The final journal version appears in
Peternell M., Gruber D., Sendra J. (2013). "Conchoid surfaces of spheres", Computer Aided Geometric Design, volume 30, issue 1, pp. 35-44.
Available by Elsevier via http://dx.doi.org/10.1016/j.cagd.2012.06.005

Conchoid surfaces of spheres

Martin Peternell[a], David Gruber[a], Juana Sendra[b]

[a] *Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria*
[b] *Dpto. Matemática Aplicada a I. T. Telecomunicación, Univ. Politécnica de Madrid, Spain*

Abstract

The conchoid of a surface F with respect to given fixed point O is roughly speaking the surface obtained by increasing the radius function with respect to O by a constant. This paper studies *conchoid surfaces of spheres* and shows that these surfaces admit rational parameterizations. Explicit parameterizations of these surfaces are constructed using the relations to pencils of quadrics in $\mathbb{R}^3$ and $\mathbb{R}^4$. Moreover we point to remarkable geometric properties of these surfaces and their construction.

Keywords: sphere, pencil of quadrics, rational conchoid surface, polar representation, rational radius function.

---

1. Introduction

The conchoid is a classical geometric construction and dates back to the ancient Greeks. Given a planar curve C, a fixed point O (focus point) and a constant distance d, the conchoid D of C with respect to O at distance d is the (Zariski closure of the) set of points Q in the line OP at distance d of a moving point P varying in the curve C,

$$D = \{Q \in OP \text{ with } P \in C, \text{ and } \overline{QP} = d\}^*, \tag{1}$$

where the asterisk denotes the Zariski closure. For a more formal definition of conchoids in terms of diagrams of incidence we refer to [12, 13].

The definition of the conchoid surface to a given surface F in space with respect to a given point O and distance d follows analogous lines.

We aim at studying real rational surfaces in 3-space whose conchoid surfaces are also rational and real. A surface $F \subset \mathbb{R}^3$ will be represented by a polar representation $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$, where $\mathbf{k}(u,v)$ is a parameterization of the unit sphere S^2. Without loss of generality we assume $O = (0,0,0)$. Consequently their conchoid surfaces F_d for varying distance d admit the polar representation $\mathbf{f}_d(u,v) = (\rho(u,v) \pm d)\mathbf{k}(u,v)$.

Since we want to determine classes of surfaces whose conchoid surfaces for varying distances are rational, we focus at rational polar surface representations. Then the 'base' surface F and its conchoids F_d correspond to the same rational parameterization $\mathbf{k}(u,v)$ of the unit sphere S^2. The following definition excludes possibly occurring cases where F and F_d are rational, but their rational parameterizations $\mathbf{f}$ and/or $\mathbf{f}_d$ are not corresponding to a rational representation $\mathbf{k}(u,v)$ of S^2.

Definition 1. *A surface F is called* rational conchoid surface *with respect to the focus point $O = (0,0,0)$ if F admits a* rational polar representation *$\rho(u,v)\mathbf{k}(u,v)$, with a rational radius function $\rho(u,v)$ denoting the distance function from O to F and a rational parameterization $\mathbf{k}(u,v)$ of S^2.*

Contribution:. The main contribution of this article is the study of the conchoid surfaces of spheres. We prove that a sphere F in $\mathbb{R}^3$ admits a rational polar representation $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$ with a rational radius function $\rho(u,v)$ and a particular rational parameterization $\mathbf{k}(u,v)$ of the unit sphere S^2, independently of the relative position of the sphere F and the focus point O. This implies that the conchoids G of F with respect to any focus in $\mathbb{R}^3$ admit rational parameterizations.

It is remarkable that an analogous result to this contribution for spheres does not exist for circles and conics in $\mathbb{R}^2$. The conchoid curves of conics C are only rational if either $O \in C$ or O coincides with one of C's focal points.

Two constructions to prove the main result are presented. The first one uses the cone model being introduced in Section 1.1 and studies a pencil of quadrics in $\mathbb{R}^4$. This construction is explicit and leads to a surprisingly simple solution and a rational polar representation of a sphere. The second approach investigates pencils of quadrics in $\mathbb{R}^3$ containing a sphere and a cone of revolution whose base locus is a rational quartic with rational distance from O.

1.1. The cone model

Let F be a surface in $\mathbb{R}^3$ and let G be its conchoid surface at distance d with respect to the origin $O = (0,0,0)$ as focal point. The construction of the conchoid surfaces G of the 'base' surface F is performed as follows. Consider Euclidean 4-space $\mathbb{R}^4$ with coordinate axis x, y, z and w, where $\mathbb{R}^3$ is embedded in $\mathbb{R}^4$ as the hyperplane $w = 0$. Consider the quadratic cone $D : x^2 + y^2 + z^2 - w^2 = 0$ in $\mathbb{R}^4$. Further, let A be the cylinder through F, whose generating lines are parallel to w. Note that A as well as D are three-dimensional manifolds in $\mathbb{R}^4$. The conchoid construction of the 'base' surface F is based on the study of the intersection $\Phi = A \cap D$, which is typically a two-dimensional surface in $\mathbb{R}^4$.

For a given parameterization $\mathbf{f}(u,v)$ of F in $\mathbb{R}^3$, the cylinder A through F admits the representation $\mathbf{a}(u,v,s) = (f_1, f_2, f_3, 0) + s(0,0,0,1)$. Let F be a rational surface and $\mathbf{f}(u,v)$ be rational. If the intersection $\Phi = A \cap D$ is a rational surface in $\mathbb{R}^4$, then it is obvious that F admits a rational polar representation. Let $\varphi(a,b) = (\varphi_1, \ldots, \varphi_4)(a,b)$ be a rational representation of Φ in $\mathbb{R}^4$, then $(\varphi_1, \varphi_2, \varphi_3)(a,b)$ is obviously a rational polar representation of F. Since $\varphi_4^2 = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$ holds, $\mathbf{k} = 1/\varphi_4(\varphi_1, \varphi_2, \varphi_3)$ is a rational parameterization of S^2 and $\rho(a,b) = \varphi_4(a,b)$ is a rational radius function of F. We summarize the construction.

