```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%
%%
%%             NOTE: The content in this template, including title, authors' name and affiliation, and etc.
%%                  is ONLY a sample for the author's guidance.
%%
%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
\documentclass{amse-new}

\numberwithin{equation}{section} %%% Equations numbered by section. If you don't want it, please delete it.

\usepackage{latexsym,enumerate}
%\usepackage{amsmath,amsthm,amsopn,amstext,amscd,amsfonts,amssymb}
\usepackage[ansinew]{inputenc}
\usepackage{verbatim}
%\usepackage{graphicx}
\usepackage{dsfont}
\usepackage{color}



\begin{document}

 \PageNum{79}
 \Volume{2014}{Jan.}{30}{1}
 \OnlineTime{May 15, 2013}
 \DOI{10.1007/s10114-013-2370-2}
 \EditorNote{Received June 19, 2012, accepted November 16, 2012}

\abovedisplayskip 6pt plus 2pt minus 2pt \belowdisplayskip 6pt
plus 2pt minus 2pt
%%%%%%%%%%%%%%%%
\def\vsp{\vspace{1mm}}
\def\th#1{\vspace{1mm}\noindent{\bf #1}\quad}
\def\proof{\vspace{1mm}\noindent{\it Proof}\quad}
\def\no{\nonumber}
\newenvironment{prof}[1][Proof]{\noindent\textit{#1}\quad }
{\hfill $\Box$\vspace{0.7mm}}
\def\q{\quad} \def\qq{\qquad}
\allowdisplaybreaks[4]
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%-------------------       Beginning of  Author's Definitions       -------------------%%
%%                     Note: You may add your own definitions here.

\newcommand{\CC}{\mathbb{C}}
\newcommand{\NN}{\mathbb{N}}
\newcommand{\PP}{\mathbb{P}}
\newcommand{\RR}{\mathbb{R}}
\newcommand{\ZZ}{\mathbb{Z}}
\newcommand{\EE}{\mathbb{E}}
\newcommand{\DD}{\mathbb{D}}
\newcommand{\UU}{\mathbb{U}}
\newcommand{\HH}{\mathbb{H}}
\newcommand{\TT}{\mathbb{T}}
\newcommand{\sop}[1]{\mbox{{\rm supp\/}}(#1)}
\newcommand{\diam}{diam}
\newcommand{\card}{card}
\newcommand{\inte}{int}
\newcommand{\cotan}{cotan}
\newcommand{\cotanh}{cotanh}
\newcommand{\sech}{sech}
\newcommand{\Arcsinh}{Arcsinh}
\newcommand{\Arccosh}{Arccosh}
```

```
\newcommand{\Arctanh}{Arctanh}
\newcommand{\p}{\partial}
\renewcommand{\a}{\alpha}
\renewcommand{\b}{\beta}
\newcommand{\e}{\varepsilon}
\renewcommand{\d}{\delta}
\newcommand{\g}{\gamma}
\newcommand{\G}{\Gamma}
\renewcommand{\l}{\lambda}
\renewcommand{\L}{\Lambda}
\renewcommand{\O}{\Omega}
\newcommand{\s}{\sigma}


%%-------------------         the end of  Author's Definitions           -------------------%%



\AuthorMark{Cant\'on A., et al.}                            %%%  appear on the head of even pages  %%%

\TitleMark{Hyperbolicity of Periodic Planar Graphs}  %%% Running Title, appear on the head of odd pages  %%%

\title{ Gromov  Hyperbolicity of Periodic Planar Graphs  %%%   Main Title of your paper  %%%
\footnote{Supported by  Ministerio de Ciencia e Innovaci\'on of
Spain (Grant No. MTM 2009-07800),   and the last author also by a
grant from CONACY of TM\'exico
(Grant No. CONACYT-UAG I0110/62/10)}}                  %%%   the Fund which you are supported by  %%%

\author{Alicia \uppercase{CANT\'ON}}             %%%  1st Author's information   %%%
    {Departamento de Ciencias Aplicadas a la Ingenier\'{i}a Naval, ETSIN, Universidad Polit\'{e}cnica de
    Madrid,  Avenida Arco de la Victoria, $4$, Ciudad Universitaria, $28040$ Madrid, Spain\\
    E-mail\,$:$ alicia.canton@upm.es }

\author{Ana GRANADOS}     %%%  2nd Author's info, if exists, or you may delete this part directly  %%%
    {Mathematics Division, St. Louis University $($Madrid Campus$)$, Avenida del Valle $34$,\\ $28003$ Madrid, Spain\\
  E-mail\,$:$  agranado@slu.edu  }

\author{Domingo PESTANA \qq Jos\'e Manuel RODR\'IGUEZ}     %%%  2nd Author's info, if exists, or you may delete this part directly  %%%
    {Departamento de Matem\'aticas, Universidad Carlos III de Madrid, Avenida de la Universidad $30$, $28911$ Legan\'es, Madrid, Spain\\
   E-mail\,$:$  dompes@math.uc$\,3$m.es \qq  jomaro@math.uc$\,3$m.es }


\maketitle%


\Abstract{The study of hyperbolic graphs is an interesting topic since the
hyperbolicity of a geodesic metric space is equivalent to the
hyperbolicity of a graph related to it. The main result in this paper
is a very simple characterization of the hyperbolicity of a large class of periodic planar graphs.}      % the abstract

\Keywords{Planar graphs, periodic graphs, Gromov hyperbolicity, infinite graphs, geodesics}        % the keywords

\MRSubClass{05C10, 05C63, 05C75, 05A20}      % MR(2000) Subject Classification

\baselineskip 14.2pt

\section{Introduction}

Hyperbolic spaces play an important role in geometric
group theory and in the geometry of negatively curved
spaces (see \cite{ABCD, GH, G1}).
The concept of Gromov hyperbolicity grasps the essence of both negatively curved
spaces like the classical hyperbolic space or Riemannian manifolds of
negative sectional curvature, and of discrete spaces like trees
and Cayley graphs of many finitely generated groups. It is remarkable
```

how a simple concept leads to such a rich
general theory (see \cite{ABCD, GH, G1}).

The study of mathematical properties of Gromov hyperbolic spaces
and their applications is a topic of recent and increasing
interest in graph theory; see, for instance,
\cite{BRS,BRST,BHB1,CPRS,CRSV,K27,K21,K22,
%K23,K24,
K56,MRSV,PRT1,RSVV,
%RT1,
RT3,T,WZ}.

The theory of Gromov spaces was used initially for the study of
finitely generated groups (see \cite{G1}
%,G2}
and the references therein), where it was observed to have
practical importance. This theory was applied mainly to the study
of automatic groups (see \cite{O}), which play a role in
computational science. The concept of hyperbolicity appears also
in discrete mathematics, algorithms and networking. For example,
it has been shown empirically in \cite{ShTa} that the internet
topology embeds with better accuracy into hyperbolic space than
into Euclidean space of comparable dimension. A few algorithmic
problems in hyperbolic spaces and hyperbolic graphs have been
considered in recent papers (see \cite{ChEs, Epp, GaLy, Kra}).
Another interesting application of these spaces is secure
transmission of information on the internet (see
\cite{K27,K21,K22}). In particular, the hyperbolicity plays a key
role in the spread of viruses through the network (see
\cite{K21,K22}). The hyperbolicity is also useful in the study of
DNA data (see \cite{BHB1}).

In \cite[Section 1.3]{BHB}, it is observed that the hyperbolicity
of a geodesic metric space is equivalent to the hyperbolicity of a
graph related to it (see also \cite{PRT1,RT3,T}). Hence,
establishing hyperbolicity criteria for graphs will be of
interest to us.

Let us state some basic facts about Gromov's spaces. If
$\g:[a,b]\rightarrow X$ is a continuous curve in a metric space
(X,d), we say that $\g$ is a \emph{geodesic} if it is an
isometry, i.e., $L(\g|_{[s,t]})=d(\g(t),\g(s))=|t-s|$ for every
$s,t\in [a,b]$. The space X is a \emph{geodesic metric space} if
for every $x,y\in X$, there exists a geodesic joining x and y;
denote by $[xy]$ any of such geodesics (since uniqueness of
geodesics is not required, this notation is ambiguous, but it is
convenient). It is clear that every geodesic metric space is
path-connected.

By a graph G, we mean a set of points called vertices connected
by (undirected) edges. The set of vertices is denoted by $V(G)$
and the set of edges by $E(G)$; we assume also that each edge has
a length assigned. In order to consider a graph G as a geodesic
metric space, identify (by an isometry) any edge $[u,v] \in E(G)$
with the real interval $[0, l]$ (if $l := L([u,v])$); therefore,
any point in the interior of an edge is a point of G. Then G
is naturally equipped with a distance defined on its points,
induced by the shortest paths.

