Using the EM algorithm to estimate the state space model for OMAX

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Abstract
This paper presents a time-domain stochastic system identification method based on Maximum Likelihood Estimation and the Expectation Maximization algorithm that is applied to the estimation of modal parameters from system input and output data. The effectiveness of this structural identification method is evaluated through numerical simulation. Modal parameters (eigenfrequencies, damping ratios and mode shapes) of the simulated structure are estimated applying the proposed identification method to a set of 100 simulated cases. The numerical results show that the proposed method estimates the modal parameters with precision in the presence of 20% measurement noise even. Finally, advantages and disadvantages of the method have been discussed.

1 Introduction

Recent advances in system identification for modal testing in civil engineering include the possibility to apply artificial (measured) forces to a structure in addition to the unmeasured ambient excitation, and to identify a model that accounts for both excitation sources. The main difference with classical forced vibration testing is that the ambient loads are not considered as noise, but as part of the excitation. Consequently, the amplitude of the artificial forces can be small compared to the amplitude of the ambient forces, and small and practical actuators can be used on relatively large structures. This approach is called Operational Modal Analysis with eXogenous inputs (OMAX).

The procedure for OMAX is: data collection (both, system input and output), system identification and modal parameter estimation. The system identification step plays a crucial role in the quality of the modal parameters that are derived from the identified system model, as well as in the number of modal parameters that can be determined. This explains the increasing interest in sophisticated system identification methods for modal analysis in general, and in particular for OMAX. The state space model can be used as the system model for OMAX because it can take into account both measured forces and unmeasured forces. This model has been estimated by mean of the well known subspace algorithms in technical literature [1]. In contrast, we propose to estimate the state space model for OMAX using maximum likelihood method and the Expectation-Maximization (EM) algorithm. Maximum likelihood has optimal statistical properties such as consistency and efficiency. Consistency is concerned with the bias of the estimates while efficiency is concerned with variance. While most subspace methods are consistent, few if any can achieve the efficiency of maximum likelihood estimate.

2 State-space model

2.1 Stochastic state-space equations

The equations of motion for an n_d degrees-of-freedom (DOF) linear, time invariant, viscously damped system subjected to external excitation is expressed as

$$M\ddot{q}(t) + H\dot{q}(t) + Kq(t) = Ju(t) \tag{1}$$

where $M, H, K \in \mathbb{R}^{n_d \times n_d}$ are the mass, damping and stiffness matrices, respectively; $J \in \mathbb{R}^{n_d \times n_i}$ is the excitation influence matrix that relates the n_i-dimensional input vector $u(t)$ to the n_d-dimensional response vector; $q(t)$ is the n_d-dimensional displacement response vector; dot denotes taking derivatives with respect to time.

By defining the state vector $x(t) = [q(t)\ \dot{q}(t)]^T$, equation (1) can be converted into the continuous state space form

$$\dot{x}(t) = A_c x(t) + B_c u(t) \tag{2}$$

where

$$A_c = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}H \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ M^{-1}J \end{bmatrix}. \tag{3}$$

In practice, only a limited number of measurements are available; therefore, the dimension of the measurement output is less than or equal to the total number of degrees of freedom. The n_o-dimensional output vector $y(t)$ can be expressed as

$$y(t) = \begin{bmatrix} C_d q(t) \\ C_v \dot{q}(t) \\ C_a \ddot{q}(t) \end{bmatrix} \tag{4}$$

where $C_d, C_v, C_a \in \mathbb{R}^{n_o \times n_d}$ are the measurement location matrices corresponding to the displacement, velocity and acceleration responses of the structural system, respectively. We can rewrite the output vector into the continuous state space form,

$$y(t) = C_c x(t) + D_c u(t) \tag{5}$$

where

$$C_c = \begin{bmatrix} C_d & 0 \\ 0 & C_v \\ -C_a M^{-1}K & -C_a M^{-1}H \end{bmatrix}. \tag{6}$$

In this work, only accelerations are considered, so

$$C_c = C_a[-M^{-1}K \quad - M^{-1}H]. \tag{7}$$

Equations (2) and (5) define the state space equation in continuous time:

$$\dot{x}(t) = A_c x(t) + B_c u(t) \tag{8a}$$
$$y(t) = C_c x(t) + D_c u(t) \tag{8b}$$

where

$y(t) \in \mathbb{R}^{n_o}$ is the measured *output vector*;

$u(t) \in \mathbb{R}^{n_i}$ is the measured *input vector*;

$x(t) \in \mathbb{R}^{n_s}$ is the *state vector*;

$A_c \in \mathbb{R}^{n_s \times n_s}$ is the *transition state matrix* describing the dynamics of the system;

$B_c \in \mathbb{R}^{n_s \times n_i}$ is the *input matrix*;

$C_c \in \mathbb{R}^{n_o \times n_s}$ is the *output matrix*, which is describing how the internal state is transferred to the the output measurements $y(t)$;

$D_c \in \mathbb{R}^{n_o \times n_i}$ is the *direct transmission matrix*;

Equation (8a) is known as the *State Equation* and equation (8b) is known as the *Observation Equation*.

