Probe Current in a Magnetized, Collisional Plasma Revisited

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Abstract. An old analysis of probe current in a strongly magnetized plasma is reconsidered. It is shown that, in the collisional limit, the plasma beyond the sheath heats up in the collection process at positive probe bias enough. The modified current is compared to the current collected in the case of collection due to Bohm diffusion.

There is no established theory of electron collection by probes in strongly magnetized plasmas. Standard collisionless[1] and turbulent[2] models do not describe how current perturbations die off far away from the probe, and do no lead to definite predictions for electron current; on the other hand, a purely collisional model may be too requiring as regards perturbation distances.[3] Actually, these models apply to regimes that differ from each other in probe bias in addition to general plasma behavior.

Here we reconsider the collisional model using a macroscopic description instead of the old kinetic description.[3] Although the present approach is more cumbersome if complete, it allows clarifying a point in the kinetic analysis that appears uncertain when reexamined: the isothermal character of the electron collection process. We prove here that a dimensionless length parameter determines whether electrons keep isothermal. The process that may account for this condition is the ion-electron energy exchange, rather than electron heat diffusion. If that process is not efficient enough, we find that electron cooling occurs at negative probe bias Φ_P, and heating at Φ_P positive enough.

We consider a fully ionized plasma in the presence of an uniform magnetic field $\overline{B}$ along the z-axis of cylindrical coordinates r, θ, and z. We assume a steady, collisional regime with strongly magnetized electrons,

$$\Omega_e\tau_{e\infty} \sim \lambda_{e\infty}/l_{e\infty} >> 1,$$

where $\Omega_e \equiv eB_e/m_e$ is the electron gyrofrequency, and $\tau_{e\infty}$, $\lambda_{e\infty}$ and $l_{e\infty}$ are unperturbed electron collision time, mean free path, and thermal gyroradius, respectively. We consider the electron current to a sphere (or disk normal to $\overline{B}$) of radius $R >> l_{e\infty}$, and moderate bias, and look for a fully consistent solution to the complete set of macroscopic equations with classical transport coefficients. The solution may depend on the dimensionless parameters $\Omega_e\tau_{e\infty}$, $R/l_{e\infty}$, $e\Phi_P/kT_{e\infty}$, $T_{i\infty}/T_{e\infty}$ and Z_i (ion charge number), and the ratios $\sqrt{m_e/m_i}$ and $l_{i\infty}/\lambda_{e\infty}$ ($l_{i\infty}$ = $l_{e\infty}\sqrt{m_iT_{i\infty}/m_eT_{e\infty}}$). Here, for simplicity, we take Z_i = 1, $T_{i\infty}/T_{e\infty}$ of order unity, and ions moderately magnetized at least,

$$l_{i\infty}/\lambda_{e\infty} \leq O(1) \rightarrow \Omega_e\tau_{e\infty} \geq O(\sqrt{m_i/m_e}).$$

As in Ref. 3 no sheath analysis is really required; we use quasineutrality throughout.

The momentum equation for ions or electrons ($\alpha = i, e$) reads

$$m_\alpha n_\alpha \overline{v}_\alpha \bullet \nabla \overline{v}_\alpha = -\nabla p_\alpha + e_\alpha n_\alpha(\overline{E} + \overline{v}_\alpha \wedge \overline{B}) + \overline{F}_{v\alpha} + \overline{R}_\alpha, \quad (1)$$

with $n_i \approx n_e \equiv n$, $\overline{E} = -\nabla\Phi$, and $\partial/\partial\theta \equiv 0$. The ion-electron force, $-\overline{R}_i = \overline{R}_e \equiv \overline{R}$, which has a nonisotropic (tensorial) character, and is made of terms linear in either $\overline{u} \equiv \overline{v}_e - \overline{v}_i$ or ∇T_e, was given by Braginskii.[4] The viscous force $\overline{F}_{v\alpha}$ is $-\nabla \bullet \overline{\overline{\Pi}}_\alpha$, with the stress tensor $\overline{\overline{\Pi}}_\alpha$ involving 5 viscous coefficients (η_0 - η_4) for each species.[4]

In solving (1) we make several ansatzen that later in the solution are shown to hold within some parametric domain. We take as negligible

i) ion and electron inertia terms on the left hand side of (1);

ii) the ion temperature gradient ∇T_i;

iii) the ion velocity against the electron velocity, $\overline{u} \approx \overline{v}_e$;

iv) F_{vir}, v) R_r, and vi) $\overline{F}_{ve}$.