If X is a geodesic metric space and $J=\{J_1,J_2,\dots,J_n\}$ is
a polygon, with sides $J_j\subseteq X$, the polygon J is
$\d$-{\it thin} if for every $x\in J_i$, one has that
$d(x,\bigcup_{j\neq i}J_{j})\le \d$. Denote by $\d(J)$ the sharp
thin constant of J, i.e., $ \d(J):=\inf\{\d: J \, \text{ is
$\d$-thin}\}. $ If $x_1,x_2,x_3\in X$, a {\it geodesic triangle}
$T=\{x_1,x_2,x_3\}$ is the union of the three geodesics
$[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is
$\d$-\emph{hyperbolic} if every geodesic triangle in X is
$\d$-thin. Let us denote by $\d(X)$ the sharp hyperbolicity
constant of X, i.e., $\d(X):=\sup\{\d(T): T \, \text{ is a
geodesic triangle in }\,X \}.$ The space X is \emph{hyperbolic}
if X is $\d$-hyperbolic for some $\d$. Note that if X is
$\d$-hyperbolic, then every geodesic polygon with n sides is
$(n-2)\d$-thin; in particular, every geodesic quadrilateral is
$2\d$-thin.

As a remark, the main examples of hyperbolic graphs are trees. In
fact, the hyperbolicity constant of a geodesic metric space can be
viewed as a measure of how ``tree-like'' the space is, since those
spaces X with $\delta(X) = 0$ are precisely the metric trees. In
many applications, one finds that the borderline between tractable
and intractable cases may be the tree-like degree of the structure
to be dealt with (see, e.g., \cite{CYY}).

It is worth pointing out that deciding whether or not a space is
hyperbolic is usually extraordinarily difficult: Note that, first
of all, one needs to consider an arbitrary geodesic triangle T,
and calculate the minimum distance from an arbitrary point P of
T to the union of the other two sides of the triangle to which
P does not belong to. And then, take the supremum over all
possible choices of P and then over all possible choices of
T. Without disregarding the difficulty of solving this minimax
problem, notice that in general, the main obstacle is that the
location of geodesics in the space is not usually known.

One of the main questions in the study of any mathematical
property is to characterize it in a simple way. If the property is
very difficult to characterize (as in the case of hyperbolicity),
a natural strategy is to do so for a subclass of objects. In this
paper, a very simple characterization of the hyperbolicity of
periodic tessellation graphs of $\RR^2$ is given (see Theorem
\ref{t:periodic} and Definitions \ref{d1} and \ref{d2}).

Theorem \ref{t:periodic} characterizes the hyperbolicity of any
periodic tessellation graph G in terms of the hyperbolicity of a
``period graph" G^* (G can be obtained by pasting infinitely
many copies of G^*, see Definition \ref{d2}). As a first
intuition, one might think that the hyperbolicity of G^*
guarantees the hyperbolicity of G; however, this does not hold
for the Cayley graph G of $\ZZ\times\ZZ$, where one can take
G^* as the Cayley graph of $\ZZ\times\ZZ_2$ (note that G^* is
hyperbolic and G is not hyperbolic). Theorem \ref{t:periodic}
states that G is hyperbolic if and only if G^* is hyperbolic
and the geodesic lines bordering G^* ``diverge" (this last
condition is not satisfied in $\ZZ\times\ZZ_2$).

The outline of the paper is as follows. In Section 2, the needed
background is collected. Section 3 contains the technical results
used in the proof of the main theorem, which appears in Section 4.

\section{Background on Gromov Hyperbolic Spaces}

Let (X,d_X) and (Y,d_Y) be two metric spaces. A map $f:
X\rightarrow Y$ is said to be an $(\alpha,
\beta)$-\emph{quasi-isometric embedding}, with constants
$\alpha\geq 1,\ \beta\geq 0$ if, for every $x, y\in X$,
$$
\alpha^{-1}d_X(x,y)-\beta\leq d_Y(f(x),f(y))\leq \alpha d_X(x,y)+\beta.
$$
The function f is ε-\emph{full} if for each $y \in
Y$, there exists $x\in X$ with $d_Y(f(x),y)\leq \varepsilon$.

A map $f: X\rightarrow Y$ is said to be a \emph{quasi-isometry},
if there exist constants $\alpha\geq 1,\, \beta,\varepsilon \geq
0$ such that f is ε-full $(\alpha,
\beta)$-quasi-isometric embedding.

Two metric spaces X and Y are \emph{quasi-isometric} if there
exists a quasi-isometry $f:X\rightarrow Y$.

An (α,β)-\emph{quasigeodesic} of a metric space X is
an (α,β)-quasi-isometric embedding $\gamma:
I\rightarrow X$, where I is an interval of $\mathbb R.$ A
\emph{quasigeodesic} is an (α,β)-quasigeodesic for some
$\alpha\geq 1,\ \beta\geq 0$. Note that a $(1,0)$-quasigeodesic is
a geodesic. A \emph{geodesic line} is a geodesic with domain
$\RR$. A \emph{geodesic ray} is a geodesic with domain
$[0,\infty)$.

Let X be a metric space, Y a non-empty subset of X and
ε a positive number. The
ε-\emph{neighborhood} of Y in X, denoted by
$\mathcal{V}_{\varepsilon}(Y)$, is the set $\{x\in X:
d_X(x,Y)\leq\varepsilon\}$.

The \emph{Hausdorff distance} between two non-empty subsets Y
and Z of X, denoted by $\mathcal{H}(Y,Z)$, is the number
defined by
\[\inf\{\varepsilon>0: Y\subset\mathcal{V}_{\varepsilon}(Z)\ \mbox{and}\ Z\subset\mathcal{V}_{\varepsilon}(Y)\}.\]

Two of the fundamental properties of hyperbolic spaces are the following:

\begin{theorem}[Invariance of hyperbolicity]\label{invarianza}
Let $f:X\rightarrow Y$ be an (α,β)-quasi-isometric
embedding between the geodesic metric spaces X and $Y.$ If Y
is hyperbolic, then X is hyperbolic.

Besides, if f is $\e$-full for some $\e\ge0$ $($a
quasi-isometry$)$, then X is hyperbolic if and only if $\,Y$ is
hyperbolic.
\end{theorem}

\begin{theorem}[Geodesic stability]\label{teoremaestabilidad}
For given constants $\a\ge 1$ and $\b,\d\ge 0$, there exists a
constant $H=H(\delta,\alpha,\beta)$ such that for every
δ-hyperbolic geodesic metric space and for every pair of
(α,β)-quasigeodesics g,h with the same endpoints,
$\mathcal{H}(g,h)\leq H$.
\end{theorem}

If X is a metric space, define the \emph{Gromov product} of
$x,y\in X$ with the base point $w\in X$ by
$$
(x,y)_w:= \frac12\, (d(x,w)+d(y,w) -d(x,y)) .
$$
If X is a Gromov hyperbolic space, it holds that

\begin{equation}
\label{eq:product}
(x,z)_w
\ge \min \big\{ (x,y)_w, (y,z)_w \big\}- \d
\end{equation}
for every $x,y,z,w\in X$ and some constant $\d\ge 0$ (see,
\mbox{e.g.}, \cite[Proposition 2.1]{ABCD} or \cite[p.\,41]{GH}).
We denote by $\d^*(X)$ the sharp constant for this inequality,
i.e.,
$$
\d^*(X):= \sup\big\{\!\min \big\{ (x,y)_w, (y,z)_w \big\}- (x,z)_w
:\, x,y,z,w\in X \big\} .
$$
\begin{remark} \label{r1}
If X is a geodesic metric space, it is known that
\eqref{eq:product} is, in fact, equivalent to our definition of
Gromov hyperbolicity; furthermore, $\d^*(X)\le 4\,\d(X)$ and
$\d(X)\le 3\,\d^*(X)$ (see, \mbox{e.g.}, \cite[Proposition
2.1]{ABCD} or \cite[p.\,41]{GH}).
\end{remark}

If D is a closed subset of X, always consider in D the
\emph{inner metric} obtained by the restriction of the metric in
X, that is
$$
d_{D}(z,w):= \inf\big\{ L_{X}(\g) : \, \g\subset D \text{ is a
continuous curve joining z and w} \big\}\ge d_{X}(z,w).
$$
Consequently, $L_{D}(\g) = L_{X}(\g)$ for every curve $\g\subset D$.

