Limit Load Instabilities In Structural Elements

Doctoral Thesis

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LIMIT LOAD INSTABILITIES IN STRUCTURAL ELEMENTS

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This present work deals with the instability of structures made of various materials. It captures and models different types of instabilities using numerical analysis.

Firstly, we consider bifurcation for anisotropic cylindrical shells subject to axial loading and internal pressure. Analysis of bifurcation and post bifurcation of inflated hyperelastic thick-walled cylinder is formulated using a numerical procedure based on the modified Riks method for an incompressible material with two preferred directions which are mechanically equivalent and are symmetrically disposed.

Secondly, bulging/necking motion in doubly fiber-reinforced incompressible nonlinearly elastic cylindrical shells is captured and we consider two cases for the nature of the anisotropy: (i) reinforcing models that have a particular influence on the shear response of the material and (ii) reinforcing models that depend only on the stretch in the fiber direction.

The different instability motions are considered. Axial propagation of the bulging instability mode in thin-walled cylinders under inflation is analyzed. We present the analytical solution for this particular motion as well as for radial expansion during bulging evolution. For illustration, cylinders that are made of either isotropic incompressible non-linearly elastic materials or doubly fiber reinforced incompressible non-linearly elastic materials are considered.

Finally, strain-softening constitutive models are considered to analyze two concrete structures: a reinforced concrete beam and an unreinforced notch beam. The bifurcation point is captured using the Riks method used previously to analyze bifurcation of a pressurized cylinder.
Este trabajo analiza distintas inestabilidades en estructuras formadas por distintos materiales. En particular, se capturan y se modelan las inestabilidades usando el método de Riks.

Inicialmente, se analiza la bifurcación en depósitos cilíndricos formados por material anisótropo sometidos a carga axial y presión interna.

El análisis de bifurcación y post-bifurcación asociados con cilindros de pared gruesa se formula para un material incompresible reforzado con dos fibras que son mecánicamente equivalentes y están dispuestas simétricamente.

Consideramos dos casos en la naturaleza de la anisotropía: (i) Fibras reforzamiento que tienen una influencia particular sobre la respuesta a cortante del material y (ii) Fibras reforzamiento que influyen sólo si la fibra cambia de longitud con la deformación.

Se analiza la propagación de las inestabilidades. En concreto, se diferencia en el abultamiento (bulging) entre la propagación axial y la propagación radial de la inestabilidad. Distintos modelos sufren una u otra propagación.

Por último, distintas inestabilidades asociadas al mecanismo de ablandamiento del material (material softening) en contraposición al de endurecimiento (hardening) en una estructura (viga) de a: hormigón y b: hormigón reforzado son modeladas utilizando una metodología paralela a la desarrollada en el análisis de inestabilidades en tubos sometidos a presión interna.
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1.1 Overview

Instability is one of the factors that limit the extent to which structures can be loaded or deformed and plays a pivotal role in the design of most structures loaded primarily in compression, but often also structures loaded in tension. The vast literature on the subject, developed over the period of 250 years since Euler’s original work, attests to the importance of the subject. The major issue of practical concern is the establishment of the critical buckling load and deformation; that is, the lowest load or smallest deformation at which alternate equilibria become possible. This is usually established through bifurcations, material non-linearities and other factors. Incorporating all thus, usually requires solution of complete nonlinear problems.

The classical approach, that is, concern about the of instability, is sufficient for the design of most structures. But there are several structural instabilities, associated with propagating instabilities, that have been identified in which the classical approach obtains the bifurcation point (or limit load instability) is not sufficient. In general, this instability, affects structures of large size in which, following the onset of instability, collapse is confined to a relatively small part of the structure i.e., it localizes. Under prevailing conditions, the initially local collapse can propagate or spread, often in a dynamic fashion, over the rest of the structure. The load required to propagate such instabilities is
often substantially lower than that required to initiate them in the original structure.

In fact, research on non linearities owing to large deformations and inelastic behaviours of materials now has to be tackled for many systematic applications in mechanical and civil engineering because the evaluation of safety margins has become computationally possible, with obvious advantages when compared with admissible stress criteria, popular in structural engineering practice.

1.2 Motivation

Recent (practical) motivation for the study of instability is due to its important role in the design of many elements that are widely being use in industrial and civil engineering. The collapse of a circular tube is of interest not only in many engineering applications, such as in the design of undersea or underground pipelines and pressure vessels, particularly submarine structures, but it is also of great interest in the biomechanics context.

The main motivation for this research is to use numerical methods to analyse and determinate different types of instabilities for different materials such as fiber reinforced material, and concrete using the Riks method that Abaqus is available for instance.

1.3 Objectives

The main goals of this work is to study deeply the instability of cylindrical tubes and to capture the types of instability and to distinguish between them by using Riks method. To this end five principal objectives have been set:

1. To study the bifurcation of thin-walled cylindrical shells made of incompressible anisotropic material under axial load and internal pressure.
2. To analyse bifurcation and post bifurcation of inflated hyperelastic thick-walled cylinders based on the modified Riks method.

3. To study Bulging/necking motion in doubly fiber-reinforced incompressible nonlinearly elastic cylindrical shells subject to axial loading and internal pressure using a numerical procedure using the modified Riks method.

4. To study the axial propagation of the bulging instability mode in thin-walled cylinders under inflation as well as the radial expansion during bulging evolution.

5. To apply Riks method on other types of materials such as concrete by analysing two samples: one is an unreinforced concrete beam and the other is a tension-pull specimen.

1.4 Content

This document is organized in seven chapters. In chapter 1, a brief introduction of the instability of structures and his role in the design of many structures, as well as the objectives of this work are in produced. In chapter 2, we speak about the bifurcation for thin-walled cylindrical shells for orthotropic materials and show the bifurcation criterion.

A numerical procedure to analyse bifurcation and post bifurcation of inflated hyperelastic thick-walled cylinders based on the Riks method is discussed in chapter 3. The analysis of necking and bulging and its propagation in anisotropic cylindrical shells under axial load and internal pressure and the effect of the materials types in the bifurcation is presented in chapter 4. Chapter 5 presents the competition between radial expansion and axial propagation in bulging of inflated cylinders. In chapter 6, further application associated with the behaviour of a strain softening structure using two models: the first one is a single notch unrefinforced concrete beam and the other one is a reinforced concrete beam. The conclusions are in chapter 7.
Chapter 2 is written in a self-contained way to get the bifurcation criterion (2.16), the reader can omit these details and jump to chapters 3, 4 and 5 that have been written accordingly and only use equation (2.16) from chapter 2.

Chapter 6 is also written in a self-contained manner since it deals with other application.
2.1 Introduction

In the last few years a big effort has been made to investigate boundary value problems in the context of stability theory. This involves a variety of different geometries and loading conditions for different materials. Analytical solutions are not easy to get but when these are found great insight into the physical interpretation of the problem at hand is shown.

The bifurcation has been studied by many authors in many problems for instance [1], [2], [3], [4] and references therein.

Solution to the problem at hand (bulging bifurcation of a tube) has been given in terms of the so called incremental equations, a sound theory originally published by Biot between years 1934 and 1940 (see [5] for a complete treatment) and later on well exploited by Ogden and co-workers, among others (see [1], [6]). Here, a transparent derivation of the bifurcation criteria is provided which was given in [7]. We consider directly equilibrium of different infinitesimal volume elements. The mode of bifurcation under study establishes the appropriate differential volume element that has to be considered. The methodology is useful to capture a better insight into the problem and is not based on complicated mathematical machinery. It is related to engineering or physics way of thinking and provides a clear interpretation of the bifurcation analysis. Needless to say that the approach developed here can be used to study
Deformation of bulging bifurcation for thin-walled cylindrical shell

any bifurcation problem and is less time consuming than the general incremental equations method. The difficulty is, of course, to single out the variables involved.

The post-bifurcation analysis of the bulging mode shown in [4] also focuses on equilibrium of infinitesimal volume elements. Furthermore, a thorough bifurcation mode analysis is carried out. The bifurcation mode having zero wave number is related to a local bulging solution and gives the initiation of instability. The results seem to be relevant to determine the initiation pressure at which bulging occurs. This zero mode is considered to be not sinusoidal or constant. Here we use the normal mode method as in [4] but only consider sinusoidal or constants solutions. The results are found to be equivalent to the ones given by the incremental equations method [1]).

The solution to the problem at hand is found for certain non-linearly elastic material models. In particular, it is associated with incompressible models for which the principal directions of deformation and stress coincide. Among these models we focus here on orthotropic materials with two preferred directions which are mechanically equivalent and are symmetrically disposed.

2.2 Basic equations

A detailed description of the equilibrium equations for a membrane tube subjected to combined axial loading and internal pressure can be found in [1], for instance. Here we just describe the main equations including both deformation and constitutive model.

We consider a circular cylindrical membrane whose middle surface in its undeformed reference configuration is described by

\[
X = R E_R(\Theta) + Z E_z, \quad 0 \leq \Theta < \pi, \quad -L/2 \leq Z \leq L/2, \quad (2.1)
\]

where \((R, \Theta, Z)\) are cylindrical polar coordinates and \(E_R, E_\Theta\) and \(E_z\) are unit vectors in the indicated directions. The length \(L\) of the cylinder may be finite or infinite.

The membrane cylinder is extended and inflated so that it remains a
circular cylinder. The axial load is denoted by $N$ and the inflated pressure by $p$. In this deformed configuration the middle surface of the membrane is described by

$$x = re_r(\theta) + ze_z, \quad 0 \leq \theta < \pi, \quad -l/2 \leq z \leq l/2,$$

(2.2)

where $(r, \theta, z)$ are cylindrical polar coordinates in this configuration. The deformed length of the cylinder is $l$. Referred to cylindrical coordinates, the associated deformation gradient tensor $F$ has components $\text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$ where $\lambda_\theta = r/R$ is the azimuthal principal stretch and $\lambda_z = l/L$. Because of the assumed incompressibility, the radial principal stretch is $\lambda_r = \lambda^{-1}_\theta \lambda^{-1}_z$.

We consider materials for which under this deformation the principal Cauchy stresses $\sigma_{ii}$ can be written as

$$\sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda_i} + q_i, \quad i = r, \theta, z,$$

(2.3)

Where $W = W(\lambda_r, \lambda_\theta, \lambda_z)$ is the strain energy function and $q$ is an arbitrary stress value associated with the incompressibility constraint. This includes isotropic materials as well as orthotropic materials with the axes of orthotropy coinciding with the principal axes of stress. In the latter case, one could think of a material with two preferred directions which are mechanically equivalent and are symmetrically disposed.

Using the membrane approximation $\sigma_{rr} = 0$ we can write

$$\sigma_{\theta \theta} = \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta} \quad \text{and} \quad \sigma_{zz} = \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z},$$

(2.4)

where we have introduced the notation

$\hat{W}(\lambda_\theta, \lambda_z) = W(\lambda^{-1}_\theta \lambda^{-1}_z, \lambda_\theta, \lambda_z)$.

The equilibrium equations for the membrane give the inflating pressure in the form

$$p = \frac{H \hat{W}_\theta}{R \lambda_\theta \lambda_z},$$

(2.5)

where $H$ is the undeformed thickness of the membrane (h in the deformed configuration) and $\hat{W}_\lambda$ denotes $\frac{\partial \hat{W}}{\partial \lambda}$ similarly, we will use the notation $\hat{W}_{\lambda_z}, \hat{W}_{\lambda_\theta \lambda_\theta}$, etc. The axial load $N$ per unit area of the undeformed cross-
2.3 Bifurcation of thin-walled cylindrical shells

To investigate the possible bifurcation modes of the cylinder we consider incremental displacements with respect to the deformed configuration (under equilibrium), of the general form

\[ \delta u = \delta u_r(\theta, z)e_r + \delta u_\theta(\theta, z)e_\theta + \delta u_z(\theta, z)e_z. \]  

(2.7)

2.3.1 Bulging mode

We consider an axisymmetric displacement field of the form

\[ \delta u = \delta u_r(z)e_r + \delta u_z(z)e_z. \]  

(2.8)

Equilibrium in the axial and radial directions of an infinitesimal volume element (see fig. 2.1) gives the differential equations

\[ \frac{\partial}{\partial z} \left( \delta (\sigma_{zz} hr) d\theta dz \right) - pr d\theta dz \frac{\partial \delta u_r}{\partial z} = 0, \]  

(2.9)

\[ \delta (pr) d\theta dz - \delta (\sigma_{\theta \theta} h) d\theta dz + \frac{\partial}{\partial z} (\sigma_{zz} hr d\theta \frac{\partial \delta u_r}{\partial z}) dz = 0, \]  

(2.10)

respectively. The left term on the left hand side of (2.9) is related to the increment of the force due to the axial stress on the infinitesimal portion of the membrane in the axial direction. It is clear that \( \delta (\sigma_{zz} hr) \) is the increment of the force due to the axial stress per unit angle acting on one end of the infinitesimal volume. The resulting incremental force due to the axial stress is obtained as the difference between the forces due to that stress acting on both ends of the infinitesimal volume and is precisely that term. The right term on the left hand side of (2.9) is the component of the force due to the pressure in the axial direction. We note that the pressure is not in the radial direction due to the incremental displacement field.
Bifurcation of thin-walled cylindrical shells

In fact, these terms are easily obtained here since now $\delta u_\theta = 0$. The right term on the left hand side of (2.10) is related to the increment of the force in the radial direction due to the axial stress. We note that the axial force on one end of the infinitesimal volume is obtained as the projection of $\sigma_{zz} h r d\theta$ on the axial direction (see Fig. 2.1).

Using (2.4) and the fact that $\delta u_r = R \delta \lambda_\theta$ the system of Eqs. (2.9) and (2.10) after some manipulations which include linearization of the resulting differential equations with respect to the incremental stretches $\delta \lambda_z$ and $\delta \lambda_\theta$ can be written as

$$(\hat{W}_{\lambda_\theta \lambda_\theta} - \hat{W}_{\lambda_\theta}) \frac{\partial \delta \lambda_\theta}{\partial z} + \hat{W}_{\lambda_z \lambda_z} \frac{\partial \delta \lambda_z}{\partial z} = 0,$$

$$\left(\hat{W}_{\lambda_\theta \lambda_\theta} R^2 \frac{\partial^2 \delta \lambda_\theta}{\partial z^2} + \left(\hat{W}_{\theta \theta \lambda_\theta} - \hat{W}_{\theta \theta} \frac{1}{\lambda_z} \lambda_z \frac{\lambda_\theta}{\lambda_z} \hat{W}_{\lambda_\theta \lambda_\theta} - \hat{W}_{\lambda_\theta \lambda_\theta} \frac{1}{\lambda_z} \lambda_z \frac{\lambda_\theta}{\lambda_z} \hat{W}_{\lambda_\theta \lambda_\theta}\right) \frac{\delta \lambda_z}{\lambda_z} = 0. \tag{2.12}$$

This system of equations allows us the following discussion. First, we note that a solution (bifurcation) for which $\delta \lambda_z = 0$ and $\delta \lambda_\theta$ is uniform is associated to (2.5). On the other hand, when $\delta \lambda_z$ and $\delta \lambda_\theta$ are uniform, using $\lambda_\theta = r/R$, $\lambda_z = l/L$ and the system of Eqs. (2.11) and (2.12) one obtains that

$$\frac{\delta r}{\delta l} = \frac{r (\lambda_z \hat{W}_{\lambda_\theta \lambda_\theta} - \hat{W}_{\lambda_\theta})}{l (\hat{W}_{\lambda_\theta \lambda_\theta} - \lambda_\theta \hat{W}_{\lambda_\theta \lambda_\theta})}.$$ \tag{2.13}

This relation establishes the change of the cylinder radius with its length for inflation at constant pressure.

We focus now on non-uniform bifurcation solutions. The characteristic equation of the system of differential equations (2.11) and (2.12) is

$$\begin{vmatrix} (\hat{W}_{\lambda_\theta \lambda_\theta} - \hat{W}_{\lambda_\theta}) \frac{\alpha}{\lambda_z} & \hat{W}_{\lambda_z \lambda_z} \frac{\alpha}{\lambda_z} \\ \hat{W}_{\lambda_\theta \lambda_\theta} R^2 \alpha^2 & \left(\hat{W}_{\theta \theta \lambda_\theta} - \hat{W}_{\theta \theta} \frac{1}{\lambda_z} \lambda_z \frac{\lambda_\theta}{\lambda_z} \hat{W}_{\lambda_\theta \lambda_\theta} - \hat{W}_{\lambda_\theta \lambda_\theta} \frac{1}{\lambda_z} \lambda_z \frac{\lambda_\theta}{\lambda_z} \hat{W}_{\lambda_\theta \lambda_\theta}\right) \frac{1}{\lambda_z} \lambda_z \frac{\lambda_\theta}{\lambda_z} \hat{W}_{\lambda_\theta \lambda_\theta} \end{vmatrix} = 0, \tag{2.14}$$

where $\alpha$ is the exponent of the assumed displacement exponential solution $e^{\alpha z}$. The value $\alpha = 0$ is associated with uniform displacement solutions and has already been discussed. Let us focus on a cylinder of infinite length. Bifurcation is associated with complex solutions of $\alpha$ which in turn give
Deformation of bulging bifurcation for thin-walled cylindrical shell

harmonic modes. Whence, eliminating $\alpha$ from the first row of the matrix in (2.14), the bifurcation criterion is

$$f(\hat{W}, \lambda_\theta, \lambda_z) = 0, \quad (2.15)$$

where

$$f(\hat{W}, \lambda_\theta, \lambda_z) = \lambda_z^2 \hat{W}_{z\lambda_z} (\lambda_\theta^2 \hat{W}_{\lambda_\theta, \lambda_\theta} - \lambda_\theta \hat{W}_{\lambda_\theta}) - (\lambda_\theta \lambda_z \hat{W}_{\lambda_\theta, \lambda_z} - \lambda_\theta \hat{W}_{\lambda_\theta})^2, \quad (2.16)$$

in agreement with the analysis shown in [1].

The analysis of the bifurcation condition (2.15) establishes the following results. Let us consider first the situation given by uni-axial stress due to extension with zero pressure. If the Hessian of $\hat{W}$ is positive definite then, it follows that (2.16) is positive and the deformed configuration is under stable equilibrium (no bifurcation). Let us turn now our attention to the general case that includes inflation. Turning points of the pressure, which are given by (2.5), obey $f \leq 0$ because the non quadratic term in (2.16) by means of (2.5) is zero. It follows that initiation of the bulging mode is prior to the pressure turning point.

Using (2.6) it is easy to show that turning points of the axial load obey

$$\hat{W}_{\lambda_z, \lambda_z} = 0. \quad (2.17)$$

Upon use of (2.17) and (2.16) is non positive and establishes that initiation of bulging is prior to turning points of the axial load.

Consider now cylinders with finite length. The first mode obeying values of zero axial incremental displacements at the boundary condition (bottom and top of the cylinder) corresponds to $\alpha = (2\pi/l)\sqrt{-1}$. With this and (2.14) we finally get that the bifurcation criterion is

$$f(\hat{W}, \lambda_\theta, \lambda_z) + \left(\frac{2\pi R}{L}\right)^2 \lambda_\theta^2 \lambda_z \hat{W}_{\lambda_z} \hat{W}_{\lambda_z, \lambda_z} = 0. \quad (2.18)$$

It follows that for a given extension bulging solutions obeying (2.18) are associated with greater values of the pressure than the bulging solutions obeying (2.15). Whence, the infinite length assumption

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establishes a pressure lower bound in regard to bulging initiation.