But measurements are taken in discrete time instants, so equations must be expressed in discrete time too. Typical for the sampling of a continuous-time equation is a Zero-Order Hold assumption, which means that the input is piecewise constant over the sampling period, that is

$$\forall t \in [t_k, t_{k+1}) = [k\Delta t, (k+1)\Delta t) \Longrightarrow$$
$$x(t) = x(t_k) = x_k,\ u(t) = u(t_k) = u_k,\ y(t) = y(t_k) = y_k. \tag{9}$$

Under this assumption, the continuous time state-space model (8a) and (8b) is converted to the discrete time state-space model:

$$x_{k+1} = Ax_k + Bu_k \tag{10a}$$
$$y_k = Cx_k + Du_k \tag{10b}$$

where x_k is the discrete time state vector containing the sampled displacements and velocities; u_k and y_k are the sampled input and output; A is the discrete state matrix; B is the discrete input matrix; C is the discrete output matrix; D is the discrete direct transmission matrix. They are related to their continuous-time counterparts as (see for instance [2]):

$$A = e^{A_c \Delta t} \tag{11}$$
$$B = (A - I)\, A_c^{-1} B_c \tag{12}$$
$$C = C_c \tag{13}$$
$$D = D_c \tag{14}$$

Up to now it was assumed that the system was only driven by a deterministic input u_k. However, besides this applied input there might be other inputs that in a more uncontrollable way contribute to the system response. This unmeasurable influence is characterized as disturbance or noise. Therefore, it is necessary to extend the state space model (10a) and (10b) including stochastic components, so *stochastic state space model* is obtained:

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{15a}$$
$$y_k = Cx_k + Du_k + v_k \tag{15b}$$

where $w_k \in \mathbb{R}^{n_s}$ is the *process noise* due to disturbances and modelling inaccuracies; $v_k \in \mathbb{R}^{n_o}$ is the measurement noise due to sensor inaccuracy. We assume they are both independent and identically distributed, zero-mean normal vectors

$$w_k \rightsquigarrow N(0, Q) \qquad u_k \rightsquigarrow N(0, R) \tag{16}$$

In the case of ambient vibration testing, only the responses of the structure y_k are measured, while the input sequence u_k remains unmeasured. Equations (15a) and (15b) result now in a purely stochastic system:

$$x_{k+1} = Ax_k + w_k \tag{17a}$$
$$y_k = Cx_k + v_k \tag{17b}$$

The input is now implicitly modelled by the noise terms w_k, v_k. However the white noise assumptions of these noise terms cannot be omitted and (16) remain still applicable in equation (17).

2.2 System identification and modal analysis in a state-space model

The system identification problem investigated here can be defined as the determination of the corresponding system matrices A, B, C, D, Q and R (up to within a similarity transformation) using the input and output measurements available for N time steps, $\{u_1, u_2, \dots, u_N\}$, $\{y_1, y_2, \dots, y_N\}$.

The natural frequencies and modal damping ratios can be retrieved from the eigenvalues of A, and the mode shapes can be evaluated using the corresponding eigenvectors and the output matrix C.

The eigenvalues of A come in complex conjugate pairs and each pair represents one physical vibration mode. Assuming low and proportional damping, the second order modes are uncoupled and the jth eigenvalue of A has the form

$$\lambda_j = \exp\left(\left(-\zeta_j\omega_j \pm i\omega_j\sqrt{1-\zeta_j^2}\right)\Delta t\right) \tag{18}$$

where ω_j are the natural frequencies, ξ_j are damping ratios, and Δt is the time step. Natural frequencies ω_j and the damping ratios ξ_j are given by

$$\omega_j = \frac{|\ln(\lambda_j)|}{\Delta t} \tag{19}$$

$$\zeta_j = \frac{-Real\left[\ln(\lambda_j)\right]}{\omega_j\Delta t} \tag{20}$$

The jth mode shape $\phi_j \in \mathbb{R}^{n_o}$ evaluated at sensor locations can be obtained using the following expression:

$$\phi_j = C\psi_j \tag{21}$$

where ψ_j is the complex eigenvector of A corresponding to the eigenvalue λ_j.

3 Maximum likelihood method with EM algorithm

In this section the algorithm for estimating the parameters of the stochastic state space model given by Equation (15) is presented, which is based on the maximum likelihood method. This method try to maximize the likelihood applying the iterative expectation maximization algorithm (EM).

