Planar Point Sets With Large Minimum Convex Decompositions

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Abstract We show the existence of sets with n points ($n \geq 4$) for which every convex decomposition contains more than $\frac{35}{32}n - \frac{3}{2}$ polygons, which refutes the conjecture that for every set of n points there is a convex decomposition with at most $n + C$ polygons. For sets having exactly three extreme points we show that more than $n + \sqrt{2(n-3)} - 4$ polygons may be necessary to form a convex decomposition.

Keywords Convex decompositions · Triangulations · Empty polygons

1 Introduction

Let V be a finite set of points in *general position* in the plane, i.e., no three points of V lie on a straight line. A convex subdivision of V is a set of convex polygons $\{P_1, \ldots, P_k\}$ with vertices in V such that $\cup_i P_i = \text{conv}(V)$ and $P_i \cap P_j$ is a (possibly empty) face of both P_i and P_j (see [6]). In this paper we impose the following additional condition: the polygons P_i must be *empty*, i.e., no element of V is contained in the interior of any of the polygons. This condition is equivalent (for points in general position) to the requirement that every element in V should be a vertex of at least one polygon in the subdivision. We use the name *convex decomposition* (or just *decomposition*) to refer to this special type of subdivision.

The number of polygons in a decomposition is called the size of the decomposition. Let $G(V)$ be the minimum size among the decompositions of V. Let $g(n)$ be the maximum value of $G(V)$ among the sets V of n points in general position in the plane. Aichholzer and Krasser [2] showed $g(n) \geq n + 2$. On the other hand we have the trivial bound $g(n) \leq 2n - 5$ for $n \geq 3$ (by considering triangulations). Hosono [5] proved $g(n) \leq \lceil (7/5)(n-3) \rceil + 1$. Sakai and Urrutia [7] announced the bound $g(n) \leq (4/3)n - 2$. Rivera-Campo and Urrutia conjectured that $g(n) \leq n + C$ for some constant C (see Conjecture 6 in Section 8.5 of [3]). It is clear that if a set V with n vertices admits a decomposition into convex quadrilaterals then $G(V) \leq n$. Thus, roughly speaking a formula of the type $g(n) \leq n + C$ for constant C would indicate that the triangles that we may be forced to use to obtain a convex decomposition of V can be offset by sufficiently many polygons with five or more faces. In this paper we refute this conjecture by showing that $g(n) > (35/32)n - 3/2$ for $n \geq 4$. We present an improved, simpler version (with corrected proofs) of the construction in our unpublished draft [4] (cited in [1,5]).

We refer to [1] for bounds on related objects such as pseudo-convex decompositions and convex and pseudo-convex partitions and coverings.

2 Basic Construction

First we review the idea of contraction in the context of decompositions. It is convenient when performing contractions to identify a decomposition S with the geometric graph consisting of the vertices and sides of the polygons in S. Let V be a finite subset of $\mathbb{R}^2$. Let $p \in \mathbb{R}^2 - V$ and suppose that $V \cup \{p\}$ is in general position. Define $\text{cell}(p, V)$ as the cell that contains p in the line arrangement determined by V, i.e., $x \in \text{cell}(p, V)$ if and only if x and p are on the same side of l for every line l through any two points in V. In other words, $V \cup \{x\}$ and $V \cup \{p\}$ have the same order type. It is clear that if S is a convex decomposition of $V \cup \{x\}$ and $x \in \text{cell}(p, V)$ then there exists a decomposition S' of $V \cup \{p\}$ which is combinatorially equivalent to S. More in general, if S is a decomposition of $V \cup V_1$ and $V_1 \subset \text{cell}(p, V)$, where p may be contained in V_1 but not in V, we obtain from S a decomposition of $V \cup \{p\}$ if we contract every element of V_1 to p.

