case, it can easily be demonstrated that the matrix X is not of complete rank and therefore $X^T X$ is not invertible, which makes the method of T-S invalid. This result can be easily proven as follows:

Supposing:

$$f : \mathfrak{R} \rightarrow \mathfrak{R}$$

$$y = f(x)$$

applying T-S method, each row of the matrix X is of the form:

$$\begin{aligned} X_k &= [\mu_k^1 \quad \mu_k^1 x_k \quad \mu_k^2 \quad \mu_k^2 x_k] \\ &= \left[\begin{array}{cccc} \frac{x_2 - x_k}{x_2 - x_1} & \frac{x_2 - x_k}{x_2 - x_1} x_k & \frac{x_k - x_1}{x_2 - x_1} & \frac{x_k - x_1}{x_2 - x_1} x_k \end{array} \right] \end{aligned}$$

verifying:

$$\left[\begin{array}{cccc} \frac{x_2 - x_k}{x_2 - x_1} & \frac{x_2 - x_k}{x_2 - x_1} x_k & \frac{x_k - x_1}{x_2 - x_1} & \frac{x_k - x_1}{x_2 - x_1} x_k \end{array} \right] \begin{bmatrix} -x_1 \\ 1 \\ -x_2 \\ 1 \end{bmatrix}$$

$$= 0$$

The rank of X in this case is 3. In other words, the columns of X are linearly dependent which in turn makes impossible the use of the identification method proposed in [29].

Analyzing another example of two variables:

$$f : \mathfrak{R}^2 \rightarrow \mathfrak{R}$$

$$y = f(x_1, x_2)$$

Each row of the matrix X is of the form:

$$\begin{aligned} X_k &= [\mu_k^{11} \mu_k^{21} \mu_k^{11} \mu_k^{21} x_{1k} \mu_k^{11} \mu_k^{21} x_{2k} \mu_k^{11} \mu_k^{22} \\ &\mu_k^{11} \mu_k^{22} x_{1k} \mu_k^{11} \mu_k^{22} x_{2k} \dots \mu_k^{12} \mu_k^{21} \mu_k^{12} \mu_k^{21} x_{1k} \\ &\mu_k^{12} \mu_k^{21} x_{2k} \mu_k^{12} \mu_k^{22} \mu_k^{12} \mu_k^{22} x_{1k} \mu_k^{12} \mu_k^{22} x_{2k}] \end{aligned}$$

It can be noticed that the columns 1, 3, 4 and 6 have the same form as in the previous example multiplied by a constant μ_k^{11} and therefore they are linearly dependent as well. The same thing happens with the columns 6, 9, 10 and 12, etc. In fact, the rank of the matrix in this case is 8.

The solution proposed in [29] avoids the occurrence of this situation. In order to identify a function in the interval $[x_{i1}, x_{i2}]$ using T-S method, certain intermediate points are chosen of the form:

$$x_{i1}^* \in [x_{i1}, x_{i2}]$$

and

$$x_{i2}^* \in [x_{i1}, x_{i2}]$$

And they use membership functions which verify:

$$\mu_i^1(x) = \begin{cases} \frac{x_i - x_{i2}^*}{x_{i1} - x_{i2}^*} & x_{i1} \leq x \leq x_{i2}^* \\ 0 & x_{i2}^* \leq x \leq x_{i2} \end{cases}$$

$$\mu_i^2(x) = \begin{cases} 0 & x_{i1} \leq x \leq x_{i1}^* \\ \frac{x_i - x_{i1}^*}{x_{i2} - x_{i1}^*} & x_{i1}^* \leq x \leq x_{i2} \end{cases}$$

And thus:

$$\begin{aligned} \mu_i^1(x_{i1}) &= 1 \\ \mu_i^1(x_{i2}^*) &= 0 \\ \mu_i^2(x_{i1}^*) &= 0 \\ \mu_i^2(x_{i2}) &= 1 \end{aligned}$$

Which impedes that the domains of these functions being overlapped and therefore it can be observed that, except for some isolated points,

$$\mu_i^1(x_i) + \mu_i^2(x_i) \neq 1$$

And thus, in general, the matrix X will be of full rank and the method is applicable.