3.1 Maximum likelihood Estimation

Given N measurements of the inputs $U_N = \{u_1, u_2, \dots, u_N\}$, and the outputs $Y_N = \{y_1, y_2, \dots, y_N\}$, one way to compute the likelihood is using the innovations $\epsilon_1, \epsilon_2, \dots, \epsilon_N$, defined by Equation (55). The innovations are independent Gaussian random vectors, $\epsilon_k \rightsquigarrow N(0, \Sigma_k)$, with covariance matrix Σ_k given by Equation (56). Thus, ignoring a constant, the logarithm of the likelihood computed from the innovations may be written as:

$$l_{Y_N}(\theta) = -\frac{1}{2}\sum_{t=1}^{N}\left(\ln|\Sigma_k(\theta)| + \epsilon_k(\theta)^T\Sigma_k(\theta)^{-1}\epsilon_k(\theta)\right) \tag{22}$$

where it has been emphasized the dependence of the innovations on the vector θ, which represent the unknown parameters of the model (17) under the assumption that the initial state is normal, $x_0 \rightsquigarrow N(\mu_0, \Sigma_0)$.

$$\theta \stackrel{def}{=} (A, B, C, D, Q, R, \mu_0, \Sigma_0)\,.$$

A wide range of numerical search algorithms are available for maximising the loglikelihood (22), and many of these are based on Newton-Raphson's algorithm. In addition to Newton-Raphson, Shumway and Stoffer [3] presented a conceptually simpler estimation procedure based on the Expectation Maximization algorithm. The EM algorithm is simple to apply since at each iteration the optimal solution for the unknown parameters can be obtained from explicit formulas.

3.2 Expectation Maximization Algorithm

In this section it is outlined the basis of the method, but a more complete description can be found in [3] and [5]. The basic idea is that if the states could be observed $X_N = \{x_0, x_1, x_2, ..., x_N\}$, in addition to the observed values, $Y_N = \{y_1, y_2, \dots, y_N\}$, and the inputs, $U_N = \{u_1, u_2, \dots, u_N\}$, then the complete data could be considered. The logarithm of the likelihood of the complete data can be expressed as

$$l_{X_N,Y_N}(\theta) = l_{X_N|Y_N}(\theta) + l_{Y_N}(\theta)$$

But $l_{X_N,Y_N}(\theta)$ and $l_{X_N|Y_N}(\theta)$ are function of the unknown states X_N, so they are replaced with its expected values. Given a value for the parameter θ at step j it is defined

$$Q(\theta|\theta_j) = \mathrm{E}[l_{X_N,Y_N}(\theta)|Y_N, \theta_j]$$

$$R(\theta|\theta_j) = \mathrm{E}[l_{X_N|Y_N}(\theta)|Y_N, \theta_j]$$

$$S(\theta|\theta_j) = \mathrm{E}[l_{Y_N}(\theta)|Y_N, \theta_j]$$

Thus

$$S(\theta_j|\theta_j) = Q(\theta_j|\theta_j) - R(\theta_j|\theta_j)$$
$$S(\theta_{j+1}|\theta_j) = Q(\theta_{j+1}|\theta_j) - R(\theta_{j+1}|\theta_j)$$

Subtracting both equations

$$S(\theta_{j+1}|\theta_j) - S(\theta_j|\theta_j) =$$
$$[Q(\theta_{j+1}|\theta_j) - Q(\theta_j|\theta_j)] - [R(\theta_{j+1}|\theta_j) - R(\theta_j|\theta_j)]$$

It can be probed that

$$R(\theta_{j+1}|\theta_j) - R(\theta_j|\theta_j) \leq 0 \quad \forall j = 1, 2, \dots$$

So if we develop a procedure which verifies

$$Q(\theta_{j+1}|\theta_j) \geq Q(\theta_j|\theta_j)$$

then automatically it is verified

$$S(\theta_{j+1}|\theta_j) - S(\theta_j|\theta_j) \geq 0$$

and we have a maximum for $l_{Y_N}(\theta)$ (Equation (22)).

In conclusion, the Expectation Maximization algorithm provides an iterative method for finding the maximum likelihood estimators of θ by successively maximizing the conditional expectation of the complete likelihood.

Each iteration of the EM algorithm consists of two steps:

1. The first step (E step) is to compute $Q(\theta|Y_N, \theta_j) = \mathrm{E}[l_{X_N,Y_N}(\theta)|Y_N, \theta_j]$.

2. The second step (M step) consists on maximizing $Q(\theta|Y_N, \theta_j)$, what is equivalent to maximize the likelihood $l_{Y_N}(\theta)$ (Equation (22)).

3.2.1 Computation of the complete likelihood $l_{X_N,Y_N}(\theta)$

The complete likelihood $l_{X_N,Y_N}(\theta)$ is computed taking into account

$$x_0 \rightsquigarrow N(\mu_0, \Sigma_0)$$

$$w_k = x_k - Ax_{k-1} - Bu_{k-1}, \quad w_k \rightsquigarrow N(0, Q)$$
$$v_k = y_k - Cx_k - Du_k, \quad v_k \rightsquigarrow N(0, R)$$

So, the log-likelihood can be written as a sum of three uncoupled functions

$$l_{X_N,Y_N}(\theta) = -\frac{1}{2}[l_1(\mu_0, \Sigma_0) + l_2(A, Q) + l_3(C, R))]$$

where, ignoring constants

$$l_1(\mu_0, \Sigma_0) = \ln|\Sigma_0| + (x_0 - \mu_0)^T \Sigma_0^{-1}(x_0 - \mu_0) \tag{23}$$

$$l_2(A, Q) = N \ln|Q| + \sum_{k=1}^{N}(x_k - Ax_{k-1} - Bu_{k-1})^T Q^{-1}(x_k - Ax_{k-1} - Bu_{k-1}) \tag{24}$$

$$l_3(C, R) = N \log|R| + \sum_{k=1}^{N}(y_k - Cx_k - Du_k)^T R^{-1}(y_k - Cx_k - Du_k) \tag{25}$$

3.2.2 Expectation Step

The function $Q(\theta|Y_N, \theta_j)$ is the conditional expectation of the sum of the Equations (23)-(25), and it depends on the parameters $\theta = (A, B, C, D, Q, R, \mu_0, \Sigma_0)$.