We work in the more general setting of $\mathbb{R}^2 \cup \{p_\infty\}$, where p_∞ is the point at infinity in the direction of the positive x-axis. An edge between a vertex v of $\mathbb{R}^2$ and p_∞ is simply an infinite ray that starts at v and extends in the positive horizontal direction. If $V \subset \mathbb{R}^2$ then $V \cup \{p_\infty\}$ is in general position if V is in general position and no two vertices of V have the same y coordinate. Suppose $V \subset \mathbb{R}^2$ and let $V \cup \{p_\infty\}$ be in general position. Define $\text{cell}(p_\infty, V)$ as the unbounded region determined by the following rule: $x \in \text{cell}(p_\infty, V)$ if and only if for every line l through two points in V, x lies above l if l has negative slope and below l if l has positive slope. If $V_1 \subset \text{cell}(p_\infty, V)$ and S is a decomposition of $V \cup V_1 \cup \{p_\infty\}$ (note p_∞ is not contained in V_1 nor V) we obtain a decomposition of $V \cup \{p_\infty\}$ by contracting V_1 to

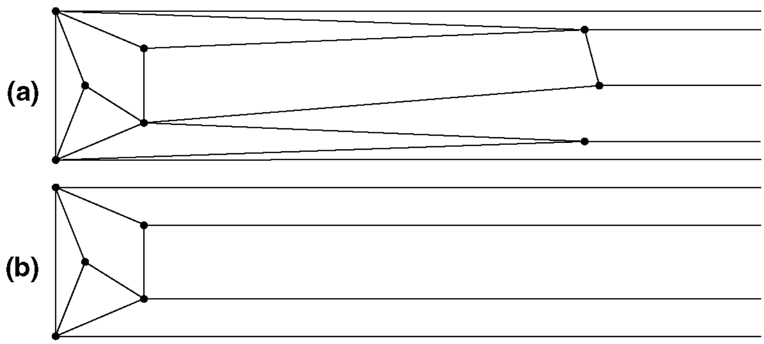


Fig. 1 **a** A decomposition of $V \cup V_1 \cup \{p_\infty\}$, where V_1 is the set containing the three points shown on the right. **b** A decomposition of $V \cup \{p_\infty\}$, obtained by contracting V_1 to p_∞ in **a**

p_∞. In this case the contraction amounts to deleting V_1 and every edge incident to it, and adding an edge (i.e., ray) between v and p_∞ for each $v \in V$ adjacent to a vertex in V_1. See Fig. 1.

Let $B = \{(0, 0), (0, 1)\}$ and $B^* = B \cup \{p_\infty\}$. We consider first decompositions of the convex hull of B^* (the "triangle" bounded by the lines $y = 0$, $x = 0$ and $y = 1$).

Inductively, define the sets $L_0, \ldots, L_{k+1}$ as follows. Let $L_0 = \emptyset$, $L_1 = \{(1, \frac{1}{2})\}$ and suppose the sets $L_0, \ldots, L_k$, $k \geq 1$, have been chosen so that the following properties hold for $0 \leq i < k$:

(i) $L_{i+1} \subset \text{cell}(p_\infty, B \cup L_1 \cup \cdots \cup L_i)$.

(ii) The vertices in L_{i+1} form a vertical layer *concave to the left*, i.e., every vertex v in L_{i+1} is contained in a line having $B \cup L_1 \cup \cdots \cup L_{i+1} - \{v\}$ on one side and p_∞ on the other side. This is equivalent (by property (i)) to requiring the set $B \cup L_{i+1}$ to be in convex position.

(iii) $|L_{i+1}| = i + 1$. Additionally, when ordered according to their y-coordinates the elements of L_i and L_{i+1} alternate.

(iv) $B^* \cup L_1 \cup \cdots \cup L_{i+1}$ is in general position.