At last, we note that this analysis is associated with both bulging and necking.

![Figure 2.1: Balance of forces for the bulging mode.][7]

### 2.4 Bulging of orthotropic models

The analysis of the previous sections involves materials for which under the deformation at hand the principal Cauchy stresses $\sigma_{ii}$ can be written as shown in (2.3). Incompressible orthotropic materials with two preferred directions which are mechanically equivalent and are symmetrically disposed follow that pattern. For a general discussion we refer to [10] and [11]. Here we summarize the main ideas. In the context of non-linear elasticity constitutive equations are given in terms of strain-energy functions that depend on certain deformation invariants. For a single family of fibers the independent invariants are, in the case of an incompressible material,

$$
I_1 = \text{tr}C, \quad I_2 = \text{tr}(C^{-1}), \quad I_4 = M.(CM), \quad I_5 = M.(C^2M),
$$

where $C = F^TF$ is the right Cauchy–Green deformation tensor, $F$ is the deformation gradient, $^T$ signifies transpose, $\text{tr}$ denotes the trace of a second-order tensor and $M$ is a unit vector defined in the reference configuration that identifies the direction of fiber reinforcement. When
there are two families of fibers, with the second family having direction $M'$ in the reference configuration, then three additional invariants arise. These are

$$I_6 = M' (CM'), \quad I_7 = M' (C^2 M'), \quad I_8 = (M.M') [M(CM')] = (M.M') I_8.$$  \hspace{1cm} \text{(2.20)}$$

Note that $I_6$ and $I_7$ are the analogues of $I_4$ and $I_5$, respectively, for $M'$, while $I_8$ introduces a coupling between the two families.

The most general strain energy function $W$ is a function of $I_1, I_2, I_4, I_5, I_6, I_7, I_8,$ and $M.M'$. In the special case in which the two preferred directions are mechanically equivalent the strain energy must be symmetric with respect to interchange of $I_4$ and $I_6$ and of $I_5$ and $I_7$. For the considered deformation at hand, we have $I_4 = I_6, I_5 = I_7$ and it follows that when $W_4 = W_6$ and $W_5 = W_7$, where $W_4 = \partial W/\partial I_4$ and similarly for the rest, then (2.3) holds. In this special case the principal axes of stress coincide with the Cartesian axes (i.e. with the Eulerian principal axes).

In certain analysis made on fiber reinforced nonlinearly elastic strain energy functions (see [12] for some cases), the strain energy is taken as the sum of two terms, one isotropic and the other anisotropic, i.e.

$$W = W_i(I_1, I_2) + W_a(I_4, I_5, I_6, I_7, I_8).$$ \hspace{1cm} \text{(2.21)}$$

This strain energy is called an augmented isotropic function with bidirectional reinforcements. The isotropic and anisotropic invariants are not coupled. Different in stabilities analysis for different energy functions following this specific pattern have been analyzed (see [13], for instance). Here we follow this idea and analyze bulging for the model (2.21). First, using $F = \text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$, (2.19) and (2.20) we introduce the notation $\hat{W}_i(\lambda_\theta, \lambda_z)$ and $\hat{W}_a(\lambda_\theta, \lambda_z)$. In analogy with (2.4), it follows easily that (see [11] for details)

$$\sigma_{\theta\theta} = \lambda_\theta \frac{\partial \hat{W}_i}{\partial \lambda_\theta} + \lambda_\theta \frac{\partial \hat{W}_a}{\partial \lambda_\theta} \quad \text{and} \quad \sigma_{zz} = \lambda_z \frac{\partial \hat{W}_i}{\partial \lambda_z} + \lambda_z \frac{\partial \hat{W}_a}{\partial \lambda_z},$$ \hspace{1cm} \text{(2.22)}$$
From now on we use the notation
\begin{align*}
\sigma_{i\theta\theta} &= \lambda_{\theta} \frac{\partial \hat{W}_i}{\partial \lambda_{\theta}}, \\
\sigma_{a\theta\theta} &= \lambda_{\theta} \frac{\partial \hat{W}_a}{\partial \lambda_{\theta}}, \\
\sigma_{izz} &= \lambda_{z} \frac{\partial \hat{W}_i}{\partial \lambda_{z}}, \\
\sigma_{azz} &= \lambda_{z} \frac{\partial \hat{W}_a}{\partial \lambda_{z}}.
\end{align*}
(2.23)

For \( \hat{W}_i, \hat{W}_a \), (2.15) using (2.16) and (2.23) after a straight forward computation gives
\begin{align*}
f(\hat{W}_a, \lambda_{\theta}, \lambda_{z}) + f(\hat{W}_i, \lambda_{\theta}, \lambda_{z}) + (\lambda_{z} \frac{\partial \sigma_{izz}}{\partial \lambda_{z}} - \sigma_{izz})(\lambda_{\theta} \frac{\partial \sigma_{a\theta\theta}}{\partial \lambda_{\theta}} - 2\sigma_{a\theta\theta}) + \\
(\lambda_{z} \frac{\partial \sigma_{azz}}{\partial \lambda_{z}} - \sigma_{azz})(\lambda_{\theta} \frac{\partial \sigma_{i\theta\theta}}{\partial \lambda_{\theta}} - 2\sigma_{i\theta\theta}) - 2(\lambda_{z} \frac{\partial \sigma_{i\theta\theta}}{\partial \lambda_{z}} - \sigma_{i\theta\theta}) \\
= 0.
\end{align*}
(2.24)

The material model anisotropy degree will give the important term in the bulging criterion (2.16). The term \( f(\hat{W}_a, \lambda_{\theta}, \lambda_{z}) \) on the left hand side of (2.16) will dominate bulging of highly anisotropic materials. On the other hand, the term \( f(\hat{W}_i, \lambda_{\theta}, \lambda_{z}) \) will dominate bulging of almost isotropic material models. The remaining crossed terms play an important role in the rest of the cases. We have expressed them in terms of the stress components to further point the dependence of these crossed terms with the mechanical response curvatures as was done in \( [2] \). We also note that the crossed terms depend specifically on the axial stretch as well as the azimuthal stress. The role of the axial stretch seems important not only in the thin-walled cylindrical case but also in the thick one as it was shown in \( [3] \).
DEFORMATION OF BULGING BIFURCATION FOR THIN-WALLED CYLINDRICAL SHELL
3.1 Introduction

In the last few years several analyses have investigated boundary value problems in the context of stability theory (see for instance [1]-[4] and [7]-[9]). In particular, a variety of authors have focused on bifurcation from a membrane tube configuration subject to axial loading and internal pressure (see, [1]-[4] and references therein) and on postbifurcation [4]. Recently, the analysis also focused on the bifurcation of isotropic thick-walled cylinders [3]. Among the instability modes of stretched pressurized cylinders bulging is likely to be the first that appears [3]. The bulging bifurcation mode of the cylinder is obtained considering axisymmetric incremental displacements with respect to a deformed configuration. All configurations are in equilibrium. It follows that incremental displacements are null along the hoop direction and do not depend on the azimuthal angle.

In this chapter we use a numerical procedure to analyse bifurcation and post bifurcation of inflated hyperelastic thick-walled cylinders based on the modified Riks method. We focus here on orthotropic materials with two preferred directions which are mechanically equivalent and are symmetrically disposed. The cylindrical shell wall is modelled with this class of constitutive equation showing significant stiffening behaviour. Analytical solutions can be found in the literature for very particular conditions involving specific material models and loads applied as well as,
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usually, a perfect geometry. Here, we provide a unified approach to both
the prediction of bifurcation and post bifurcation propagation that can
be applied to more general conditions. We show that bulging instability
under the conditions at hand propagates axially. In chapter 6 the model
will be further analyzed.

3.2 Description of the problem

In this section we pay attention to the mathematical description of
the problem and describe the bulging instability.

3.2.1 Geometry and loading conditions

Consider a right circular tube that occupies the region

\[ A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -L/2 \leq Z \leq L/2, \]  

(3.1)
in its undeformed reference configuration where \((R, \Theta, Z)\) are cylindrical
coordinates. The position vector of a material point can then be written
as:

\[ X = RE_R(\Theta) + ZE_z, \]  

(3.2)

where \(E_R, E_\Theta\) and \(E_Z\) are unit vectors in the indicated directions. The
length \(L\) of the cylinder may be finite or infinite. We denote the initial
thickness of the material as \(e\).

The cylinder is inflated and extended so that it remains as a circular
cylinder. The inflating pressure is denoted by \(p\) and the axial load by \(N\).
In this deformed configuration the cylinder is described by

\[ x = re_r(\theta) + ze_z, \quad a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad -l/2 \leq z \leq l/2, \]  

(3.3)

where \(r, \theta = \Theta\) and \(z\), are cylindrical coordinates in this configuration.
The deformed length of the cylinder is \(l\).

Referred to cylindrical coordinates, the associated deformation
gradient tensor \(F\) for an incompressible material has components \(\text{diag}\)
Description of the problem

\((\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z)\), where \(\lambda = \lambda_{th} = r/R > 0\) is the azimuthal principal stretch and \(\lambda_z\) is the axial stretch, so that \(l = \lambda_z L\). Because of the assumed incompressibility, the principal radial stretch is \(\lambda_r = \lambda^{-1}\lambda_z^{-1}\). The displacement \(u\) is given by \(u = x - X\).

In passing, we just note that for a membrane tube, we should consider a circular cylindrical membrane whose middle surface in its undeformed reference configuration is described by (3.2). In the deformed configuration the middle surface of the membrane is described by

\[ x = re_r(\theta) + ze_z, \quad 0 \leq \theta \leq 2\pi, \quad -l/2 \leq z \leq l/2, \tag{3.4} \]

Referred to cylindrical coordinates, the associated deformation gradient tensor has the components just given above for the thick case.

### 3.2.2 Material model

The cylinder is made up of an incompressible hyperelastic material with a constitutive equation given by a strain energy function \(W\). In the literature (see [3], [14], for instance), analytical solutions consider materials for which under the deformation at hand the principal Cauchy stresses \(\sigma_{ii}\) can be written as

\[ \sigma_{ii} = \lambda_i \frac{\partial W}{\partial \lambda} + q, \quad i = r, \theta, z, \tag{3.5} \]

Where \(W = W(\lambda_r, \lambda_{th}, \lambda_z)\) is the strain energy function and \(q\) is an arbitrary stress value associated with the incompressibility constraint. This includes isotropic materials as well as orthotropic materials with the axes of orthotropy coinciding with the principal axes of stress. In the latter case, one could think of a material with two preferred directions which are mechanically equivalent and are symmetrically disposed.

In the context of non linear elasticity constitutive equations are given in terms of strain energy functions that depend on certain deformation invariants. For a single family of fibers the independent invariants are, in the case of an incompressible material,

\[ I_1 = trC, \quad I_2 = tr(C^{-1}), \quad I_4 = M.(CM), \quad I_5 = M.(C^2M), \tag{3.6} \]
where $C = F^T F$ is the right Cauchy–Green deformation tensor, $F$ is the deformation gradient, $^T$ signifies transpose, $\text{tr}$ denotes the trace of a second-order tensor and $M$ is a unit vector defined in the reference configuration that identifies the direction of fiber reinforcement. When there are two families of fibers, with the second family having direction $M'$ in the reference configuration, then three additional invariants arise. These are

$$I_6 = M'.(CM'), \quad I_7 = M'.(C^2M'), \quad T_8 = (M.M')[M(CM')] = (M.M')I_8.$$  \hfill (3.7)

Note that $I_6$ and $I_7$ are the analogues of $I_4$ and $I_5$, respectively, for $M'$, while $I_8$ introduces a coupling between the two families \cite{10}.

The wall of the cylindrical shell has been modeled as an incompressible orthotropic material with two preferred directions (defined by two unit vectors $M$ and $M'$) which are mechanically equivalent and are symmetrically disposed. The fibers which are inside the wall act as fiber reinforcement directions. The elastic wall is composed of three layers and each layer is modeled with in the same constitutive framework. In particular, each layer can be considered as a composite reinforced by two families of fibers. The ground matrix of a layer is modeled by a neo-Hookean material \cite{15}

$$W_g = \frac{\mu}{2}(I_1-3),$$  \hfill (3.8)

where $\mu$ is the ground state shear modulus of the material. The strain energy stored in each reinforced direction is taken to depend on the stretch in that direction. It follows that two independent invariants need to be considered, one for each single family of fibers, and these are $I_4$ and $I_6$. The total strain energy of an cylindrical wall layer is written as

$$W_e(I_1, I_4, I_6) = \frac{\mu}{2}(I_1-3)+\frac{k_1}{2k_2}\sum_{i=1,6}\{\exp[k_2(k(I_1-3)+(1-3k)(I_i-1))^2] - 1\},$$  \hfill (3.9)

where $\kappa$ is related to the fiber dispersion and $k_1$ and $k_2$ are positive.
constants. It is assumed that fibers contribute to the energy function when these are elongated. Furthermore, the fibers mechanical response gives rise to those called strain stiffening. The fiber stretch invariants using $F=\text{diag}(\lambda^{-1}\lambda_z^{-1}, \lambda, \lambda_z)$, (3.6) and (3.7) are

$$I_4 = I_6 = \lambda^2 \cos^2 \varphi + \lambda^2 \sin^2 \varphi,$$

(3.10)

where $\varphi$ is the angle of the fiber with respect to the axial direction of the cylinder.

### 3.2.3 Equilibrium equations of the model

The equilibrium equations for the models at hand can be obtained, for instance, by minimizing the energy functional of this purely mechanical analysis. In the case of thin-walled cylinders, this analysis has been given in [14]. Results show that curves giving the applied pressure against the azimuthal stretch (in a more general sense load–displacement curves) reach instability points which are close to maximum points of these curves (see for instance [7]).

Finite element analyses are developed from the so-called weak formulations of the problem, i.e. energy functionals. The problem at hand is transformed into a system of algebraic equations in which the displacements are involved. This is a well established procedure and we will not develop the formulation.

We summarize the equations for cylindrical shells [1]. It follows that one can write

$$\sigma_{\theta\theta} = \lambda_{\theta} \frac{\partial \hat{W}}{\partial \lambda_{\theta}} \quad \text{and} \quad \sigma_{zz} = \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z},$$

(3.11)

where we have introduced the notation $\hat{W}(\lambda_{\theta}, \lambda_z) = W(\lambda_{\theta}^{-1}\lambda_z^{-1}, \lambda_{\theta}, \lambda_z)$. The equilibrium equations for the membrane give the inflating pressure in the form

$$p = \frac{H\hat{W}_{\theta}}{R\lambda_{\theta} \lambda_z},$$

(3.12)

where $\hat{W}_{\theta}$ denotes $\partial \hat{W}/\partial \lambda_{\theta}$. In what follows, we use the notation $\hat{W}_{z}$, $\hat{W}_{\theta\theta}$, etc.
To investigate the possible bifurcation modes of the cylinder one should consider incremental displacements with respect to the deformed configuration (under equilibrium), of the general form

$$\delta \mathbf{u} = \delta u_r(\theta, z)e_r + \delta u_\theta(\theta, z)e_\theta + \delta u_z(\theta, z)e_z,$$

where $\delta$ is the increment. We are interested in the bulging instability mode in which case one should consider an axisymmetric displacement field of the form

$$\delta \mathbf{u} = \delta u_r(z)e_r + \delta u_z(z)e_z,$$

The bulging condition is (see [7])

$$f(\hat{W}, \lambda_\theta, \lambda_z, L/R) = \lambda_\theta^2 \hat{W}_{zz}(\lambda_\theta^2 \hat{W}_{\theta\theta} - \lambda_\theta \hat{W}_\theta) - (\lambda_\theta \lambda_z \hat{W}_{\theta z} - \lambda_\theta \hat{W}_\theta)^2 + \frac{2\pi R}{L}\lambda_\theta^2 \lambda_z \hat{W}_z \hat{W}_{zz} = 0. \tag{3.15}$$

For a cylinder of infinite length, the last term on the RHS of (3.15) is zero.

The neo-Hookean material model is recovered considering $k_1 = 0$ in (3.9). In this case, as shown in [7], using (3.15), bulging occurs when

$$-\lambda_\theta^2 \lambda_z^4 + 6\lambda_\theta^4 \lambda_z^2 + 4\lambda_\theta^2 \lambda_z^4 + 3 = 0. \tag{3.16}$$

It was also shown in [7] that if the isotropic part is negligible in (3.9) (i.e. $k_1 \gg \mu$) then, for any $\lambda_\theta \geq 1$ and $\lambda_z \leq 1$, it is necessary that $k_2 > k_{2c} = 1/(288 \cos^4 \varphi)$ to avoid bulging instability.

These analytical results are based on the so-called normal mode method (see, for instance, [4]) which considers only sinusoidal or constant solutions. The results are found to be equivalent to the ones given by the incremental equations method [1]. It follows that the onset of bifurcation coincides with the maximum point that appears in load–displacement curves.

To determine the bifurcation and post bifurcation behaviour of the problem means to capture the deformation of the initial circular thick-walled cylinder under the conditions at hand. In the next section we focus
on the specific details of the technique developed and more in particular on
the necessary details to implement the numerical analysis in a commercial
finite element code.

3.3 Finite element analysis

In this section we describe the numerical technique developed, and, for the purpose of this work, we focus on the commercial code Abaqus. We use a commercial software package for professional purposes since one of the main objectives of this work is to disseminate and promote non linear computational modelling techniques.

3.3.1 Model data: geometry, material model and loading conditions

The circular tube is generated by means of a (thin or thick) sector of
finite axial length with one (or more) finite element in the hoop direction. An axisymmetric element model is not possible since it prevents the use of an isotropic hyperelastic constitutive equations. The axial length of the sector is a parameter that can be modified in the analysis as well as the inner and outer radius. A small imperfection should be introduced in the geometry to obtain a smooth solution during loading (more details are given in the next section). This can be achieved in several ways, for instance, varying the outer and/or inner diameter along the axial length of the sector. Sensitivity analyses should be done to verify that those non-perfect geometries do not modify substantially the results. On the other hand, it may be that the problem at hand has a non-perfect geometry. In that case, the model has to account for that geometry.

We will consider for the purpose of this communication the constitutive equation [16] but more general material models can be used. The model is just a prototype and is useful to show qualitative results.

Mixed three-dimensional reduce integrated linear elements were selected (Abaqus element type C3D8RH). The (hybrid) mixed pressure displacement formulation is recommended to impose the material incompressibility constraint. A minimum of five elements are recommended in the radial direction and a sufficient number in the
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longitudinal direction to get well-shaped elements. The angle of the sector should also be adequate to generate elements with reasonable shape factors.

In our analysis longitudinal displacements (axial stretch) at the end faces of the cylinder are imposed. Similarly, since we try to capture the bulging instability, hoop displacements are prevented during the analysis following (3.14). This is obtained by defining that particular condition in a nodal local cylindrical system.