Taking into account ansatzen (i)-(v), the ion r-momentum equation reads

$$kT_i\frac{\partial n}{\partial r} = -en\frac{\partial \Phi}{\partial r}\ ; \quad (2)$$

using $n \rightarrow n_\infty$, $\Phi \rightarrow 0$, as $r \rightarrow \infty$ at fixed z, Eq.(2) gives

$$n = n_\infty \exp(\frac{-e\Phi}{kT_i}). \quad (2')$$

The electron momentum equations read

$$\frac{\partial p_e}{\partial z} - en\frac{\partial \Phi}{\partial z} = R_z \rightarrow -\alpha_0\frac{m_e n}{\tau_e}v_{ez} - \beta_0 nk\frac{\partial T_e}{\partial z}, \quad (3)$$

$$-eBnv_{er} = R_\theta \rightarrow -\frac{m_e n}{\tau_e}v_{e\theta} - \frac{\beta_1''}{\Omega_e\tau_e}nk\frac{\partial T_e}{\partial r}, \quad (4)$$

$$eBnv_{e\theta} = -\frac{\partial p_e}{\partial r} + en\frac{\partial \Phi}{\partial r}. \quad (5)$$

In equations (3) and (4) we used ansatzen (i), (iii), and (vi); in (5) we used ansatzen (i), (v), and (vi). In Eq. (3), $\alpha_0(Z_i = 1)$ and $\beta_0(Z_i = 1)$ are Ohm and Seebeck (thermoelectric) coefficients given by Braginskii. On the right hand side of (4) there are only Ohm (nonisotropic) and Nernst terms (with β_1''=3/2); there is no Seebeck term because $\partial T_e/\partial\theta$ = 0, while the collisional Hall effect ends out to be smaller than the Ohm effect by a factor of order $1/(\Omega_e\tau_{e\infty})^2$. (Corrections to Braginskii's coefficients given by Epperlein and Haines only affect the Seebeck and collisional Hall coefficients.[5]) The electron collision time, given by Braginskii, takes the local form $\tau_e = \tau_{e\infty}T_e^{3/2}n_\infty/T_{e\infty}^{3/2}n$.

Equations (2') - (5) are now used to eliminate n and $nv_{e\theta}$ and to obtain the particle fluxes nv_{ez} and nv_{er} in terms of Φ, T_e, and their z and r gradients. The electron continuity equation

$$\nabla\bullet n\bar{v}_e = 0, \qquad (6)$$

then provides a first, elliptic equation relating Φ and T_e. Introducing dimensionless variables,

$$\tilde{r} \equiv r/R, \quad \tilde{z} \equiv z\sqrt{\alpha_0}/R\Omega_e\tau_{e\infty},$$
$$\tilde{\Phi} \equiv 2e\Phi/kT_{i\infty}, \quad \tilde{T}_e \equiv T_e/T_{e\infty}, \quad \tilde{T}_{i\infty} \equiv T_{i\infty}/T_{e\infty},$$

Eq. (6) finally read

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left[\tilde{r}\frac{\exp(-\tilde{\Phi})}{\tilde{T}_e^{3/2}}\left\{(\tilde{T}_{i\infty}+\tilde{T}_e)\frac{\partial\tilde{\Phi}}{\partial\tilde{r}}+\frac{\partial\tilde{T}_e}{\partial\tilde{r}}\right\}\right]+ \\ +\frac{\partial}{\partial\tilde{z}}\left[\tilde{T}_e^{3/2}\left\{(\tilde{T}_{i\infty}+\tilde{T}_e)\frac{\partial\tilde{\Phi}}{\partial\tilde{z}}-2(1+\beta_0)\frac{\partial\tilde{T}_e}{\partial\tilde{z}}\right\}\right]. \qquad (6')$$