Theorem 1 *Given the value of the parameters θ for iteration j, Properties 2 and 3 can be used to obtain the desired conditional expectations as smoothers:*

$$x_k^N = E[x_k|Y_N, \theta_j] \tag{26}$$

$$P_k^N = E[(x_k - x_k^N)(x_k - x_k^N)^T|Y_N, \theta_j] \tag{27}$$
$$P_{k,k-1}^N = E[(x_k - x_k^N)(x_{k-1} - x_{k-1}^N)^T|Y_N, \theta_j] \tag{28}$$

and from them it is possible to compute $Q(\theta|Y_N, \theta_j)$ as follows

$$Q(\theta|Y_N, \theta_j) = E[l_{X_N,Y_N}(\theta)|Y_N, \theta_j] =$$
$$= E[l_1(\mu_0, \Sigma_0)|Y_N, \theta_j] + E[l_2(A, B, Q)|Y_N, \theta_j] + E[l_3(C, D, R)|Y_N, \theta_j]$$

with

$$E[l_1(\mu_0, \Sigma_0)|Y_N, \theta_j] = \ln|\Sigma_0| + tr\left(\Sigma_0^{-1}\left[P_0^N + (x_0^N - \mu_0)(x_0^N - \mu_0)^T\right]\right) \tag{29}$$

$$E[l_2(A, B, Q)|Y_N, \theta_j] = N \log|Q| + tr\big(Q^{-1}\big[S_{xx} - S_{xb}A^T - AS_{bx} - S_{xu1}B^T - BS_{ux1} + AS_{bu}B^T + BS_{ub}A^T + AS_{bb}A^T + BS_{uu1}B^T\big]\big) \tag{30}$$

$$E[l_3(C, D, R)|Y_N, \theta_j] = N \log|R| + tr\big(R^{-1}\big[S_{yy} - S_{yx}C^T - CS_{xy} + S_{yu}D^T - DS_{uy} + CS_{xu2}D^T + DS_{ux2}C^T + CS_{xx}C^T + DS_{uu2}D^T\big]\big) \tag{31}$$

where it has been used

$$S_{xx} = \sum_{k=1}^{N} \left(P_k^N + x_k^N (x_k^N)^T \right) \tag{32}$$

$$S_{xb} = \sum_{k=1}^{N} \left(P_{k,k-1}^N + x_k^N (x_{k-1}^N)^T \right), \quad S_{bx} = S_{xb}^T \tag{33}$$

$$S_{bb} = \sum_{k=1}^{N} \left(P_{k-1}^N + x_{k-1}^N (x_{k-1}^N)^T \right) \tag{34}$$

$$S_{yy} = \sum_{k=1}^{N} \left(y_k y_k^T \right) \tag{35}$$

$$S_{yx} = \sum_{k=1}^{N} \left(y_k (x_k^N)^T \right), \quad S_{xy} = S_{yx}^T \tag{36}$$

$$S_{yu} = \sum_{k=1}^{N} \left(y_k u_k^T \right), \quad S_{uy} = S_{yu}^T \tag{37}$$

$$S_{xu1} = \sum_{k=1}^{N} \left(x_k^N u_{k-1}^T \right), \quad S_{ux1} = S_{xu1}^T \tag{38}$$

$$S_{bu} = \sum_{k=1}^{N} \left(x_{k-1}^N u_{k-1}^T \right), \quad S_{ub} = S_{bu}^T \tag{39}$$

$$S_{uu1} = \sum_{k=1}^{N} \left(u_{k-1} u_{k-1}^T \right), \tag{40}$$

$$S_{xu2} = \sum_{k=1}^{N} \left(x_k^N u_k^T \right), \quad S_{ux2} = S_{xu2}^T \tag{41}$$

$$S_{uu2} = \sum_{k=1}^{N} \left(u_k u_k^T \right). \tag{42}$$

3.2.3 Maximization Step

Maximizing $Q(\theta|Y_N, \theta_j)$ with respect of the parameters θ, at iteration j, constitutes the M-step. This is the strong point of the EM algorithm because the maximum values are obtained from explicit formulas.