3.3.2 Riks analysis: Modified method (See appendix A for more details)

In our methodology, we first stretch the cylindrical model imposing axial displacements. The cylindrical model deforms and remains as a cylinder. Then, we apply pressure to capture both the instability and the postcritical behaviour. The modified Riks method introduces two concepts: the load proportional factor (LPF) and the arc length, for details see [17]. The arc length is a quantity related to the evolution of the structure that combines displacements and loads. The load proportional factor determines the load applied and it may decrease. In fact, Riks method is not very useful when the load proportional factor is monotonically increasing. Fig. 3.1 shows a typical curve of the load proportional factor vs. the arc length. In this curve there is a maximum which is an indicator that the system exhibits an instability. The way to understand this curve is the following. LPF is related to the load applied. As loading is applied the structure will reach the maximum point shown in the curve in a sequence of states under static equilibrium. It is not possible to increase the load above the maximum LPF under static equilibrium. Furthermore, a sudden and new catastrophic rearrangement could be obtained by the structure for loads close to the maximum LPF. This is an indicator of an instability but it should not be understood as a bifurcation point. To capture bifurcation, an imperfection has to be introduced in the geometry of the problem. In that way, the structure will behave smoothly as loading approaches the instability point.
A linear buckle estimation is not the methodology to be used since it is only accurate for relatively stiff structures. A general nonlinear analysis (other than the Riks method) is not recommended because it may not estimate properly the maximum point. The method may stop at a point in which no more loading can be applied but there is no certainty that this point is the maximum point. The modified Riks method overcomes this difficulty [17]. Using this technique the loading (internal pressure) can be decreased capturing the maximum point. A smooth (continuum) bifurcated and postbifurcated solution is obtained as loading is applied. It should be noted that in some cases the load–displacement (pressure-azimuthal stretch) response shows a very high curvature at the maximum point and the solution may reverse the path. In any case, the peak load proportional factor (which is a function of the arc length, see [17] for specific details) can be accurately determined and the postcritical behaviour can also be captured.

3.3.2.1 Numerical validation of the bulging bifurcation point

To validate the numerical procedure, in this section, we compare analytical results for a membrane tube with regard to solutions obtained using the numerical procedure established here. In Section 3.4, we specifically will focus on details of the commercial code Abaqus but for
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now we elaborate the methodology and validate the analysis.

Under perfect conditions, the onset of instability coincides with the maximum point of the load–displacement curve (which is the one obtained by the Riks method). Special caution has to be taken with the imperfection that is introduced in the analysis. This will be clarified in the next section. It follows that to capture the onset of bifurcation and validate the numerical procedure one needs to identify that point precisely. It may be that for certain values of the displacement in a given load–displacement curve the slope of the curve is nearly zero which makes difficult to identify the maximum point. One possible procedure to identify that point for the cylinder problem at hand is to follow the displacement of different points located at different altitudes of the cylinder. If there is no bifurcation, those points (actually two points are just sufficient) will have the same displacement. When bifurcation occurs the displacements of those points will differ. Let us illustrate the analysis and show different problems that one can face.

Consider the anisotropic model (3.9) with $\varphi = 45^\circ$, $k_2 = 0.02 > k_{2c}$, and two different values for $k_1 = 0.02\mu$ and $0.2\mu$, with $\mu = 300$ kPa. Bulging is expected in the former case ($k_1$ is not large enough, and $0.06\mu$ would be needed to avoid bifurcation) but is not expected in the latter (see [7] for details).

A three dimensional sector of a cylinder is defined with only one element in the hoop direction. The length of the cylinder is $50R$, where $R$ is the external radius, and the thickness $0.025R$. We take the value $R=5$ mm. To avoid a strict numerical problem and capture the bifurcation, a small imperfection of maximum deviation $4 \times 10^{-5}R$ is introduced: at one end of the cylinder the external radius is $R$ while at the other end the external radius is $R$ minus the maximum deviation, i.e. the external radius changes along the axis of the cylinder while thickness is kept constant along the axis. Four elements are generated in the thickness direction, and 10,000 elements along the longitudinal direction to obtain a good-shaped mesh. The cylinder is stretched imposing relative axial displacements, i.e. $\lambda_z$ is applied and considered fixed during the rest of the analysis. Once this is completed, internal pressure is applied using the modified Riks analysis (i.e. the arc length procedure). Hoop
Finite element analysis

displacements (in a nodal local cylindrical system) are prevented during the simulation. Using one CPU core each analysis (i.e. for each $\lambda_z$) needs around 100 min to be completed. After this period of time, enough results are obtained to process and clearly identify bulging and postbulging which also adds more time. It is time consuming but it is necessary to accurately determine the onset of bifurcation.

![Graph showing curves for applied pressure vs. arc length for different models](image)

Figure 3.2: Curves give values of applied pressure vs. arc length when $\lambda_z = 1.5$ for a cylinder made up of a neo-Hookean material (solid line) and two anisotropic models (each model given by a different dashed line). The curve associated with the anisotropic model for which $k_1 = 0.02\mu$ is monotonically increasing and no instability is expected. The other two curves show a maximum point which gives the onset of bifurcation.

Results are shown for $\lambda_z = 1.5$. The evolution of the applied pressure with the arc length (which gives the evolution of the cylinder as it is deformed) is shown in Fig. 3.2. Bulging should occur at the maximum point in each curve. Thus, no bifurcation is expected for the anisotropic model with $k_1 = 0.20\mu$ since the curve is monotonically increasing. On the other hand, bulging is expected for both the neo-Hookean material and the anisotropic model with $k_1 = 0.02\mu$.

It is important to remark (as previously indicated) that the position of the maximum point in a given curve may not be clear (see Fig. 3.2). This may occur for some simulations. In any case, the bifurcation point can be
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captured with the following methodology. One can follow the evolution (with deformation) of the hoop stretch of two points which are located in the middle of the thickness of the cylinder at two different cross sections. One point is at one end of the cylinder and the other point is at the other end. Bifurcation is identified when the hoop stretches of these two points differ. In particular, we take the bifurcation point when the deviation of the hoop stretches of these two points is a small fraction of the hoop stretch average of both points (for instance, 1 %). Fig. 3.3 shows these results for the three cases under consideration. For each case, as loading is applied from the undeformed configuration (in the figure it corresponds with values in each curve for which $\lambda_\theta$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, Fig. 3.3 shows for each case just one line. Furthermore, as expected, for the anisotropic model with $k_1 = 0.20\mu$, Fig. 3.3 just shows one line. For the other two cases, when bifurcation occurs, from each curve in Fig. 3.3 two lines emanate. It follows that each line gives the hoop stretch of the two points considered. In each cylinder, the bulge appears at the cross section for which hoop stretch increases which is the end of the cylinder with the greater external radius.

Fig. 3.4 shows values of pressure vs. hoop stretch up to bifurcation for each cylinder as given by the FEM solution (for each model the thick curve) and the analytical one (for each model the thin curve) which is obtained using $f(\hat{W}, \lambda_\theta, \lambda_z, L/R)$, [17] and the corresponding material model. In the three cases, the FEM solution and the analytical one do not coincide since the thin and thick curves show small differences. This is due, mainly, to the thickness used to get the results with the FEM computational model as opposed to the results obtained for the analytical model which is based on the theoretical cylindrical shell assumption.

Values of hoop stretch vs. axial stretch associated with the onset of bulging bifurcation as given by the FEM solution (points) and the analytical one (line) are shown in Fig. 3.5 for the neo-Hookean material and the anisotropic model with $k_1 = 0.02\mu$. The results show a small disagreement as it has been explained previously. To complete the analysis, values of pressure vs. axial stretch at bifurcation are shown in Fig. 3.6 for the neo-Hooke.
Figure 3.3: For $\lambda_z = 1.5$, the plot gives values of the hoop stretch vs. arc length for two points located at different altitudes of a cylinder made up of a neo-Hookean material (solid line) and made up of two anisotropic models (each model given by a different dashed line). Prior to bifurcation, both points have a non-distinguishing hoop stretch difference, and, for the three materials, there is only one curve. The material for which there is no bifurcation shows that the two points have the same hoop stretch as loading is applied (the figure shows only one curve). For the other two materials, in each case, the onset of bifurcation is identified when the hoop stretch of these two points differ giving rise to two curves as the cylinder deforms.

Figure 3.4: For $\lambda_z = 1.5$, values of pressure vs. hoop stretch up to bifurcation for each cylinder as given by the FEM solution and the analytical one which is obtained using $f(\hat{W}, \lambda_0, \lambda_z, L/R) = 0$, Eq. (3.15) and the corresponding material model. No bifurcation is found in the anisotropic model for which $k_2 = 0.20 \mu$.
Computational modelling of bulging of inflated cylindrical shells

Figure 3.5: Values of hoop stretch $\lambda_\theta$ vs. axial stretch $\lambda_z$ associated with the onset of bulging bifurcation as given by the FEM solution (points) and the analytical one (line).

Figure 3.6: The figure shows values of pressure vs. axial stretch at bifurcation for the neo-Hookean material and the anisotropic model with $k_1 = 0.02\mu$ as given by the FEM solution (points) and the analytical one (lines).
3.3.2.2 Sensitivity analysis of the imperfection: instability and the maximum point

As it has been previously mentioned, instability is close to the maximum point that appears in plots such as the one shown in Fig. 3.1. The existence of these maximum points are indicators of instability and, conversely, if there is no maximum point then instability is not to be expected. But the interpretation of the instability is not obvious. To capture the (bulging) bifurcation, an imperfection is introduced in the (perfect) geometry of the problem. This imperfection may modify the interpretation of the results in the following way. A tiny imperfection should not modify the results and the maximum point obtained with the Riks method should coincide with the onset of bifurcation. A non-small imperfection may modify the perfect conditions for bifurcation. A maximum would appear in the load–displacement curve but the bigger the imperfection the more sudden the rearrangement of the structure at the maximum point. One can conclude that the solution departs from bifurcation to snapping at the maximum point. This has to be clearly identified if the goal of the analysis is to accurately determine the onset of bifurcation. Nevertheless, critical and postcritical behaviour can still be captured and furthermore, under the conditions at hand, a bulge is captured as the instability mode. For a given problem, the bigger the imperfection that is introduced the easier that the Riks method captures an instability since the maximum point appears more clearly. On the other hand, the maximum point does not coincide with the breakdown of bifurcation (which occurs prior to the maximum point). Nevertheless, the bifurcation point can be identified, for the problem at hand, following the displacement of two points as explained in the previous section.

To accept a solution, it is important to check that the maximum of the load proportional factor corresponds with an instability and it has a physical meaning. In particular, it is important to show that it is not related to, for instance, problems with the mesh (one of them, the so-called propagated hourglass modes or singular modes). Several precaution measurements should be taken into account before accepting the solution. First, it could happen that the bifurcation point does not correspond with the peak load proportional factor. This could be either
due to the geometrical imperfection introduced in the model (as explained earlier) or due to excessive arc length increments. Second, with this methodology we are restricting the analysis to capture the bulging bifurcation mode. Other instability modes could appear prior to the bulging bifurcation mode during the loading process but these will not be captured with this analysis. All these aspects should be verified.

For the purpose of this communication in the analysis of the next section, in which an imperfection is introduced, we capture the maximum point with the Riks method and consider that one as the instability point or critical point. Our main objective is to show the possibilities of the procedure. The constitutive model in this paper (as well as the geometry, loading conditions, etc.) is just a prototype and is useful to show qualitative results. Suffice it to say, with the data at hand, to accurately determine the onset of bifurcation in addition to being time consuming would not provide information useful for clinical applications. Physiological conditions are much more complex and involve, among other aspects, residual stresses, growth, remodelling, age, etc.

3.4 Numerical results: Instability and propagation in cylindrical shells

The undeformed geometry of the cylinder shell is chosen following: Basic values include $A=4$ mm, $e=0.2$ mm and $L=20$ mm. An imperfection has to be introduced in the model to capture the bulging mode of bifurcation. With this in mind, we introduce an inner radius and an outer one that are dependent on the coordinate $Z$. In particular, at the top and at the bottom of the undeformed perfect cylinder the inner radius is $A=4$ mm while the inner and outer radius vary along the axial length of the cylinder by means of a very small curvature keeping the thickness constant. The maximum value of the imperfection deviation is $3 \times 10^{-2} A$. As a result the model is axisymmetric and the greatest values of the inner and outer radii are at the cross section that is in the middle of the axial length. As opposed to the analysis in the previous section, here the imperfection is of a greater value. This makes easier for the Riks method to identify the maximum point and it is also less time-consuming.
than with a tiny imperfection.

In Abaqus the constitutive model includes the parameter $C_{10} = \mu / 2$ and we choose $C_{10} = 7.64$ kPa. These data allow us to examine a cylinder made up of a neo-Hookean material (model (3.8)). Analyses were carried out and results are shown in Fig. 3.3 and Fig. 3.4. It is important to remark that we capture the bulging bifurcation mode. Results shown in [2] and [3] for the neo-Hookean material do not distinguish between bulging and necking. Furthermore, in some cases, those results are associated with necking. Our results here are only associated with bulging. Various analyses were run for different lengths including $L = 20$ mm, $L = 80$ mm and $L = 200$ mm as well as for different values of the shear modulus $\mu$ and thickness $e$. In all cases, it was found that in the competition between radial expansion and axial propagation during the evolution of bulging, bifurcation evolves due to radial expansion. This is in agreement with the result given in [4] for thin-walled cylinders. Fig. 3.7 gives this qualitative result for a cylinder made up of a neo-Hookean material. We also point out that for this analysis it is not necessary to use the element C3D8RH. The element CAX4RH of Abaqus can be used, for instance.

The geometry of the tube in all cases is given by $A = 4$ mm and $e = 0.1$ mm. Elastic constants $k_1$ and $k_2$ take the values $k_1 = 0.005$ kPa and $k_2 = 0.012$. The angle $\varphi$ is taken to be $\varphi = 15^o$ and the fiber dispersion $k = 0$. 
Figure 3.7: Plot showing bulging evolution of a cylinder made up of an incompressible neo-Hookean material. In the competition between radial expansion and axial propagation during the evolution of bulging the results show that bifurcation evolves due to radial expansion. The radius of the bulge grows without bound.

Postbifurcation results were also obtained with regard to bulging propagation. To clearly identify the propagation mechanism three tube lengths were taken, namely, $L=20$ mm, $L=80$ mm and $L=200$ mm. Fig. 3.8 shows a bulge in each of these three tubes, the one on the left has $L=20$ mm, the one in the middle has $L=80$ mm and the one on the right has $L=200$ mm. In all the examples considered which include different lengths and thicknesses as well as different material moduli the propagation of the bulging bifurcation mode is axial. In the competition between radial expansion and axial propagation during the evolution of bulging the bifurcation evolves due to axial expansion. Fig. 3.9 shows the qualitative behaviour of this mechanism. Each line in Fig. 3.9 represents the outer contour of the cylinder. The line on the left gives the non-bifurcated cylinder. Next lines to the right of that one show the formation of bulging and the axial propagation of the bulging mechanism.
Figure 3.8: This figure shows a bulge in each of the three cylindrical tubes. The tube on the left has $L=20$ mm, the one in the middle has $L=80$ mm and the one on the right has $L=200$ mm. The propagates in the axial direction of the cylindrical tubes.

Figure 3.9: Shows the formation of bulging and the axial propagation of the bulging mechanism, where each line represents the outer contour of the cylinder.
Analysis of neck and propagation in anisotropic cylindrical shells under axial load and internal pressure

4.1 Introduction

The onset of diverse phenomena can be triggered at large stress (strains) under both tension and compression loading conditions. This includes, for instance, kink band formation [18] and cavitation [19] in fiber reinforced materials; localization phenomena as nucleated by cracks and other stress-raisers (see the seminal work of [16] and [20]), propagating instabilities in structures (see [21]–[23] and references therein), etc. Different researchers have analyzed these mechanisms combining analytical, numerical and experimental works.

Lately, a lot of attention has been given to the study of different boundary value problems in the context of large (strain) deformation stability theory [1]–[4] and [6]–[9]. More in particular, the onset of (bulging) bifurcation as well as the (quasi-static) propagation of different instability modes in thin-walled as well as thick-walled cylinders subject to axial loading and internal pressure has been the focus of different investigations (see, [1]–[4] and [6]–[9]). The analysis is well understood for closed-end tubes (to be more precise, one of the ends is closed but at the other end air is forced into the tube) made of isotropic elastomers as
opposed to the analysis for open ended ones. It has been found that a necessary condition for a bulge to propagate axially is that the pressure-change in volume response of a section of a tube constrained to inflate uniformly has an up–down–up behavior (see for instance [8] and [21]). The so-called Maxwell line plays an important rule to determine the propagation pressure. For some materials, the response does not recover to a second stable branch, in which case the bulge keeps growing radially. The propagation of a bulge along a long party balloon has been described in [21], among others. There too, the initiation of the localized bulge occurs with the attainment of the peak pressure. Neck propagation has also been studied under different loading conditions for isotropic nonlinearly elastic materials among others (see [21], [22] and [23] and references therein). Here, we extend those analyses for closed-end tubes to open ended tubes made of anisotropic materials under more complex loading conditions using a unified approach able to capture bifurcation and postbifurcation. We show that the qualitative description of the different motions for open ended tubes mimics the results given for closed-end tubes.

Bifurcation conditions for open ended tubes were studied in [1] and [7]. Among the mechanical instabilities modes of hyperelastic cylinders, the bulging one is most likely to occur prior to other instability modes (see e.g. [7]). Formation of a localized bulge is a nonlinear (bifurcation) phenomenon and cannot be described by any linear theory. The analytical procedure that gives the onset of bulging does not distinguish between a bulge and a neck. It is necessary to follow the propagation of the instability to completely characterize the bifurcation mode. Indeed, the analysis of the motion has important applications (see [23] and [24]). Nevertheless, analytical postcritical behaviour is difficult to be determined and in the literature there exist solutions only for very simple cases and are based on complicated mathematical machinery [4] and [9]. Postcritical behaviour requires combination of theoretical and experimental analyses as well as numerical simulation not only to predict the instability at hand but more importantly to describe qualitatively the mechanics of the nonlinear phenomena associated with this process as we show in this chapter.
We consider a bifurcation phenomenon (see for instance [2]) in which the zero mode is sinusoidal with axisymmetric deformations. Under these circumstances, the onset of bifurcation has previously been related to a maximum pressure. Other authors have considered bifurcations in which the zero mode is not sinusoidal (see [4]). In that case, the onset of bifurcation may not be related to a maximum pressure.

In [24] a numerical procedure to analyze bulging bifurcation of inflated hyperelastic (open ended) thick-walled cylinders based on the modified Riks method [17] was developed. Our attention was focused not only on the possibility to obtain an appropriate methodology but also on the possibility to implement that numerical methodology in a commercial finite element code. For that purpose, we specifically focus on the finite element code Abaqus/Standard which has implemented the Riks procedure. Indeed, this analysis enlightens new possibilities in several ways including scientific and professional practice. In [24], it was mentioned that the procedure can also follow postbifurcation and, furthermore, axial propagation of bulging was captured. Here we continue our previous investigations in this area (see [2], [3], [7] and [24]) and focus now on different material models. Our goal is to capture propagation of bulges and necks. Furthermore, by looking at different (prototypes) models available in the literature, we provide a qualitative description of necking as well as radial propagation of bulging. This assesses the (nonlinear) method capability and, furthermore, completes the previous analyses developed that focused on axial propagation of bulging [24].

In the last few years there have been different studies in which the macroscopic response of fiber-reinforced materials has been analyzed in the context of anisotropic nonlinear elasticity. We focus here on orthotropic materials with two preferred directions (two families of fibers) which are mechanically equivalent and are symmetrically disposed.