Theorem 2. *The rational conchoid surfaces $F \subset \mathbb{R}^3$ are in bijective correspondence to the rational 2-surfaces in the quadratic cone $D : x^2 + y^2 + z^2 - w^2 = 0$ in $\mathbb{R}^4$.*

Proof: We proved already that for a rational surface $\Phi \subset D$, its orthogonal projection $(\varphi_1, \varphi_2, \varphi_3)$ onto $\mathbb{R}^3$ is a rational conchoid surface with rational radius function φ_4. Conversely, any rational conchoid surface F with respect to O is defined by a rational polar parameterization $\rho(u,v)\mathbf{k}(u,v)$, with $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{R}(u,v)^3$ and $\|\mathbf{k}\| = 1$. The corresponding surface $\Phi \subset D$ is represented by $\varphi(u,v) = \rho(k_1, k_2, k_3, 1)(u,v)$. $\square$

The quadratic cone D possesses universal parameterizations and we may use them to specify all possible rational parameterizations of rational conchoid surfaces. The construction starts with rational universal parameterizations of the unit sphere S^2. Following [4] we choose four arbitrary polynomials $a(u,v)$, $b(u,v)$, $c(u,v)$ and $d(u,v)$ without common factor. Let

$$\alpha = 2(ac + bd), \beta = 2(bc - ad), \gamma = a^2 + b^2 - c^2 - d^2, \delta = a^2 + b^2 + c^2 + d^2,$$

then $\mathbf{k}(u,v) = \frac{1}{\delta}(\alpha, \beta, \gamma)$ is a rational parameterization of the unit sphere S^2. Thus $\varphi(u,v) = \rho(u,v)(\alpha, \beta, \gamma, \delta)$ with a non-zero rational function $\rho(u,v)$ is a rational parameterization of a two-dimensional surface $\Phi \subset D$. Consequently

$$\mathbf{f}(u,v) = \rho(u,v) \left(\frac{\alpha}{\delta}, \frac{\beta}{\delta}, \frac{\gamma}{\delta} \right)(u,v) = \rho(u,v)\mathbf{k}(u,v),$$

is a rational polar representation of a rational conchoid surface F in $\mathbb{R}^3$, with $\rho(u,v)$ as radius function and $\mathbf{k}(u,v)$ as rational parameterization of the unit sphere S^2. It is sufficient to consider polynomials and the construction reads as follows.

Corollary 3. *Given six relatively prime polynomials $a(u,v)$, $b(u,v)$, $c(u,v)$, $d(u,v)$, and $r(u,v)$ and $s(u,v)$, a universal parameterization of a rational 2-surface $\Phi \subset D$ in $\mathbb{R}^4$ is given by*

$$\varphi(u,v) = \frac{r}{s}\left(2(ac+bd), 2(bc-ad), a^2+b^2-c^2-d^2, a^2+b^2+c^2+d^2\right)(u,v). \tag{2}$$

Consequently, a universal rational parameterization of a rational conchoid surface reads

$$\mathbf{f}(u,v) = \frac{r}{s(a^2+b^2+c^2+d^2)}\left(2(ac+bd), 2(bc-ad), a^2+b^2-c^2-d^2\right)(u,v). \tag{3}$$

This is a general result about all rational parameterizations of rational conchoid surfaces. For a particular given rational surface F it is difficult to decide whether the intersection $\Phi = D \cap W$ admits rational parameterizations or not. Typically the surface Φ is not rational. Nevertheless, there are interesting non-trivial cases where Φ admits rational parameterizations.

In [7] it has been proved that conchoids of rational ruled surfaces F are rational. We give a hint how this result can be proved with help of the cone model D and Theorem 2. If F is a ruled surface, the cylinder $A \subset \mathbb{R}^4$ carries a one-parameter family of planes parallel to the w-axis. These planes pass through the generating lines of F. This implies that typically the intersection $\Phi = A \cap D$ carries a one-parameter family of conics obtained as intersections of the mentioned planes with D. This family of conics is rational, and it is known ([6, 9]) that there exist rational parameterizations $\varphi(u,v)$ of Φ. Thus the conchoids of real rational ruled surfaces are rational.

In this context we mention a trivial but useful statement which we prove for completeness.

Lemma 4. *Given a rational curve C with parameterization $\mathbf{c}(t)$ on a rotational cone D, then the distance $\|\mathbf{c}(t) - \mathbf{v}\|$ between the curve C and the vertex $\mathbf{v}$ of D is a rational function.*

Proof: We use a special coordinate system with $\mathbf{v}$ at the origin, and z as rotational axis of D. This implies that D is the zero set of $x^2+y^2-\gamma^2 z^2$. Without loss of generality we let $\gamma = 1$. The given curve C admits therefore a rational parameterization $\mathbf{c}(t) = (c_1, c_2, c_3)(t)$ satisfying $c_1^2 + c_2^2 = c_3^2$. Obviously one obtains $\|\mathbf{c}(t)\| = \sqrt{2}c_3(t)$ being rational. □

2. Conchoids of spheres

Given a sphere F in $\mathbb{R}^3$ and an arbitrary focus point O, the question arises if there exists a rational representation $\mathbf{f}(u,v)$ of F with the property that $\|\mathbf{f}(u,v)\|$ is a rational function of the parameters u and v. To give a constructive answer to this question we describe an approach using the cone-model presented in Section 1.1. Later on in Section 3 we study a different method working in $\mathbb{R}^3$ directly. There are several relations between these methods which will be discussed along their derivation.

Let F be the sphere with center $\mathbf{m} = (m,0,0)$ and radius r, and let $O = (0,0,0)$. Thus F is given by

$$F : (x-m)^2 + y^2 + z^2 - r^2 = 0. \tag{4}$$

If $m = 0$, the center of F coincides with O. In this trivial situation the conchoid surface of F is reducible and consists of two spheres, where one might degenerate to F's center if $d = r$. If $m^2 - r^2 = 0$, the focal point O is contained in F. To construct a rational polar representation, we make the ansatz $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$ with $\mathbf{k}(u,v) = (k_1,k_2,k_3)(u,v)$ and $\|\mathbf{k}(u,v)\| = 1$ and an unknown radius function $\rho(u,v)$. Plugging this into (4), we obtain a rational polar representation with rational radius function $\rho(u,v) = 2mk_1(u,v)$. Note that in this case the conchoid is irreducible and rational.

2.1. Pencil of quadrics in $\mathbb{R}^4$

Consider the Euclidean space $\mathbb{R}^4$ with coordinate axes x, y, z and w and let $\mathbb{R}^3$ be embedded as the hyperplane $w = 0$. Let a sphere $F \subset \mathbb{R}^3$ be defined by (4) and $O = (0, 0, 0)$. To study the general case we assume $m \neq 0$ and $m^2 \neq r^2$. The equation of the cylinder $A \subset \mathbb{R}^4$ through F with w-parallel lines agrees with the equation of F in $\mathbb{R}^3$,

$$A : (x - m)^2 + y^2 + z^2 - r^2 = 0. \tag{5}$$

Consider the pencil $Q(t) = A + tD$ of quadrics in $\mathbb{R}^4$, spanned by A and the quadratic cone $D : x^2 + y^2 + z^2 = w^2$ from Section 1.1. Any point $\overline{X} = (x, y, z, w) \in D$ has the property that the distance from $X = (x, y, z)$ to O in $\mathbb{R}^3$ equals w. We study the geometric properties of the del Pezzo surface $\Phi = A \cap D$ of degree four, the base locus of the pencil of quadrics $Q(t)$. According to Theorem 2, the sphere F is a rational conchoid surface exactly if Φ admits rational parameterizations.