Theorem 2 *The maximum of* $E[l_1(\mu_0, \Sigma_0)|Y_N, \theta_j]$ (29) *is attained at*

$$\hat{\mu}_0 = x_0^N \tag{43}$$

$$\hat{\Sigma}_0 = P_0^N \tag{44}$$

Theorem 3 *The maximum of* $E[l_2(A, Q)|Y_N, \theta_j]$ (30) *is attained at*

$$\begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} = \begin{bmatrix} S_{xb} & S_{xu1} \end{bmatrix} \begin{bmatrix} S_{bb} & S_{bu} \\ S_{ub} & S_{uu1} \end{bmatrix}^{-1} \tag{45}$$

$$\hat{Q} = \frac{1}{N} \left(S_{xx} - S_{xb}A^T - AS_{bx} - S_{xu1}B^T - BS_{ux1} + AS_{bu}B^T + BS_{ub}A^T + AS_{bb}A^T + BS_{uu1}B^T \right) \tag{46}$$

Theorem 4 *The maximum of* $E[l_3(C,R)|Y_N,\theta_j]$ (31) *is attained at*

$$\begin{bmatrix}\hat{C} & \hat{D}\end{bmatrix} = \begin{bmatrix}S_{yx} & S_{yu}\end{bmatrix}\begin{bmatrix}S_{xx} & S_{xu2}\\ S_{ux2} & S_{uu2}\end{bmatrix}^{-1} \tag{47}$$

$$\hat{R} = \frac{1}{N}\left(S_{yy} - S_{yx}C^T - CS_{xy} + S_{yu}D^T - DS_{uy} + CS_{xu2}D^T + DS_{ux2}C^T + CS_{xx}C^T + DS_{uu2}D^T\right) \tag{48}$$

The above properties can be obtained equating to zero the corresponding derivatives.

3.3 EM procedure

The overall method can be summarized as an iterative procedure as follows:

1. Initialize the procedure by selecting starting values for the parameters

$$\theta_0 = (A_0, B_0, C_0, D_0, Q_0, R_0, \mu_0, \Sigma_0)$$

2. Start iteration j ($j = 1, 2, \ldots$).
3. Use Property 1 to compute the innovations (Equation 55) and the incomplete-data likelihood, $l_{Y_N}(\theta_{(j-1)})$ (Equation 22).
4. Perform the E-Step.
   - Use Properties 1, 2 y 3 to obtain the smoothed values x_k^N, P_k^N, and $P_{k,k-1}^N$, for $k = 1, 2, \ldots, N$, using the parameters $\theta_{(j-1)}$.
   - Use the smoothed values to calculate $S_{xb}, S_{bb}, S_{xx}, \ldots, S_{uu2}$ given in (32)-(42).
5. Perform the M-Step.
   - Update the parameters $\theta_j = \left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{Q}, \hat{R}, \hat{\mu}, \hat{\Sigma}\right)$ using (43)-(48).
6. Repeat Steps 2-5 to convergence. Two options can be considered in the algorithm:
   - Perform a predefined number of iterations j_{max}.
   - Stop when the values of $l_{Y_N}(\theta_j)$ differs from $l_{Y_N}(\theta_{(j-1)})$ by some predetermined, but small amount δ.

$$\frac{|l_{Y_N}(\theta_j) - l_{Y_N}(\theta_{j-1})|}{|l_{Y_N}(\theta_{j-1})|} < \delta \tag{49}$$

4 Numerical examples

We are going to use a 8-DOF simulated system to show the performance of the proposed method (Figure 1). We have the following parameters:

- The mass matrix M is equal to the identity matrix;

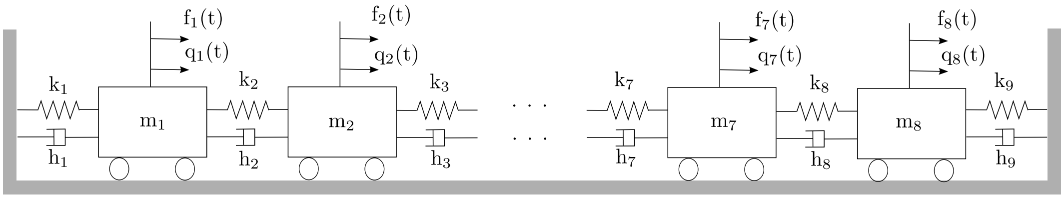

Figure 1: Simulated system.

- the stiffness matrix K is equal to

$$K = \begin{bmatrix} 2400 & -1600 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1600 & 4000 & -2400 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2400 & 5600 & -3200 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3200 & 7200 & -4000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4000 & 8800 & -4800 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4800 & 10400 & -5600 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5600 & 12000 & -6400 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6400 & 13600 \end{bmatrix} \, (N/m);$$

- The damping matrix is equal to $H = 0.6798M + 1.7431 \cdot 10^{-4}K \ (N \cdot s/m)$ (Rayleigh damping).
- The input is a Gaussian white noise vector with zero mean and variance equal to one.
- The input location matrix is $[B_u] = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T$, that is, the input is applied to all DOF.
- The output vector is

$$y_t = \begin{bmatrix} \ddot{q}_{1,t} \\ \ddot{q}_{2,t} \\ \ddot{q}_{3,t} \\ \ddot{q}_{4,t} \\ \ddot{q}_{5,t} \\ \ddot{q}_{6,t} \\ \ddot{q}_{7,t} \\ \ddot{q}_{8,t} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \\ e_{4,t} \\ e_{5,t} \\ e_{6,t} \\ e_{7,t} \\ e_{8,t} \end{bmatrix} \quad (m/s^2)$$

  where $\ddot{q}_{j,t}$ is the acceleration of DOF j at time instant t computed by mean of Equation (1), and $e_{j,t}$ is a noise term. In this case, the noise are Gaussian white noise series with zero mean and variance equal to 0.20 times the maximum variance of the system accelerations, $\ddot{q}_j, \ j = 1, \dots, 8$.
- We have simulated 100 seconds with a sampling frequency of 50 Hz.
- The modal characteristics of the simulated system are given in Figure 2.