In an informal way, a tessellation, T, on $\RR^2$ is a partition
of $\RR^2$ by geometric shapes (called tiles) with no overlaps and
no gaps. The tessellation graph associated with T is the union
of the boundaries of the tiles. More precisely, for $n\ge 1$, an
n-cell is a topological space homeomorphic to the open ball in
$\RR^n$. A 0-cell is a singleton space. A \emph{tesselation} on
$\RR^2$ is a CW 2-complex on $\RR^2$ such that every point on
$\RR^2$ is contained in some n-cell of the complex for some
$n\in\{0,1,2\}$. A \emph{tessellation graph} is the 1-skeleton
(the set of 0-cells and 1-cells). The edges (1-cells) of a
tessellation graph are just rectifiable paths (paths with finite
Euclidean length) in $\RR^2$ and have the length induced by the
Euclidean metric. Note that this class of graphs contains as
particular cases many planar graphs.

\section{Technical Lemmas}

Since the proof of our main result (Theorem \ref{t:periodic})
is long and technical, in order to make the arguments more transparent, we
collect some results needed along the proof in technical lemmas.
Let us start with the definition of periodic graph.

\begin{definition}
Let G be a tessellation graph of $\RR^2$. Then, G is
\emph{periodic} if there exists $(u,v) \in\RR^2 \backslash
\{(0,0)\}$ such that $T(G)=G$, where $T:\RR^2 \rightarrow \RR^2$
is defined as $T (x,y) = (x,y) + (u,v)$. In this case, T is a

\emph{periodic transformation of} G.
\end{definition}

\begin{definition} \label{d1}
A periodic tessellation graph G of $\RR^2$ is \emph{normalized}
if $T (x,y) = (x+k_0,y)$ is a periodic transformation of G for some positive constant $k_0>0$,
and $\s:=\RR\times\{0\}$ is contained in G.
\end{definition}

Note that in order to study the hyperbolicity of a periodic tessellation graph G of $\RR^2$, by applying a rotation and/or a lift,
without loss of generality one can assume that
$T(x,y)=(x+k_0,y)$ with $k_0>0$.
Furthermore, one can assume that $\s:=\RR\times\{0\} \subset G$, since otherwise the tessellation graph obtained by adding $\s$ to G
is quasi-isometric to G.
Hence, in order to study the hyperbolicity, assume that every periodic tessellation graph of $\RR^2$ is normalized.

\begin{definition} \label{d2}
Denote by U $($respectively, $L)$ the closed upper
$($respectively, lower$)$ half-plane in $\RR^2$. If G is a
normalized periodic tessellation graph of $\RR^2$, then $\g$ is a
\emph{fundamental ray} of G in U $($respectively, in $L)$ if
it is a geodesic ray of G in U $($respectively, in $L)$
starting in $(0,0)$ satisfying the following property$\,:$ if B
is the closed connected set in U $($respectively, in $L)$
bounded by $\g$ and $T(\g)$, then $\bigcup_{n\in\ZZ} T^n(B)=U$
$($respectively, $\bigcup_{n\in\ZZ} T^n(B)=L)$. In this case,
$\g_0$ is a \emph{fundamental line} of G if $\g_0=\g_1 \cup
\g_2$, where $\g_1$ is a fundamental ray of G in U and $\g_2$
is a fundamental ray of G in $L;$ if B is the closed connected
set in $\RR^2$ bounded by $\g_0$ and $T(\g_0)$, then
$\bigcup_{n\in\ZZ} T^n(B)=\RR^2$. The \emph{period graph} G^* of
G $($with respect to $\g_0)$ is the subgraph $G^*:=G \cap B.$
Then $\bigcup_{n\in\ZZ} T^n(G^*)=G$.
\end{definition}

The following result is a main tool to state our main result.

\begin{lemma} \label{l0}
For any normalized periodic tessellation graph G of $\RR^2$,
there exists a fundamental line.
\end{lemma}

\begin{prof}
By symmetry, it suffices to show that there exists a fundamental
ray of the graph G in~U.

Given $s>0$, let E_s be the set of geodesics starting in
$p=(0,0)$ and finishing at some point q with $d_G(p,q) \le s$;
hence, any geodesic in E_s is contained in the closed ball
$\overline{B_G(p,s)}$ (if a geodesic in E_s has length $s'<s$,
 it can be considered as a map defined on $[0,s]$ which is
constant in the interval $[s',s]$). Let us consider the uniform
convergence topology in E_s. Since the closed ball
$\overline{B_G(p,s)}$ is compact, E_s is compact by
Arzel\`a--Ascoli's theorem. If $\g: J \rightarrow G$ is a geodesic
with $J=[0,a]$, define $J^{s}:=J\cap [0,s]$ and denote by
$\g^{(s)}$ the restriction of $\g$ to J^s (hence, $\g^{(s)} \in
E_s$).

Note that if a geodesic starts at $p=(0,0)$ and leaves $\s$, it
does not return to $\s$. For each natural number $n>s$, choose a
point r_n in the open upper half-plane with $d_G(r_n,\s) \ge n$
and a geodesic $[pr_n]$. The geodesic $[pr_n]$ exits from $\s$ in

the point p_n; let u be such that $T^{-u}(p_n)\in [pr_n]$ and
$d_G(T^{-u}(p_n),p) \le s$. Then $\g_n':=[pT^{-u}(p_n)] \cup
T^{-u}([p_nr_n])$ is a geodesic; note that $L(\g_n'\cap \s) \le s$
and $L(\g_n') \ge n$. Iterating this argument, one can obtain a
geodesic $\g_n$ with the following property: for every horizontal
line $\s'$ one gets $L(\g_n \cap \s') \le s$ and $L(\g_n) \ge n$.

Since E_s is compact, for each $s>0$, there exists a subsequence
$\{\g_{s,m}\}_m$ of $\{\g_{n}\}_n$ such that
$\{\g_{s,m}^{(s)}\}_m$ converges uniformly. Cantor's diagonal
argument gives a subsequence $\{g_{n}\}_n$ of $\{\g_{n}\}_n$ such
that the sequence $\{g_{n}^{(s)}\}_n$ converges uniformly to a
geodesic g^s; since $(g^s)^{(s')}=g^{s'}$ if $s'<s$, these
geodesics g^s define a geodesic ray g starting at p. One can
check that g is contained in U and that $L(g\cap \s') \le s$
for every horizontal line $\s'$. If $g=(u,v)$, let us check that
$\limsup_{t\to\infty}v(t) = \infty$: since g does not contain a
horizontal ray, if $v(t)\le M$ for some constant M and every
$t\ge 0$, then $\{T^{-n}(g)\}_{n\ge 0}$ accumulates, which is a
contradiction.

Let us denote by B the closed connected set in U bounded by
g and $T(g)$. Since $$\limsup_{t\to\infty}v(t) = \infty,$$ then
$\bigcup_{n\in\ZZ} T^n(B)=U$.
\end{prof}

\begin{lemma}\label{l1}
Let G be a $\d$-hyperbolic graph and let $\g_0$ be either a
geodesic line or a geodesic ray in G. For any $x\in G$, denote
by x' a point on $\g_0$ with $d_G(x,\g_0)=d_G(x,x')$. Then, for
any $w\in \g_0$,
$$
d_G(x,x')+d_G(x',w)-8\d \le d_G(x,w) \le d_G(x,x')+d_G(x',w).
$$
\end{lemma}

\begin{prof}
The upper bound is just the triangle inequality. Let us prove the
lower bound. Given any geodesic triangle $T=\{x,x',w\}$, let
$a\in[x'w],b\in[xw],c\in[xx']$ be the points that satisfy
$d_G(x,b)=d_G(x,c)$, $d_G(x',a)=d_G(x',c)$, $d_G(w,a)=d_G(w,b)$.
Since G is $\d$-hyperbolic, it is well known that
$d_G(a,b),d_G(a,c),d_G(b,c)\le 4\d$ (see, e.g., \cite[Proposition
2.1]{ABCD}). Since $d_G(x,\g_0)=d_G(x,x')$ and $a\in\g_0$, it
follows that $d_G(x',a)=d_G(c,x')\le d_G(c,a)\le 4\d$ and one
deduces $d_G(x,w)=d_G(x,b)+ d_G(b,w)=d_G(x,c)+ d_G(a,w)\ge
d_G(x,c)+ d_G(c,x')-4\d+d_G(x',a)-4\d+ d_G(a,w)
=d_G(x,x')+d_G(x',w)-8\d$.
\end{prof}

A subgraph $\G$ of G is \emph{isometric} if
$d_{\G}(x,y)=d_{G}(x,y)$ for every $x,y\in \G.$
The following result appears in \cite[Lemma 5]{RSVV}.