The analysis leading to Eq.(6') also provides values for the characteristic length along z, and for all components of the electron velocity,

$$L_z = R\ \Omega_e\tau_{e\infty}/\sqrt{\alpha_0} \qquad (7)$$
$$v_{ez}\sim v_{e\theta}\sim v_{er}\times\Omega_e\tau_{e\infty}\sim v_e(\text{thermal})\times l_{e\infty}/R. \qquad (8)$$

Results (7) and (8) are then used to verify the ansatzen. Using ansatzen (i) - (iii) in the two missing ion momentum equations,

$$R_z + en\frac{\partial\Phi}{\partial z} + T_i k\frac{\partial n}{\partial z} = F_{viz} \to \frac{1}{r}\frac{\partial}{\partial r}\left(r\eta_2\frac{\partial v_{iz}}{\partial r}\right), \qquad (9)$$

$$R_\theta = F_{vi\theta} \to \frac{1}{r}\frac{\partial}{\partial r}\left(r\eta_1\frac{\partial v_{i\theta}}{\partial r}\right), \qquad (10)$$

we first obtain characteristic values for v_{iz} and $v_{i\theta}$. In the viscous force F_{viz}, the η_1 and η_3 terms vanish, while η_0 and η_4 terms are smaller than the dominant η_2 term by factors $l_{e\infty}^2/l_{i\infty}^2 \sim m_e/m_i$ and $l_{e\infty}^2/l_{i\infty}\lambda_{e\infty} \sim \sqrt{m_e/m_i}/\Omega_e\tau_{e\infty}$ respectively; in $F_{vi\theta}$, η_0 and η_2 terms vanish and η_3 and η_4 terms are small by a factor $\sqrt{m_e/m_i}$. We then use the ion continuity equation

$$\nabla\bullet n\bar{v}_i = 0,$$

to find a characteristic value for v_{ir}. We finally arrive at

$$\frac{v_{iz}}{v_{ez}} \sim \frac{v_{i\theta}}{v_{e\theta}} \sim \frac{v_{ir}}{v_{er}} \sim \frac{R^2}{l_{e\infty}^2}\left(\frac{m_e}{m_i}\right)^{3/2}. \qquad (11)$$

Use of Eqs.(8) and (11) now determines characteristic values for all inertia terms in Eq.(1). We find

$$\frac{e-inertia\ terms}{dominant\ terms} = O\left(\frac{l_{e\infty}^2}{R^2}\right), \qquad (12a)$$

$$\frac{i-inertia\ terms}{dominant\ terms} = O\left[\frac{R^2}{l_{e\infty}^2}\left(\frac{m_e}{m_i}\right)^2\right]. \qquad (12b)$$

Next, we determine characteristic values for R_r,

$$\frac{Ohm, Seebeck\ and\ collisional\ Hall\ R_r-terms}{terms\ in\ Eqs.(2)\ and\ (5)} = O\left[\frac{1}{(\Omega_e\tau_{e\infty})^2}\right] \qquad (13)$$

again, there is no Nernst term in R_r because $\partial T_e/\partial\theta = 0$. We also find

$$\frac{F_{vir}-term}{terms\ in\ (2)} = O\left(\sqrt{\frac{m_e}{m_i}}\right) \qquad (14)$$

where

$$F_{vir} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\eta_3\frac{\partial v_{i\theta}}{\partial r}\right),$$

(the η_4 term in F_{vir} vanishes; the η_0 term is small by a factor $\sqrt{m_e/m_i}$; η_1 and η_2 terms are small by a factor $l_{e\infty}l_{i\infty}/\lambda_{e\infty}^2$). Concerning the electron viscous terms we find

$$F_{vez}/F_{viz}\sim F_{ve\theta}/F_{vi\theta}\sim F_{ver}/(\partial p_e/\partial r)\sim l_{e\infty}^2/R^2. \qquad (15)$$

To test ansatz ii) we next consider the ion entropy equation[4]

$$nkT_i\bar{v}_i\bullet\nabla\ln\frac{T_i^{3/2}}{n} = -\nabla\bullet\bar{q}_i - \bar{\bar{\Pi}}_i:\nabla\bar{v}_i + \\ +\frac{3m_e}{m_i}nk\frac{T_e-T_i}{\tau_e}. \qquad (16)$$