Besides A and D there exist two further singular quadrics in $Q(t)$. These quadrics are obtained for the zeros $t_1 = -1$ and $t_2 = r^2/\gamma^2$ of the characteristic polynomial

$$\det(A + tD) = -(1 + t)^2 t(\gamma^2 t - r^2), \text{ with } \gamma^2 = m^2 - r^2 \neq 0. \tag{6}$$

The quadric corresponding to the twofold zero $t_1 = -1$ is a cylinder

$$R : w^2 - 2mx + m^2 - r^2 = 0. \tag{7}$$

Its directrix is a parabola in the xw-plane and its two-dimensional generators are parallel to the yz-plane. The singular quadric S corresponding to $t_2 = r^2/\gamma^2$ is a quadratic cone and reads

$$S : \left(x - \frac{m^2 - r^2}{m}\right)^2 + y^2 + z^2 = \frac{r^2}{m^2} w^2.$$

Its vertex is the point $O' = (\frac{m^2-r^2}{m}, 0, 0, 0)$. The intersections of S with three-spaces $w = c$ are spheres $\sigma(c)$, whose top view projections in $w = 0$ are centered at O' and their radii are rc/m. The intersections of D with three-spaces $w = c$ are spheres $d(c)$ whose top view projections in $w = 0$ are centered at O with radii c. The intersections $k(c) = s(c) \cap d(c)$ of these spheres ($w = c$) are circles in planes $x = (c^2 + m^2 - r^2)/(2m)$. Thus Φ contains a family of conics, whose top view projections are the circles $k(c)$. The conics in Φ are contained in the planes

$$\varepsilon(c) : x = \frac{c^2 + m^2 - r^2}{2m}, w = c.$$

The half opening angle δ of D with respect to the w-axis is $\pi/4$, thus $\tan\delta = 1$. The half opening angle σ of S is given by $\tan\sigma = r/m$, see Figure 1(a). Applying the scaling

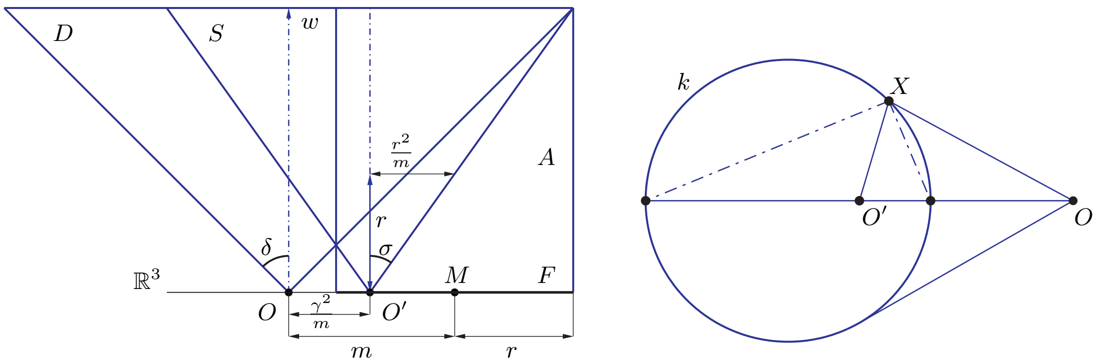
$$(x', y', z', w') = (fx, fy, fz, w), \text{ with } f = \frac{r}{m}$$

in $\mathbb{R}^4$ maps D to a congruent copy of S. Consider a point $\overline{X} = (x, y, z, w)$ in $\Phi = A \cap D$ and its projection $X = (x, y, z)$ in F. The distance $\operatorname{dist}(X, O)$ of X to O in $\mathbb{R}^3$ is w. For the distance $\operatorname{dist}(X, O')$ between X and O' we consequently obtain

$$\operatorname{dist}(X, O') = \frac{r}{m} \operatorname{dist}(X, O), \text{ for all } X \in F. \tag{8}$$

Remark on the circle of Apollonius. Note that O' is the inverse point of O with respect to the sphere F. It is an old result by Apollonius Pergaeus (262–190 b.c.) that the set of points X in the plane having constant ratio of distances $f = d/d'$, with $d = \operatorname{dist}(O, X)$ and $d' = \operatorname{dist}(O', X)$, from two given fixed points O and O', respectively, is a circle k, see Fig. 1(b). Rotating k around the line OO' gives the sphere F and O and O' are inverse points with respect to F (and the circle k).

If we consider a varying constant ratio f, one obtains a family of spheres $F(f)$ with inverse points O and O' which form an elliptic pencil of spheres. Their centers are on the line OO'. Ratio 1 ($d = d'$) corresponds to the bisector plane of O and O'.



(a) Sketch of the quadric pencil in $\mathbb{R}^4$ (b) Apollonius circle

Figure 1: Pencil of quadrics in $\mathbb{R}^4$ and Apollonius circle

2.2. A rational quartic on the sphere

The pencil of quadrics $Q(t)$ in $\mathbb{R}^4$ spanned by the sphere F and the cone D contains the cylinder R. Expressing the variable x from (7) one gets

$$x = \frac{w^2 + m^2 - r^2}{2m}, \tag{9}$$

and inserting this into D results in the polynomial

$$\alpha(w) : 4m^2(y^2 + z^2) + p(w) = 0, \text{ with } p(w) = w^4 - 2w^2(m^2 + r^2) + (m^2 - r^2)^2. \tag{10}$$

Considering y and z as variables, $\alpha(w)$ is a one-parameter family of conics (circles) in the yz-plane, depending *rationally* on the parameter w. The circles $\alpha(w)$ do not possess real points for all w, but there exist intervals determining families of real circles $\alpha(w)$. To obtain real circles one has to perform a re-parameterization $w(u)$ within an appropriate interval. The factorization of $p(w)$ reads

$$p(w) = (w + a)(w - a)(w + b)(w - b), \text{ with } a = m + r, \text{ and } b = m - r.$$