\begin{lemma}
\label{l:subgraph} If $\G$ is an isometric subgraph of G, then $\d(\G) \le \d(G)$.
\end{lemma}

\begin{lemma}\label{l2}

Let G be a graph and let $\g_0$ be either a geodesic line or a geodesic ray in G such that $\g_0$ disconnects G.
Let G_1',G_2' be two connected components of $G\setminus \g_0$ and define $G_1:=G_1'\cup \g_0$ and $G_2:=G_2'\cup \g_0$.
Then G_1,G_2 are isometric subgraphs of G.
For any $x\in G$, denote by x' a point in $\g_0$ with $d_G(x,\g_0)=d_G(x,x')$.
If G_1,G_2 are $\d$-hyperbolic and $x_1\in G_1,x_2\in G_2$,
then
$$
d_G(x_1,x_1')+d_G(x_1',x_2')+d_G(x_2',x_2)-16\d \le d_G(x_1,x_2) \le d_G(x_1,x_1')+d_G(x_1',x_2')+d_G(x_2',x_2).
$$
\end{lemma}

\proof First of all, it will be shown that G_1,G_2 are isometric
subgraphs of G. The inequality $d_{G}(x,y) \le d_{G_i}(x,y)$ for
every $x,y \in G_i$ is straightforward. In order to prove the
reverse inequality, let us fix $x,y \in G_i$ and a geodesic $\s$
in G joining them, $\s:[0,l] \rightarrow G$. If $\s$ is
contained in G_i, then $d_G(x,y) = d_{G_i}(x,y)$. Otherwise,
$\s$ intersects $\g_0$. If $\s'$ is a subcurve of $\s$ joining two
points u,v on $\g_0$ and g is the subcurve of $\g_0$ joining
$u,v,$ then $L(\s')=L(g)$ since $\g_0$ is a geodesic.
Consequently, $d_G(x,y) \ge d_{G_i}(x,y)$ and conclude
$d_{G_i}(x,y) = d_{G}(x,y)$ for every $x,y \in G_i$.

Let w be a point in $\g_0 \cap [x_1x_2]$.
Since G_1,G_2 are $\d$-hyperbolic, Lemma \ref{l1} implies
\begin{align*}
d_G(x_1,x_2)
& = d_{G_1}(x_1,w)+d_{G_2}(w,x_2)
\\
& \ge d_{G_1}(x_1,x_1')+d_{G_1}(x_1',w) -8\d+d_{G_2}(w,x_2') +d_{G_2}(x_2',x_2) -8\d
\\
& = d_{G}(x_1,x_1')+d_{G}(x_1',w) +d_{G}(w,x_2') +d_{G}(x_2',x_2) -16\d
\\
& \ge d_G(x_1,x_1')+d_G(x_1',x_2')+d_G(x_2',x_2)
-16\d.\tag*{$\Box$}
\end{align*}

\begin{lemma}\label{l3}
Let G be a graph and let $\g_0$ be either a geodesic line or a geodesic ray in G such that $\g_0$ disconnects G.
Let G_1',G_2' be two connected components of $G\setminus \g_0$ and define $G_1:=G_1'\cup \g_0$ and $G_2:=G_2'\cup \g_0$.
For any $x\in G$, denote by x' a point in $\g_0$ with $d_G(x,\g_0)=d_G(x,x')$.
If G_1,G_2 are $\d$-hyperbolic, $x_1\in G_1,x_2\in G_2$ and $w\in \g_0$,
then
$$
d_G(x_2',w)-d_G(x_1,x_2')-16\d \le d_G(x_2,w)-d_G(x_1,x_2)\le d_G(x_2',w)-d_G(x_1,x_2')+16\d .
$$
\end{lemma}

\proof Since $\g_0$ is a geodesic line, G_1,G_2 are isometric
subgraphs in G. Then Lemma \ref{l1} (applied to G_1 and G_2)
and Lemma \ref{l2} imply
\begin{align*}
d_G(x_2,w)-d_G(x_1,x_2)
& \ge d_G(x_2,x_2') + d_G(x_2',w)-8\d -d_G(x_1,x_1')-d_G(x_1',x_2')-d_G(x_2',x_2)
\\
& \ge d_G(x_2',w)-d_G(x_1,x_2')-16\d ,
\\
d_G(x_2,w)-d_G(x_1,x_2)
& \le d_G(x_2,x_2') + d_G(x_2',w) -d_G(x_1,x_1')-d_G(x_1',x_2')-d_G(x_2',x_2)+16\d
\\
& \le d_G(x_2',w)-d_G(x_1,x_2')+16\d. \tag*{$\Box$}
\end{align*}

The following result appears in \cite[Corollary 1.1B]{G1} and \cite[Proposition 2.2]{ABCD}.

\begin{lemma}\label{l4}
Let X be a metric space such that, for some fixed $w_0\in X$,
$$
(x,z)_{w_0} \ge \min \{ (x,y)_{w_0}, (y,z)_{w_0} \}- \d
$$
for every $x,y,z\in X$ and some constant $\d\ge 0$. Then
\eqref{eq:product} holds with constant $2\d$ for every $x,y,z,w\in
G$.
\end{lemma}

\begin{lemma}\label{l5}
Let G be a graph and let $\g_0$ be either a geodesic line or a
geodesic ray in G such that $G\backslash \g_0$ has two connected
components G_1',G_2'. Define $G_1:=G_1'\cup \g_0$ and
$G_2:=G_2'\cup \g_0$. If G is $\d$-hyperbolic, then G_1,G_2
are $\d$-hyperbolic. If G_1,G_2 are $\d$-hyperbolic, then G is
$120\d$-hyperbolic.
\end{lemma}

\begin{prof}
Note that Lemma \ref{l2} gives that G_1,G_2 are isometric subgraphs of G.
Therefore, if G is $\d$-hyperbolic, then G_1,G_2 are $\d$-hyperbolic by Lemma \ref{l:subgraph}.

Assume now that G_1,G_2 are $\d$-hyperbolic. We will prove that
G is $120\d$-hyperbolic by Remark \ref{r1} and Lemmas \ref{l3}
and \ref{l4}. Let us fix $w\in \g_0$ and $x,y,z \in G$. Without
loss of generality, one can assume that either $x,y,z\in G_1$, or
$x,y\in G_1$ and $z\in G_2$, or $x,z\in G_1$ and $y\in G_2$ (in
our argument x and z play a symmetric role, but y plays
another role since it appears in a different place in the
inequalities).

If $x,y,z\in G_1$,
since G_1 is $\d$-hyperbolic, then Remark \ref{r1} gives
$$
(x,z)_{w} \ge \min \{ (x,y)_{w}, (y,z)_{w} \}- 4\d.
$$