The dominant term in Eq.(16) is that part of the heat flux divergence arising from the radial flux, q_{ir}, with corrections of order $(m_e/m_i)^{3/2}R^2/l_{e\infty}^2$ arising from the ion-electron heat exchange term, Q_i [last term in (16)],

$$0 = \frac{1}{r}\frac{\partial}{\partial r}rq_{ir}\left[to\_terms\_of\_order\_O\left\{\frac{R^2}{l_{e\infty}^2}\left(\frac{m_e}{m_i}\right)^{3/2}\right\}\right] \qquad (16')$$

$$\to q_{ir}(r,z)\propto\partial T_i/\partial r = 0 \ \to T_i(r,z) = constant = T_{i\infty} \qquad (17)$$

The last macroscopic (electron energy) equation reads[4]

$$\nabla\bullet\left[\left(\frac{mv_e^2}{2}+\frac{5}{2}kT_e\right)n\bar{v}_e+\bar{q}_e+\bar{\bar{\Pi}}_e\bullet\bar{v}_e\right] = -en\bar{E}\bullet\bar{v}_e+\bar{R}\bullet\bar{v}_e+Q_e. \qquad (18)$$

The first and last term in the lef-hand-side give corrections of order $l_{e\infty}^2/R^2$. On the right-hand-side we have [4]

$$\bar{R}\bullet\bar{v}_e + Q_e = \bar{R}\bullet\bar{v}_i - Q_i \approx -Q_i \quad \left[\textit{to order}\ \frac{R^2}{l_{e\infty}^2}\left(\frac{m_e}{m_i}\right)^{3/2} \right] \cdot \quad (19)$$

We thus have

$$\nabla \bullet \left[\left(\frac{5}{2} kT_e - e\Phi \right) n\bar{v}_e + \bar{q}_e \right] = -Q_i\,, \quad (18')$$

with

$$q_{ez} = -\gamma_0 \frac{\tau_e}{m_e} nkT_e\, k \frac{\partial T_e}{\partial z} + \beta_0 nkT_e v_{ez}\,, \quad (20)$$

$$q_{er} = -\frac{\gamma_1'}{\Omega^2 \tau_e^2} \frac{\tau_e}{m_e} nkT_e\, k \frac{\partial T_e}{\partial r} - \frac{\beta_1''}{\Omega_e \tau_e} nkT_e v_{e\theta}\,. \quad (21)$$

In (20), $\gamma_0(Z_i = 1)$ and $\beta_0(Z_i = 1)$ are Fourier and Peltier coefficients.[4] Only Fourier and Ettinghausen terms appear in (21); there is no Righi-Leduc term because $\partial T_e/\partial\theta = 0$, while the Peltier effect is smaller than the Ettinghausen effect by a factor $1/(\Omega_e \tau_{e\infty})^2$. Taking n, nv_{ez} and nv_{er} from Eqs. (2'), (3) and (4), and using dimensionless variables, (18') becomes

$$\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}} \tilde{r} \frac{\exp(-\tilde{\Phi})}{\tilde{T}^{3/2}} \cdot$$
$$\left[(\tilde{T}_{i\infty} + \tilde{T}_e)\left(\tilde{T}_e - \frac{\tilde{T}_{i\infty}}{2}\tilde{\Phi}\right)\frac{\partial \tilde{\Phi}}{\partial \tilde{r}} + \left\{ \left(\frac{11}{2} - 2\gamma_1'\right)\tilde{T}_e - \frac{\tilde{T}_{i\infty}}{2}\tilde{\Phi} \right\} \frac{\partial \tilde{T}_e}{\partial \tilde{r}} \right]$$
$$+ \quad \frac{\partial}{\partial \tilde{z}} \tilde{T}_e^{3/2} [(\tilde{T}_{i\infty} + \tilde{T}_e) \left\{ \left(\frac{5}{2} + \beta_0\right)\tilde{T}_e - \frac{\tilde{T}_{i\infty}}{2}\tilde{\Phi} \right\} \frac{\partial \tilde{\Phi}}{\partial \tilde{z}} -$$
$$- \left\{ 2\left(\alpha_0\gamma_0 + \frac{5}{2} + \frac{7}{2}\beta_0 + \beta_0^2\right)\tilde{T}_e - (1+\beta_0)\tilde{T}_{i\infty}\tilde{\Phi} \right\} \frac{\partial \tilde{T}_e}{\partial \tilde{z}}] =$$
$$-3\frac{m_e R^2}{m_i l_{e\infty}^2} \frac{\exp(-\tilde{\Phi})}{\tilde{T}_e^{3/2}} (\tilde{T}_e - \tilde{T}_{i\infty}) \cdot \quad (18'')$$