In general (in three dimensions), two independent invariants are sufficient to characterize the nature of each family of fibers. For one family, these invariants are denoted by $I_4$ and $I_5$ while for the other family these invariants are denoted by $(I_6)$ and $(I_7)$. Nevertheless, we note that $(I_6)$ and $(I_7)$ are the analogues of $I_4$ and $(I_5)$, respectively. The
invariant $I_4$ ($I_6$) is related directly to the fiber stretch. The standard reinforcing model is a quadratic function that depends only on this invariant. The other invariant, $I_5$ ($I_7$), is also related to the fiber stretch but introduces an additional effect that relates to the behaviour of the reinforcement under shear deformations. The coupling between the two families is characterized by an additional invariant denoted by ($I_8$) in which we will not focus.

Here in this chapter, we investigate bifurcation and postbifurcation of a cylinder under inflation and axial load made of a neo-Hookean material reinforced with two families of fibers which are mechanically equivalent and are symmetrically disposed. We consider a reinforcing model that depends on $I_4$ ($I_6$), taken as a quadratic model, and compare these results with those for the corresponding $I_5$ ($I_7$) reinforcement. Our purpose is twofold: on the one hand, to analyze the influence of the anisotropic invariants on bulging and on the other hand, to describe the mechanics underlying propagation of the instability mode. Furthermore, results show a very different behaviour for the models at hand. Necking is captured for the model depending on $I_5$ ($I_7$) reinforcement. It is shown that the axial stretch in the necking zone is smaller than the one outside the necking area. In addition, it is shown that motion of necking is a combination of both axial propagation and radial propagation. These are new results, although expected. On the other hand, axial stretch in the bulge zone is greater than outside the bulge zone for both axial and radial propagation of bulging. Indeed these simulations have to guide appropriate theoretical analyses able to describe these mechanisms.

4.2 Basic equations

In this chapter we consider the strain energy given by

$$W(I_1, I_4, I_6) = c(I_1 - 3) + \frac{g}{2} \sum_{i=4,6} (I_i - 1)^2,$$

where $c$ and $g$ are the material constants and the quadratic standard reinforcing model $F(I_4) = \frac{1}{2}(I_4 - 1)^2$ has been introduced. This reinforcing model depends only on the stretch in the fiber direction [25]. The standard reinforcing function is a prototype that was introduced in
the 90s and that has been widely used by several authors to model qualitative mechanical behavior of fiber reinforced materials (see [16], [19] and [26]. Nevertheless, more recently, micromechanics based approaches that enable one to account for the behaviors of the individual constituents have been developed to obtain macromechanical strain energies for composite materials that have also the features of augmented isotropic functions with the so-called standard reinforcing model (see [27] for one fiber reinforcement and [28] for two families of fibers, as well as references therein). The fiber stretch invariants using $F = \text{diag}(\lambda^{-1}, \lambda^{-1}, \lambda, \lambda_z)$, (3.6) and (3.7) are

$$I_4 = I_6 = \lambda^2 \cos^2 \varphi + \lambda^2 \sin^2 \varphi,$$

(4.2)

In addition, we also consider the strain energy function:

$$W(I_1, I_5, I_7) = c(I_1 - 3) + \frac{g}{2} \sum_{i=5,7} (I_i - 1)^2,$$

(4.3)

which includes the counterpart of the standard reinforcing model with respect of $I_5(I_7)$, i.e. $G(I_5) = \frac{1}{2}(I_5 - 1)$. This reinforcing model has a particular influence on the shear response of the material (see [16] and [25] and [26]). The value of the invariants using $F = \text{diag}(\lambda^{-1},\lambda^{-1},\lambda,\lambda_z)$, and the expression of $I_5$ shown in (3.6) as well as the expression of $I_7$ given in (3.7) are

$$I_5 = I_7 = \lambda^4 \cos^2 \varphi + \lambda^4 \sin^2 \varphi.$$

(4.4)

Here, our purpose is to investigate and compare the behaviour of tubes made of either material model (4.1) and (4.3) subject to axial loading and internal pressure with regard to bulging/necking motion.

### 4.3 Analytical results: Effect of the materials types in the bifurcation

In this section we completely characterize the behaviour of the two models at hand with respect to the onset of bulging/necking. This will guide our numerical analysis of post bifurcation in the next Section.
4.3.1 Influence of the parameters $I_4$ and $I_6$

We focus first on the material model (4.1). Expanding the bulging condition (3.15) using (4.1) we find that asymptotic expressions for the associated $\lambda_\theta$ in the limit as $\lambda_z = \infty$ is given by $-\lambda_\theta^8 g^2/16$. It follows that for any $g > 0$ the bulging instability criterion is satisfied for values $\lambda_\theta \geq 1$ when the cylinder is elongated with an axial stretch $\lambda_z \geq 1$. More specifically, the bulging condition (3.15) with the assumption of infinite length, i.e. when $L/R \to \infty$ can be written as

$$f = -\frac{1}{16} g^2 \lambda_\theta^8 + \frac{3}{4} g^2 \lambda_z^6 \lambda_\theta^4 + \frac{45}{16} g^2 \lambda_z^4 \lambda_\theta^4 + \left( \frac{1}{2} g^2 - gc \right) \lambda_\theta^8 + (4gc - 3g^2) \lambda_\theta^4 \lambda_\theta^4 + (4gc - g^2 - 4c^2) \lambda_\theta^4 - 3gc + 6gc \lambda_z^2 \lambda_\theta^2 + 54gc \frac{\lambda_z^2}{\lambda_\theta^2} + \frac{16c^2 - 24gc}{\lambda_z^2} \frac{\lambda_z^2}{\lambda_\theta^2} + \frac{24c^2 - 12gc}{\lambda_z^2} \frac{\lambda_z^2}{\lambda_\theta^2} + \frac{12c^2}{\lambda_z^2} \lambda_z^4 \lambda_\theta^4.$$ (4.5)

Furthermore, by our previous argument, we can determine $\alpha$ such that the critical hoop stretch is $\alpha \lambda_z$ as $\lambda_z \to \infty$. Substituting in (4.5) it is sufficient to consider higher order terms in $\lambda_z$, in particular the first three terms, so that one can write

$$-\frac{1}{16} g^2 \alpha^8 \lambda_z^8 + \frac{3}{4} g^2 \alpha^6 \lambda_z^8 + \frac{45}{16} g^2 \alpha^4 \lambda_z^8 = 0,$$ (4.6)

From which one obtains $\alpha = \sqrt{15}$ (apart from the non-physical solutions $\alpha \leq 0$). Whence, and since $m=FM$, the angle of the deformed fibers with respect to the axis of the tube for highly axially stretched cylinders tends to

$$\tan^{-1} \left( \frac{\sqrt{15}}{\tan \varphi} \right),$$ (4.7)

at the onset of bulging bifurcation.

To show some numerical results we consider (4.3) with $g = 0.01c$ and $\theta = 30^\circ$. In Fig. 4.1 we display the bifurcation curves in the $\lambda_z - \lambda_\theta$ plane for a tube with(i) $L/R=6$ (solid line) and with(ii) $L/R=12$ (dashed curve). Values of pressure associated with the onset of the instability are shown in Fig. 4.2. We note, as expected, that the longer the length of the cylinder,
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the smaller the values of the hoop stretch and the pressure associated with the onset of bifurcation. These results are not highly modified for other values of \( g \) such as \( g = 0.001c \), although for greater values bifurcation can be avoided.

The angle of the fibers associated with bifurcation is shown in Fig. 4.3. According to (4.7) the angle tends to \( \tan^{-1}(\sqrt{15} \tan 30^\circ) = 65.9^\circ \) for infinitely long tubes as \( \lambda_z \to \infty \). Convergence to this value is slow as it is shown in the figure. For instance, when \( L/R = 12 \) one obtains an angle of 64.4° for \( \lambda_z = 20 \).

### 4.3.2 Influence of the parameters \( I_5 \) and \( I_7 \)

We now turn to analysis of the resistance generated by the reinforcing model \( G(I_5) \) in comparison with that generated by the model \( F(I_4) \). We follow in a parallel way the analysis developed previously.

Expanding the bulging condition (3.15) using (4.3) we find that asymptotic expressions for the associated \( \lambda_\theta \) in the limit as \( \lambda_z \to \infty \) is given by \( -\lambda_\theta^6 g^2/4 \). It follows, as it has been obtained for (4.1), that for any \( g > 0 \) the bulging instability criterion is satisfied for values \( \lambda_\theta \geq 1 \) when the cylinder is elongated with an axial stretch \( \lambda_z \geq 1 \). The bulging condition (3.15) with the assumption of infinite length can be written as

\[
f = \left( -\frac{1}{4} g^2 \lambda_\theta^6 + 18g^2 \lambda_z^4 \lambda_\theta^4 + \frac{351}{4} g^2 \lambda_z^8 \lambda_\theta^8 + \frac{189}{2} g^2 \lambda_z^{12} \lambda_\theta^4 \right) + 2g^2 \lambda_z^8 - 90g^2 \lambda_z^4 \lambda_\theta^8 - 18g^2 \lambda_z^8 \lambda_\theta^4 - 2g^2 \lambda_z^{10} + 6g^2 \lambda_z^8 \lambda_\theta^4 - 18g^2 \lambda_z^4 \lambda_\theta^8 + 6g^2 \lambda_z^8 \lambda_\theta^4 - 4g^2 \lambda_z^6 + 72g^2 \lambda_z^4 \lambda_\theta^4 + 8g^2 \lambda_z^8 \lambda_\theta^4 - 8g^2 \lambda_z^4 \lambda_\theta^8 - 4 \lambda_\theta^4 \right) + \frac{24g \lambda_\theta^6}{\lambda_z^2} + \
\frac{252g \lambda_\theta^6}{\lambda_z^2} - \frac{48g \lambda_\theta^6}{\lambda_z^2} - \frac{144g \lambda_\theta^6}{\lambda_z^2} + \frac{24c^2}{\lambda_\theta^2} + \frac{16c^2}{\lambda_\theta^2} + \frac{12c^2}{\lambda_z^2 \lambda_\theta^2}, \tag{4.8}
\]

By the same argument given in Section 4.3.1, it is easy to show that the angle of the fibers at bifurcation for highly stretched cylinders tends to \( \tan \tan^{-1}(\alpha \tan \varphi) \). Now, taking the higher order terms in 4.8, it yields that \( \sqrt{150 + 6\sqrt{681}/\sqrt{2}} \). For the particular case of \( \varphi = 30^\circ \) this implies
an angle at bifurcation of $\simeq 59.7^\circ$

For $g = 0.001c$ and $\varphi = 30^\circ$ bulging curves are shown in Figs. 4.4 and 4.5. Plots are provided for (4.3) in Figs. 4.4 and 4.5 which are corresponding to Figs. 4.1 and 4.2, respectively. Now, jumps appear in the plots. This is fully explained by looking at the values of the bulging criterion $f$ vs $\lambda_\theta$ for fixed values of $\lambda_z$. In Fig. 4.6, values of $f$ vs $\lambda_\theta$ are shown for $\lambda_z \simeq 1.55$ and $L/R=6$. In this plot, it is shown that $f$ is tangent to the abscissa axis. This feature gives the jump in Figs. 4.4 and 4.5. Similar jumps were captured in Figs. 4.7 and 4.8 of [2]. Nevertheless, since that paper provided an analysis to capture just the bifurcation, that jump could not be interpreted. Indeed, we show later here that for values of the axial stretch greater than the one associated with the jump the bifurcation mode changes from bulging to necking. More in particular, for values of the axial stretch less than the one associated with the jump, the mode of bifurcation has been found to be bulging. On the other hand, for values of the axial stretch greater than the one associated with the jump, the mode of bifurcation is either necking or bulging.

For completeness, values of $I_5(I_7)$ against $\lambda_z$ associated with the onset of the bulging mode are given in Fig. 4.7. At last, the angle of the fibers at bifurcation is shown in Fig. 4.8. The angle tends to be $\simeq 59.7^\circ$ for an infinitely long cylinder as $\lambda_z \to \infty$. Convergence is much faster in this case than for the model with invariants $I_4$ and $I_6$ because the order of the stretches with the higher order terms in the bulging criterion is now greater (16 for (4.3) as opposed to 8 for (4.1)).
Analytical results: Effect of the materials types in the bifurcation

Figure 4.1: Plot of the bifurcation curves for the bulging mode for a tube made of (4.1) (the material with $I_4$ and $I_6$) with (i) $L/R=6$ (solid line) and with (ii) $L/R=12$ (dashed curve) in the $\lambda_z-\lambda_\theta$ plane. In both cases $g=0.01c$ and $\varphi = 30^\circ$.

Figure 4.2: Values of the (normalized) pressure against $\lambda_z$ associated with the onset of the bulging mode for a tube made of (4.1) (the material with $I_4$ and $I_6$) with (i) $L/R=6$ (solid line) and with (ii) $L/R=12$ (dashed curve) in the $\lambda_z-\lambda_\theta$ plane. In both cases $g=0.01c$ and $\varphi = 30^\circ$. 

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Figure 4.3: Values of the fiber angle (given with respect to the axis of the cylinder) against $\lambda_z$ associated with the onset of the bulging mode for a tube made of (4.1) (the material with $I_4$ and $I_6$) with (i) $L/R=6$ (solid line) and with (ii) $L/R=12$ (dashed curve) in the $\lambda_z - \lambda_\theta$ plane. In both cases $g = 0.01c$ and $\varphi = 30^\circ$. 

\[
\begin{align*}
L/R &= 6 & L/R &= 12
\end{align*}
\]
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Figure 4.4: Plot of the bifurcation curves for the bulging mode for a tube made of (4.3) (the material with $I_5$ and $I_7$) with (i) $L/R=6$ (solid line) and with (ii) $L/R=12$ (dashed curve) in the $\lambda_z - \lambda_\theta$ plane. In both cases $g = 0.001c$ and $\varphi = 30^\circ$.

Figure 4.5: Values of the (normalized) pressure against $\lambda_z$ associated with the onset of the bulging mode for a tube made of (4.3) (the material with $I_5$ and $I_7$) with (i) $L/R=6$ (solid line) and with (ii) $L/R=12$ (dashed curve). In both cases $g = 0.001c$ and $\varphi = 30^\circ$. 
Figure 4.6: For $\lambda_z = 1.55$, values of $f/c$ against $\lambda_\theta$ for a tube made of (4.3) (the material with $I_5$ and $I_7$) with $L/R = 6$, $g = 0.001c$ and $\varphi = 30^\circ$.

Figure 4.7: Each curve in this Figure has its corresponding curve in Fig. 4.5. Here, values of $I_5$ ($I_7$) against $\lambda_z$ associated with the onset of the bulging mode are given.
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Figure 4.8: Values of the fiber angle (given with respect to the axis of the cylinder) against $\lambda_z$ associated with the onset of the bulging mode for a tube made of (4.3) (the material with $I_5$ and $I_7$) with (i) $L/R=6$ (solid line) and with (ii) $L/R=12$ (dashed curve). In both cases $g=0.001c$ and $\varphi = 30$.

Figure 4.9: The solid line in this figure is the one given in Fig. 4.2. Additionally, and for comparison, in this figure each point gives the solution that is obtained by means of the finite element model.
Figure 4.10: For $\lambda_z = 1.4$, the figure gives values of pressure of inflation vs values of hoop stretch for two points (A and B) located at different altitudes of a cylinder with $L/R = 6$ made of (4.1) (the material with $I_4$ and $I_6$) for which $g = 0.01c$ and $\varphi = 30^\circ$. As loading is applied from the undeformed configuration (in the figure it corresponds with values for which $\lambda_\theta$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows one solid line. When bifurcation occurs (close to the maximum point), the structure is not able to support higher values of pressure and the hoop stretches of these two points differ. Furthermore during postbifurcation, as pressure decreases, point A decreases its hoop stretch while point B increases its hoop stretch. The hoop stretch of point B is given by the dashed line which is at the right of the bifurcation point. This description is associated with radial propagation of bulging. Point B is located in the bulge.
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Figure 4.11: History of configurations, (1) is the beginning and (6) is the ending, associated with radial propagation of bulging for the material with $I_4$ and $I_6$ and axial stretch 1.4.

Figure 4.12: From left to right, the figure shows a graphic representation of radial propagation of bulging. Points A and B are shown to further clarify the curves of Fig. 4.10.
Figure 4.13: The solid line in this figure is the one given in Fig. 4.5. Additionally, and for comparison, in this figure each point gives the solution that is obtained by means of the finite element model.

Figure 4.14: This plot mimics the plot of Fig. 4.10. In particular, for $\lambda_z = 1.3$, the figure gives values of pressure of inflation vs values of hoop stretch for two points (A and B) located at different altitudes of a cylinder with $L/R = 6$ made of (4.3) (the material with $I_5$ and $I_7$) for which $g = 0.001c$ and $\varphi = 30^\circ$. As loading is applied from the undeformed configuration (values for which $\lambda_\theta$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows one solid line. When bifurcation occurs (close to maximum point), the structure is not able to support higher values of pressure and the hoop stretches of these two points differ. Furthermore during postbifurcation, as pressure decreases, point A decreases its hoop stretch while point B increases its hoop stretch. The hoop stretch of point B is given by the dashed line which is at the right of the bifurcation point. This description is associated with radial propagation of bulging. Point B is located in the bulge.
4.4 Bifurcation and postbifurcation: Finite element analysis

4.4.1 Methodology

We have developed a finite element model to show the quantitative and qualitative capabilities of this methodology, able to capture bifurcation and postcritical behaviour [4]. We describe here the main aspects.

The type of element used in the simulations is a 8-node linear brick with reduced integration, including hourglass control, and hybrid with constant pressure (it is the C3D8RH in Abaqus notation). Incompressibility is imposed strictly (not by penalization). The deformation is axisymmetric, but the problem can not be solved in an axisymmetric space because Abaqus supports anisotropic hyperelastic materials with fibre directions that are in planes containing the axis of the cylinder or fibers that are along the hoop direction. Instead, to avoid this problem, only a wedge of small angle is included in the model with symmetry conditions in planes containing the axis of the cylinder. One element in the hoop direction is enough to capture accurately the response. More in particular, a three dimensional sector of the cylinder with an angle of 0.18° is defined with only one element in the hoop direction. The length of the cylinder is 6R and the thickness is 0.05R. First, we stretch the cylinder imposing relative axial displacements. Then, inner pressure in an Abaqus Riks analysis (arc length procedure) is applied. To avoid a strict numerical bifurcation, a small imperfection of a maximum deviation $10^{-4}R$ increasing from one end of the tube to the other end of the tube is introduced. Four elements are generated in the thickness direction and 2500 elements along the longitudinal direction for a good-shaped mesh. Hoop displacements (in a nodal local cylindrical system) are prevented during the simulation.

One can follow the evolution (with deformation) of the stretches of two points which are located in the middle of the thickness of the cylinder at two different cross sections (at two different altitudes). One point is at one end of the cylinder, point A, and the other point is at the other end of the cylinder, point B. Bifurcation is estimated when the relevant displacement...
history of both points differs. In particular, we take the bifurcation point when the deviation of the hoop stretches of these two points is a small fraction of the hoop stretch average of both points (for instance, 1% ). It is worth to clarify that with this strategy we do not determine exactly the bifurcation point. This point corresponds with the state for which the tangent stiffness matrix becomes singular. In fact, we avoid the precise bifurcation point introducing the imperfection. When the solution violates significantly a symmetry (in our case based on the history of two control points), one estimates the associated state as critical.