Then L_{k+1} can be chosen so that properties (i)–(iv) are satisfied for $0 \leq i < k + 1$: on account of the general position $\text{cell}(p_\infty, B \cup L_1 \cup \cdots \cup L_k)$ contains an infinite rectangle of height 1, i.e., there exists a number M such that the convex hull of $\{(M, 0), (M, 1), p_\infty\}$ is contained in $\text{cell}(p_\infty, B \cup L_1 \cup \cdots \cup L_k)$. For $i \in \{1, \ldots, k+1\}$ choose y_i in the open interval $(\bar{y}_{i-1}, \bar{y}_i)$ where $\bar{y}_1, \ldots, \bar{y}_k$ are the y-coordinates in increasing order of the k points in L_k and $\bar{y}_0 = 0$, $\bar{y}_{k+1} = 1$. Also choose the numbers y_i to be distinct from all the y-coordinates of the points in $L_1 \cup \cdots \cup L_k$ to ensure general position. Now choose $x_i > M$ such that the points (x_i, y_i) form a vertical layer concave to the left (as defined in (ii)) while preserving the general position, and let $L_{k+1} = \{(x_1, y_1), \ldots, (x_{k+1}, y_{k+1})\}$.

For every $k \geq 1$, denote the set $L_1 \cup \cdots \cup L_k$ by C_k. See Fig. 2.

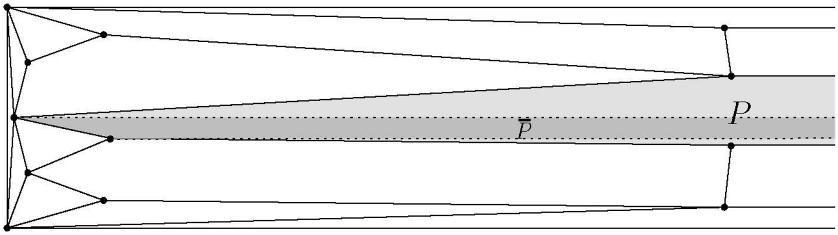


Fig. 2 A decomposition of $B^* \cup C_4$. $\bar{P}$ is obtained from P when L_4 is contracted to p_∞. Note that P is incident to p_∞, hence $\bar{P} \subseteq P$

3 Results

We are interested in the sets $B^* \cup C_k$ because the discrepancy between its size and the size of its decompositions is large. Define $\Delta(S)$ for any decomposition S of a set of vertices V to be $|S| - |V_{\text{int}}|$, i.e., the difference between the number of polygons in the decomposition and the number of vertices of V that lie in the interior of $\text{conv}(V)$. Let $R_\infty(S)$ be the set of polygons in S incident to p_∞.

Theorem 1 *If S is a decomposition of $B^* \cup C_k$ then $\Delta(S) \geq |R_\infty(S)|$.*

Proof 1 By induction on k. The case $k = 1$ is clear since the only decomposition S of $B^* \cup L_1$ satisfies $|S| = 3$, $|R_\infty(S)| = 2$ and there is one interior vertex.

Assume the theorem is true for $k \geq 1$. Let S be a decomposition of $B^* \cup C_{k+1}$. Since $L_{k+1} \subset \text{cell}(p_\infty, B \cup C_k)$ we obtain a decomposition $\bar{S}$ of $B^* \cup C_k$ by contracting L_{k+1} to p_∞. For $P \in S$ let $\bar{P}$ be the polygon obtained from P after contracting L_{k+1} to p_∞. Note that $\bar{P}$ may be a degenerate polygon (i.e., with empty interior) consisting only of the point p_∞ or of a horizontal ray joining a point to p_∞. Clearly $\bar{P}$ is degenerate if and only if P is incident to at most one point in $B \cup C_k$. Let $R^{\text{new}}(S)$ be the subset of S consisting of the polygons P for which $\bar{P}$ is degenerate. The map $P \to \bar{P}$ restricted to $S - R^{\text{new}}$ establishes a bijection with $\bar{S}$, so $|S| = |\bar{S}| + |R^{\text{new}}(S)|$. If we think of the decomposition S as being obtained from $\bar{S}$ by reversing the contraction then the polygons in R^{new} are the new polygons that have not been counted in $\bar{S}$. Applying the inductive hypothesis we obtain

$$\begin{aligned}\Delta(S) &= |S| - |C_{k+1}| = |\bar{S}| + |R^{\text{new}}(S)| - |C_k| - |L_{k+1}| \\ &= \Delta(\bar{S}) + |R^{\text{new}}(S)| - k - 1 \geq |R_\infty(\bar{S})| + |R^{\text{new}}(S)| - k - 1 \qquad (1)\end{aligned}$$