This solution can be clearly seen in [29] where the authors find the optimum membership functions minimizing the performance index and reducing the problem to a nonlinear programming one. For this reason, they use the well-known complex method for the minimization. This can obviously be observed in the illustrative examples selected by the authors in [29] where all the identified memberships are non overlapping ones.

4 Proposed Approach

The restriction of T-S identification method for the case presented in the previous section does not mean the non-existence of solutions. The problem comes from the fact that the solution should fulfil:

$$\nabla J = X^t Y - X^t X P = 0 \tag{5}$$

But as it was shown above, the columns of the matrix X are linearly dependent and consequently $X^t X$ is not an invertible matrix, therefore it is impossible to calculate P through:

$$P = (X^t X)^{-1} X^t Y \tag{6}$$

Nevertheless, as the rows of X^t are linearly dependent, the independent term in equation (5) will have the same dependence among its rows and thereupon the rank of the system matrix will be the same as the rank of the extended matrix by the independent term.

$$rank(X^t X) = rank(X^t X | X^t Y) \tag{7}$$

And so the system has solution. In other words, the system is a compatible indeterminate one, that is, if P is a solution of (5) and K is a Kernel of $X^t X$.

$$K = Ker(X^t X) \tag{8}$$

Then $P^* = P + K\beta$ will also be a solution, where β is any real vector whose dimension is same to that of the Kernel. Therefore, the problem is not the lack of a solution rather the existence of infinite solutions and the key idea is the ability to find one of them. Several proposals can be made to select a solution. In our case, we propose a solution with lower norm.

4.1 Reduced Matrix Approach-Optimum Solution

The approach implies the search for an exact and optimum solution at the expense of increasing the degree of complexity and computational cost. The problem is stated as follows:

$$minimize \|P\|^2 \tag{9}$$

subject to

$$minimize J = \|Y - X P\|^2$$

which can be re-described as follows:

$$minimize \|P\|^2$$

subject to

$$\nabla J = X^t Y - X^t X P = 0$$

As already stated, the system of restriction equations is a compatible undetermined one, and therefore, there are linearly dependent restrictions

upon others. What is proposed is to eliminate the linear restrictions until obtaining a system with all its restrictions being linearly independent. We obtain a reduced system of equations linearly independent

$$Y_r - X_r P = 0 \tag{10}$$

Then, Lagrange theorem can be applied, defining a lagrangian function

$$L = P^t P + \lambda^t (Y_r - X_r P) \tag{11}$$

that fulfils:

$$\nabla_P L = 2P^t - \lambda^t X_r = 0 \Rightarrow 2P - \lambda X_r^t = 0$$

$$\nabla_\lambda L = (Y_r - X_r P)^t = 0 \Rightarrow Y_r - X_r P = 0$$

This can also be represented in matrix form:

$$\begin{bmatrix} 2I & -X_r^t \\ X_r & 0 \end{bmatrix} \begin{bmatrix} P \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ Y_r \end{bmatrix}$$

And the solution will be:

$$\begin{bmatrix} P \\ \lambda \end{bmatrix} = \begin{bmatrix} 2I & -X_r^t \\ X_r & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ Y_r \end{bmatrix}$$

5 Examples

In the following examples, we will compare this method with the one proposed by T-S. Different non-linear functions will be proposed and the fuzzy models will be obtained assigning an interval $[x_{i1}, x_{i2}]$ for each variable x_i . In this interval, two fuzzy sets are defined whose membership functions are:

$$\mu_i^1 = \frac{x_i - x_{i2}}{x_{i1} - x_{i2}}$$

$$\mu_i^2 = \frac{x_i - x_{i1}}{x_{i2} - x_{i1}}$$

However, as demonstrated above, these membership functions can not be used directly in the method of T-S, as the resulting matrix X would not be complete rank. In order to compare these methods, we

use a factor $0 \leq \alpha \leq 1$, which determines two points within the interval, this means:

$$x_{i1}^* = x_{i2} - \alpha(x_{i2} - x_{i1})$$

$$x_{i2}^* = x_{i1} + \alpha(x_{i2} - x_{i1})$$

And so, we define two fuzzy sets whose membership functions are:

$$\mu_i^1(x) = \begin{cases} \frac{x_i - x_{i2}^*}{x_{i1} - x_{i2}^*} & x_{i1} \leq x \leq x_{i2}^* \\ 0 & x_{i2}^* \leq x \leq x_{i2} \end{cases}$$

$$\mu_i^2(x) = \begin{cases} 0 & x_{i1} \leq x \leq x_{i1}^* \\ \frac{x_i - x_{i1}^*}{x_{i2} - x_{i1}^*} & x_{i1}^* \leq x \leq x_{i2} \end{cases}$$