If $x,y\in G_1$ and $z\in G_2$, then Lemma \ref{l3} gives
$d_G(z,w)-d_G(x,z) \ge d_G(z',w)-d_G(x,z')-16\d$ and
$d_G(z',w)-d_G(y,z') \ge d_G(z,w)-d_G(y,z) -16\d$. Since G_1
satisfies \eqref{eq:product} with constant $4\d$, then
$$
\begin{aligned}
2(x,z)_{w}
& = d_G(x,w) + d_G(z,w)-d_G(x,z)
\ge d_G(x,w) + d_G(z',w)-d_G(x,z')-16\d
\\
& = 2(x,z')_{w}-16\d \ge \min \{ 2(x,y)_{w}, 2(y,z')_{w} \}- 8\d
-16\d
\\
& = \min \{ d_G(x,w) + d_G(y,w)-d_G(x,y), d_G(y,w) +
d_G(z',w)-d_G(y,z') \}- 24\d
\\
& \ge \min \{ d_G(x,w) + d_G(y,w)-d_G(x,y), d_G(y,w) +
d_G(z,w)-d_G(y,z) \}- 16\d -24\d
\\
& = 2 \min \{ (x,y)_{w}, (y,z)_{w} \}- 40\d.
\end{aligned}
$$

```
If $x,z\in G_1$ and $y\in G_2$, then Lemma \ref{l3} gives
$d_G(y',w)-d_G(x,y') \ge d_G(y,w)-d_G(x,y) -16\d$
and $d_G(y',w)-d_G(y',z) \ge d_G(y,w)-d_G(y,z) -16\d$.
Since $G_1$ satisfies \eqref{eq:product} with constant $4\d$,  one concludes
$$
\begin{aligned}
2(x,z)_{w} & \ge \min  \{ 2(x,y')_{w}, 2(y',z)_{w}  \}- 8\d
\\
& = \min \{ d_G(x,w) + d_G(y',w)-d_G(x,y'), d_G(y',w) +
d_G(z,w)-d_G(y',z) \}- 8\d
\\
& \ge \min  \{ d_G(x,w) + d_G(y,w)-d_G(x,y), d_G(y,w) +
d_G(z,w)-d_G(y,z)  \}- 24\d
\\
& = 2 \min  \{ (x,y)_{w}, (y,z)_{w}  \}- 24\d.
\end{aligned}
$$
Hence,  in any case,  $(x,z)_{w} \ge \min  \{ (x,y)_{w}, (y,z)_{w}
 \}- 20\d$ if $w\in \g_0$. Consequently, Lemma \ref{l4} shows
that \eqref{eq:product} holds with constant $40\d$, and Remark
\ref{r1} gives that $G$ is $120\d$-hyperbolic.
\end{prof}



\begin{lemma}\label{l:geodesic}
Let $G$ be a normalized periodic tessellation graph of $\RR^2$
such that $G^*$ is $\d^*$-hyperbolic and $\lim_{|z| \to \infty, z
\in \g_0} d_G(z,T(z)) = \infty$ for some choice of the fundamental
line $\g_0$. Assume also that $\g_0$ is a geodesic line. Let $x\in
T^j(G^*)$ and $y\in T^k(G^*)$ with $j \le k$. Then there exist
constants $M$, $N$, which just depend on $G^*$ and $\d^*$\!, with
the following properties$\,:$

$(1)$ For each geodesic $\g$ joining $x$ and $y$, there exists
another geodesic $\g'$ joining $x$ and $y$, with $\g'$ contained
in $\bigcup_{i=j}^k T^i(G^*)$ and such that
$\mathcal{H}(\g,\g')\leq M$. Furthermore, $\g'\cap T^i(G^*)$ is a
connected set for each $j\le i \le k$.
%$d_G(z,\g') \le M$ for every $z \in \g$.

$(2)$ If $j+2\le k$, then for each $j<i<k$, there exists a point
$z_i \in \g'$ with $d_{T^i(G^*)}(z_i,\s\cap T^i(G^*)) \le N$.
\end{lemma}


\begin{remark}\label{r:geodesic}
The proof of Lemma \ref{l:geodesic} gives that the same result holds for periodic tessellation graphs of $U$ or $L$.
\end{remark}


\begin{prof}[Proof of Lemma $\ref{l:geodesic}$]
Note that $G^*$ is an isometric subgraph, since $\g_0$ (and $T(\g_0)$) is a geodesic line.
Therefore, if $x,y\in G^*$ and $A,B\subset G^*$, then $d_{G^*}(x,y)=d_{G}(x,y)$ and
$\mathcal{H}_{G^*}(A,B)=\mathcal{H}(A,B)$.

In order to prove $(1)$, start by showing that there exists a
constant $M$, which just depends on $G^*$ and $\d^*$, with the
following property: if $u,v\in \g_0$, $\eta_0$ is the subset of
$\g_0$ joining $u$ and $v$, and $\eta$ is any geodesic joining $u$
and $v$, then $\mathcal{H}(\eta_0,\eta)\leq M$.

Let $H$ be the constant $H=H(\d^*,1,0)$ in Theorem \ref{teoremaestabilidad}.
One can check that if
```

$$
\lim_{|z| \to \infty, z \in \g_0} d_G(z,T(z)) = \infty,
$$
then $\lim_{|z| \to \infty, z \in \g_0} d_G(z,T(\g_0)) = \infty$ and $\lim_{|z| \to \infty, z \in \g_0} d_G(z,T^{-1}(\g_0)) = \infty$.
Then there exists a constant R such that if $z \in \g_0$ and $d_{G^*}(z,\s\cap G^*) > R$, then
$d_{G^*}(z,T(\g_0)) > H$ and $d_{T^{-1}(G^*)}(z,T^{-1}(\g_0)) > H$.

Assume first that η is contained either in G^* or in
$T^{-1}(G^*)$. Without loss of generality, assume that η is
contained in G^*. Then $\{\eta_0,\eta\}$ is a geodesic bigon in
G^*, and Theorem \ref{teoremaestabilidad} gives
$\mathcal{H}(\eta_0,\eta)\leq H$. If η is contained in
$G^*\bigcup T^{-1}(G^*)$, apply the previous argument to each
subset of η contained either in G^* or in $T^{-1}(G^*)$,
obtaining also $\mathcal{H}(\eta_0,\eta)\leq H$.

Assume now that η is not contained in $G^*\bigcup
T^{-1}(G^*)$ and that it is contained either in $\bigcup_{i<0}
T^i(G^*)$ or in $\bigcup_{i\ge 0} T^i(G^*)$. Without loss of
generality, one can assume that η is contained in
$\bigcup_{i\ge 0} T^i(G^*)$. If $\eta: [0,l] \rightarrow
\bigcup_{i\ge 0} T^i(G^*)$, then define
$$
a:= \max\{s\in [0,l]/\, \eta(t)\in G^* ,\ \forall\, t\in [0,s]
\}, \quad b:= \min\{s\in [0,l]/\, \eta(t)\in G^* ,\ \forall\,
t\in [s,l] \}.
$$

If η_1 is the subset of $T(\g_0)$ joining $\eta(a)$ and
$\eta(b)$, then define $\eta_2:=\eta([0,a]) \cup \eta_1 \cup
\eta([b,l])$. Hence, $\{\eta_0,\eta_2\}$ is a geodesic bigon in
G^*, and Theorem \ref{teoremaestabilidad} gives
$\mathcal{H}(\eta_0,\eta_2)\leq H$. Since $d_G(\eta_0,\eta(a)),
d_G(\eta_0,\eta(b)) \leq H$ and $\eta(a), \eta(b) \in T(\g_0)$,
one has $d_{G^*}(\eta(a),\s\cap G^*), d_{G^*}(\eta(b),\s\cap G^*)
\le R$; thus, $d_{G^*}(\eta(a),\eta(b)) \le 2R + L(\s\cap G^*)$.
Therefore, $\mathcal{H}(\eta_1,\eta|_{[a,b]})\leq R + L(\s\cap
G^*)/2$ and $\mathcal{H}(\eta_0,\eta)\leq M:= H + R + L(\s\cap
G^*)/2$.

In the general case, one can apply the previous argument to each
subset of η contained either in $\bigcup_{i<0} T^i(G^*)$ or
in $\bigcup_{i\ge 0} T^i(G^*)$, to obtain also
$\mathcal{H}(\eta_0,\eta)\leq M$.

Assume now that $\g$ is contained either in $\bigcup_{i\ge j}
T^i(G^*)$ or in $\bigcup_{i\le k} T^i(G^*)$. Without loss of
generality, one can assume that $\g$ is contained in
$\bigcup_{i\ge j} T^i(G^*)$. If $\g: [0,L] \rightarrow
\bigcup_{i\ge j} T^i(G^*)$, then define
\begin{align}
a:& = \max\bigg\{s\in [0,L]/\, \eta(t)\in \bigcup_{i=j}^k T^i(G^*), \; \forall\, t\in [0,s] \bigg\}, \notag \\
b:& = \min\bigg\{s\in [0,L]/\, \eta(t)\in \bigcup_{i=j}^k
T^i(G^*), \; \forall\, t\in [s,L] \bigg\}. \notag
\end{align}
If $\g_1$ is the subset of $T^{k+1}(\g_0)$ joining $\g(a)$ and
$\g(b)$, then define $\g':=\g([0,a]) \cup \g_1 \cup \g([b,L])
\subset \bigcup_{i=j}^k T^i(G^*)$. But
$\mathcal{H}(\g_1,\g|_{[a,b]})\leq M$ and, hence,
$\mathcal{H}(\g,\g')\leq M$.