If O is outside of F, thus $m > r$, the polynomial $-p(w)$ is positive in the interval $[m - r, m + r]$. Thus a possible re-parameterization is

$$w(u) = \frac{au^2 + b}{1 + u^2} = \frac{u^2(m + r) + m - r}{1 + u^2}. \tag{11}$$

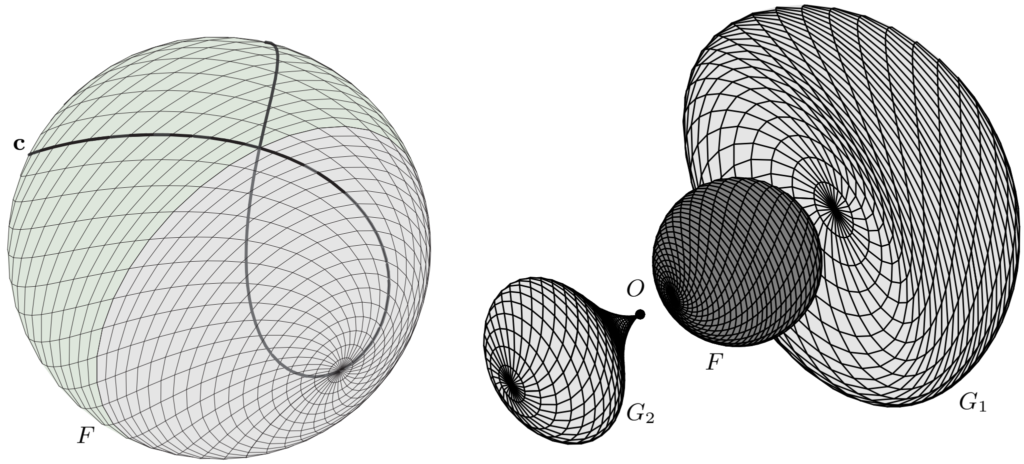
Otherwise we could re-parameterize over another appropriate interval. Additionally we note that if O is inside of F, the inverse point O' is outside of F. Since equation (8) holds for the distances of a point $X \in F$ to O and O', we can exchange roles and perform the computation for the point O'.

We return to the family of conics $\alpha(w)$. Substituting (11) into (10) leads to a family of real conics

$$\alpha(u) : y^2 + z^2 = \frac{4r^2u^2}{m^2(1 + u^2)^4}(au^2 + m)(mu^2 + b). \tag{12}$$

We are looking for rational functions $y(u)$ and $z(u)$ satisfying (12) identically. Therefore we introduce auxiliary variables $\tilde{y}$ and $\tilde{z}$ by the relations $y = 2\tilde{y}ru/(m(1+u^2)^2)$ and $z = 2\tilde{z}ru/(m(1+u^2)^2)$. We obtain $\tilde{y}^2+\tilde{z}^2 = (au^2+m)(mu^2+b)$. Factorizing left and right hand side of this equation results in a linear system to determine $\tilde{y}$ and $\tilde{z}$,

$$\begin{aligned} \tilde{y} + \mathrm{i}\tilde{z} &= (\sqrt{a}u + \mathrm{i}\sqrt{m})(\sqrt{m}u + \mathrm{i}\sqrt{b}), \\ \tilde{y} - \mathrm{i}\tilde{z} &= (\sqrt{a}u - \mathrm{i}\sqrt{m})(\sqrt{m}u - \mathrm{i}\sqrt{b}). \end{aligned}$$



(a) Base curve of the pencil in $\mathbb{R}^3$ (b) Sphere F and both conchoid surfaces G_1 and G_2 w.r.t. O

Figure 2: Rational polar representation of a sphere and its conchoid surfaces

The solution $\tilde{y} = \sqrt{m}(\sqrt{a}u^2 - \sqrt{b})$, $\tilde{z} = u(m + \sqrt{ab})$ finally leads to

$$y(u) = \frac{2r\sqrt{m}u}{m(1+u^2)^2}\left(\sqrt{a}u^2 - \sqrt{b}\right), \text{ and } z(u) = \frac{2ru^2}{m(1+u^2)^2}\left(m + \sqrt{ab}\right), \tag{13}$$

which is a rational parameterization of a curve in the yz-plane, following the family of conics $\alpha(w)$.

We note that any real rational family of conics possesses real rational parameterizations, see for instance [6, 9]. The solution (13) together with (9) determines a curve $C \subset F$ which possesses the rational distance function

$$\|\mathbf{c}(u)\| = w(u) = \frac{u^2(m+r) + (m-r)}{1+u^2} \tag{14}$$

with respect to O. Its parameterization is

$$\mathbf{c}(u) = \frac{1}{m(1+u^2)^2}\begin{pmatrix} u^4 m(m+r) + 2u^2(m^2 - r^2) + m(m-r) \\ 2r\sqrt{m}u(u^2\sqrt{m+r} - \sqrt{m-r}) \\ 2ru^2(m + \sqrt{m^2 - r^2}) \end{pmatrix}. \tag{15}$$

Theorem 5. *Let F be a sphere and let O be an arbitrary point in $\mathbb{R}^3$. Then there exists a rational quartic curve $C \subset F$ and a rational parameterization $\mathbf{c}(u)$ of C such that the distance of C to O is a rational function in the curve parameter u.*

Rotating C around the x-axis leads to a rational polar representation $r(u,v)\mathbf{k}(u,v)$ of F with rational distance function $\rho(u,v) = w(u)$ from O. The quartic curve C together with this parameterization is illustrated in Fig. 2(a). Fig. 2(b) displays a sphere F together with both conchoid surfaces G_1 and G_2 for distances d and $-d$ with respect to O. We summarize the presented construction.

Theorem 6. *Spheres in $\mathbb{R}^3$ admit rational polar representations with respect to any focus point O. This implies that the conchoid surfaces of spheres admit rational parameterizations. The construction is based on rational quartic curves on F with rational distance from O.*

Rationality and Uni-Rationality. The construction performed in Section 2.2 yields a rational parameterization $\mathbf{f}(u,v)$ of the sphere F with rational radius function $\rho(u,v)$, given by (14), such that $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$, where $\mathbf{k}(u,v)$ is an improper parameterization of the unit sphere S^2. This means that typically a point $X \in F$ corresponds to two points (u, v_1) and (u, v_2) in the parameter domain. Rotating the curve C around the x-axis, the sphere F is double covered.