These membership functions are those used with the direct method of T-S. As a measure of error for comparing these methods, the maximum of the absolute values of the errors is used, the same method applied in by T-S in [29].

Example 1

Consider the following simple nonlinear system:

$$\dot{x} = x^2$$

It is aimed to estimate this system:

$$y = x^2 \quad x \in [0,1]$$

Let us suppose that we define in this interval two fuzzy sets with their corresponding membership functions as follows:

$$\mu_1(x) = 1 - x$$

$$\mu_2(x) = x$$

The objective is to calculate the corresponding fuzzy model in an optimum form:

$$S^1 : \text{if } x \text{ is } M_1^1 \text{ then } y = p_0^1 + p_1^1 x$$

$$S^2 : \text{if } x \text{ is } M_1^2 \text{ then } y = p_0^2 + p_1^2 x$$

In order to identify the nonlinear function, we take 20 points uniformly distributed in the interval [0, 1].

Applying the reduced matrix method, the product $X^T X$ is calculated as follows:

$$\begin{array}{cccc} 6.8421 & 1.5789 & 3.1579 & 1.5789 \\ 1.5789 & 0.6333 & 1.5789 & 0.9456 \\ 3.1579 & 1.5789 & 6.8421 & 5.2632 \\ 1.5789 & 0.9456 & 5.2632 & 4.3175 \end{array}$$

Its rank is 3 and can be reduced to X_r ,

$$\begin{array}{cccc} 6.8421 & 1.5789 & 3.1579 & 1.5789 \\ 1.5789 & 0.6333 & 1.5789 & 0.9456 \\ 3.1579 & 1.5789 & 6.8421 & 5.2632 \end{array}$$

The resultant P vector of the parameters in the consequent part is:

$$\begin{array}{c} 0.0000 \\ -0.3333 \\ 0.3333 \\ 0.6667 \end{array}$$

This means that the resultant fuzzy rules are:

$$S^1: \text{if } x \text{ is } M_1^1 \text{ then } y = -0.3333x$$

$$S^2: \text{if } x \text{ is } M_1^2 \text{ then } y = 0.3333 + 0.6667x$$

The result in this case is obtained with an error of $7.0031e-016$, which is practically zero. In the method of T-S, let us suppose that $\alpha = 0.7$. The fuzzy model is:

$$S^1: \text{if } x \text{ is } M_1^1 \text{ then } y = -0.0389 + 0.5238x$$

$$S^2: \text{if } x \text{ is } M_1^2 \text{ then } y = -0.5151 + 1.4762x$$

The identification error is $2.01e-002$, which is higher than the one obtained in the proposed method. Increasing $\alpha = 0.8$, the error is reduced to $1.60e-02$, i.e., the identification is improved. When $\alpha = 0.9$, the error becomes $8.1e-003$, where the identification is again improved. The same occurs for $\alpha = 0.95$ where the error is reduced to $4.4e-003$.

In other words, as the factor α is approaching unity the identification is improved, but can not reach the optimum which precisely occurs when $\alpha = 1$. Since at this value, the matrix X is not of complete rank and therefore the matrix $X^T X$ is not invertible.

Even when α is approaching unity, the condition number of the matrix X starts increasing which indicates that $X^T X$ is approaching the singularity and therefore its inverse is no longer numerically reliable.

For instance, when $\alpha = 0.999$, the condition number of X is $5.2523e+004$ which shows clearly a non reliable result.