The materials (4.1) and (4.3) are not provided by the material library in Abaqus but can easily be defined in a user subroutine defining the first and second derivatives of the strain energy function with respect to the invariants. We choose for (4.1) the value $g = 0.01c$ and for (4.3) the value $g = 0.001c$, as it has been taken previously, and for both $\varphi = 30^\circ$. 

Figure 4.15: For $\lambda_z = 1.7$, the figure gives values of pressure of inflation vs values of hoop stretch for two points (A and B) located at different altitudes of a cylinder with $L/R = 6$ made of (4.3), (the material with $I_5$ and $I_7$) for which $g = 0.001c$ and $\varphi = 30^\circ$. As loading is applied from the undeformed configuration (values for which $\lambda_\theta$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows one solid line. During postbifurcation, as pressure decreases, the hoop stretch of both points (A and B) decrease. This description is associated with necking motion. Point B is located in the necking area.
Figure 4.16: The values shown in the curves of this figure have their corresponding values in Fig. 4.15. In particular, for $\lambda_z = 1.7$, the figure gives values of pressure of inflation vs values of axial stretch for two points (A and B) located at different altitudes of a cylinder with $L/R = 6$ made of (4.3) (the material with $I_5$ and $I_7$) for which $g = 0.001c$ and $\varphi = 30^\circ$. As loading is applied from the undeformed configuration (in the figure it corresponds with the vertical line), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows one solid line (the vertical line). During postbifurcation, as pressure decreases, point B decreases its axial stretch while point A increases its axial stretch. The axial stretch of point B is given by the dashed line which is at the left of the bifurcation point. This description is associated with necking motion. Point B is located in the necking area.
4.4.2 Results

Values of pressure associated with bifurcation for (4.1) are shown in Fig. 4.9 together with the corresponding analytical results.

The difference between the numerical results and the thin-wall analytical results is mainly due to the fact that the finite element model has finite thickness. For $\lambda_z = 1.4$, the inflation pressure vs the hoop stretch of the two points A and B are plotted in Fig. 4.10. A bulge develops after the critical state. Some of the cross sections increase their diameter while other sections decrease their diameter. Furthermore, during postbifurcation pressure decreases. This description is associated with radial propagation of bulging. The peak pressure corresponds to the bifurcation point. A sequence of configurations showing radial propagation of bulging for $\lambda_z = 1.4$ is given in Fig. 4.11 while Fig. 4.12 shows a scheme of that motion. This qualitative behaviour has been captured for the values of $\lambda_z$ that have been analysed.

Values of pressure associated with bifurcation for (4.3) are shown in Fig. 4.13. For $\lambda_z = 1.3$, the inflation pressure vs the hoop stretch of the two points A and B are plotted in Fig. 4.14. This result mimics
qualitatively the results obtained for (4.1) since radial propagation of bulging is captured. On the other hand, it has been detected that the imperfection has a strong influence on the results for values of \( \lambda_z \) close to the jump in Fig. 4.13, for instance for \( \lambda_z = 1.7 \). A sensitivity analysis has been performed and for this particular simulation (\( \lambda_z = 1.7 \)) the sector angle and the maximum deviation have been reduced to 0.045° and \( 10^{-7}R \), respectively. In addition, attention has been paid to avoid excessive element distortion during the simulation. The inflation pressure vs the hoop stretch of the two control points are plotted in Fig. 4.15 for \( \lambda_z = 1.7 \). As in the previous cases, the peak pressure corresponds to the bifurcation point. Nevertheless, during postbifurcation, as pressure decreases, the hoop stretch of both points (A and B) decreases. This description is associated with necking motion. Point B is located in the necking area. To complete the qualitative description of necking motion, the inflation pressure vs the axial stretch of the two control points are plotted in Fig. 4.16. It follows from the figure that the axial stretch at the necking zone (point B) is smaller than outside the necking region (point A). Fig. 4.17 shows a detail of the original and critical (at the onset of bifurcation) meshes. A history of configurations showing necking motion is given in Fig. 4.18 for \( \lambda_\theta = 1.7 \). A graphic representation of the motion is given in Fig. 4.19. During necking motion, all the cross sections decrease their diameter and furthermore localized deformation appears at one end of the tube, the one in which necking does not develop. This behaviour is obtained because while the distance between both ends of the tube is kept fixed during the simulation, in the necking zone both the axial stretch and the hoop stretch decrease.

We have considered tubes of finite length. An imperfection is introduced in the system and a sensitivity analysis is necessary to confirm that the initial imperfection is small enough to have little effect upon the final results. In our simulations, it is found that bifurcation always initiates from one of the two ends of the tube, which is expected. Similarly, for an infinitely long tube, bifurcation should initiate at no particular place. In fact, it depends on the geometry of the imperfection.
Figure 4.18: History of configurations, (1) is the beginning and (6) is the ending, associated with necking motion for the material with $I_5$ and $I_7$ and axial stretch 1.7.

Figure 4.19: From left to right, the figure shows a graphic presentation of necking propagation. Points A and B are shown to further clarify the curves of Fig 4.15.
5.1 Introduction

In this chapter, the axial propagation of the bulging instability mode in thin-walled cylinders under inflation is analysed. We present the analytical solution for this particular motion as well as for radial expansion during bulging evolution. For illustration, cylinders that are made of either isotropic incompressible non-linearly elastic materials or doubly fiber reinforced incompressible non-linearly elastic materials are considered. The former model is studied to assess the analytical methodology described in this chapter. In particular, it is shown that axial propagation of bulging involves two periods: firstly, pressure remains essentially fixed during the ensuing propagation of the bulging instability mode beyond the onset of bifurcation until a suitable configuration is obtained; secondly, in subsequent motion, for further axial propagation of bulging pressure of inflation must be increased. Hence, the structure in subsequent motion can support higher pressures than the pressure associated with the onset of bulging bifurcation. Configurations can be described by two uniform cylinders joined by a transition zone. On the other hand, radial expansion of bulging is related to a decrease of pressure beyond the onset of bulging. Finite element simulations of some thick-walled cylinders made of both models are also
performed. Comparison between numerical and analytical results shows a
good agreement qualitatively.

To show a complete picture of the analysis we focus first on an isotropic
model. In particular, we consider a mixed version of the neo-Hookean and
Demiray strain energy functions, namely,

\[ W = \frac{\mu}{2(1 - k + k\alpha)} \left\{ (1 - k)I_1 + k\exp[\alpha(I_1 - 3)] + 2k - 3 \right\}, \quad (5.1) \]

where \( k \) is a parameter such that if \( k = 0 \) then the neo-Hookean model
is obtained while if \( k = 1 \) then the Demiray material is obtained. It
follows that the material parameter \( \mu \) is the initial tangent shear modulus
associated with the neo-Hookean model and \( \alpha \) is the parameter (exponent)
associated with the Demiray model.

### 5.2 Axial propagation of bulging

Firstly, an undeformed (membrane) cylinder in which the ends are
open is stretched, i.e. the axial stretch \( \lambda_z \) is given and kept constant
throughout the analysis. Pressure is then applied and it is assumed that
bulging bifurcation conditions are met. Furthermore, we consider that
axial motion of bulging takes place. Following the experimental results
for closed ended tubes we consider that pressure remains essentially fixed
during the ensuing propagation of the bulging instability mode beyond
the onset of bifurcation until a suitable configuration in equilibrium is
obtained. This particular configuration can be described by two uniform
cylinders joined by a transition zone. Each cylinder is characterized by a
hoop stretch \( \lambda_{\theta i} \) and an axial stretch \( \lambda_{z i} \), where we use the subindex
\( i \in \{1, 2\} \) to denote the two cylinders. Furthermore, to characterize this
configuration, it is sufficient to know the two stretches of each cylinder
(axial and azimuthal) a total of 4 unknowns since the (propagation)
pressure is determined by either set of the stretches of one of the
cylinders using (3.12).
5.2.1 The solution given by the differential equations

For a very long tube the analytical solution can be deduced from Fu et al. [4] although it is not clearly stated in that paper. The authors did not focus on the two periods of the bulging motion, the first one at an essentially fixed pressure (the propagation pressure) while the second one is obtained as pressure increases. In Fu et al. [4], it was shown that there is a bifurcation with zero mode number and that the associated axial variation of near-critical bifurcated configuration admits a locally bulging or necking solution. We extend those results here. The propagation pressure is obtained as follows. Eqs. (3.2) and (3.3) of Fu et al. [4] are the two integrated equations of the two differential equations that govern the problem at hand. Under the conditions at hand, the axial stretch of one of the cylinders (cylinder 1) is known and the three other stretches are obtained using Eqs. (3.2) and (3.3) of Fu et al. [4] together with the condition that pressure at infinity and at the center of the bulge coincide, which is a system of equations that can be written as

\[
\frac{\partial \hat{W}}{\partial \lambda_z} \bigg|_{\lambda_\theta=\lambda_{\theta_2}} - \frac{\lambda_{\theta_2}^2 - \lambda_{\theta_1}^2}{2\lambda_{\theta_1}\lambda_{\theta_2}} \frac{\partial \hat{W}}{\partial \lambda_\theta} \bigg|_{\lambda_z=\lambda_{z_2}} - \frac{\partial \hat{W}}{\partial \lambda_z} \bigg|_{\lambda_\theta=\lambda_{\theta_1}} \lambda_z=\lambda_{z_1} = 0
\]

\[
\hat{W} \bigg|_{\lambda_\theta=\lambda_{\theta_2}} - \lambda_{z_2} \frac{\partial \hat{W}}{\partial \lambda_z} \bigg|_{\lambda_\theta=\lambda_{\theta_2}} - \hat{W} \bigg|_{\lambda_\theta=\lambda_{\theta_1}} + \lambda_{z_1} \frac{\partial \hat{W}}{\partial \lambda_z} \bigg|_{\lambda_\theta=\lambda_{\theta_1}} \lambda_\theta=\lambda_{\theta_1} = 0
\]

\[
\lambda_{\theta_2} \lambda_{z_2} \frac{\partial \hat{W}}{\partial \lambda_\theta} \bigg|_{\lambda_z=\lambda_{z_2}} - \lambda_{\theta_1} \lambda_{z_1} \frac{\partial \hat{W}}{\partial \lambda_\theta} \bigg|_{\lambda_z=\lambda_{z_1}} = 0,
\]

respectively. In addition, once the solution is obtained, it is necessary to check that cylinders 1 and 2 are stable, i.e. that \( f_i > 0 \) for \( \lambda_{\theta_i} \) and \( \lambda_{z_i} \), where \( i \in \{1, 2\} \). These conditions are sufficient to establish that the solution is related to axial propagation of bulging. Results show that bifurcation pressure (which is obtained using (2.16)) is just a little greater than propagation pressure (obtained once the two uniform cylinders are characterized with their stretches). For the material model 5.1 with \( k = 0.5 \) and \( \alpha = 0.25 \) when \( \lambda_z = 1.1 \) the two cylinders have the stretches (\( \lambda_{\theta_i}; \)
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\( \lambda_{zi} = (1.7987,1.1) \) and \( (2.0396,1.2630) \). Bifurcation pressure is \( pR/H = 0.9416\mu \) and, similarly, propagation pressure is 0.926\mu. For the material model (3.9) with \( \varphi = 45^\circ; k_2 = 0.1 \) and \( k_1 = 0.05\mu \) when \( \lambda_z = 1.1 \) the two cylinders have the stretches \( \lambda_{\theta{i}}, \lambda_{zi} = (1.845,1.1) \) and \( (1.9451,1.168) \). Bifurcation pressure is 0.9905\mu and propagation pressure is 0.985\mu.

The solution of the system (5.2) gives the configuration associated with the end of the first period of motion. In subsequent motion, for further axial propagation of the bulge the pressure of inflation must be increased beyond the propagation pressure. It follows that for a given pressure greater than the propagation pressure both cylinders obey (3.12). These are two equations, since \( p \) is known, that we write as

\[
\frac{H\hat{W}_{\theta i}}{R\lambda_{\theta i}\lambda_{zi}} = p, \ i \in \{1,2\}.
\]  

(5.3)

Whence, in this second period of bulging motion, the four unknowns \( (\lambda_{\theta i}, \lambda_{zi}, \text{ where } i \in \{1,2\} ) \) are solved using four equations, namely, (5.2)\_1,2 and (5.3). This completes the analytical solution for axial motion of bulging using the differential equations that govern the problem at hand. This analysis does not provide information about the transition zone, which is neglected. In the next section we provide an alternative approach in which the solution is obtained by construction. This methodology shows great insight into the mechanics of the problem at hand.

### 5.2.2 The solution using an energy analysis

We take long tubes for which their axial length is much greater than the diameter of the cross section. It follows that, for mathematical convenience, one can use (2.16) to determine the onset of bifurcation. In addition, this makes possible to compare the results we obtain here with the numerical results obtained in the next section for long tubes. In the numerical simulations we will consider that \( L = 100R \). We consider a deformed configuration in equilibrium at a given pressure greater than the propagation pressure that consists of long cylinders joined by transition zones. More in particular, for this mathematical description, it is sufficient to consider two cylinders joined by one transition zone. In
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Fig. 5.1, such a configuration is shown and for now the important point is that the structure is in equilibrium at a pressure $p(p^\ast$ in Fig. 5.1) that is known and that is greater than the propagation pressure since two cylinders have been formed. A developed cylinder tube with regard to Fig. 5.1 is assumed to be symmetric with respect to the plane containing the bottom section of cylinder 2. We further consider that the axial length of the transition zone can be neglected so that one can write that.

$$L\lambda_z = L\beta\lambda_{z1} + L(1 - \beta)\lambda_{z2},$$

(5.4)

where $\beta(0 \leq \beta \leq 1)$ is associated with the length of each cylinder in the reference configuration. Furthermore, it is easy to obtain using (5.4) that

$$\beta = \frac{\lambda_z - \lambda_{z2}}{\lambda_{z1} - \lambda_{z2}},$$

(5.5)

It is clear that $\lambda_z$ must be a value in the interval between $\lambda_{z1}$ and $\lambda_{z2}$. Equilibrium in the axial direction (see Fig. 5.1) implies

$$2\pi r_1\sigma_{zz1} - p\pi r_1^2 = 2\pi r_2\sigma_{zz2} - p\pi r_2^2,$$

(5.6)
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\[
\begin{align*}
\text{Axis} & \quad r_1 = \lambda_\theta_1 R \\
\text{Cylinder 1 after bifurcation} & \quad \text{with stretches } (\lambda_\theta_1, \lambda_z_1) \\
\text{Cylinder at bifurcation with} & \quad \text{stretches } (\lambda'_\theta, \lambda'_z) \text{ under pressure } p^* \\
\text{Transition} & \quad \text{pressure } p^* \\
& \quad r_2 = \lambda_\theta_2 R \\
& \quad \text{Cylinder 2 after bifurcation} \\
& \quad \text{with stretches } (\lambda_\theta_2, \lambda_z_2)
\end{align*}
\]

Figure 5.1: The figure shows the bulge in equilibrium beyond bifurcation at a pressure which is precisely the one associated with the onset of bulging. The structure is made of two cylinders joined by one transition zone. A developed cylinder tube with regard to this figure is assumed to be symmetric with respect to the plane containing the bottom section of cylinder 2. By construction, the bifurcation condition \( f = 0 \) is obeyed at the transition zone for values \( \lambda'_\theta \) and \( \lambda'_z \). On the other hand, each cylinder, characterized by a hoop stretch \( \lambda_\theta_i \) and an axial stretch \( \lambda_z_i \) where \( i \in \{1, 2\} \), is stable and obeys \( f_i > 0 \). This is a graphic representation associated with axial propagation of bulging.

It follows that (5.3) is obeyed. In summary, we look for four unknowns, \( \lambda_\theta_1, \lambda_z_1; \lambda_\theta_2 \) and \( \lambda_z_2 \) such that \( f > 0 \) for both cylinder 1 and cylinder 2 obeying three equations: (5.3) and (5.6). The length of each cylinder is solved using (5.4) and (5.5). It is clear that there exist infinite solutions for such a problem. Physically, this means that not all the configurations are stable. We further clarify this argument by means of an energy analysis that is able to determine the energetically admissible solution among all the possible solutions. The configuration that will be adopted by the structure will be the one that minimizes the total energy \( E \) of the system. It is clear that \( E = U - V \), where \( U \) is the internal energy and \( V \) is the potential energy of external loads. In our case

\[
U = 2\pi RHL[\beta W_1 + (1 - \beta)W_2], \quad (5.7)
\]

where \( W_i \) is the strain energy density function of each cylinder. The potential energy of external loads is
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\[ V = \int_0^l \int_R^r p2\pi r dr dz = \int_0^{\lambda_z \beta L} \int_R \int_R^r p2\pi r dr dz + \int_0^{\lambda_z (1-\beta) L} \int_R \int_R^r p2\pi r dr dz = \pi p R^2 L [\lambda_{\theta 1}^2 \lambda_z \beta + \lambda_{\theta 2}^2 \lambda_z (1-\beta)]. \] (5.8)

Using (5.7) and (5.8), it is convenient to define

\[ \bar{E} = \frac{E}{\pi R HL} = 2[W_1 \beta + W_2 (1-\beta)] - \bar{p}[\lambda_{\theta 1}^2 \lambda_z \beta + \lambda_{\theta 2}^2 \lambda_z (1-\beta)], \] (5.9)

where \( \bar{p} = p R / H \). Thus, among all the solutions that can be obtained using (5.3) and (5.6) the structure adopts the configuration that minimizes \( \bar{E} \).

For a given pressure greater that the propagation pressure, it is easy to show numerically that the solution obtained using (5.2) and (5.3) and the solution that obeying (5.3) and (5.6) minimizes \( \bar{E} \) are the same. For instance, it has been found that the two cylinders formed at the end of the propagation pressure are associated with \( \beta = 1 \) since it gives the minimum value of the energy (5.9). It is an open question to show that these two approaches are equivalent. In what follows, we give some results that show the robustness of this alternative approach that does not make use of Eq. (5.2).