The conchoid surface G of F at distance d typically consists of two surfaces G_1 and G_2, which admit the rational parameterizations

$$\mathbf{g}_1 = (\rho(u,v) + d)\mathbf{k}(u,v), \text{ and } \mathbf{g}_2 = (\rho(u,v) - d)\mathbf{k}(u,v), \tag{16}$$

for positive and negative distance. The conchoid $G = G_1 \cup G_2$ is an irreducible algebraic surface of order six. It is *not bi-rational* equivalent to the projective plane but each component G_1 as well as G_2 admits improper *rational parameterizations*. These components G_1 and G_2 are called *uni-rational*. This is not a contradiction to Castelnuovo's theorem since we are not working over an algebraically closed field but over the field of real numbers $\mathbb{R}$.

Let us consider an example to illustrate these properties. We consider the sphere F with center $\mathbf{m} = (3/2, 0, 0)$ and radius $r = 1$, and compute its conchoid G for variable distance d. We obtain parameterizations $\mathbf{g}_1(u,v)$ and $\mathbf{g}_2(u,v)$ from equation (16) for the real uni-rational varieties G_1 and G_2. The algebraic variety $G = G_1 \cup G_2$ is given by the equation

$$\begin{aligned} G: \quad & (x^2+y^2+z^2)(4(x^2+y^2+z^2) - 12x + 5)^2 \\ & + d^2(40(x^2+y^2+z^2) - 144x^2 + 96x(x^2+y^2+z^2) - 32(x^2+y^2+z^2)^2) \\ & + 16d^4(x^2+y^2+z^2) = 0. \end{aligned} \tag{17}$$

Remarks on the parameterization. The rational quartic C on F is of course not unique but depends on the re-parameterization (11). An admissible rational re-parameterization of a real interval is of even degree. Let us consider a quadratic re-parameterization. Since α is of degree four in w, the re-parameterized family is typically of degree ≤ 8 in u. This implies that the solutions $y(u)$ and $z(u)$ are of degree ≤ 4, which holds also for $x(u)$ because of (9). The coefficient functions $\mathbf{c}(u) = (x,y,z)(u)$ determine a rational quartic C on F, with rational norm $\|\mathbf{c}\| = w(u)$.

Different choices of the interval and a quadratic re-parameterization will typically result in different quartic curves on F. In (11) we have chosen the largest possible interval and a rational function satisfying $w(-u) = w(u)$ and obtained the curve C through antipodal points of F. By rotating we obtain the full sphere, doubly covered.

For any quadratic re-parameterization, the quartic C is the base locus of a pencil of quadrics $Q(t) = F + tK$, spanned by the sphere F and, for instance, the quadratic projection cone K with vertex at C's double point.

The particular choice (11) implies that the quartic C is symmetric with respect to the xz-plane. This holds since u appears only with even powers in x and z, thus we have $x(-u) = x(u)$ and $z(-u) = z(u)$. The orthogonal projection of C to the xz-plane is doubly covered, thus a conic. In this case $(x,z)(u)$ parameterizes a parabola, because of the factor $(1+u^2)^2$ in $\mathbf{c}(u)$'s denominator. This implies that the pencil $Q(t)$ can also be spanned by the sphere F and the parabolic cylinder P passing through C, whose generating lines are parallel to y. It can be proved that all quadrics in $Q(t)$ except P are rotational quadrics with parallel axes. This implies that K is a rotational cone, and the remaining singular quadric L is a rotational cone, too. For the particular choice (11) and for the generalized construction performed in Section 3, the rotational cone L has the vertex O. We note that for any admissible re-parameterization L's vertex is typically different from O.

2.3. Pencil of quadrics in $\mathbb{R}^3$

The quartic curve C from (15) on the sphere F is the base locus of a pencil of quadrics $F + \lambda K$ in $\mathbb{R}^3$, spanned by F and the projection cone K of C from its double point $\mathbf{s}$, see Fig. 3. The double point $\mathbf{s}$ is located in the symmetry plane of C and in the polar plane of the origin O with respect to F. Its coordinates are

$$\mathbf{s} = \frac{1}{m}(\gamma^2, 0, r\gamma) \text{ with } \gamma^2 = m^2 - r^2. \tag{18}$$

The pencil $F+\lambda K$ contains two further singular quadrics which are obtained for the zeros $\lambda_1 = 1/m$ and $\lambda_2 = -1/\gamma$ of the characteristic polynomial

$$\det(F + \lambda K) = r^2(m\lambda - 1)(\gamma\lambda + 1).$$

Corresponding to λ_1 there is a parabolic cylinder P with y-parallel generating lines passing through C. Corresponding to λ_2 we find the rotational cone L through C with vertex O.

To give explicit representations for the quadrics we use homogeneous coordinates $\mathbf{y} = (1, x, y, z)^T$. Since there should not be any confusion, we use same notations for the quadric F and its coordinate matrix appearing in the homogeneous quadratic equation $\mathbf{y}^T \cdot F \cdot \mathbf{y} = 0$. The coefficient matrices F and K read

$$F = \begin{pmatrix} m^2 - r^2 & -m & 0 & 0 \\ -m & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; K = \begin{pmatrix} \gamma^3 & -\gamma m & 0 & 0 \\ -\gamma m & \gamma & 0 & r \\ 0 & 0 & -m & 0 \\ 0 & r & 0 & -\gamma \end{pmatrix}. \tag{19}$$

An elementary computation shows that K is a cone of revolution with opening angle $\pi/2$ and $\mathbf{a} = (m + \gamma, 0, r)$ denotes a direction vector of its axis.

The cone L through C with vertex at O is again a cone of revolution, whose axis is parallel to $\mathbf{a}$. The parabolic cylinder P through the quartic C has y-parallel generating lines. The axis of the cross section parabola in the xz-plane is orthogonal to $\mathbf{a}$, see Fig. 3(a). The coefficient matrices L and P are

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \\ 0 & 0 & m+\gamma & 0 \\ 0 & -r & 0 & 2\gamma \end{pmatrix}, \; P = \begin{pmatrix} \gamma^2(m+\gamma) & -m(m+\gamma) & 0 & 0 \\ -m(m+\gamma) & m+\gamma & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & r & 0 & m-\gamma \end{pmatrix}. \tag{20}$$

A trigonometric parameterization of the quartic C is obtained by intersecting the cone K with one quadric of the pencil $F + \lambda K$, for instance F. Let $\mathbf{a}$ be a unit vector in direction of K's axis, and $\mathbf{b}$ and $\mathbf{c}$ complete it to an orthonormal basis in $\mathbb{R}^3$. A trigonometric parameterization of K is given by

$$\begin{aligned} &\mathbf{k}(t, v) := \mathbf{s} + v(\mathbf{a} + (\mathbf{b}\cos t + \mathbf{c}\sin t)), \text{ with} \\ &\mathbf{a} = \tfrac{1}{\sqrt{2m(m+\gamma)}}(m+\gamma, 0, r), \; \mathbf{b} = (0, -1, 0), \text{ and } \mathbf{c} = \tfrac{1}{\sqrt{2m(m+\gamma)}}(r, 0, -(m+\gamma)). \end{aligned}$$