Example 2

Consider the following nonlinear system

$$y = x_1^2 + x_2^2 \quad x_1 \in [0,1] \quad x_2 \in [0,1]$$

Applying the reduced matrix method, the resultant fuzzy rules are:

$$S^{11}: \text{if } x_1 \text{ is } M_1^1 \text{ and } x_2 \text{ is } M_2^1$$

$$\text{then } y = -0.3333x_1 - 0.3333x_2$$

$$S^{12}: \text{if } x_1 \text{ is } M_1^1 \text{ and } x_2 \text{ is } M_2^2$$

$$\text{then } y = 0.3333 - 0.2500x_1 + 0.6667x_2$$

$$S^{21}: \text{if } x_1 \text{ is } M_1^2 \text{ and } x_2 \text{ is } M_2^1$$

$$\text{then } y = 0.3333 + 0.6667x_1 - 0.2500x_2$$

$$S^{22}: \text{if } x_1 \text{ is } M_1^2 \text{ and } x_2 \text{ is } M_2^2$$

$$\text{then } y = 0.5000 + 0.7500x_1 + 0.7500x_2$$

The identification model is obtained with an error of $7.1497e-015$. Applying the method presented by T-S, with $\alpha = 0.9$ for x_1 and x_2 , the resultant fuzzy model is:

$$S^{11}: \text{if } x_1 \text{ is } M_1^1 \text{ and } x_2 \text{ is } M_2^1$$

$$\text{then } y = -0.0287 + 0.5473x_1 - 4.6928x_2$$

$$S^{12}: \text{if } x_1 \text{ is } M_1^1 \text{ and } x_2 \text{ is } M_2^2$$

$$\text{then } y = 4.6843 - 0.5473x_1 - 3.2235x_2$$

$$S^{21}: \text{if } x_1 \text{ is } M_1^2 \text{ and } x_2 \text{ is } M_2^1$$

$$\text{then } y = -0.4814 + 1.4527x_1 - 5.5301x_2$$

$$S^{22}: \text{if } x_1 \text{ is } M_1^2 \text{ and } x_2 \text{ is } M_2^2$$

$$\text{then } y = 5.1964 + 4.4527x_1 - 4.7962x_2$$

The resultant error is 0.0515 , but the condition number for the matrix X is $3.4381e+015$ which is very close to singularity.

Example 3

Let us analyze the following non linear system

$$y = x_1^3 + x_2^3 \quad x_1 \in [0,1] \quad x_2 \in [0,1]$$

Applying the reduced matrix method, the fuzzy model is:

S^{11} :if x_1 is M_1^1 and x_2 is M_2^1

then $y = -0.0016 + 0.9510x_1 - 0.6985x_2$

S^{12} :if x_1 is M_1^1 and x_2 is M_2^2

then $y = 0.1030 + 9510x_1 + 0.8015x_2$

S^{21} :if x_1 is M_1^2 and x_2 is M_2^1

then $y = -1.2998 + 2.3090x_1 - 1.1312x_2$

S^{22} :if x_1 is M_1^2 and x_2 is M_2^2

then $y = -0.7625 + 2.3090x_1 + 0.3688x_2$

The identification error is 2.14e-02. But in the TS method, using $\alpha = 0.8$, which is the value that after several attempts provides better results, we get:

S^{11} :if x_1 is M_1^1 and x_2 is M_2^1

then $y = -0.0085 + 0.5910x_1 - 1.0541x_2$

S^{12} :if x_1 is M_1^1 and x_2 is M_2^2

then $y = 0.4578 + 0.5910x_1 + 0.4273x_2$

S^{21} :if x_1 is M_1^2 and x_2 is M_2^1

then $y = -0.9482 + 1.9274x_1 - 1.3607x_2$

S^{22} :if x_1 is M_1^2 and x_2 is M_2^2

then $y = -0.1880 + 1.9274x_1 + 0.1465x_2$

The identification error is 3.07e-02. In this case, with $\alpha = 0.9$ the results becomes worse because the condition number for the matrix X is 3.4381e +015 which is the same as that in the previous case, where the matrix X does not depend upon the function rather on the interval.

6 Conclusions

A new optimization method has been developed to improve the local and global approximation and modelling capability of Takagi-Sugeno (T-S) identification methodology model. The main problem encountered is that T-S model can not be applied when the membership functions are overlapped by pairs. This restricts the application of the T-S model because this type of membership function has been widely used during the last two decades in fuzzy control applications. An optimal solution has been proposed to reduce the error between the original system and the identified one. The results obtained have shown tangible improvement compared to the solution offered by T-S. Several illustrative examples have been presented to evaluate the validity and performance of the proposed method and the high accuracy obtained in approximating nonlinear systems locally and globally in comparison with the original

T-S model. The results obtained by applying the proposed method have demonstrated better results in comparison with the original T-S.