In the general case, apply the argument above to each subset of
$\g$ contained either in $\bigcup_{i> k} T^i(G^*)$ or in

```
$\bigcup_{i< j} T^i(G^*)$, therefore $\mathcal{H}(\g,\g')\leq M$.

Applying again the  argument above, $\g'\cap T^i(G^*)$ is a
connected set for each $j\le i \le k$.



In order to prove $(2)$,  consider the set
$$
W:= \{ z\in G^* /\, d_{G^*}(z,\g_0) \le 2\d^*, \,
d_{G^*}(z,T(\g_0)) \le 2\d^*  \} = G^* \cap
\mathcal{V}_{2\d^*}(\g_0) \cap \mathcal{V}_{2\d^*}(T(\g_0)) .
$$
Since $\lim_{|z| \to \infty, z \in \g_0} d_G(z,T(\g_0)) =
\lim_{|z| \to \infty, z \in \g_0} d_G(z,T^{-1}(\g_0)) = \infty$,
the set $W$ is compact, so define $ N_0:=\max \{ d_{G^*}(z,\s\cap
G^*) /\, z\in W  \} . $ Given any geodesic $g: [0,\ell]
\rightarrow G^*$ joining a point on $\g_0$ with a point on
$T(\g_0)$ (and contained in $G^*$),
 define
$$
\a:= \max\{s\in [0,\ell]/\, \eta(t)\in \mathcal{V}_{2\d^*}(\g_0),
\; \forall\, t\in [0,s] \}.
$$
\indent Let us consider the geodesic quadrilateral $Q$ in $G^*$
with sides $g=:[a_gb_g]$, $\s\cap G^*=:[a_{\s} b_{\s}]$, $[a_g
a_{\s}] \subset \g_0$ and $[b_g b_{\s}] \subset T(\g_0)$. Since
$Q$ is $2\d^*$-thin, $d_{G^*}(g(\a),\s\cap G^*) \le
N:=\max\{N_0,2\d^*\}$. Consequence $(2)$ follows directly from
this inequality.
% applied to a connected component of $\g'\cap T^i(G^*)$ joining $T^i(\g_0)$ and $T^{i+1}(\g_0)$.
\end{prof}


\section{Main Result}


\begin{theorem} \label{t:periodic}
Let $G$ be a normalized periodic tessellation graph of $\RR^2$.
Then the following statements are equivalent$\,:$

$(1)$ $G$ is hyperbolic.

$(2)$  $G^*$ is hyperbolic and $\lim_{|z| \to \infty, z \in \g_0} d_G(z,T(z)) = \infty$ for some choice of the fundamental line $\g_0$.

$(3)$  $G^*$ is hyperbolic and $\lim_{|z| \to \infty, z \in \g_0} d_G(z,T(z)) = \infty$ for every choice of the fundamental line $\g_0$.
\end{theorem}

\begin{prof}
Let us define $\g_u:=\g_0\cap U$, $G_u:=G\cap U$ and
$G_u^*:=G^*\cap U$. Then, by symmetry and Lemma \ref{l5} applying
to the geodesic line $\s$, it suffices to show the statement of
Theorem \ref{t:periodic} replacing $\g_0$, $G$ and $G^*$ by
$\g_u$, $G_u$ and $G_u^*$, respectively.

Let us prove the implication $(1) \Rightarrow (3)$.
Assume that $G_u^*$ is not hyperbolic for some choice of the fundamental ray $\g_u$.
Since $\g_u$ and $T(\g_u)$ are geodesic rays, $G_u^*$ is an isometric subgraph of $G_u$.
Hence, Lemma \ref{l:subgraph} gives that $G_u$ is not hyperbolic.

Assume now that for some choice of the fundamental ray $\g_u$,
there exist a constant $c_0$ and a sequence $\{z_n\} \subset \g_u$
with $\lim_{n \to \infty} |z_n| = \infty$ and $d_G(z_n,T(z_n)) \le
c_0$ for every $n$. Since $\bigcup_{n\in\ZZ} T^n(G_u^*)=U$, if
$z_n = (x_n,y_n)$, then the sequence $\{y_n\}$ goes to infinity.
```

Let $\s_n$ be a geodesic in G_u joining z_n and $T(z_n)$, and
given $m\in \NN$ let $\s_n^m$ be the continuous curve in G_u
joining z_n and $T^m(z_n)$ given by $\s_n^m:=\bigcup_{j=0}^{m-1}
T^j(\s_n)$. Next, it will be shown that $\s_n^m$, with its
arc-length parametrization, is a $(c_0/k_0, 2k_0)$-quasigeodesic
(recall that $T(x,y)=(x+k_0,y)$ and then $c_0 \ge d_G(z_n,T(z_n))
\ge d_{\RR^2}(z_n,T(z_n)) = k_0$ and $c_0/k_0\ge 1$). If $s<t$,
then $d_G(\s_n^m(t),\s_n^m(s)) \le L(\s_n^m|_{[s,t]})=t-s$. If
$\s_n^m(s) \in T^j(\s_n)$ and $\s_n^m(t) \in T^{j+r}(\s_n)$, with
$r\ge 0$, then
$$
\begin{aligned}
& t-s
 \le (r+1)L(\s_n)
 = (r+1)d_{G_u}(z_n,T(z_n)) \le (r+1)c_0 ,
\\
& d_{G_u}(\s_n^m(t),\s_n^m(s)) \ge d_{\RR^2}(\s_n^m(t),\s_n^m(s))
\ge (r-1)k_0 = (r+1)c_0 \frac{k_0}{c_0} - 2k_0 \ge
\frac{k_0}{c_0}\,(t-s) - 2k_0 ,
\end{aligned}
$$
thus concluding $\s_n^m$ is a $(c_0/k_0, 2k_0)$-quasigeodesic (for every n,m).
If $\g_u^n$ is the subcurve of $\g_u$ joining z_1 and z_n, then let us choose a natural number $m=m(n)$
with $d_{\RR^2}(\g_u^n,T^m(\g_u^n)) \ge n$.
Hence,
$Q_n:=\{ \g_u^n, \s_n^m, T^m(\g_u^n), \s_1^m \}$ is a
$(c_0/k_0, 2k_0)$-quasigeodesic quadrilateral.

Seeking for a contradiction, let us assume that G_u is
hyperbolic. Let Q_n'
%$Q_n':=\{ g_0^n, s_n^n, T^n(g_0^n), s_1^n \}$
be a geodesic quadrilateral in G_u with the same vertices as
Q_n. By Theorem \ref{teoremaestabilidad}, the Hausdorff distance
between a quasigeodesic side in Q_n and its corresponding
geodesic side in Q_n' is less or equal than a constant
$H=H(\d(G_u),c_0/k_0, 2k_0)$. Let us show now that Q_n is
$(2\d(G_u)+2H)$-thin. If p belongs to a side of Q_n, then
there exists a point p' in its corresponding geodesic side in
Q_n' at distance from p less or equal than H; since Q_n'
is a geodesic quadrilateral, there exists a point q' on the
union of the other three geodesic sides in Q_n' at distance from
p' less or equal than $2\d(G_u)$; then, there exists a point q
on the union of the corresponding three quasigeodesic sides in
Q_n at distance from q' less or equal than H, and
$d_G(p,q)\le 2\d(G_u)+2H$. Hence, Q_n is $(2\d(G_u)+2H)$-thin.

Consider a point $p_n:=(a_n,b_n) \in \g_u^n$ with $b_n=(y_n+y_1)/2$.
Since $d_{\RR^2}(z_n,T(z_n)) \le d_G(z_n,T(z_n)) \le c_0$,
\begin{align*}
&d_{G_u}(p_n,\s_n^m\cup \s_1^m) \ge d_{\RR^2}(p_n,\s_n^m\cup
\s_1^m) \ge \frac12 \, (y_n-y_1) - c_0 ,
\\
&d_{G_u}(p_n,T^m(\g_0^n)) \ge d_{\RR^2}(p_n,T^m(\g_0^n)) \ge n .
\end{align*}
And therefore
$$
\min \!\bigg\{ \frac12 \, (y_n-y_1) - c_0 , \, n \bigg\} \le
2\d(G_u)+2H
$$
for every n.
This is a contradiction
since $\{y_n\}$ goes to infinity,
thus obtaining that G_u is not hyperbolic.

The implication $(3) \Rightarrow (2)$ is direct.

Let us prove the implication $(2) \Rightarrow (1)$.
Define $\d^*:=\d(G_u^*)$.
Let us consider any geodesic triangle $\mathcal{T}=\{x_1,x_2,x_3\}$ with $x_i\in T^{j_i}(G_u^*)$ and $j_1\le j_2\le j_3$.

Since the constant M in Lemma \ref{l:geodesic} just depends on
G^* and $\d^*$, one can assume that $[x_1x_2]\subset
\bigcup_{j=j_1}^{j_2} T^j(G_u^*)$, $[x_2x_3]\subset
\bigcup_{j=j_2}^{j_3} T^j(G_u^*)$, $[x_1x_3]\subset
\bigcup_{j=j_1}^{j_3} T^j(G_u^*)$, and that $[x_1x_2]\cap
T^i(G_u^*)$, $[x_2x_3]\cap T^i(G_u^*)$, $[x_1x_3]\cap T^i(G_u^*)$
are either the empty set or a connected set for each i.

Applying at most four times Lemma \ref{l5}, one obtains that if
$b-a\le 4$, then $\bigcup_{j=a}^{b} T^j(G_u^*)$ is
$\d_0$-hyperbolic, with $\d_0=(120)^4\d^*$.