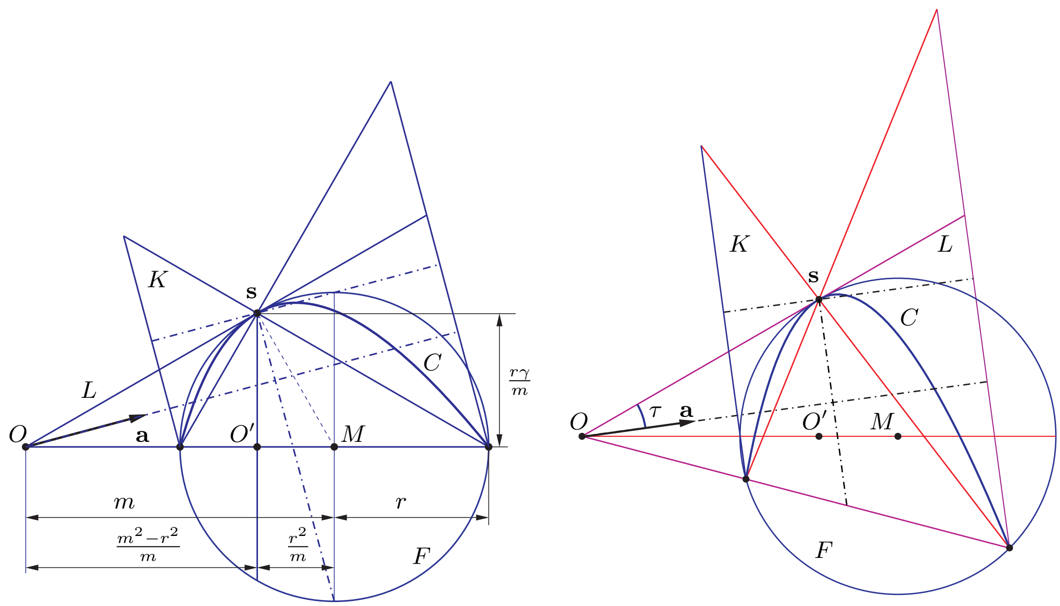
Thus K admits the explicit parameterization

$$\mathbf{k}(t, v) = \frac{1}{2m\sqrt{m(m+\gamma)}} \begin{pmatrix} 2\gamma^2\sqrt{m(m+\gamma)} + v\sqrt{2}m(m+\gamma+r\sin t) \\ -2vm\sqrt{m(m+\gamma)}\cos t \\ 2r\gamma\sqrt{m(m+\gamma)} + v\sqrt{2}m(r - (m+\gamma)\sin t) \end{pmatrix}.$$

Finally, a trigonometric parameterization of the quartic C follows by

$$\mathbf{c}(t) = \frac{1}{2m} \begin{pmatrix} (m + r\sin t)^2 + \gamma^2 \\ \sqrt{2}\sqrt{m(m+\gamma)}\cos t(\gamma - m - r\sin t) \\ r(m+\gamma)\cos^2 t \end{pmatrix}, \text{ with } \|\mathbf{c}(t)\| = m + r\sin t. \tag{21}$$

The correspondence of the trigonometric parameterization and its norm with the expressions (15) and (14) in terms of rational functions is realized by the substitutions $\sin t = (u^2-1)/(u^2+1)$ and $\cos t = 2u/(u^2+1)$ and some rearrangement of the equations. Section 2.4 discusses relations to Viviani's curve (or Viviani's window). This particular quartic has a similar shape and its pencil of quadrics has similar properties. Viviani's curve has an additional symmetry.



(a) Symmetric Case

(b) General Case

Figure 3: Geometric properties of the conchoid construction

Remark. The inversion with center O at the sphere which intersects the given sphere F perpendicularly, maps the sphere F onto itself. Analogously this inversion fixes the rotational cone L. Thus the quartic intersection curve $C = F \cap L$ remains fixed as a whole, but of course not point-wise. The product of the distances $\mathrm{dist}(O, P)$ and $\mathrm{dist}(O, P')$ of two inverse points $P \in F$ and $P' \in F$ equals $\sqrt{m^2 - r^2}$. This property follows from the elementary tangent-secant-theorem of a circle.

2.4. Relations to Viviani's curve

The quartic curve C, the base locus of the pencil of quadrics $F + tK$, can be considered as generalization of Viviani's curve V. This particularly well known curve V is the base locus of a pencil of quadrics, spanned by a sphere F and a cylinder of revolution L touching F and passing through the center of F. The pencil of quadrics of Viviani's curve also contains a right circular cone K with vertex in V's double point and opening angle $\pi/2$, and further a parabolic cylinder P. Viviani's curve V is obtained from C by letting $O \to \infty$. Consequently, the inverse point O' becomes the center of the sphere F.

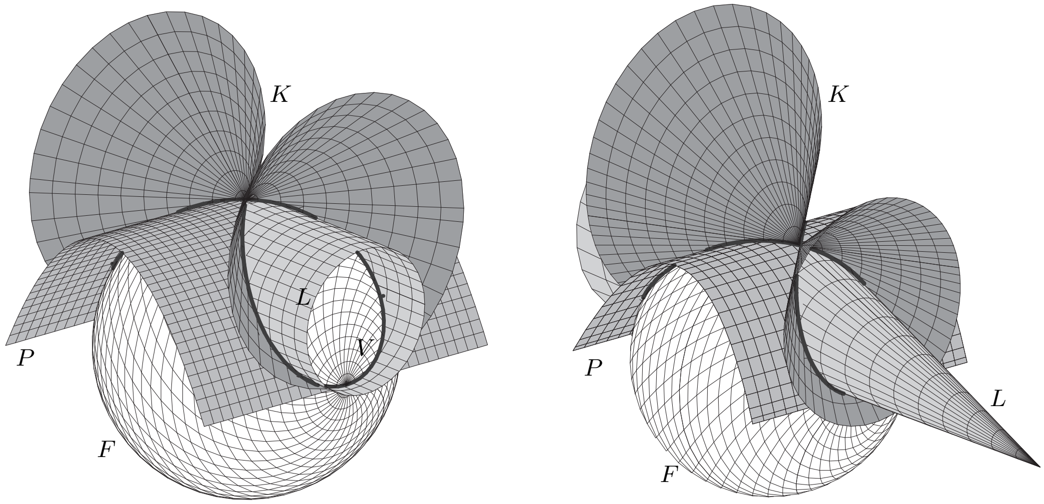
Choosing the inverse point $O' = (\frac{m^2-r^2}{m}, 0, 0)$ as origin, the parameterization (21) of C becomes

$$\mathbf{c}(t) = \frac{1}{2m}\begin{pmatrix} r^2(1+\sin^2 t) + 2mr\sin t \\ \sqrt{2}\sqrt{m(m+\gamma)}\cos t(\gamma - m - r\sin t) \\ r(m+\gamma)\cos^2 t \end{pmatrix}. \tag{22}$$