References

- [1] R. Babuska, *Fuzzy Modelling for Control*. Boston, MA: Kluwer, 1998.
- [2] P. Baranyi, "SVD-based complexity reduction to TS fuzzy models," *IEEE Trans. Ind. Electron.*, vol. 49, no. 2, pp. 433–443, Apr. 2002.
- [3] S. G. Cao, N. W. Rees, and G. Feng, "Analysis and design of fuzzy control systems using dynamic fuzzy global models," *Fuzzy Sets Syst.*, vol. 75, pp. 47–62, 1995.
- [4] S. G. Cao, N. W. Rees, and G. Feng, "Stability analysis and design for a class of continuous-time fuzzy control systems," *Int. J. Control*, vol. 64, pp. 1069–1087, 1996.
- [5] S. G. Cao, N. W. Rees, and G. Feng, "Analysis and design for a class of complex control systems— Part I: Fuzzy modeling and identification," *Automatica*, vol. 33, pp. 1017–1028, 1997.
- [6] F. Delmotte, T. M. Guerra, and M. Ksantini, "Continuous Takagi–Sugeno's Models: Reduction of the Number of LMI Conditions in Various Fuzzy Control Design Techniques," *IEEE Trans. Fuzzy Syst.* 15(3), 426–438, June, 2007.
- [7] C. Fantuzzi and R. Rovatti, "On the approximation capabilities of the homogeneous Takagi-Sugeno model," in *Proc. 5th IEEE Int. Conf. Fuzzy Systems*, New Orleans, LA, 1996, pp. 1067–1072.
- [8] G. Feng, "Approaches to quadratic stabilization of uncertain fuzzy dynamic systems," *IEEE Trans. Circuits Syst. I*, vol. 48, no. 6, pp. 760–769, Jun. 2001.
- [9] F. Gang., "A survey on analysis and design of model-based fuzzy control systems," *IEEE Trans. Fuzzy Syst.* 14(5), 676–697, Oct. 2006.
- [10] Y. Hou, Jacek M. Zurada, W. Karwowski, W. S. Marras, and K. Davis, "Identification of Key Variables Using Fuzzy Average With Fuzzy Cluster Distribution," *IEEE Trans. Fuzzy Syst.* 15(4), 673–685, Aug. 2007.
- [11] T. Hseng S. Li and S.-H. Tsai, "T–S Fuzzy Bilinear Model and Fuzzy Controller Design for a Class of Nonlinear Systems," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 494–506, Jun. 2007.
- [12] T. A. Johansen, R. Shorten, and R. Murray-Smith, "On the interpretation and identification of dynamic Takagi–Sugeno models," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 3, pp. 297–313, Jun. 2000.