5.2.2.1 The bifurcated configuration at the pressure of bifurcation

We consider a very special configuration, namely, the one in equilibrium beyond bifurcation at a pressure which is precisely the one associated with the onset of bulging. Beyond the propagation pressure, as pressure increases, the value of the pressure reaches the one associated with the onset of bulging. This is an illustrative configuration that gives further insight into the problem. The values of the hoop stretch and the axial stretch satisfying the bulging condition (2.16) are denoted as \( \lambda_{\theta}^* \) and \( \lambda_z^* \), respectively. Following that notation, the structure is in equilibrium at pressure \( p^* \) and axial stretch \( \lambda_z^* = \lambda_z \). It is clear that
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\[ p^* = \frac{H\dot{W}_\theta^*}{RA_\theta^*\lambda_z^*}. \]  (5.10)

On the other hand, equilibrium in the axial direction that is given by 5.6 where \( p \) has to be replaced by \( p^* \) (see Fig. 5.1) can be rewritten using (2.4) and (5.10) as

\[ 2(\dot{W}_{z2} - \dot{W}_{z1}) = \frac{\dot{W}_\theta^*}{\lambda^*_\theta \lambda^*_z} (\lambda^2_{\theta2} - \lambda^2_{\theta1}). \]  (5.11)

Let us study further these equations. We analyze the conditions under which the transition zone propagates axially. During this mechanism we assume that cylinder 2 nucleates, i.e. bulging advances into cylinder 1. This means that for a very long tube with a bulging perturbation of finite size (much smaller than the rest of the structure) one can write \( \lambda_{\theta1} = \lambda^*_\theta \) and \( \lambda_{z1} = \lambda^*_z \). Under these circumstances, (5.3) and (5.11) can be rewritten as

\[ \frac{\dot{W}_{\theta2}}{\lambda_{\theta2} \lambda_{z2}} = \frac{\dot{W}_\theta^*}{\lambda^*_\theta \lambda^*_z}, \]

\[ 2(\dot{W}_{z2} - \dot{W}_{z1}^*) = \frac{\dot{W}_\theta^*}{\lambda^*_\theta \lambda^*_z} (\lambda^2_{\theta2} - \lambda^2_{\theta1}), \]  (5.12)

respectively. It follows that a necessary condition for axial propagation of bulging is that there exists \( \lambda_{\theta2} \) and \( \lambda_{z2} \) (different to \( \lambda^*_\theta \) and \( \lambda^*_z \), respectively) obeying the system of Eq. (5.12). This is now a problem with two unknowns and two equations. In addition, it has to be checked that

\[ f(\dot{W}, \lambda_{\theta2}, \lambda_{z2}) > 0. \]  (5.13)

The above conditions are necessary but not sufficient in general.

Now we show that axial nucleation is related to the bifurcation criterion (2.16). Bulging bifurcation of a single very long cylinder implies that the system of Eq. (5.12) has a nontrivial solution (\( \lambda_{\theta2}, \lambda_{z2} \)) when it is linearized for \( \lambda_{\theta2} = \lambda^*_\theta + d\lambda_\theta \) and \( \lambda_{z2} = \lambda^*_z + d\lambda_z \). Thus, differentiating (5.12), it yields
Application to representative materials

\[ \lambda^* \lambda^* \left( \hat{W}_{\theta \theta} d\lambda_\theta + \hat{W}_{\theta z} d\lambda_z \right) - \hat{W}_{\theta \theta} (\lambda^* d\lambda_\theta + \lambda^* d\lambda_\theta) = 0, \]

\[ \hat{W}_{\theta \theta} d\lambda_\theta + \hat{W}_{\theta z} d\lambda_z = \frac{\hat{W}_{\theta \theta}}{\lambda^*} = d\lambda_\theta. \quad (5.14) \]

The matrix of coefficients of the differentials must have null determinant for a non-trivial solution to exist, i.e. one can write that

\[ \text{det} \begin{pmatrix} \lambda^* \lambda^* \hat{W}_{\theta \theta} - \hat{W}_{\theta \theta} \lambda^* & \lambda^* \lambda^* \hat{W}_{\theta z} - \hat{W}_{\theta z} \lambda^* \\ \hat{W}_{\theta \theta} - \hat{W}_{\theta \theta} / \lambda^* & \hat{W}_{\theta z} \end{pmatrix} = 0. \quad (5.15) \]

Expanding (5.15), the condition (2.16) is recovered. This last result can be seen as a simple consequence of our argument since it has been built by construction. Nevertheless, the important connection which is between (5.3) and (5.11) with (2.16) is not obvious. Furthermore, in a more complex situation with heterogeneous deformations in each of the cylinders, the requirement (5.13) can be relaxed in subsequent motion. Under those circumstances, cylinder 2 may not keep its cylindrical shape in certain parts of the structure. Nevertheless, motion would be associated with axial propagation of bulging as long as cylinder 2 is nucleated from cylinder 1. On the other hand, cylinder 2 may further develop a new bulge that may propagate either axially or radially.

5.3 Application to representative materials

Numerical simulations are developed in Section 5.4 for thick-walled tubes. Now, we further exploit the analytical approach developed in the latter section for cylindrical membranes made of the isotropic model (5.1) and the anisotropic model (3.9).

5.3.1 Isotropic material models: Onset and propagation of a bulge

The model (5.1) has not been analyzed previously with regard to bulging and in what follows we show for this strain energy function results
of both bifurcation and postbifurcation

5.3.1.1 Onset of bifurcation

Using the expression of $f$ given in (2.16) and a pure Demiray material model, i.e. (5.1) with $k=1$, it yields

$$
\frac{f}{\mu} = \exp\left[2\alpha \frac{\lambda^4 \lambda_z^2 + \lambda_0^2 \lambda^2 + 1 - 3\lambda_0^2 \lambda^2 (6\alpha \lambda^2 - \lambda_0^2) \lambda_z^0 \lambda_0^0}{\lambda_0^2 \lambda_z^2} \right] + \ldots,
$$

(5.16)

where ... replaces complicated expressions which are too lengthy to include here. In what follows we describe the main results. To obey $f > 0$ for any $\lambda_z > 1$ it is necessary that $\alpha \geq 1/6 \approx 0.167$. A detailed behaviour of (5.16) for $\alpha \geq 1/6$ is given in Fig. 5.2, where it is shown that no bulging instability is expected.

We further show other results for the material model (5.1). In particular, we consider the neo-Hookean material ((5.1) with $k = 0$), the Demiray material ((5.1) with $k = 1$) when $\alpha = 25$ as well as a mixed material with $k = 0.5$ and $\alpha = 0.25$. For $\lambda_z = 1.1$ values of pressure as given by (2.5) vs hoop stretch are shown in Fig. 5.3 for all these models.

![Figure 5.2: Values of $f$ for the Demiray model with $\alpha = 1/6$ as given by (5.16) vs hoop stretch for different values of the axial stretch. In all cases $f > 0$ and no instability is expected.](image-url)
Figure 5.3: Values of pressure vs hoop stretch when $\lambda_z = 1.1$ for different material models. There is no instability for the Demiray model. For the mixed material, the bulging condition $f = 0$ is obeyed when $(\lambda_\theta, \lambda_z) = (1.839, 1.1)$ and is associated with a normalized pressure $pR/\mu H = 0.942$. We note that the curve associated with this model in Fig. 5.4 has two zeros (the first one gives the onset of bifurcation). For the neo-Hookean model, $f = 0$ at the pair of values $(\lambda_\theta, \lambda_z) = (1.615, 1.1)$ and is associated with a normalized pressure $pR/\mu H = 0.799$. The curve associated with this model in Fig. 5.4 has no other zero beyond the first one (which gives the onset of bifurcation).
For each one of these materials, values of $f/\mu$ when $\lambda_z = 1.1$ are shown in Fig. 5.4 vs hoop stretch. The values of $f$ are obtained using its expression given in (2.16) particularized for each model. The different mathematical expressions of $f$ are very long to be displayed here and, furthermore, no insight is gained by writing them down. It follows that no instabilities are expected for the Demiray material model since $f > 0$. On the other hand, the bulging condition ($f = 0$) is satisfied for both the neo-Hookean material and the mixed material. It is found that $f = 0$, when $(\lambda_\theta, \lambda_z) = (1.839, 1.1)$ for the mixed material while $f = 0$ is obeyed when $(\lambda_\theta, \lambda_z) = (1.615, 1.1)$ for the neo-Hookean material. These bifurcation points are also shown, for completeness, in Fig. 5.3. Note that for the mixed material $f$ has a second zero as shown in Fig. 5.4. The first zero gives the onset of bifurcation. The second zero in some sense establishes that there are configurations beyond the first zero for which a cylindrical shape may be stable. On the other hand, the neo-Hookean curve in Fig. 5.4 has only one zero.
To show a complete picture, the evolution of pressure as given by (5.10) vs values of axial stretch $\lambda_z$ for both the neo-Hookean material and the mixed material are shown in Fig. 5.5. The curve in this figure associated with the mixed material shows that when $\lambda_z > 1.232$ bulging is not possible.

5.3.1.2 Propagation of bulge

Let us focus on the analysis of axial propagation. For a given pressure beyond the propagation pressure, the solution is obtained using (5.2) together with (5.3). Nevertheless, the purpose here is to use the (alternative) procedure developed in Section 5.2.2. One constructs all possible structures made of two cylinders at a given pressure. Among all the possible solutions, we search for the one that minimizes the energy (5.9). For simplicity and clarity, we focus on the bulged configuration that is in equilibrium at pressure $p^*$.

For a neo-Hookean material, if $\lambda_z^* = 1.1$ the system (5.12) has no solution. It follows that axial propagation is not possible, in agreement with the results given in Fu et al. [4]. Bulging motion is associated with radial expansion. On the other hand, for the mixed material under study, if $\lambda_z = \lambda_z^* = 1.1$, then the system of equations (5.12) is satisfied when $\lambda_{\theta 2} = 2.133$ and $\lambda_{z2} = 1.307$. In addition, under these conditions it can be shown that $f_2/\mu = 3.569 > 0$, i.e. cylinder 2 is stable. If two cylinders exist (as shown in Fig. 5.1), then, since $\lambda_{z2} > \lambda_z^* = 1.1$ it follows that $\lambda_{z2} < \lambda_z^*$. We note that in Fig. 5.5, the pressure that would be associated with these possible values of $\lambda_z^*$ is greater than the bulging pressure associated with $\lambda_z^* = 1.1$.

We obtain pairs of contiguous cylinders using Maple. Specifically, for each value of $\lambda_{\theta 1}$ we solve numerically the system of three equations defined by (5.2) and (5.11). The unknowns are now $\lambda_{\theta 1}$, $\lambda_{\theta 2}$ and $\lambda_{z2}$. This is accomplished invoking the $f$ solve command in Maple whose main ingredient is the Newton’s algorithm. Pairs of contiguous cylinders that are solution of Eqs. (5.2) and (5.11) at pressure $p^*$ are shown in Fig. 5.6. The two cylinders associated with one solution are drawn using the same symbol (a circle, a triangle, etc). The circle represents the solution given by (5.12).
Among all possible solutions the structure tends to adopt the configuration that minimizes the total energy $\bar{E}/\mu$ as given by (5.9) for each model. The expression of $\bar{E}/\mu$ is too long to display here and adds no additional meaning to the analysis. The value of the total energy has been obtained for some configurations, i.e. for some pairs of cylinders and these values are shown in Fig. 5.7. A thorough energy analysis has been carried out and it has been found that the minimum value of the energy is $\bar{E}/\mu = -1.56832$. In this particular configuration, the stretches of cylinder 1 are $(\lambda_{\theta_1}, \lambda_{z_1}) = (1.769, 1.054)$ while the stretches of cylinder 2 are $(\lambda_{\theta_2}, \lambda_{z_2}) = (2.107, 1.285)$, which is the solution obtained using (17) together with (5.3). Now, using (5.5) and the values obtained for the two uniform cylinders, it follows that the length of cylinder 1 in this (bulged) configuration is 80% of the total initial length (the length in the reference configuration).

![Figure 5.5: Values of pressure vs axial stretch at the onset of bifurcation for different models.](image)

For the mixed material model, the curve shows that if $\lambda_z > 1.232$ bulging is not possible. On the other hand, for the neo-Hookean material there always exists a value of pressure associated with the onset of instability for any value of the axial stretch.
Figure 5.6: The curve in this figure gives all possible pairs of values \((\lambda_\theta, \lambda_z)\) associated with the pressure \(p^*\) which, in turn, corresponds to \(\lambda_z^* = 1.1\) and \(\lambda_\theta^* = 1.839\) for the mixed material. In the figure one solution is formed by two cylinders that are labeled using the same symbol (a triangle, a circle, etc). The circle corresponds to the solution of the system of Eq. (5.12). In this solution, the values of stretch of cylinder 1 are \(\lambda_{z1} = \lambda_z^* = 1.1\) and \(\lambda_{\theta1} = \lambda_\theta^* = 1.839\) while the values of stretch of cylinder 2 are \(\lambda_{z2} = 1.307\) and \(\lambda_{\theta2} = 2.133\). The other solutions (pairs of cylinders) obey the system of Eqs. (5.3) and (5.11).

Figure 5.7: Magnification of Fig. 5.7 near \((\lambda_\theta, \lambda_z) = (2, 1.2)\). The two cylinders corresponding to one solution are drawn with the same symbol (a circle, a triangle, etc). Near the symbol used to mark each solution, the numerical value of the normalized energy associated with that solution is given. The configuration with minimum energy is marked with white squares.
5.3.2 Anisotropic material model: propagation of bulge

Now we consider the strain energy function (3.9) which is used to model the cylindrical wall. It has been shown in Rodriguez and Merodio [7] that \( k_2 > 1/(288 \cos^4 \varphi) \) is sufficient to avoid bulging bifurcation when the isotropic part is negligible. We take \( \varphi = 45^\circ, k = 0, k_2 = 0.1 \) and \( k_1 = 0.05 \mu \). Under these circumstances, for an axial stretch \( \lambda_z = \lambda_z^* = 1.1 \) the onset of bulging instability (i.e. \( f = 0 \) using (2.16)) occurs when \( \lambda_z^* = 1.858 \) and it is associated with a normalized internal pressure \( p^*R/\mu H = 0.991 \). The analysis of axial propagation of bulging follows closely the one given in the previous section for the mixed isotropic model. The system (5.12) gives the solution \( \lambda_{\theta_2} = 1.984 \) and \( \lambda_{z_2} = 1.187 \), for which one can obtain using the expression of \( f(\tilde{W}, \lambda_{\theta_2}, \lambda_{z_2}) \) in (2.16) that \( f_2 \geq 0.713 > 0 \). Hence, bulging instability is expected for (3.9). The analysis is qualitatively similar to the one given for the mixed material model and we do not repeat the procedure.

5.4 Finite element simulations

We have modeled several cylinders using different wall thicknesses as well as the material models introduced previously. The major objectives of this analysis are twofold. First, to capture the analytical procedure introduced in Section 5.2. Second, to study the sensitivity of the analytical procedure with respect to thick-walled cylinders. The computational analysis is performed using the general purpose finite element program Abaqus.

For a complete description of the computational methodology developed we refer to [24]. In [24], attention was paid to the onset of bifurcation.

Nevertheless, and in passing, it was obtained that bulging propagates axially for the material model (3.9). Here we establish these results according to the analytical work described in Section 5.2 and further developed in Section 5.3.

We consider a radius \( R = 5 \) mm and a length \( L = 100R \) which is sufficiently long to accept infinite length conditions for the initial cylindrical shell. For a particular material model, two different tubes are
analyzed since they have a different wall thickness, namely, $H/R = 0.05$ and $H/R = 0.10$. In the Figures that follow, the solutions obtained by means of numerical simulations are labeled using the term FEM (see for instance Figs. 5.9, 5.10, etc). For comparison, these solutions are plotted together with the analytical one that is obtained using the theoretical thin-walled cylinder assumption. For clarity, the analytical solution is shown just up to the onset of bifurcation since the focus is to estimate the validity of the numerical approach qualitatively. We are using these models as prototypes and it makes not much sense to look for high accuracy, which would demand a high effort in the modeling and simulation. The analytical solution is labeled using the terms analytical thin wall (see for instance Figs. 5.9, 5.10, etc).

The numerical procedure developed in [24] is just summarized. Mixed (hybrid) pressure–displacement elements are used to impose incompressibility. Other details depend on the material model at hand and are specified later on for the different models. First we stretch the cylinder by imposing relative axial displacements. Hoop displacements (in a nodal local cylindrical system) are prevented during the whole simulation. Then we apply internal pressure in a modified Riks analysis (arc length procedure, Crisfield, [17], [29]). To avoid a strict numerical bifurcation, we introduced a small imperfection in the cylindrical shell with a maximum deviation of $0.002R$ with respect to the perfect circular cylinder (see [24]).

One can follow the evolution (with deformation) of the hoop stretch of two points which are located in the middle of the thickness of the cylinder at two different cross sections (at two different altitudes). One point is at one end of the cylinder, point A, and the other point is at the other end of the cylinder, point B. The two points are shown in the cylinders of Fig. 5.10. The figure shows two cylinders, each one with a bulge. Point A is located at a cross section of the cylinder where a bulge develops while point B is at the top of the cylinder where a bulge is not going to appear. As loading is applied from the undeformed configuration, the onset of bifurcation is identified when the hoop stretches of these two points differ. In particular, we take the bifurcation point when the deviation of the hoop stretches of these two points is a small fraction of the hoop stretch.
average of both points (for instance, 1%). In what follows, if the onset of bifurcation is reached point A clearly will increase its hoop stretch since it is in the bulge.

5.4.1 Isotropic material models

Abaqus material model library does not include the mixed material model (5.1). We implemented the material in a subroutine in which the strain energy function as well as its derivatives with respect to the invariants of the right Cauchy–Green tensor were specified. In the initial configuration, the modeling space is axisymmetric with five elements in the thickness direction. In the axial direction square elements are used. An adequate number of elements needs to be used to avoid problems associated with elements that are not well-shaped, among others (see [24]).

For the pure Demiray material with $\alpha = 0.25$ no bulging instability appears during the simulation as it is expected. Values of pressure of inflation vs values of the hoop stretch of points A and B are shown in Fig. 5.8 when $\lambda_z = 1.1$. Results for the different thicknesses studied are shown together with the results obtained assuming a theoretical thin-walled cylinder (analytical results). The figure shows one line for the solution associated with the thin-walled cylinder since there is no bifurcation. This indicates that both points have the same hoop stretch. For the other two cases, the numerical simulations, as loading is applied from the undeformed configuration (in the figure it corresponds with values in each curve for which $\lambda_\theta$ is close to 1), the figure shows two lines: one line is associated with the hoop stretch of point A while the other line is associated with the hoop stretch of point B. Both lines are very close. This means that both points have a non-distinguishing hoop stretch difference and that there is no bifurcation.

Let us focus now on the neo-Hookean model. Values of pressure of inflation vs values of the hoop stretch of points A and B are shown in Fig. 5.9 when $\lambda_z = 1.1$. Analytical results (solid line) are just shown up to the onset of bifurcation to validate the results obtained numerically for the cylinders with different thicknesses (dashed lines). We describe the mechanism in the following way. As loading is applied from the undeformed
configuration (in the figure it corresponds with values in each curve for which $\lambda_\theta$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, Fig. 5.9 shows for each case two lines that are very close. When bifurcation occurs (close to maximum points in each curve), in both cases, the structure is not able to sustain higher values of pressure. Furthermore, as pressure decreases, point B reverses the original path decreasing its hoop stretch while point A increases its hoop stretch (i.e. in each cylinder the hoop stretch of point A is given by the dashed line at the right of the bifurcation point). A deformed configuration of the tube is given by the left cylinder of Fig. 5.10. This is in agreement with our analytical results of Section 5.2. It follows that bulging motion is associated with radial expansion of the bulge.