Note that if $P \in R_\infty(S) - R^{\text{new}}(S)$ then $\bar{P} \in R_\infty(\bar{S})$ since an unbounded polygon (i.e., incident to p_∞) either becomes degenerate or remains unbounded after the contraction. But we also have elements in $R_\infty(\bar{S})$ that come from bounded polygons in S. Let $R^{\text{bnd}}(S)$ be the set of polygons P in $S - R_\infty(S)$ such that $\bar{P} \in R_\infty(\bar{S})$. Then $|R_\infty(\bar{S})| = |R^{\text{bnd}}(S)| + |R_\infty(S) - R^{\text{new}}(S)|$. Substituting in (1) we obtain

$$\begin{aligned}\Delta(S) &\geq |R^{\text{bnd}}(S)| + |R_\infty(S) - R^{\text{new}}(S)| + |R^{\text{new}}(S)| - k - 1 \\ &\geq |R^{\text{bnd}}(S)| + |R_\infty(S)| - k - 1 \qquad (2)\end{aligned}$$

Therefore the result follows if we show that $|R^{\text{bnd}}(S)| \geq k+1$. Now, for each vertex v in L_{k+1} there is a polygon Q_v in $R_\infty(\bar{S})$ that contains v in its interior (since the polygons in $\bar{S}$ cover conv(B^*) and $L_{k+1} \subset$ cell$(p_\infty, B \cup C_k)$). Let P_v be the polygon in S such that $\bar{P}_v = Q_v$. We want to show that $P_v \notin R_\infty(S)$ and that $P_v \neq P_{v'}$ for $v \neq v'$. Suppose $P_v \in R_\infty(S)$. The two rays incident to p_∞ on the boundary of P_v are horizontal, therefore, by convexity, every vertex of P_v lies either between the two parallel lines defined by these rays or at the endpoint of one of the rays. Hence, $P_v \in R_\infty(S)$ implies $\bar{P}_v \subseteq P_v$ (see Fig. 2). Since v belongs to the interior of $\bar{P}_v$ and $\bar{P}_v \subseteq P_v$ it follows that v belongs to the interior of P_v, which contradicts the fact that P_v is an *empty* polygon. Therefore $P_v \in R^{\text{bnd}}(S)$ for all $v \in L_{k+1}$.

Finally suppose $P_v = P_{v'}$ for some $v, v' \in L_{k+1}, v \neq v'$. Then $\bar{P}_v = \bar{P}_{v'}$ so $\bar{P}_v$ contains both v and v' in its interior. By its convexity $\bar{P}_v$ contains the segment joining v and v'. But there is a vertex w in L_k whose y coordinate lies between the y coordinates of v and v' (the coordinates of L_k and L_{k+1} alternate). Since L_k is concave to the left w has to be incident to p_∞ in the decomposition $\bar{S}$. Thus, the ray coming out of w intersects the segment between v and v' which is supposed to be contained in the interior of $\bar{P}_v$. This contradiction shows that $P_v \neq P_{v'}$ for $v \neq v'$. Hence $|R^{\text{bnd}}(S)| \geq k+1$ and the result follows from (2). □

Corollary 2 *For any decomposition S of $B^* \cup C_k$, $\Delta(S) \geq k+1$.*

Proof 2 L_k is concave to the left. Therefore every $v \in L_k$ is incident to p_∞, so $R_\infty(S) \geq k+1$. □