Relations (11)- (16') concerning ansatzen (i)-(vi) show that our model applies for conditions

$$l_{e\infty}^2 << R^2 << l_{e\infty}^2 (m_i/m_e)^{3/2} \quad (27)$$
$$l_{e\infty}^2 << \lambda_{e\infty}^2 \ (\Omega_e^2 \tau_{e\infty}^2 >> 1). \quad (28)$$

Actually, in our discussion on $\overline{F}_{vi}$ we assumed a condition more stringent than (28),

$$l_{i\infty}/\lambda_{e\infty} \le O(1) \rightarrow \ \Omega_e \tau_{e\infty} \ge O(\sqrt{m_i/m_e}),$$

as we had advanced. Equations (6') and (18'') contain the dimensionless parameters $T_{i\infty}/T_{e\infty}$ and $3m_e R^2/m_i l_{e\infty}^2$. Condition (27) allows for large and small values of this last parameter.

For $3m_e R^2/m_i l_{e\infty}^2 >> 1$, Eq.(18'') just yields $T_e(z, r) = T_{i\infty}$, and thus $T_{e\infty} = T_{i\infty}$. Now a) electrons keep isothermal under collection and b) the large collection length L_z in (7) requires a plasma so extensive that collisions make for equal undisturbed temperatures, $T_{e\infty} = T_{i\infty}$. For $3m_e R^2/m_i l_{e\infty}^2$ of order unity collection is non-isothermal, although $T_{e\infty}$ and $T_{i\infty}$ are equal. Finally, for $3m_e R^2/m_i l_{e\infty}^2$ small, L_z is short enough to allow for $T_e(z, r) \neq T_{e\infty} \neq T_{i\infty}$.

Equations (6') and (18'') determine Φ and T_e when boundary conditions are given. These conditions are

$$\tilde{\Phi} \rightarrow 0,\ \tilde{T}_e \rightarrow 1 \ \ as\ \tilde{r} \rightarrow \infty \ \ or\ \tilde{z} \rightarrow \infty, \quad (22a, b)$$

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{r}} = \frac{\partial \tilde{T}_e}{\partial \tilde{r}} = 0 \quad at \quad \tilde{r} = 0, \quad (23a,b)$$

$$\frac{\partial \tilde{\Phi}}{\partial \tilde{z}} = \frac{\partial \tilde{T}_e}{\partial \tilde{z}} = 0 \quad at \quad \tilde{r} > 1,\ \ \tilde{z} = 0; \quad (24a, b)$$

also, as $z/L_z \rightarrow 0$ at $r < R$,

$$nv_{ez} + \sqrt{\frac{kT_e}{2\pi m_e}}\, n_\infty \exp\left(-\frac{e\Phi}{kT_{i\infty}}\right) \exp\left(\frac{e(\Phi_P - \Phi)}{kT_e}\right) \rightarrow 0 \quad (25)$$

$$q_{ez} + nv_{ez}\left[e(\Phi_P - \Phi) + \frac{kT_e}{2} \right] \rightarrow 0 \cdot \quad (26)$$

Boundary condition (25), taken from Ref. 3, just states that the (electron) particle z-flux is conserved over distances short compared with L_z in (7), through a transitional layer and a sheath next to the probe. Boundary condition (26) establishes a similar conservation of (electron) energy z-flux. Note that conservation of momentum flux would involve $\nabla\Phi$ (unknown at the probe) rather than Φ itself.