First, the state space model (15) has been estimated using the EM algorithm (see [5] for details). This model corresponds to output-only modal analysis because the inputs are not used in the estimation process (other names used in literature are ambient modal analysis and operational modal analysis). Then, the modal parameters have been computed from Equations (19), (20) and (21). The results for mode 2 and mode 5 are presented in Figures 3 and 4. We observe that the estimation of eigenfrequencies and mode shapes are good, but not the estimation of damping ratios.

Next, the state space model (17) has been estimated using the EM algorithm as proposed in Section 3, that is, taking into account the system inputs. The modal parameters are computed using the corresponding

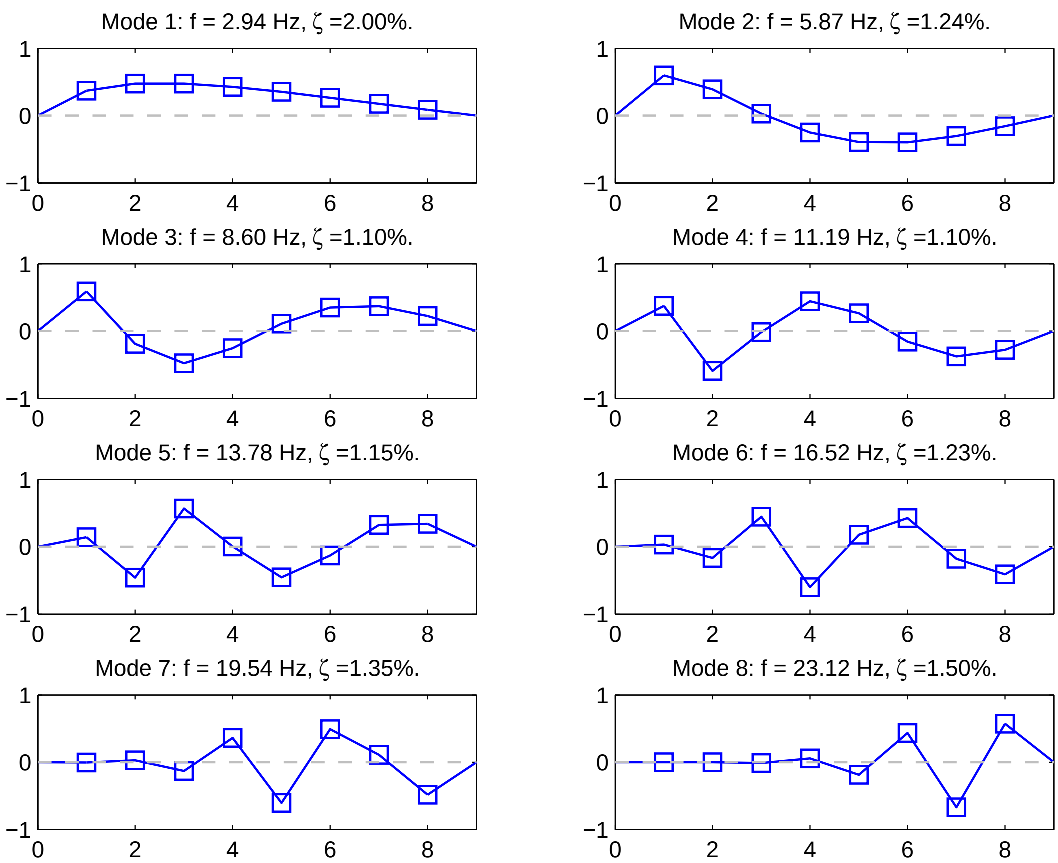

Figure 2: Simulated system.

equations (19), (20) and (21). The results for mode 2 and mode 5 are presented in a new column of Figures 3 and 4. We observe that the estimation of eigenfrequencies, damping ratios and mode shapes are greatly improved.

One important aspect of the EM algorithm, both with and without inputs, is the need of a starting point. In this case we have used the Stochastic Subspace Identification (SSI) algorithm to build the starting points as follows:

1. Estimate the state space model (17) using the SSI algorithm (see [1]). The estimated matrices are called $\{A_s, C_s, Q_s, R_s\}$.

2. The starting point $\theta_0 = \{A_0, C_0, Q_0, R_0, \mu_0, \Sigma_0\}$ for the EM algorithm to estimate the model (17) is $\theta_0 = \{A_s, C_s, Q_s, R_s, [0], [0]\}$ ($[0]$ stands for a matrix a zeros with the appropriate size).