- [13] P. Julian, A. Desages, and O. Agamennoni, "High-level canonical piecewise linear representation using a simplicial partition," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 46, no. 4, pp. 463–480, Apr. 1999.
- [14] Jae-Hun Kim, Chang-Ho Hyun, Euntai Kim, and Mignon Park, "New Adaptive Synchronization of Uncertain Chaotic Systems Based on T–S Fuzzy Model," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 359–369, Jun. 2007.
- [15] J. Kim, Yasuo Suga, and Sangchul Won, "A New Approach to Fuzzy Modeling of Nonlinear Dynamic Systems With Noise: Relevance Vector Learning Mechanism," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 222–231, Jun. 2006.
- [16] E. Kim, C.-H. Lee, and Y.-W. Cho, "Analysis and Design of an Affine Fuzzy System Via Bilinear Matrix Inequality," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 3, pp. 115–123, Feb. 2005.
- [17] K. Kobayashi, K. Cheok, K. Watanabe, Estimation of the absolute vehicle speed using fuzzy logic rule-based Kalman filter, *American Control Conf.*, Seattle, 1995, pp. 3086–3090.
- [18] Lagrat, A. El Ougli, I Boumhidi, "Optimal Adaptive Fuzzy Control for a Class of Unknown Nonlinear Systems", WSEAS Transactions on systems and control, Issue 2, Volume 3, February 2008.
- [19] K.-Y. Lian, C.-H. Su, and C.-S. Huang, "Performance Enhancement for T–S Fuzzy Control Using Neural Networks," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 5, pp. 619–627, Oct. 2006.
- [20] Ji-Chang Lo and Min-Long Lin, "Existence of Similarity Transformation Converting BMIs to LMIs", *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 5, pp. 840–851, Oct. 2007.
- [21] H. Mazinan and N. Sadati, "Multiple Modeling and Fuzzy Predictive Control of A Tubular Heat Exchanger System", WSEAS Transactions on systems and control, Issue 4, Volume 3, April 2008.
- [22] A. Mencattini, Marcello Salmeri and Adelio Salsano, "Sufficient Conditions to Impose Derivative Constraints on MISO Takagi–Sugeno Fuzzy Logic Systems", *IEEE Trans. Fuzzy Syst.*, Vol. 13, No. 4, Aug. 2005
- [23] S. Mollov, R. Babuska, J. Abonyi, and H. B. Verbruggen, "Effective optimization for fuzzy model predictive control," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 5, pp. 661–675, Oct. 2004.
- [24] Ramirez Canelon and Eliezer Colina Morles, "Fuzzy Clustering Based Models Applied To Petroleum Processes", WSEAS Transactions on systems and control, Issue 3, Volume 3, March 2008.
- [25] Ginalber Serra and Celso Bottura, "An IV-QR Algorithm for Neuro-Fuzzy Multivariable Online Identification," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 2, pp. 200–210, Apr. 2007.
- [26] D. Simon, "Sum normal optimization of fuzzy membership functions", *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Vol. 10, No. 4, Aug. 2002, pp.363–384.
- [27] M. Sugeno, K. Tanaka, Successive identification of a fuzzy model and its applications to prediction of a complex system, *Fuzzy Sets and Systems* 42 (1991) 315–334.
- [28] M. Sugeno and G. Kang, "Structure identification of fuzzy model," *Fuzzy Sets Syst.*, vol. 28, pp. 15–33, 1988.
- [29] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-15, no. 1, pp. 116–132, Jan. 1985.
- [30] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York: Wiley, 2001.
- [31] K. Tanaka, H. Ohtake, and H. O. Wang, "A Descriptor System Approach to Fuzzy Control System Design via Fuzzy Lyapunov Functions," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 331–341, Jun. 2007.
- [32] S. G. Tzafestas and K. C. Zikidis, "NeuroFAST: On-line neuro-fuzzy ART-based structure and parameter learning TSK model," *IEEE Trans. Syst., Man, Cybern., B, Cybern.*, vol. 31, no. 5, pp. 797–802, Oct. 2001.
- [33] L. X. Wang and J. M. Mendel, "Generating fuzzy rules by learning from examples," in *Proc. 6th IEEE Int. Symp. Intell. Contr.*, 1991, pp. 263–268.
- [34] L. Wang and J. Mendel, "Back-propagation of Fuzzy Systems as Nonlinear Dynamic System Identifiers," *IEEE International Conference on Fuzzy Systems*, San Diego, California, 1992, pp. 1409–1418.
- [35] H. Ying, "General SISO Takagi–Sugeno fuzzy systems with linear rule consequent are universal approximators," *IEEE Trans. Fuzzy Syst.*, vol. 6, no. 4, pp. 582–587, Nov. 1998.
- [36] W. Yu and X. O. Li, "Fuzzy identification using fuzzy neural networks with stable learning algorithms," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 3, pp. 411–420, Jun. 2004.
- [37] W. Yu and X. O. Li, "Sufficient Conditions to Impose Derivative Constraints on MISO Takagi–Sugeno Fuzzy Logic Systems," *IEEE Trans.*

Fuzzy Syst., vol. 13, no. 4, pp. 454–467, Aug. 2005.

- [38] K. Zeng, N. Y. Zhang, and W. L. Xu, “A comparative study on sufficient conditions for Takagi–Sugeno fuzzy systems as universal approximators” *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 773–780, Dec. 2000.