With nv_{ez} and q_z taken from (2), (3), and (20), Eqs. (25) and (26) provide two relations among Φ, T_e, $\partial\Phi/\partial z$, and $\partial T_e/\partial z$ as $z/L_z \rightarrow 0$ (at $r < R$). These values do not correspond to the probe, Eqs. (6') and (18'') applying at distances z from the probe large compared with $\lambda_{e\infty}$. In particular we have

$$\Phi_0(r < R) \equiv \Phi(r/R < 1,\ z/L_z \rightarrow 0) \neq \Phi_P.$$

As we shall see, one has $\Phi_0 > \Phi_P$ for some Φ_P range; this is the potential "overshoot" first noticed in Ref. 3. Our analysis is valid for that Φ_P range, the overshoot allowing to reduce the analysis of both transitional layer and sheath to the derivation of conditions (25) and (26), based on the fact that the incoming electrons are then repelled when travelling from distances $z/L_z << 1$ to the probe.[3]

Figure 1 shows the average

$$\langle T_{e0} \rangle \equiv \int_0^R \frac{2\pi\, r dr}{\pi R^2} T_{e0}(r < R)$$
$$T_{e0}(r < R) \ \equiv T_e(r/R,\ z/L_z \rightarrow 0).$$

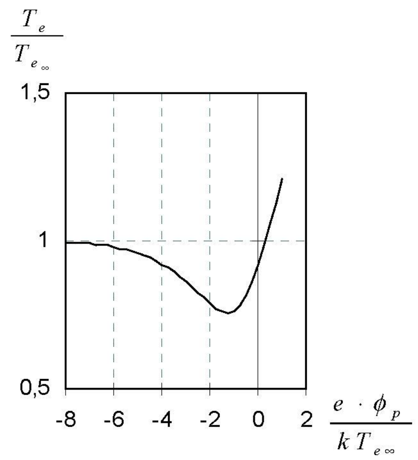

FIG.1 Average electron temperature

Heating occurs for positive Φ_P . Figure 2 shows the average overshoot $<\Phi_0>$.

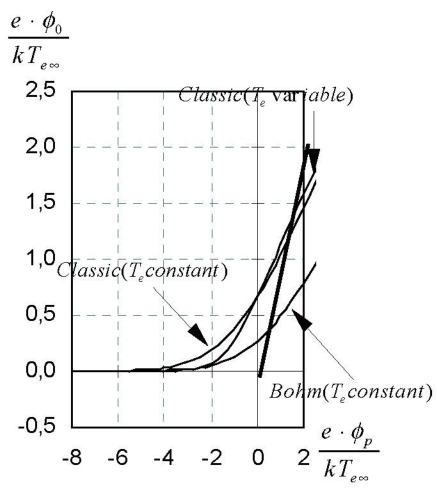

FIG.2 Average potential

Finally Fig. 3 compares normalized results for the current,

$$I = \int_0^R 2\pi\, r dr\, e n v_{ez}(r / R,\, z / L_z \to 0),$$

with results obtained assuming isothermal electrons, [3] and the Bohm diffusion model.

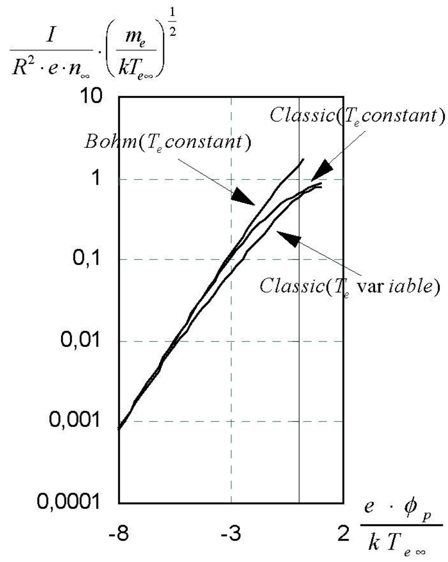

FIG.3 Electron current

In all figures, we consider the following values for the parameters

$$Z = 1\ ;\ \frac{R}{l_{e_\infty}} = 10\ ;\ \frac{\lambda_{e_\infty}}{l_{e_\infty}} = 300\ ;\ \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} = 0.01$$

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