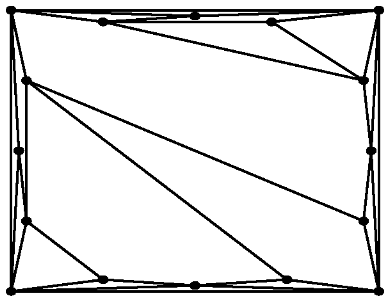
Corollary 3 *For any decomposition S of $B^* \cup C_k$, the number of polygons not incident to p_∞ is at least $|C_k|$.* □

Recall that $G(V)$ is defined as the minimum size of a decomposition of V and $g(n)$ as the maximum value of $G(V)$ among all the sets with n elements in general position. Let $g_3(n)$ be the maximum value of $G(V)$ among the sets $V \subset \mathbb{R}^2$ in general position having n elements of which exactly three lie on the boundary of conv(V). In order to obtain a bound for g_3 we replace p_∞ by an actual point on $\mathbb{R}^2$. Let $z_k \in$ cell$(p_\infty, B \cup C_k)$. It is clear that any decomposition of $B \cup C_k \cup \{z_k\}$ yields a decomposition of $B^* \cup C_k$ with the same number of polygons when z_k is contracted to p_∞. Combined with Corollary 2 we obtain the following bound on g_3:

Theorem 4 $g_3(3+|C_k|) \geq |C_k| + k + 1$. □

In order to obtain a result valid for every n, observe that $g_3(n+1) \geq g_3(n)+1$ (i.e., g_3 is strictly monotonic). Indeed, let V be a set of n points of which exactly three, b_1, b_2 and b_3, lie on the boundary of conv(V) and such that $G(V) = g_3(n)$. Let V' be the set $V \cup \{w\}$ where w satisfies $w \in$ conv$(V) \cap$ cell$(b_1, V - \{b_1\})$ and the line through b_1 and w separates b_2 from the rest of the points in V (w is very close to the side b_1b_2 and to the vertex b_1). Then every decomposition of V' contains the triangle with vertices b_1, b_2 and w. Moreover, every decomposition S of V' produces a decomposition $\bar{S}$ of V when w is contracted to b_1. Since the triangle b_1b_2w collapses after the contraction we obtain $|S| > |\bar{S}| \geq |G(V)|$. Therefore, $g_3(n+1) \geq G(V') > G(V) = g_3(n)$. Taking V to be a set such that $g(n) = G(V)$ and proceeding in a similar way we also obtain the monotonicity of the function $g(n)$.

Fig. 3 A decomposition of the set $D_{4,2}$



Theorem 5 *$g_3(n + j) \geq g_3(n) + j$ and $g(n + j) \geq g(n) + j$.* □

Let $n_k = 3 + |C_k| = 3 + k(k + 1)/2$. Solving for k we obtain $k = (1/2)(-1 + \sqrt{1 + 8(n_k - 3)})$. Therefore, taking $n_k \leq n < n_{k+1}$ we have $k \leq (1/2)(-1 + \sqrt{1 + 8(n - 3)}) < k + 1$. From this inequality and Theorems 4 and 5 we obtain the following bound on g_3:

Theorem 6 *For all $n \geq 3$, $g_3(n) > n - 4 + \sqrt{2(n - 3)}$.*

Proof 3 Let $n_k \leq n < n_{k+1}$. By Theorems 4 and 5, $g_3(n) \geq g_3(n_k) + n - n_k \geq n_k - 3 + k + 1 + n - n_k$. But $k + 1 > (1/2)(-1 + \sqrt{1 + 8(n - 3)})$, so $g_3(n) > n - 3 + (1/2)(-1 + \sqrt{1 + 8(n - 3)}) > n - 4 + \sqrt{2(n - 3)}$. □

Now we consider a more general construction that yields a bound for the function $g(n)$. Let $P = \{v_0, \ldots, v_{h-1}\}$ be the vertices of a regular h-gon listed in clockwise order. Let B_i denote the set $\{v_{i-1}, v_i\}$ (all indices from now on are computed modulo h). Let $\bar{B}_i$ be the segment with endpoints v_{i-1} and v_i. Let p^i_∞ be the point at infinity in the direction perpendicular to $\bar{B}_i$ on the side containing conv(P). We work with p^i_∞ in the same way we did with p_∞. For example, cell(p^i_∞, V) (for $V \subset \mathbb{R}^2$) is the unbounded region that corresponds to cell$(p_\infty, \bar{V})$ when the plane is rotated so that p^i_∞ corresponds to p_∞ and V to $\bar{V}$.