On the other hand, for the mixed material model (5.1) with $k = 0.5$ and $\alpha = 0.25$ the structure shows configurations as the one given in Fig. 5.1 beyond the onset of bifurcation. Results are shown in Fig. 5.11. The two cylinders (the one with $H/R = 0.05$ and the one with $H/R = 0.10$) show qualitatively the same behaviour. For each cylinder there are two dashed lines: one gives values of pressure vs hoop stretch of point A and the other gives values of pressure vs hoop stretch of point B. Bifurcation is identified (for each cylinder) when these two (dashed) lines start to diverge. Just beyond the onset of bifurcation, the hoop stretch of point A increases (it is given by the dashed line at the right of the bifurcation point) while the hoop stretch of point B decreases (it is given by the dashed line at the left of the bifurcation point). The figure shows that the pressure, in subsequent bulging motion just after the onset of bifurcation, maintains a (relatively constant) plateau for a range of values of $\lambda_\theta$ associated with point A. This is clearly appreciated for each cylinder in the features of the dashed line at the right of the bifurcation point, i.e. in the dashed line that gives values of pressure vs hoop stretch of point A (see Fig. 5.10). These results capture the analytical description of bulging motion given in Section 5.2. It also follows that pressure during postbifurcation can be increased. The structure can support values of pressure greater than the pressure associated with the onset of bulging as it is shown in Fig. 5.11. In particular, we conclude
that axial propagation of bulging involves two periods: firstly, pressure remains essentially fixed during the ensuing propagation of the bulging instability beyond the onset of bifurcation until a suitable configuration is obtained; secondly, in subsequent motion, for further axial propagation of bulging pressure of inflation must be increased.

Figure 5.8: For $\lambda_z = 1.1$, the figure gives values of pressure of inflation vs values of hoop stretch for two points located at different altitudes of (i) a cylinder with $H/R = 0.05$ (dashed thick line) (ii) a cylinder with $H/R = 0.10$ (dashed thin line) and (iii) a thin-walled cylinder (solid line). All the cylinders are made of a Demiray material with $\alpha = 0.25$. There is only one line associated with the thin-walled cylinder (analytical results) because points A and B have the same hoop stretch. Whence, there is no bifurcation. The values associated with the other two cylinders have been obtained using numerical simulations. In each case, the figure shows two lines: one line is associated with the hoop stretch of point A while the other line is associated with the hoop stretch of point B. Both lines are very close. This means that both points have a non-distinguishing hoop stretch difference and that there is no bifurcation.
Figure 5.9: For $\lambda_z = 1.1$, the figure gives values of pressure of inflation vs values of hoop stretch for two points located at different altitudes of (i) a cylinder with $H = R = 0.05$ (dashed thick line) (ii) a cylinder with $H/R = 0.10$ (dashed thin line) and (iii) a thin-walled cylinder (solid line). All the cylinders are made of a neo-Hookean material. Analytical results (solid line) are just shown up to the onset of bifurcation to validate the results obtained numerically for the cylinders with different thicknesses (dashed lines). As loading is applied from the undeformed configuration (in the figure it corresponds with values in each curve for which $\lambda_\theta$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows in each numerical simulation (each cylinder) two lines that are very close. When bifurcation occurs (close to maximum points in each curve), the structure is not able to support higher values of pressure. Furthermore, as pressure decreases, point B reverses the original path decreasing its hoop stretch while point A increases its hoop stretch. It follows that, in each case, the hoop stretch of point A is given by the dashed line which is at the right of the bifurcation point.
A Competition between radial expansion and axial propagation in bulging of inflated cylinders

Figure 5.10: Details of the deformed meshes around the bulge in the cylinder made of a neo-Hookean material (left) and around the (half) bulge in the cylinder made of a mixed material (right). Bulging motion in the cylinder on the left is associated with radial expansion while bulging motion in the cylinder on the right is associated with axial propagation. Points A and B are specified in both cases. The bulge appears at the cross sections for which hoop stretch increases (where point A is located).
Finite element simulations

Figure 5.11: For $\lambda_z = 1.1$, the plot gives values of pressure vs hoop stretch for two points (A and B) located at different altitudes of (i) a cylinder with $H/R = 0.05$ (dashed thick line), (ii) a cylinder with $H/R = 0.10$ (dashed thin line) and (iii) a thin-walled cylinder (solid line). All cylinders are made of the mixed material 5.1 with $k = 0.5$ and $\alpha = 0.25$. Analytical results (solid line) are just shown up to the onset of bifurcation to validate the results obtained numerically for the other two cylinders. For each cylinder there are two dashed lines. Each one of them gives values of pressure vs hoop stretch for one point (either point A or point B). The onset of bifurcation is identified when the hoop stretch of these two points differ. Under these circumstances, the two (dashed) lines start to diverge. Prior to bifurcation, both points have a non-distinguishing hoop stretch difference and the two (dashed) lines are very close. After the onset of bifurcation, for each case, the dashed line to the right gives the hoop stretch of point A (where the bulge appears) while the dashed line to the left gives the hoop stretch of point B. Axial propagation of bulging involves two periods: firstly, pressure remains essentially fixed during the ensuing propagation of the bulging instability mode beyond the onset of bifurcation until a suitable configuration is obtained; secondly, in subsequent motion, for further axial propagation of bulging pressure of inflation must be increased. This is clearly shown in the evolution of the hoop stretch of point A beyond the onset of bifurcation.

5.4.2 Anisotropic material model

It is not possible in Abaqus to define helical fibers in an axisymmetric model. Thus, a three-dimensional sector of length $100R$ is defined with only one element in the hoop direction. As in the isotropic
cases, a small imperfection of maximum deviation 0.002R is introduced in the geometry. Five elements are generated in the thickness direction and a sufficient number in the longitudinal direction for a good-shaped mesh. Longitudinal displacements at the end faces are imposed and then internal pressure is applied in a modified Riks analysis.

Figure 5.12: For the strain energy function (3.9), the figure shows parallel curves to the ones given in Fig. 5.11 for the mixed material model.

As expected, and similarly to bulging for mixed isotropic materials, a bulge appears and the structure adopts configurations that allow long cylinders to be formed with increasing values of pressure. Results are shown in Fig. 5.12. These results are qualitatively identical to the ones shown in Fig. 5.11 for the mixed material model.
6.1 Introduction

Strain-softening is a decline of stress at increasing strain or, more generally, a situation where the matrix of tangential elastic moduli ceases to be positive-definite. This phenomenon has been well documented for concrete, rocks and some soils, and it probably also exists in many other materials, including sea ice, filled elastomers, wood, particle boards, paper, some tissues and fabrics, fiber-reinforced composites, fiber-reinforced concretes, asphalt concretes, polymer concretes, various refractory concretes and ceramics, and also some metals.

Strain-softening hampers finite element applications as calculations are repeatedly reported to be unstable and the analyst often finds it difficult to obtain a truly converged solution. Another major difficulty in applying finite element models to strain-softening materials is the fact that the results are not objective with regard to mesh refinement, i.e. the results do not converge to the true solution when the finite element mesh is refined [30], [31]. To remedy this disease, it has been proposed to make the softening modulus a function of the element size. In doing so, objective results can again be obtained [31], [32], [33], [34].

Starting from a different point of view, Crisfield [35] arrived at similar conditions. He also demonstrated using the simple example of a bar loaded in pure tension that solutions in the strain-softening regime are non-unique, and moreover, that some equilibrium branches emanating
from the bifurcation point lead to snap-back behaviour. These observations have severe consequences for analyzing concrete structures, either reinforced or not. It implies for instance that truly converged solutions may not always be obtained by customary displacement control, and that we often have to resort to more sophisticated techniques like arc-length control in order to obtain a proper solution [33], [36], [58], [59].

Zdenek P. and others [37], attempt to model softening of beams and frames using material behaviour assumptions that are consistent with those used in a number of other problems. Strain softening associated with a certain characteristic length of the material has been used successfully to describe the fracture test data for concrete.

In this work we will outline a numerical approach to detect bifurcation points and to trace post-bifurcation and post-failure paths in strain softening solids by using finite element program Abquse. The procedures are subsequently applied to a numerical bifurcation analysis to two limit problems involving snap-back behaviour, whereby it is noted that in the latter case the snap-behaviour is believed to be physical and does not have a numerical cause as was probably the case in some previous analyses [38], [39].

### 6.2 Strain softening and snap-back behaviour

We consider an unreinforced bar subject to pure tension [35] and the material of the bar is modeled as elastic-softening with an ultimate strain $\epsilon_u$ at which the tensile strength has vanished completely (see Fig. 6.1). $\epsilon_u$ is assumed to be equal to $n$ times the strain at the tensile strength $\epsilon_e$. The bar is modeled with $m$ elements (see Fig. 6.2). If we have a perfect bar, so that all elements have exactly the same tensile strength and so on, the bar deforms uniformly throughout the loading process and the load-deflection curve is simply a copy of the imposed stress-strain law.
Strain softening and snap-back behaviour

Figure 6.1: Stress-strain law for concrete in tension. The stress is plotted against the total strain, i.e. the sum of crack strain and concrete strain.[40]

Figure 6.2: Model of bar composed of strain-softening material.[40]

However, if one element has a slight imperfection, only this element will show loading while the other elements will show unloading. In this situation, the imposed stress-strain law at local level is not reproduced. Instead, an average strain is calculated in the post-peak regime which is smaller than the strain of the stress-strain law. This may be explained as follows. The element which shows loading will follow the path A-B in Fig. 6.1, while the other elements will follow the path A-C. This implies that when all elements have the same dimensions, we have for the average strain increment $\Delta \bar{\varepsilon}$...
\[ \Delta \varepsilon = \frac{1}{m} \left[ -\frac{m-1}{E} \Delta \sigma + \frac{\Delta \sigma}{E/n} \right] = \left[ \frac{n}{m} - 1 \right] \frac{\Delta \sigma}{E}. \quad (6.1) \]

Consequently, when we increase the number of elements while keeping the length of the bar fixed, the average strain in the post-peak regime gradually becomes smaller and for \( m > n \) the average strain in the post peak regime even becomes smaller than the strain at peak load (Fig. 6.3). This implies that for \( m > n \), the load-deflection curve shows a snap-back [35]. It is noted that similar results have also been derived for beams composed of strain-softening material [41].

The above results imply that computational results for materials with a local softening constitutive law are not objective upon mesh refinement. To remedy this disease, it has been proposed to make the softening modulus dependent on the element size [31], [42]. Numerical experiments have confirmed that numerical results are then objective with regard to mesh refinement [31], [32], [33], [34]. The problem of making the softening modulus dependent on the element size is that, for an arbitrary concrete structure, the spread of the softening region is not known in advance. Consequently, the observation that use of a local softening law may involve snap-back behaviour on structural level may hold even when we use a model in which the softening modulus has been adapted to some structural size.
In spite of its apparent simplicity the example of a bar loaded in tension is a challenging problem for a numerical simulation. To achieve this goal, current numerical techniques must either be modified or be employed very judiciously. First, we must have a criterion to detect a bifurcation point in a discrete mechanical model. In the present case, the bifurcation point is obvious as it simply coincides with the limit point, but this seldom happens. In general, bifurcation will be possible when the lowest eigenvalue vanishes [43]. In numerical applications, the lowest eigenvalue will never become exactly zero owing to round-off errors. Therefore, it has been assumed that bifurcation is possible when the lowest eigenvalue becomes slightly negative. It is noted in passing that the above statement holds rigorously for mechanical systems with a symmetric stiffness matrix, but that for systems with a non-symmetric stiffness matrix the situation is less clear-cut [44].

Having determined the bifurcation point by an eigenvalue analysis
of the tangent stiffness matrix, continuation on the localization branch instead of on the fundamental branch can be forced by adding a part of the eigenmode $v_1$, which belongs to the vanishing eigenvalue, to the incremental displacement field of the fundamental path $\Delta a^*$ \cite{44}, \cite{61}:

$$\Delta a = \alpha(\Delta a^* + \beta v_1),$$ (6.2)

with $\alpha$ and $\beta$ scalars. The magnitude of these scalars is fixed by second-order terms or by switch conditions for elastoplasticity or for plastic-fracturing materials. The most simple way to determine $\beta$ numerically is to construct a trial displacement increment $\Delta a$ such that it is orthogonal to the fundamental path:

$$\Delta a^T \Delta a^* = 0,$$ (6.3)

where the symbol T is employed to denote a transpose. Substituting equation 6.2 in this expression yields, for $\beta$,

$$\beta = -\frac{(\Delta a^*)^T \Delta a^*}{(\Delta a^*)^T v_1},$$ (6.4)

so that we obtain for $\Delta a$

$$\Delta a = \{\Delta a^* - \frac{(\Delta a^*)^T \Delta a^*}{(\Delta a^*)^T v_1}\}. \quad (6.5)$$

Equation 6.5 fails when $(\Delta a^*)^T v_1 = 0$, i.e. when the bifurcation mode is orthogonal to the basic path. A simple remedy is to normalize $\Delta a$ such that

$$(\Delta a^*)^T \Delta a^* = \Delta a^T \Delta a.$$ (6.6)

This results in:

$$\Delta a = \frac{1}{\sqrt{(\Delta a^*)^T \Delta a^* - [(\Delta a^*)^T v_1]^2}} \{((\Delta a^*)^T v_1) \Delta a^* - (\Delta a^*)^T \Delta a^* v_1\}. \quad (6.7)$$

The denominator of this expression never vanishes, since this would imply that the eigenmode is identical with the fundamental path.

In general, the bifurcation path will not be orthogonal to the
fundamental path, but when we add equilibrium iterations, the orthogonality condition 6.3 will maximize the possibility that we converge on a bifurcation branch and not on the fundamental path, although this is not necessarily the lowest bifurcation path when more equilibrium branches emanate from the bifurcation point. When we do not converge on the lowest bifurcation path, this will be revealed by negative eigenvalues of the bifurcated solution. The above described procedure can then be repeated until we ultimately arrive at the lowest bifurcation path.

The procedure described above is well suited for assessing post-bifurcation behaviour. Bifurcations, however, are rather rare in normal structures owing to imperfections, and even if a bifurcation point exists in a structure, numerical round-off errors and spatial discretization usually transfer the bifurcation point into a limit point unless we have a homogeneous stress field. This observation does not render the approach to bifurcation problems worthless as it provides a thorough insight which is of importance for the associated limit problems, but it is obvious that numerical procedures must also be capable of locating limit points and tracing post-limit behaviour.

To control the incremental-iterative solution procedure, we have analogous to experiments load control and (direct) displacement control. However, either of these procedures may fail in particular circumstances. With load control, we are not able to overcome limit points at all, and with direct displacement control it is not possible to properly analyze snap-back behaviour (see e.g. Fig. 6.3). Fortunately, a very general and powerful method has been developed within the realm of geometrically non-linear analysis. In this method, the incremental-iterative process is controlled indirectly using a norm of incremental displacements [36], [58], [59], [61]. For this reason, the name arc-length method has been coined for the procedure. For materially non-linear analysis, a global norm on incremental displacements is often less successful due to localization effects and it may be more efficient to employ only one dominant degree of freedom or to omit some degrees of freedom from the norm of incremental displacements. The name arc-length control then no longer seems very appropriate. Instead we will use the term indirect displacement control.
In a nonlinear finite element analysis, the load is applied in a number of small increments (e.g. [46]). Within each load increment, equilibrium iterations are applied and the iterative improvement $\delta a_i$, in iteration number $i$ to the displacement increment $\Delta a_{i-1}$ is given by

$$\delta a_i = K_{i-1}^i [p_{i-1} + \Delta \mu_i q^*], \quad (6.8)$$

where $K_{i-1}^i$ is the possibly updated stiffness matrix, $q^*$ is a normalized load vector, $\Delta \mu_i$ is the value of the load increment which may change from iteration to iteration and $p_{i-1}$, is defined by

$$p_{i-1} = \mu_0 q^* - \int_v B^T \sigma_{i-1}^1 dV. \quad (6.9)$$

In equation 6.9 the symbols $\mu_0$, $B$ and $\sigma_{i-1}^1$ have been introduced for, respectively, the value of the scalar load parameter at the beginning of the current increment, the strain-nodal displacement matrix and the stress vector at iteration number $i - 1$.

The essence of controlling the iterative solution procedure indirectly by displacements is that $\delta a_i$ is conceived to be composed of two contributions:

$$\delta a_i = \delta a_i^I + \Delta \mu_i \delta a_i^{II} \quad (6.10)$$

with

$$\delta a_i^I = K_{i-1}^{-1} p_{i-1} \quad (6.11)$$

and

$$\delta a_i^{II} = K_{i-1}^{-1} q^*. \quad (6.12)$$

After calculating the displacement vectors $\delta a_i^I$ and $\delta a_i^{II}$, the value for $\Delta a_i$ is determined from some constraint equation on the displacement increments and $\Delta a_i$ is subsequently calculated from

$$\Delta a_i = \Delta a_{i-1} + \delta a_i. \quad (6.13)$$

Crisfield [58], for instance, uses the norm of the incremental displacements as constraint equation.
\[ \Delta a_i^T \Delta a_i = \Delta l^2, \] (6.14)

where \( \Delta l \) is the arc-length of the equilibrium path in the n-dimensional displacement space. The drawback of this so-called spherical arc-length method is that it yields a quadratic equation for the load increment. To circumvent this problem, one may linearize equation 6.14, yielding [59]

\[ \Delta a_i^T \Delta a_{i-1} = \Delta l^2. \] (6.15)

This method, known as the updated normal path method, results in a linear equation for the load increment. Equation 6.15 may be simplified by subtracting the constraint equation of the previous iteration. This gives

\[ \Delta a_i^T (\Delta a_i - \Delta a_{i-2}) = 0. \] (6.16)

When we furthermore make the approximation

\[ \delta a_i \approx 2(\Delta a_i - \Delta a_{i-2}), \] (6.17)

we obtain

\[ \Delta a_{i-1}^T \delta a_i = 0. \] (6.18)

Substituting equation 6.10 then gives for \( \Delta u_i \):

\[ \Delta \mu_i = -\frac{\Delta a_{i-1}^T \delta a_i^I}{\Delta a_{i-1}^T \delta a_i^T}. \] (6.19)

Both equations 6.14 and 6.15 have been employed very successfully within the realm of geometrically non-linear problems, where snapping and buckling of thin shells can be traced very elegantly. Nevertheless, for physically non-linear problems the method sometimes fails, which may be explained by considering that for physically non-linear problems, failure or bifurcation modes are often highly localized. Hence, only a few nodes contribute to the norm of displacement increments, and failure is not sensed accurately by such a global norm. As straightforward application of equations 6.14 or 6.15 is not always successful, we may amend these constraint equations by applying weights to the different degrees of
freedom or omitting some of them from the constraint equation. The constraint equation 6.15 then changes into

$$\Delta u_i^T \Delta u_{i-1} = \Delta l^2,$$  \hspace{1cm} (6.20)

where \( \Delta u_i \) contains only a limited number of the degrees of freedom of those of \( \Delta a_i \), and equation 6.19 changes in a similar manner. The disadvantage of modifying the constraint equation is that the constraint equation becomes problem dependent. As a consequence, the method loses some of its generality and elegance.