By symmetry, it suffices to deal with the following cases:

(a) $j_2-j_1\le 2$ and $j_3-j_2\le 2$.

(b) $j_2-j_1\le 2$ and $j_3-j_2\ge 3$.

(c) $j_2-j_1\ge 3$ and $j_3-j_2\ge 3$.

\th{Case (a)} Since $\mathcal{T}\subset
\bigcup_{j=j_1}^{j_3}T^j(G_u^*)$, with $j_3-j_1\le 4$,
$\mathcal{T}$ is $\d_0$-thin.

\th{Case (b)} Let y_1 be the endpoint of $[x_1x_3]\cap
(\bigcup_{j=j_1}^{j_2+1} T^j(G_u^*))$ with $y_1\in
T^{j_2+2}(\g_u)$, and let y_2 be the endpoint of $[x_2x_3]\cap
(\bigcup_{j=j_1}^{j_2+1} T^j(G_u^*))$ with $y_2\in
T^{j_2+2}(\g_u)$. Consider the geodesic quadrilateral
$\mathcal{Q}=\{x_1,x_2,y_2,y_1\}$ in $\bigcup_{j=j_1}^{j_2+1}
T^j(G_u^*)$.

Let us bound $d_G(y_1,y_2)$. By Lemma \ref{l:geodesic} and Remark
\ref{r:geodesic}, for each $j_2<j<j_3$, there exists a constant
N, which just depends on G^* and $\d^*$, and points $z_1^j\in
[x_1x_3]\cap T^{j}(G_u^*)$, $z_2^j\in [x_2x_3]\cap T^{j}(G_u^*)$,
such that $d_{T^{j}(G_u^*)}(z_1^j,\s\cap T^{j}(G_u^*)),
d_{T^{j}(G_u^*)}(z_2^j,\s\cap T^{j}(G_u^*)) \le N$. Consider $z\in
[x_1x_3]\cap T^{j}(G_u^*)$ and $w\in [x_2x_3]\cap T^{j}(G_u^*)$,
with $j_2+2\le j\le j_3 -2$. If $\ell:=L(\s\cap G_u^*)$, then
$d_{G}(z,z_1^j) \le \max\{ d_{G}(z_1^{j-1},z_1^{j}),
d_{G}(z_1^{j},z_1^{j+1}) \}\le 2N+2\ell$ and $d_{G}(w,z_2^j) \le
\max\{ d_{G}(z_2^{j-1},z_2^{j}), d_{G}(z_2^{j},z_2^{j+1}) \}\le
2N+2\ell$. Since $d_{G}(z_1^j,z_2^j) \le 2N+\ell$, one obtains
$d_{G}(z,w) \le 6N+5\ell$ and, in particular, $d_{G}(y_1,y_2) \le
6N+5\ell$.

Since $\bigcup_{j=j_1}^{j_2+1} T^j(G_u^*)$ is $\d_0$-hyperbolic,
$\mathcal{Q}$ is $2\d_0$-thin; therefore, given any point p on
any side of $\mathcal{Q}$, there exists a point q on another
side of $\mathcal{Q}$ with $d_G(p,q) \le 2\d_0$. Hence, if $p\in
\mathcal{Q}\cap \mathcal{T}$, then there exists a point q' on
another side of $\mathcal{T}$ with $d_G(p,q) \le 2\d_0+6N+5\ell$.

```
If $z\in [x_1x_3]\cap  (\bigcup_{j=j_2+2}^{j_3 -2} T^j(G_u^*) )$,
it was shown above $d_{G}(z,[x_2x_3]) \le 6N+5\ell$. The same
argument gives that if $w\in [x_2x_3]\cap  (\bigcup_{j=j_2+2}^{j_3
-2} T^j(G_u^*) )$, then $d_{G}(w,[x_1x_3]) \le 6N+5\ell$.

Let $y_1'$ be the endpoint of $[x_1x_3]\cap
(\bigcup_{j=j_3-1}^{j_3} T^j(G_u^*) )$ with $y_1'\in
T^{j_3-1}(\g_u)$, and let $y_2'$ be the endpoint of $[x_2x_3]\cap
 (\bigcup_{j=j_3-1}^{j_3} T^j(G_u^*) )$ with $y_2'\in
T^{j_3-1}(\g_u)$. Consider the geodesic triangle
$\mathcal{T}_3=\{y_1',x_3,y_2'\}$ in $\bigcup_{j=j_3-1}^{j_3}
T^j(G_u^*)$. Then $d_{G}(y_1',y_2') \le 6N+5\ell$.

Since $\bigcup_{j=j_3-1}^{j_3} T^j(G_u^*)$ is $\d_0$-hyperbolic,
$\mathcal{T}_3$ is $\d_0$-thin; therefore, given any point $p$ on
any side of $\mathcal{T}_3$, there exists a point $q$ on another
side of $\mathcal{T}_3$ with $d_G(p,q) \le \d_0$. Hence, if $p\in
\mathcal{T}_3\cap \mathcal{T}$, then there exists a point $q'$ on
another side of $\mathcal{T}$ with $d_G(p,q) \le \d_0+6N+5\ell$.



\th{Case (c)} Let $a_1$ be the endpoint of $[x_1x_2]\cap
 (\bigcup_{j=j_2-1}^{j_2+1} T^j(G_u^*) )$ with $a_1\in
T^{j_2-1}(\g_u)$, and let $a_3$ be the endpoint of $[x_2x_3]\cap
 (\bigcup_{j=j_2-1}^{j_2+1} T^j(G_u^*) )$ with $a_3\in
T^{j_2+2}(\g_u)$. Let $b_1$ and $b_3$ be, respectively, the
endpoints of $[x_1x_3]\cap  (\bigcup_{j=j_2-1}^{j_2+1} T^j(G_u^*)
)$ with $b_1\in T^{j_2-1}(\g_u)$ and $b_3\in T^{j_2+2}(\g_u)$.
Consider the geodesic pentagon
$\mathcal{P}=\{a_1,x_2,a_3,b_3,b_1\}$ in
$\bigcup_{j=j_2-1}^{j_2+1} T^j(G_u^*)$.

The argument in  Case (b) gives that $d_{G}(a_1,b_1),
d_{G}(a_3,b_3) \le 6N+5\ell$.

Since $\bigcup_{j=j_2-1}^{j_2+1} T^j(G_u^*)$ is $\d_0$-hyperbolic,
$\mathcal{P}$ is $3\d_0$-thin; therefore, given any point $p$ on
any side of $\mathcal{P}$, there exists a point $q$ on another
side of $\mathcal{P}$ with $d_G(p,q) \le 3\d_0$. Hence, if $p\in
\mathcal{P}\cap \mathcal{T}$, then there exists a point $q'$ on
another side of $\mathcal{T}$ with $d_G(p,q) \le 3\d_0+6N+5\ell$.

Finally,  deal with the other situations as in  Case (b).



Hence,  $\d(G_u) \le 3\d_0+6N+5\ell$.
\end{prof}
```


```
\acknowledgements{\rm The authors would like to thank the referees
for their careful reading of the manuscript and several useful
comments.}
```



```
\begin{thebibliography}{99}


\bibitem{ABCD}
Alonso, J., Brady, T., Cooper, D., et al.: Notes on word
```

hyperbolic groups. In: Group Theory from a Geometrical Viewpoint,
E. Ghys, A. Haefliger, A. Verjovsky Eds., World Scientific,
Singapore, 1992

\bibitem{BRS} Bermudo, S., Rodr\'{\i}guez, J. M., Sigarreta, J. M.:
Computing the hyperbolicity constant. \emph{Comput. Math. Appl.},
\textbf{62}, 4592--4595 (2011)

\bibitem{BRST} Bermudo, S., Rodr\'{\i}guez, J. M., Sigarreta, J. M.,
et al.: Hyperbolicity and complement of graphs. \emph{Appl. Math.
Lett.}, \textbf{24}, 1882--1887 (2011)

\bibitem{BHB} Bowditch, B. H.: Notes on Gromov's hyperbolicity criterion for path-metric spaces.
Group theory from a geometrical viewpoint, Trieste, 1990 (ed. E.
Ghys, A. Haefliger and A. Verjovsky; World Scientific, River Edge
NJ, 1991, 64--167)

\bibitem{BHB1} Brinkmann, G., Koolen J., Moulton, V.: On the hyperbolicity of chordal
graphs. \emph{Ann. Comb.}, \textbf{5}, 61--69 (2001)