Now apply appropriate affine transformations to C_k in order to obtain, for each i, sets C^i_k satisfying the following property:

$$\bigcup_{j \neq i} C^j_k \cup P - B_i \subset \text{cell}(p^i_\infty, B_i \cup C^i_k). \tag{3}$$

In other words, rotate and scale C_k until the set B_i plays the role of B in our previous construction and then affinely compress C_k as much as necessary toward the segment $\bar{B}_i$ to obtain a set C^i_k such that cell$(p^i_\infty, B_i \cup C^i_k)$ contains all of conv(P) except for $\bar{B}_i$ and a small neighborhood around it. Let $D_{h,k} = P \cup \bigcup_i C^i_k$. See Fig. 3 for an example.

Let S be a decomposition of $D_{h,k}$. From (3) it follows that if we contract every vertex *not* in $B_i \cup C^i_k$ to p^i_∞ we obtain a decomposition which is combinatorially equivalent to a decomposition of $B^* \cup C_k$. In particular, from Corollary 3 we see that in any decomposition S of $D_{h,k}$ and for each i the number of polygons incident *only* to vertices in $B_i \cup C^i_k$ is at least $k(k+1)/2$. In addition to these $hk(k+1)/2$ polygons,

S must contain at least $hk/2$ more polygons which are incident to the diagonals that connect the last layer of each C_k^i with vertices outside $B_i \cup C_k^i$ (there are at least $hk/2$ such diagonals). Therefore, for every decomposition S of $D_{h,k}$, $|S| \geq hk(k+2)/2$. Hence,

Theorem 7 $G(D_{h,k}) \geq hk(k+2)/2$, *for* $h \geq 3, k \geq 1$. □

The quotient between $hk(k+2)/2$ and $|D_{h,k}|$ does not depend on h and attains a maximum, among integer values of k, at $k = 5$. Since $|D_{h,5}| = 16h$ the previous theorem yields $g(n) \geq (35/32)n$ when $n = 16h, h \geq 3$. For arbitrary n, we obtain the following theorem:

Theorem 8 $g(n) > \frac{35}{32}n - \frac{3}{2}$, *for* $n \geq 4$. □

Proof 4 For $n \geq 48$, let $n = 16h + c$ with $0 \leq c < 16$. By Theorem 5 we get $g(n) \geq g(16h) + c \geq (35/32)16h + (35/32)c - (35/32 - 1)c = (35/32)n - (3/32)c > (35/32)n - 3/2$. For $9 \leq n \leq 47$, note that the bound on $g_3(n)$ in Theorem 6 is better than the bound on $g(n)$ in Theorem 8 for these values of n, i.e., $n - 4 + \sqrt{2(n-3)} > (35/32)n - 3/2$ for $9 \leq n \leq 47$. By definition $g(n)$ and $g_3(n)$ are the maximum values of $G(V)$ over two classes of sets, one containing the other, hence $g(n)$ and $g_3(n)$ satisfy $g(n) \geq g_3(n)$ for all n. Therefore in this case $g(n) > (35/32)n - 3/2$ by virtue of Theorem 6. The case $4 \leq n \leq 8$ can be verified using Theorem 4 for $k = 1, 2$ and the monotonicity of g. □

4 Conjectures

We conjecture that our bound for g_3 (Theorem 6) is tight up to a constant. Also we note that our basic construction seems to admit a direct generalization to higher dimensions and thus it is likely that the results and proofs concerning this construction (i.e., up to Theorem 6) might also have a generalization.

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