3. The starting point $\theta_0 = \{A_0, B_0, C_0, D_0, Q_0, R_0, \mu_0, \Sigma_0\}$ for the EM algorithm to estimate the model (15) is $\theta_0 = \{A_s, [0], C_s, [0], Q_s, R_s, [0], [0]\}$.

5 Conclusions

In this paper we have explored the use of the EM algorithm for Operational Modal Analysis with eXogenous inputs, that is, when a known input is applied to the system and this input is used in the estimation of the modal parameters. The results obtained in the simulation example show that the modal parameters are estimated with precision using the proposed algorithm.

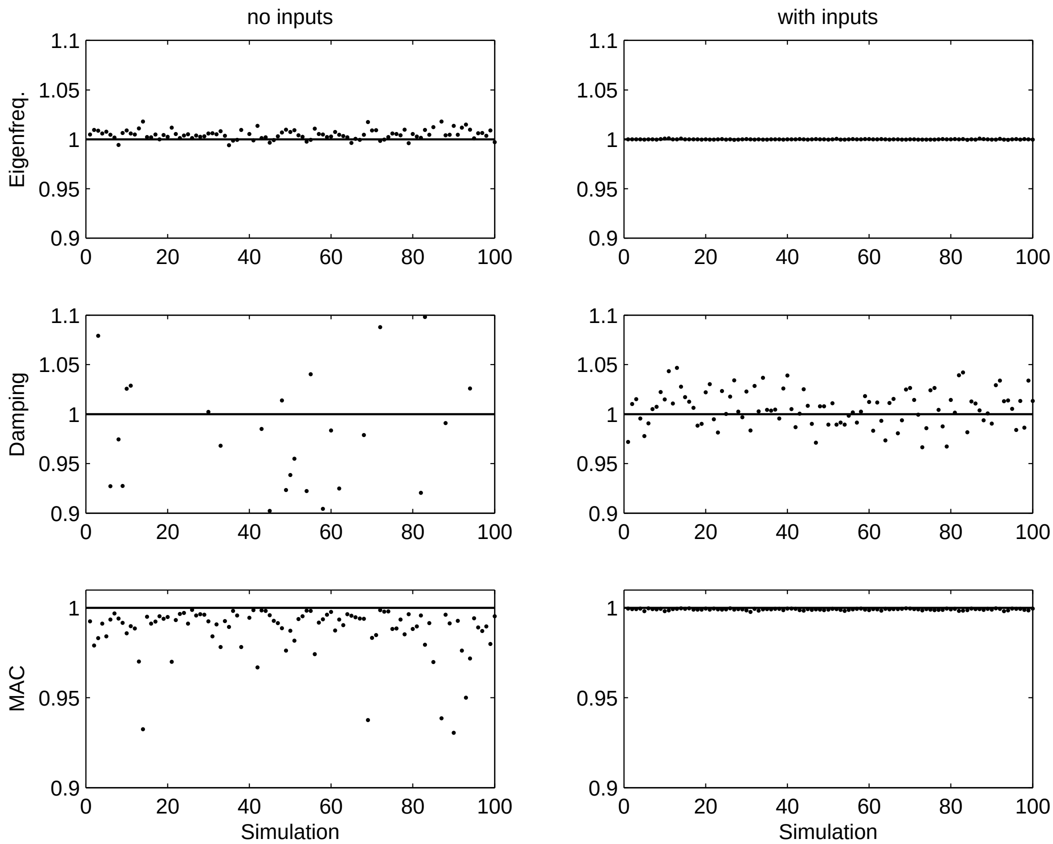

Figure 3: Modal parameters of mode 2 estimated from 100 simulations: left: inputs are not taken into account (state space model (17)); right: inputs are taken into account (state space model (15)).

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A Three important properties

The following results are used in the Expectation step (the proof of these properties can be found in [3]). First we include some notation used in the properties:

Given the output data for s time steps $Y_s = \{y_1, y_2, \ldots, y_s\}$, it is defined

$$x_t^s = \mathrm{E}\left[x_t | Y_s\right]$$

$$P_{t_1,t_2}^s = \mathrm{E}\left[(x_{t_1} - x_{t_1}^s)(x_{t_2} - x_{t_2}^s)^T | Y_s\right]$$

where $\mathrm{E}\left[\bullet|\bullet\right]$ is the conditional expected operator. When $t_1 = t_2 = t$, P_{t_1,t_2}^s will be written P_t^s:

$$P_t^s = \mathrm{E}\left[(x_t - x_t^s)(x_t - x_t^s)^T | Y_s\right] = \mathrm{Var}\left[x_t | Y_s\right]$$