\bibitem{CPRS} Carballosa, W., Pestana, D., Rodr\'{\i}guez, J. M., et al.:
Distortion of the hyperbolicity constant of a graph. {\em Electr.
J. Comb.}, \textbf{19}, p67 (2012)

\bibitem{CRSV} Carballosa, W., Rodr\'{\i}guez, J. M., Sigarreta, J. M., et al.:
Gromov hyperbolicity of line graphs. \emph{Electr. J. Comb.},
\textbf{18}, p210 (2011)

\bibitem{CYY} Chen, B., Yau, S.-T., Yeh, Y.-N.: Graph homotopy and Graham
homotopy. \emph{Discret. Math.}, \textbf{241}, 153--170 (2001)

\bibitem{ChEs} Chepoi, V., Estellon, B.: Packing and covering $\d$-hyperbolic
spaces by balls. APPROX-RANDOM 2007, 59--73

\bibitem{Epp} Eppstein, D.: Squarepants in a tree: sum of subtree clustering
and hyperbolic pants decomposition. SODA 2007, 29--38

\bibitem{GaLy} Gavoille, C., Ly, O.: Distance labeling in hyperbolic
graphs. ISAAC 2005, Volume LNCS 3827, 1071--1079

\bibitem{GH} Ghys, E., de la Harpe, P.: Sur les Groupes Hyperboliques d'apr\`es
Mikhael Gromov, Progress in Mathematics 83, Birkh\"auser Boston
Inc., Boston, MA, 1990

\bibitem{G1} Gromov, M.:
Hyperbolic groups. In: Essays in Group Theory, Edited by S. M.
Gersten, M. S. R. I. Publ., 8, Springer, 1987, 75--263

\bibitem{K21} Jonckheere, E. A.: Contr\^ole du traffic sur les r\'eseaux \`a
g\'eom\'etrie hyperbolique--Vers une th\'eorie g\'eom\'etrique de
la s\'ecurit\'e l'acheminement de l'information. \emph{J. Europ.
Syst. Autom.}, \textbf{8}, 45--60 (2002)

\bibitem{K27} Jonckheere, E., Lohsoonthorn, P.: A hyperbolic geometry approach to
multipath routing. In: Proceedings of the 10th Mediterranean
Conference on Control and Automation (MED 2002), Lisbon, Portugal,
July 2002, FA5-1

\bibitem{K22} Jonckheere, E. A., Lohsoonthorn, P.: Geometry of network
security.
\emph{Amer. Control Conf.,} \textbf{6}, 976--981 (2004)

\bibitem{Kra} Krauthgamer, R., Lee, J. R.: Algorithms on negatively curved
spaces. FOCS 2006

\bibitem{K56} Koolen, J. H., Moulton, V.: Hyperbolic Bridged
Graphs. \emph{Europ. J. Combin.}, \textbf{23}, 683--699 (2002)

\bibitem{MRSV} Michel, J., Rodr\'{\i}guez, J. M., Sigarreta, J. M., et al.:
Hyperbolicity and parameters of graphs. \emph{Ars Comb.}, Volume
\textbf{C}, 43--63 (2011)

\bibitem{O} Oshika, K.: Discrete Groups, AMS Bookstore, Providence, RI, 2002

\bibitem{PRT1} Portilla, A., Rodr\'{\i}guez, J. M., Tour{\'\i}s, E.:
Gromov hyperbolicity through decomposition of metric spaces II.
\emph{J. Geom. Anal.}, \textbf{14}, 123--149 (2004)

\bibitem{RSVV} Rodr\'{\i}guez, J. M., Sigarreta, J. M., et al.,
Villeta, M.: On the hyperbolicity constant in graphs.
\emph{Discret. Math.}, \textbf{311}, 211--219 (2011)

\bibitem{RT3} Rodr\'{\i}guez, J. M., Tour{\'\i}s, E.:
Gromov hyperbolicity of Riemann surfaces. \emph{Acta Math. Sin.,
Engl. Series}, \textbf{23}(2), 209--228 (2007)

\bibitem{ShTa} Shavitt, Y., Tankel, T.: On internet embedding in hyperbolic
spaces for overlay construction and distance estimation. INFOCOM
2004

\bibitem{T} Tour{\'\i}s, E.:
Graphs and Gromov hyperbolicity of non-constant negatively curved
surfaces. \emph{J. Math. Anal. Appl.}, \textbf{380}, 865--881
(2011)

\bibitem{WZ} Wu, Y., Zhang, C.:
Chordality and hyperbolicity of a graph. \emph{Electr. J.
Combin.}, \textbf{18}, p43 (2011)

\end{thebibliography}

```
\end{document}




\subsection{A Subsection}

Please make sure that your paper contains correct reference
sequence (If NOT, please resort them according to its appearance
sequence, not alphabetical order. Moreover, please make sure that
each bibliographical item is labelled and that these items are
recalled using the command \verb|\cite{...}|, such as \cite{ref1},
and \cite{ref2,ref3,ref4,ref5})

All equations, theorems, definitions, lemmas, propositions,
corollaries, examples, remarks etc. would be better to be numbered
consecutively and unrepeatedly within each section. For example,
Definition 2.1, Lemma 2.2, Theorem 2.3 \ldots.

Use \verb|\label| and \verb|\ref| or \verb|\eqref| to
automatically cross-reference sections, equations, theorems and
theorem-like environments, tables, figures, etc.

\begin{theorem}[\cite{ref2}]\label{th:1.1} %Theorem 1.1 could be recalled by using Theorem \ref{th:1.1}
The statements of theorems, lemmas, definitions, propositions,
corollaries, conjectures, etc. are set in italics, by using
\begin{verbatim}
\begin{theorem/lemma/definition/proposition/corollary/conjecture}
\end{theorem/lemma/definition/proposition/corollary/conjecture}.
\end{verbatim}
\end{theorem}

\begin{prof}  %you can also use the environment \begin{proof}\end{proof}
Observe that
\begin{align}\label{E:1.1}
AAAAAAAAAA &= BBBBBBBBBBB\nonumber \\
           &\quad + CCCCCCCCCC\nonumber \\
           &= DDDDDDDDDDDDD.
\end{align}
Now apply induction on $n$ to \eqref{E:1.1}\ldots
\end{prof}

\begin{remark}\label{re:1.2}
Remarks, examples, problems, etc. are set in roman type.
\end{remark}


\subsection{Table}

\begin{table}
\begin{tabular}{|c|c|c|l|c|}
\hline $P(x)$ & $i$& $(e(1),e(2),e(4))$ & $(e(3),e(6),e(12),e(24))$ & $T(E)$ \\
\hline $P_1$  &    & & &$\emptyset$ \\
\hline $P_2$  & 4  & & $(1,1,1,0)\rightarrow(0,0,0,1)$ &2\\
\hline $P_3$  & 2  & &$(1,1,1,0)\rightarrow(0,0,2,0)$ &1\\
\hline $P_4$  & 2  & $(0,1,1)\rightarrow(1,2,0)$ & &1\\
\hline $P_5$  & 2  & $(0,1,1)\rightarrow(1,2,0)$ &$(1,1,1,0)\rightarrow(0,0,0,1)$ &$1,2$\\
\hline $P_6$  & 6  & $(0,1,1)\rightarrow(1,2,0)$ &$(1,1,1,0)\rightarrow(2,2,0,0)$ &1\\
\hline $P_7$  & 3  & $(0,1,1)\rightarrow(1,0,1)$ &$(1,1,1,0)\rightarrow(2,0,1,0)$ &0\\
\hline $P_8$  & 3  & $(0,1,1)\rightarrow(2,1,0)$ &$(1,1,1,0)\rightarrow(2,0,1,0) \rightarrow(3,1,0,0)$ &$0,1$\\
\hline
\end{tabular}
\caption{Aaa bbb ccc\label{tab}}
\end{table}
```

```
\subsection{Figure}

%\centerline{\includegraphics[scale=1.2]{actmark.eps}}
%\centerline{\small Figure 1\quad Journal mark}

\acknowledgements{\rm Thanks \ldots}



\begin{thebibliography}{99}

\bibitem{ref1} %% Journals %% surname(s), initial(s), article, journal, volume, relevant page numbers, year
Lorenzini, D., Tucker, T. J.: The equations and the method of
Chabauty--Coleman. \emph{Invent. Math.}, \textbf{148}, 1--46
(2002)

\bibitem{ref2} %% Books %% surname(s), initial(s), title, publisher, place of publication, year
Huppert, B., Blackburn, N.: Finite Groups II, Springer-Verlag, New
York, 1982

\bibitem{ref3}
Test

\bibitem{ref4}
Test

\bibitem{ref5}
Test

\end{thebibliography}


\end{document}
```