By letting $m \to \infty$ one obtains V as limit curve

$$\mathbf{v}(t) = \left(r\sin t, -r\sin t\cos t, r\cos^2 t\right). \tag{23}$$

Fig. 4(a) illustrates Viviani's curve V, together with the sphere and the singular quadrics belonging to the pencil. The generalized Viviani curve C being the base locus of the pencil appearing in the conchoid construction of the sphere is illustrated in Fig. 4(b). In contrast to the classical Viviani curve V whose single parameter r is the radius of the sphere F, the quartic curve C has two parameters r and m.



(a) Quadric pencil of Viviani's curve (b) Quadric pencil of the quartic C

Figure 4: Quadric pencils of Viviani's curve and its generalization

3. Rotational quadrics with parallel axes

We consider the mentioned pencil of quadrics $Q(t) = A + tD$ from Section 2.1, and a hyperplane $E : ax + by + cz - dw = 0$ passing through $O = (0, 0, 0, 0)$. The intersection $D \cap E$ is a quadratic cone whose projection onto $\mathbb{R}^3$ is a cone of revolution L with axis in direction of $\mathbf{a} = (a, b, c)$. Assuming $\|\mathbf{a}\| = 1$, the opening angle 2τ of L is determined by $d = \cos\tau$.

Consider the quartic intersection curve $C = F \cap L$ of a sphere F and the cone of revolution L. It is rational exactly if the cone L is touching F at a single point. Since this touching point has to be contained in the polar plane of $O = (0, 0, 0)$ with respect to F, we choose $\mathbf{s} = (\gamma^2/m, 0, r\gamma/m)$ (compare 18) and prescribe an arbitrary opening angle 2τ for L. Thus the unit direction vector of L's axis is

$$\mathbf{a} = \frac{1}{m}(\gamma\cos\tau - r\sin\tau, 0, \gamma\sin\tau + r\cos\tau) = (a, b, c).$$

The quartic C is real if the axis is contained in the wedge formed by $\mathbf{s}$ and the x-axis, see Figure 3(b). Thus $-r/\gamma \le \tan\tau \le 0$, because the rotation from $\mathbf{s}$ to $\mathbf{a}$ by $\tau \le 0$ is counterclockwise. In the following we use the abbreviations $ct := \cos\tau$ and $st := \sin\tau$. The quadrics of the pencil with base locus C are denoted similarly to Section 2.3. The coefficient matrix of the projection cone L reads

$$L(\tau) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r^2(ct^2 - st^2) + 2\gamma r st ct & 0 & -\gamma r(ct^2 - st^2) + (r^2 - \gamma^2) st ct \\ 0 & 0 & m^2 ct^2 & 0 \\ 0 & -\gamma r(ct^2 - st^2) + (r^2 - \gamma^2) st ct & 0 & \gamma^2(ct^2 - st^2) - 2\gamma r st ct \end{pmatrix}.$$

Rewriting $L(\tau)$ in terms of the double angle 2τ and substituting

$$\cos 2\tau = \gamma/m, \text{ and } \sin 2\tau = -r/m \tag{24}$$

we obtain L from equation (20). This holds for all equations and parameterizations in this section in an analogous way.

The pencil of quadrics $F + tL(\tau)$ contains two further singular quadrics. The first is a parabolic cylinder $P(\tau)$ passing through C. It corresponds to the eigenvalue $\frac{-1}{m^2 ct^2}$ and its generating lines are parallel to the y-axis. Its coefficient matrix of cylinder reads

$$P(\tau) = \begin{pmatrix} \gamma^2 m^2 ct^2 & -m^3 ct^2 & 0 & 0 \\ -m^3 ct^2 & \gamma^2(ct^2 - st^2) + m^2 st^2 - 2r\gamma stct & 0 & (\gamma^2 - r^2)stct + r\gamma(ct^2 - st^2) \\ 0 & 0 & 0 & 0 \\ 0 & (\gamma^2 - r^2)stct + r\gamma(ct^2 - st^2) & 0 & m^2 ct^2 - \gamma^2(ct^2 - st^2) + 2r\gamma stct \end{pmatrix}.$$

Our goal is not only to characterize the pencil of quadrics but to provide an explicit parameterization of the quartic curve C on F whose distance from O is rational. This is performed by using a parameterization of the second singular quadric K which corresponds to the zero $\frac{r}{\gamma m^2 ctst}$ of the characteristic polynomial $\det(F + tL(\tau))$. K is a cone of revolution with axis parallel to $\mathbf{a}$, and its coefficient matrix reads

$$K(\tau) = \begin{pmatrix} \gamma^2 & -m & 0 & 0 \\ -m & \frac{\gamma(m^2+2r^2)stct+r^3(ct^2-st^2)}{\gamma m^2 stct} & 0 & \frac{-r((\gamma^2-r^2)stct+\gamma r(ct^2-st^2))}{\gamma m^2 stct} \\ 0 & 0 & \frac{\gamma st+rct}{\gamma st} & 0 \\ 0 & \frac{-r((\gamma^2-r^2)stct+\gamma r(ct^2-st^2))}{\gamma m^2 stct} & 0 & \frac{\gamma(\gamma^2-r^2)stct+r\gamma^2(ct^2-st^2)}{\gamma m^2 stct} \end{pmatrix}.$$

A parameterization of the cone of revolution K with respect to its vertex $\mathbf{s}$ is

$$\mathbf{k}(u, v) = \mathbf{s} + v(\mathbf{a} + R(\mathbf{b}\cos u + \mathbf{c}\sin u)),$$

where $\mathbf{a}$ is a unit vector in direction of its axis, and $\mathbf{b}$ and $\mathbf{c}$ complete $\mathbf{a}$ to an orthonormal basis in $\mathbb{R}^3$, and R denotes the radius of the cross section circle at distance 1 from $\mathbf{s}$ which has still to be determined. In detail this reads

$$\mathbf{k}(u, v) = \begin{pmatrix} \frac{\gamma^2}{m} + v(\frac{\gamma ct - rst}{m} + R\frac{\sin u(\gamma st + rct)}{m}) \\ -vR\cos u \\ \frac{\gamma r}{m} + v(\frac{\gamma st + rct}{m} + R\frac{\sin u(-\gamma ct + rst)}{m}) \end{pmatrix}.$$