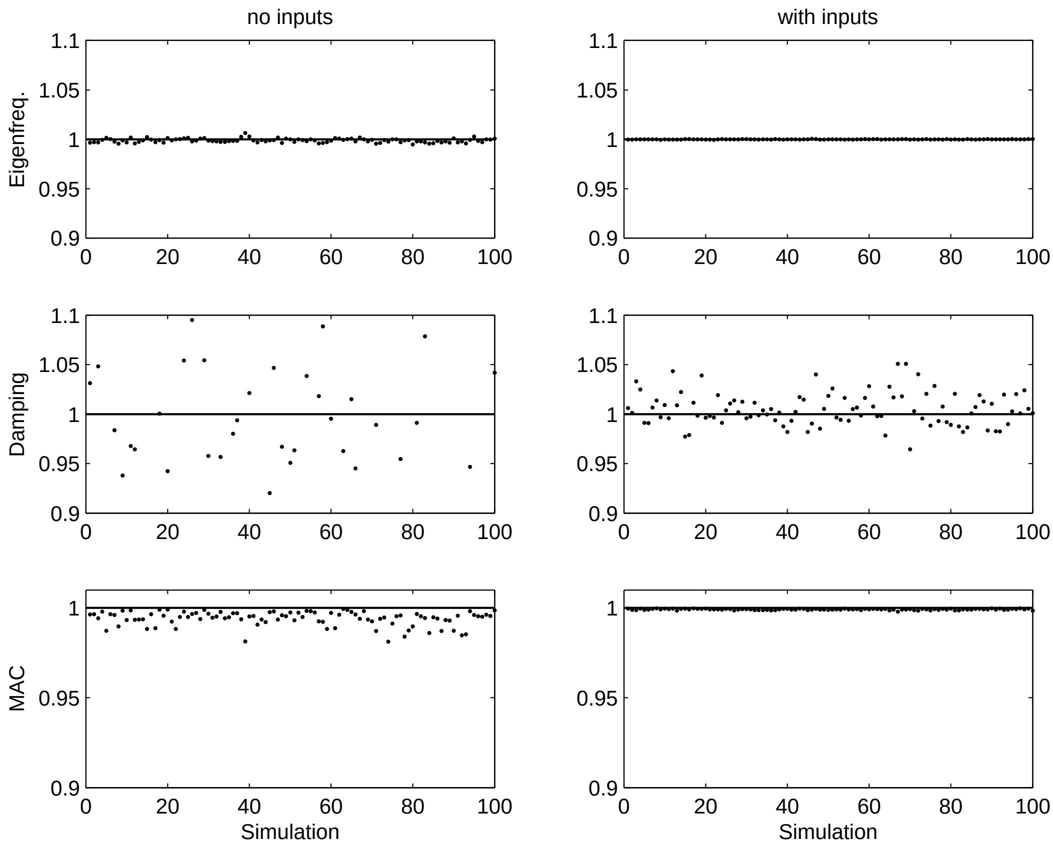

Figure 4: Modal parameters of mode 5 estimated from 100 simulations: left: inputs are not taken into account (state space model (17)); right: inputs are taken into account (state space model (15)).

Property 1 (The Kalman Filter) *For the state space model specified in* (17) *with initial conditions* $x_0^0 = \mu_0$ *and* $P_0^0 = \Sigma_0$*, for* $k = 1, 2, \ldots, N$*,*

$$x_k^{k-1} = Ax_{k-1}^{k-1} + Bu_{k-1} \tag{50}$$

$$P_k^{k-1} = AP_{k-1}^{k-1}A^T + Q \tag{51}$$

with

$$x_k^k = x_k^{k-1} + K_k\epsilon_k \tag{52}$$

$$P_k^k = (I - K_kC)P_k^{k-1} \tag{53}$$

where

$$K_k = P_k^{k-1}C^T\Sigma_k^{-1} \tag{54}$$

$$\epsilon_k = y_k - \mathrm{E}\left[y_k|Y_{k-1}\right] = y_k - Cx_k^{k-1} - Du_k \tag{55}$$

$$\Sigma_k = \mathrm{Var}(\epsilon_k) = \mathrm{Var}[C(x_k - x_k^{k-1}) + v_k] = CP_k^{k-1}C^T + R \tag{56}$$

K_k *is called the Kalman gain and* ϵ_k *are the innovations.*

Property 2 (The Kalman Smoother) *For the state space model specified in* (17) *with initial conditions* x_N^N *and* P_N^N *obtained via Property 1, for* $k = N, N-1, \ldots, 1$*,*

$$x_{k-1}^N = x_{k-1}^{k-1} + J_{k-1}\left(x_k^N - x_k^{k-1}\right) \tag{57}$$

$$P_{k-1}^N = P_{k-1}^{k-1} + J_{k-1}\left(P_k^N - P_k^{k-1}\right)J_{k-1}^T \tag{58}$$

where

$$J_{k-1} = P_{k-1}^{k-1} A^T \left[P_k^{k-1} \right]^{-1} \tag{59}$$

Property 3 (The Lag-One Covariance Smoother) *For the state space model specified in* (17)*, with* K_k*,* J_k $(k = 1, 2, \ldots, N)$*, and* P_N^N *obtained from Properties 1 and 2, with initial condition*

$$P_{N,N-1}^N = (I - K_N C) \, A P_{N-1}^{N-1} \tag{60}$$

for $k = N, N-1, \ldots, 2$

$$P_{k-1.k-2}^N = P_{k-1}^{k-1} J_{k-2}^T + J_{k-1} \left(P_{k,k-1}^N - A P_{k-1}^{k-1} \right) J_{k-2}^T \tag{61}$$

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