Inserting $\mathbf{k}(u, v)$ into the equation $\mathbf{y}^T \cdot K(\tau) \cdot \mathbf{y} = 0$ defines the radius

$$R = \frac{\sqrt{-ctst(\gamma st + rct)(\gamma ct - rst)}}{ct(\gamma st + rct)} = \sqrt{\frac{-st(\gamma ct - rst)}{ct(\gamma st + rct)}}.$$

The final parameterization of the quartic curve C is obtained for $v = \frac{2r(R\sin uct - st)}{1+R^2}$ and is a bit lengthy. It reads

$$\mathbf{c}(u) = \begin{pmatrix} \frac{(4Rr\sin uct(\gamma ct - rst) + 2rct(\gamma st + rct)(R^2\sin^2 u - 1) + m^2 + r^2 + R^2(m^2 - r^2) - 2Rr\gamma\sin u)}{m(1+R^2)} \\ \frac{-2Rr\cos u(Rct\sin u - st)}{1+R^2} \\ \frac{r(2R^2 ct\sin^2 u(rst - \gamma ct) + 4R\gamma\sin uctst - \gamma(1-R^2) - 2rctst + 2\gamma ct^2 + 2Rr\sin u(ct^2 - st^2))}{m(1+R^2)} \end{pmatrix}, \tag{25}$$

and its norm is

$$\|\mathbf{c}(u)\| = \frac{\gamma ct(1 + R^2) - 2rst + 2rRct\sin(u)}{ct(1 + R^2)}.$$

This is proved by using the incidence $\mathbf{c} \subset E$, thus $a\mathbf{c}_1 + b\mathbf{c}_2 + c\mathbf{c}_3 = ctw$, with $w = \|\mathbf{c}\|$. Note that R is *not rational* in any rational substitution for the trigonometric functions $\cos\tau$ and $\sin\tau$. Rotating C around the x-axis gives a rational polar representation $\mathbf{f}(u, v)$ of the sphere F. The resulting parameterization $\mathbf{f}$ of F is not proper, but almost all points of F are traced twice, therefore belonging to two parameter values (u_1, v) and (u_2, v). We summarize the construction.

Corollary 7. *There exists a one-parameter family of quartic curves $C(\tau) \subset F$ with double point at $\mathbf{s}$ and symmetry plane $y = 0$. The corresponding pencils of quadrics $Q(t) = F + \lambda L(\tau)$ contain rotational cones $K(\tau)$ and $L(\tau)$, where the vertex of the latter is at O, and a parabolic cylinder $P(\tau)$. Besides $P(\tau)$ all quadrics have rotational symmetry with parallel axes $\mathbf{a}(\tau)$. The distance function* $\mathrm{dist}(OC) = \|\mathbf{c}(u)\|$ *is rational in the curve parameter, but not rational in the angle-parameter τ.*

4. Conclusion

We have discussed the conchoid construction for spheres and have shown that a sphere in $\mathbb{R}^3$ admits a rational polar representation with respect to an arbitrary chosen focus point, which implies that the conchoid surfaces of spheres possess rational parameterizations. Additionally we have given a geometric construction for these parameterizations which are based on a rational curve of degree four being the base locus of a pencil of quadrics in $\mathbb{R}^3$. Relations to the classical Viviani curve have been addressed. The construction of the rational parameterization of the conchoids is also based on a pencil of quadrics in $\mathbb{R}^4$.

Acknowledgments

This work has been partially supported by the Spanish 'Ministerio de Ciencia e Innovación' under the Project MTM2008-04699-C03-01, and by the 'Ministerio de Economia y Competitividad' under the project MTM2011-25816-C02-01.The third author is member of the Research Group asynacs (Ref. CCEE2011/R34).

References

[1] M. Aigner, Bert Jüttler, Laureano Gonzalez-Vega, Josef Schicho, Parameterizing surfaces with certain special support functions, including offsets of quadrics and rationally supported surfaces, Journal of Symbolic Computation **44**, 2009, 180–191.

[2] A. Albano, M. Roggero: Conchoidal transform of two plane curves, Applicable Algebra in Engineering, Communication and Computing, Vol.21, No.4, 2010, pp. 309–328.

[3] D. Cox D., Little J. and O'Shea D., 2010. Ideals, Varieties, and Algorithms. Springer-Verlag, New York.

[4] Dietz, R., Hoschek, J., and Jüttler, B., An algebraic approach to curves and surfaces on the sphere and other quadrics, Comp. Aided Geom. Design 10, pp. 211229, 1993.

[5] Peternell, M. and Pottmann, H., 1998. A Laguerre geometric approach to rational offsets, Comp. Aided Geom. Design **15**, 223–249.

[6] Peternell, M., 1997. Rational Parametrizations for Envelopes of Quadric Families, Thesis, University of Technology, Vienna.

[7] Peternell, M., Gruber, D. and Sendra, J., 2011: Conchoid surfaces of rational ruled surfaces, Comp. Aided Geom. Design **28**, 427–435.

[8] Pottmann, H., and Peternell, M. 1998. Applications of Laguerre geometry in CAGD, offsets, Comp. Aided Geom. Design **15**, 165–186.

[9] Schicho, J.,1997. Rational Parametrization of Algebraic Surfaces. Technical report no. 97-10 in RISC Report Series, University of Linz, Austria, March 1997, PhD Thesis.

[10] Schicho, J., 2000. Proper Parametrization of Real Tubular Surfaces, J. Symbolic Computation **30**, 583–593.

[11] Schicho, J., 1998. Rational Parametrization of Surfaces, J. Symbolic Computation **26**, 1–29.

[12] J.R. Sendra and J. Sendra, 2008. An algebraic analysis of conchoids to algebraic curves, Applicable Algebra in Engineering, Communication and Computing, Vol.19, No.5, pp. 285–308.

[13] J. Sendra and J.R. Sendra, 2010. Rational parametrization of conchoids to algebraic curves, Applicable Algebra in Engineering, Communication and Computing, Vol.21, No.4, pp. 413–428.

[14] Shafarevich, R.I., 1994. Basic Algebraic Geometry, Vol.I, Springer, Heidelberg.