### 6.4 Numerical examples

#### 6.4.1 Single notch unreinforced concrete beam

The example which we consider, is an unreinforced notched beam which has been analyzed by R. de Borst [40]. The geometry and loading arrangement of the notched beam is shown in Fig. 6.4, where the finite element mesh is shown in Fig. 6.5, the beam has been analyzed using a four-noded bilinear plane stress quadrilateral elements (CPS4). The concrete has been modeled as linearly elastic in compression with a Young’s modulus \( E_c = 24800 \text{ N/mm}^2 \) and a Poisson’s ratio \( \nu = 0.18 \). This approach is justified in this case, because the compressive stresses remain low enough to avoid yielding in compression. In tension, the crack model as developed by R. de Borst [47], [48], de Borst and Nauta [49] and Rots et al. [33] has been employed. The crack parameters have been taken as: tensile strength \( f_c = 2.8 \text{ N/mm}^2 \) and fracture energy \( G_f = 0.055 \text{ N/mm} \). The width of the crack band was assumed to be \( h = 10.167 \text{ mm} \), and linear as well as nonlinear softening relations [50] have been employed.
Modified Riks method is used to analyze the notch beam by using numerical program (Abaqus), where Fig. 6.6 shows the relation between the arc length and the load proportional factor (LPF). We note from this relation the maximum point of load as an indicator of an instability. Figs 6.7 and 6.8 show the load - deflection at points A and B respectively. Fig. 6.9 gives the incremental displacement at ultimate load and Fig. 6.10 shows the crack pattern at ultimate load. The crack pattern at ultimate load reveals that the crack arising from the notch has developed fully.
Figure 6.6: plot of LPF vs. arc length showing a maximum

Figure 6.7: Load - deflection curve of point A
Figure 6.8: Load - deflection curve of point B

Figure 6.9: Incremental displacement at ultimate load
6.4.2 Single notch reinforced concrete beam

It is a widespread belief that reinforcement stabilizes the numerical process. However, this is not generally true, as addition of reinforcement not only gives rise to stiffness differences in the structure, thus leading a deterioration of the condition of the stiffness matrix, but also adds to the possibility of the occurrence of spurious alternative equilibrium states and of snap-back behaviour [38], [39]. We will demonstrate this by means of perhaps the most simple reinforced structure, namely an axisymmetric specimen with an axial reinforcing bar.

Specifically, we will consider the tension-pull specimen which is shown in Fig. 6.11 [40] for axisymmetry only used one quarter in the analysis. The reinforcing bar is given by the line AB and a linear bond-slip law is assumed between the concrete and the reinforcement, i.e. the relation between the slip and the shear stress between concrete and steel has been assumed to be linear. In fact, the element which is employed for the reinforcement is a combined steel-bond slip element [51]. The concrete has been modeled as linearly elastic in compression just as in the preceding example with a Young’s modulus $E_c = 25000 \text{ N/mm}^2$ and a Poisson’s ratio $\nu = 0.2$. Also in this case the approach is justified because of the relatively low compressive stresses. The tensile strength has been assumed as $f_{ct} = 2.1 \text{ N/mm}^2$ and a non-linear softening curve has been employed after crack formation with a fracture energy $G_f = 0.06 \text{ N/mm}^2$. The shear retention factor $\beta$ was taken equal to 0.1. The reinforcing bar was assigned
a Young’s modulus $E_s = 177000 \, N/mm^2$ and a yield strength $\sigma_y = 210 \, N/mm^2$.

The beam has been analyzed by R. de Borst [40], the uniform displacement is applied at the end of the bar as shown in Fig. 6.11. The mesh element shown in Fig. 6.12 which has been used four node bilinear axisymmetric quadrilateral reduced integration hourglass control (CAX4R). The beam has been analyzed using Riks method where the load-displacement relationship at point A is given in Fig. 6.13.

Figure 6.11: Tension pull specimen[40]

Figure 6.12: Finite element mesh of specimen
Figure 6.13: Load vs displacement at point A
1. To determine the onset of bifurcation instabilities is generally a challenge in soft structures. We have provided here a methodology to study bulging of stretched and inflated cylinders. Needless to say that parallel analysis can be carried out to capture other surface instabilities. The methodology is based on the modified Riks finite element analysis.

2. Bifurcation and post bifurcation results show a very different qualitative behaviour for the reinforcing models. Radial propagation of bulging has been captured for both models. Furthermore, in general, localized bulging occurs first. Nevertheless, necking solutions have been found for the material with $I_5$ and $I_7$ during inflation. It has been shown that necking propagates both axially and radially. Furthermore, axial stretch at the necking zone is smaller than outside the necking region. In addition, necking motion is related to a decrease of pressure beyond the onset of necking. These features are identified with the necking solutions found during deflation from a highly inflated tube.

3. Axial propagation of bulging instabilities have been reported in axially stretched and pressurized cylinders made of isotropic and non-isotropic materials. Axial propagation of bulging is needed to support (statically) internal pressures greater than the one at bifurcation. Here we have studied necessary conditions for axial propagation of bulging. Numerical simulation with finite element
models have also been conducted to capture the work described. Axial propagation of bulging involves two periods: firstly, pressure remains essentially fixed during the ensuing propagation of the bulging instability mode beyond the onset of bifurcation until a suitable configuration is obtained; secondly, in subsequent motion, for further axial propagation of bulging pressure of inflation must be increased.

4. On the other hand, as it has been captured for the neo-Hookean model, if conditions for axial propagation are not obeyed then the structure does not withstand values of pressure greater than the one corresponding to the onset of bifurcation. It follows that bulging motion is associated with radial expansion of bulging.

5. Use of strain-softening models may lead to snap-back behaviour on a structural level. This observation, which has probably been made first by Crisfield [35], implies that structures composed of strain-softening material cannot always be analyzed using standard displacement control.
A Arc length methods

An incremental iterative Newton Raphson based solution scheme may not appropriate to obtain a solution for unstable problems. Therefore, arc length procedures are typical applied and turn out to be advantages and robust enough to obtain elastic equilibrium states during unstable stages of the response of a force-driven deformation problem and can conveniently enable post-peak branches of load-displacement curves. The method was originally proposed by Riks [56] and Wempner [57] and, as modified by Crisfield [58] and Ramm [59] can be easily introduced into a standard finite element computer program.

In this appendix, we summarise some basics of arc length controlled, moreover, we related and specify the equations with regard to the arc-length algorithm implemented in Abaqus, emphasise issues of implementation in UEL-subroutines and comment on specific control parameters to be specified by the user [52].

For general survey of arc-length methods, we refer to the monographs by crisfield [53], [54] and references cited therein.

A.1 Basic equations

According to general arc length schemes, the loading parameter $\lambda_{n+1}$ at time $t_{n+1}$, introduced to incrementally increase the load magnitude, is allowed to vary during the iteration process. The particular value of the loading parameter is governed by a non-linear scalar constrain equation of the form $f(\phi_{n+1}, \lambda_{n+1}) = 0$ in terms of the current displacement $\phi_{n+1}$. For the sake of clarity, we restrict the following description to uncoupled purely mechanical problems; the extension to coupled problems is straightforward. Consequently, the residual format of the non-linear balance of momentum enhanced by the constraint equation reads

$$r^\phi(\phi, \lambda) = f_{int}^\phi(\phi) - \lambda f_{ext}^\phi = 0, \quad (A.1)$$

$$f(\phi, \lambda) = 0, \quad (A.2)$$
where here and in the following, we often omit the subscript index \( n + 1 \) associated with time \( t_{n+1} \) as well as the dependencies on \( \varphi \) and \( \lambda \) for the sake of readability. A Taylor series expansion around the solution at the current iteration step \( l \)-terms of quadratic and higher order being neglected gives

\[
\begin{align*}
    r_{l+1}^\varphi &= r_l^\varphi + \Delta r_l^\varphi = 0, \\
    f_{l+1} &= f_l + \Delta f = 0,
\end{align*}
\]  

(A.3) 

(A.4) 

with the increments of the residuals

\[
\begin{align*}
    \Delta r_l^\varphi &= \frac{d r_l^\varphi}{d \varphi} \Delta \varphi + \frac{d r_l^\varphi}{d \lambda} \Delta \lambda, \\
    \Delta f_l &= \frac{d f_l}{d \varphi} \Delta \varphi + \frac{d f_l}{d \lambda} \Delta \lambda.
\end{align*}
\]  

(A.5) 

(A.6) 

Herein, the increments \( \Delta \varphi = \varphi_{l+1} - \varphi_l \) and \( \Delta \lambda = \lambda_{l+1} - \lambda_l \) represent the difference between the values at iteration step \( l+1 \) and \( l \), while the derivatives are abbreviated by

\[
\begin{align*}
    K^{\varphi \varphi} &:= \frac{d r_l^\varphi}{d \varphi}, \\
    r_{\varphi \lambda} &:= \frac{d r_l^\varphi}{d \lambda} = -f_{ext}^\varphi, \\
    f_{\varphi} &:= \frac{d f_l}{d \varphi}, \\
    f_{\lambda} &:= \frac{d f_l}{d \lambda}.
\end{align*}
\]  

(A.7) 

(A.8) 

This results in the following linearised system of equations on element level

\[
\begin{bmatrix}
    K^{\varphi \varphi} & -f_{ext}^\varphi \\
    f_{\varphi}^l & f_{\lambda}^l
\end{bmatrix}
\begin{bmatrix}
    \Delta \varphi \\
    \Delta \lambda
\end{bmatrix}
= \begin{bmatrix}
    \lambda f_{ext}^\varphi - f_{int}^\varphi \\
    -f
\end{bmatrix},
\]  

(A.9) 

which is neither symmetric nor banded. According to Batoz and Dhatt [55], the solution of this system of equations can be obtained by means of so called block solutions

\[
\begin{align*}
    \Delta \varphi_r &= K^{\varphi \varphi}^{-1} \lambda f_{ext}^\varphi - f_{int}^\varphi, \\
    \Delta \varphi_\lambda &= K^{\varphi \varphi}^{-1} f_{ext}^\varphi,
\end{align*}
\]  

(A.10) 

(A.11) 

which enable us to give an explicit representation for the increment of the loading factor

\[
\Delta \lambda = \frac{f + f_{\varphi} \Delta \varphi_r}{f_{\varphi} \Delta \varphi_\lambda + f_{\lambda}}.
\]  

(A.12) 

Based on this, the increment in placement is calculated as
\[ \Delta \varphi = \Delta \varphi_r + \Delta \lambda \Delta \varphi_\lambda, \] (A.13)

and the updates of displacements and loading parameter then read \( \varphi_{l+1} = \varphi_l + \Delta \varphi \) and \( \lambda_{l+1} = \lambda_l + \Delta \lambda \). Note, that within the predictor step at \( t_0 \), constraint condition A.4 is not determined. As a remedy, the loading parameter \( \lambda_n \) is increased by 1 providing the placement increment based on the last equilibrium state \( \{ \varphi_n, \lambda_n \} \) reads as

\[ \Delta \varphi_\lambda = K^{\varphi_\varphi} \cdot \Delta \varphi_\lambda f_{ext}. \] (A.14)

Figure A.1: Illustration of arc-length methods according to Crisfield [53]. (a) Iteration on the current normal plane (Ramm’s method), (b) Iteration on the initial normal plane (Riks-Wemper method). Quantities without subscript \( n \) are associated with \( t_{n+1} \)

and the distance to the last equilibrium point reads as

\[ s_0 = \sqrt{\Delta \varphi_\lambda \Delta \varphi_\lambda + 1}. \] (A.15)

Comparing the distance \( s_0 \) and the prescribed arc length \( d \), the increments of the loading parameter and the placement can be scaled by \( \Delta \lambda = s/s_0 \) and \( \Delta \varphi = [s/s_0] \). The updates of the displacements \( \varphi_{l+1} \) and the loading-parameter \( \lambda_{l+1} \) can then be calculated for \( l = 0 \).

The constraint equation A.2 remains to be specified. Different solution strategies are suggested in the literature as summarised, for instance, One prominent example is the iteration on the current normal plane according to Ramm [59], [60], which in Crisfield [54] is denoted as Ramm’s method. Following this approach, which is illustrated in Fig. A.1(a), the equation

\[ f = [\varphi - \varphi_n] \cdot [\varphi_{l+1} - \varphi_l] + [\lambda_l - \lambda_n] \cdot [\lambda_{l+1} - \lambda_l], \] (A.16)
constrains the solution to the normal plane with respect to the current iteration step. According to equation A.12, the related increment can be specified as

\[
\Delta \lambda = -\frac{[\phi - \phi_n] \cdot \phi_r}{[\phi - \phi_n] \cdot \phi_\lambda + \lambda_l - \lambda_n}.
\] (A.17)

Within this approach, the increment size is limited by moving a given distance along the tangent of the current solution point. Equilibrium is then sought in the plane that passes through the point thus obtained and that is normal to the same tangent. Moreover, the constraint A.16 ensures that the iterative change is always normal to the secant change, which causes the equilibrium-search to be normal to the tangent of the previous iteration step, rather than to the tangent at the beginning of the increment. This method can be considered as an extension of the iteration on the initial normal plane according to the original work by Riks [56], [61] and Wempner [57] which, following [53], is accordingly denoted as Riks-Wemper method; see Fig. A.1(b) for an illustration.

A.2 Application within general user-elements via Abaqus subroutine UEL

Along with the UEL-subroutine, Abaqus is capable of using self-implemented element-formulations in combination with the Abaqus-internal arc-length procedure, referred to as the modified Riks algorithm, see [62] for general usage aspects. According to the Abaqus Theory Manual [63], the modified Riks algorithm is a version of the aforementioned iteration on the current normal plane established by Ramm [59], [60], see also Crisfield [58]. In order to be able to use the Abaqus-internal modified Riks algorithm in combination with a UEL-subroutine, we have to meet the following requirements. As described above, the essential quantities to be specified within the UEL-subroutine are the element residual vector RHS, the element Jacobian- respectively stiffness-matrix AMATRX, and the internal-solution dependent state variables SVARS. Regarding AMATRX and SVARS, no modifications due to the Riks method have to be provided. However, in contrast to the purely Newton-Raphson-based solution scheme, RHS now consists of NRHS=2 columns. The first column RHS(:,1) is still associated with the residual vector. The additional second column RHS(:,2) contains the increments of external loads of the respective element, also referred to as the incremental load vector. In summary,

\[
AMATRX := K_{\varepsilon \varphi}
\]  

(A.18)
\[ \text{RHS}(; 1) := \lambda f^{\varphi}_{\text{ext}} - f^{\varphi}_{\text{inte}} \quad (\text{A.19}) \]

\[ \text{RHS}(; 2) := \Delta \lambda f^{\varphi}_{\text{ext}}. \quad (\text{A.20}) \]

cf. the system of equations A.9. Here, definitions A.18–A.20 require that all (external) force loads \( f^{\varphi}_{\text{ext}} \) must be passed into UEL by means of distributed load definitions so that the force loads are available for the definition of incremental load vectors; the load key \( U_n \) must be used consistently, as discussed in the Abaqus Analysis User’s Manual [64]. The coding in subroutine UEL must distribute the loads into consistent equivalent nodal forces and account for them in the calculation of the RHS array.

Furthermore, the UEL-subroutine provides the two variables ADLMAG and DDLMAG, where ADLMAG contains the total load magnitudes of the distributed loads at the end of the current increment and where DDLMAG represents the increments in the magnitudes of the distributed loads currently active on this element. With regard to relations A.19 and A.20, we consequently note that

\[ \text{ADLMAG} := \lambda \quad \text{and} \quad \text{DDLMAG} := \Delta \lambda. \quad (\text{A.21}) \]

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆( l_{\text{in}} )</td>
<td>initial arc-length increment</td>
</tr>
<tr>
<td>( l_{\text{period}} )</td>
<td>total arc-length scale factor</td>
</tr>
<tr>
<td>∆( l_{\text{min}} )</td>
<td>minimum arc-length increment</td>
</tr>
<tr>
<td>∆( l_{\text{max}} )</td>
<td>maximum arc-length increment</td>
</tr>
<tr>
<td>( \lambda_{\text{end}} )</td>
<td>maximum load proportionality factor (LPF) to end increment</td>
</tr>
<tr>
<td>( \text{nod}_{\text{mon}} )</td>
<td>monitored node number</td>
</tr>
<tr>
<td>( \text{dof}_{\text{mon}} )</td>
<td>monitored degree of freedom (DOF)</td>
</tr>
<tr>
<td>( u_{\text{end}} )</td>
<td>maximum displacement at node and DOF to end increment</td>
</tr>
</tbody>
</table>

Table A.1: Control parameters to be used within the input-file accompanied by the keyword STATIC, RIKS for the modified Riks method as implemented in Abaqus. The initial load proportionality factor (LPF) is computed as \( \Delta \lambda_{\text{in}} = \Delta l_{\text{in}} / l_{\text{period}}, [62] \) and [65].

The input-file-setting is identical compared to a conventional modified Riks static analysis without any user-coding, i.e. introduced by STATIC, RIKS followed by eight partially optional parameters which are summarised in Table A.1.


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4.13 The solid line in this figure is the one given in Fig. 4.5. Additionally, and for comparison, in this figure each point gives the solution that is obtained by means of the finite element model.
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4.15 For \( \lambda_z = 1.7 \), the figure gives values of pressure of inflation vs values of hoop stretch for two points (A and B) located at different altitudes of a cylinder with L/R = 6 made of (4.3), (the material with \( I_5 \) and \( I_7 \)) for which \( g = 0.001c \) and \( \varphi = 30^\circ \). As loading is applied from the undeformed configuration (values for which \( \lambda_\theta \) is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows one solid line. During postbifurcation, as pressure decreases, the hoop stretch of both points (A and B) decrease. This description is associated with necking motion. Point B is located in the necking area.
4.16 The values shown in the curves of this figure have their corresponding values in Fig. 4.15. In particular, for $\lambda_z = 1.7$, the figure gives values of pressure of inflation vs values of axial stretch for two points (A and B) located at different altitudes of a cylinder with $L/R = 6$ made of (4.3) (the material with $I_5$ and $I_7$) for which $g = 0.001c$ and $\varphi = 30^\circ$. As loading is applied from the undeformed configuration (in the figure it corresponds with the vertical line), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows one solid line (the vertical line). During postbifurcation, as pressure decreases, point B decreases its axial stretch while point A increases its axial stretch. The axial stretch of point B is given by the dashed line which is at the left of the bifurcation point. This description is associated with necking motion. Point B is located in the necking area.

4.17 Details of the reference mesh (left) and the deformed one at the onset of bifurcation (right) for material with $I_5$ and $I_7$ and axial stretch 1.7. Postcritical behaviour is related to necking.

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4.19 From left to right, the figure shows a graphical presentation of necking propagation. Points A and B are shown to further clarify the curves of Fig 4.15.

5.1 The figure shows the bulge in equilibrium beyond bifurcation at a pressure which is precisely the one associated with the onset of bulging. The structure is made of two cylinders joined by one transition zone. A developed cylinder tube with regard to this figure is assumed to be symmetric with respect to the plane containing the bottom section of cylinder 2. By construction, the bifurcation condition $f = 0$ is obeyed at the transition zone for values $\lambda_\theta$ and $\lambda_z^*$. On the other hand, each cylinder, characterized by a hoop stretch $\lambda_{\theta_i}$ and an axial stretch $\lambda_{zi}$ where $i \in \{1, 2\}$, is stable and obeys $f_i > 0$. This is a graphic representation associated with axial propagation of bulging.

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5.4 Values of $f/\mu$ for different materials vs hoop stretch when axial stretch $\lambda_z = 1.1$. It is found that the onset of bulging, i.e. $f = 0$, occurs at the pair of values $(\lambda_\theta, \lambda_z) = (1.839, 1.1)$ for the mixed material. Similarly, $f = 0$ when $(\lambda_\theta, \lambda_z) = (1.615, 1.1)$ for the neo-Hookean material. For the Demiray model $f > 0$.

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5.6 The curve in this figure gives all possible pairs of values $(\lambda_\theta, \lambda_z)$ associated with the pressure $p^*$ which, in turn, corresponds to $\lambda_\theta^* = 1.1$ and $\lambda_z^* = 1.839$ for the mixed material. In the figure one solution is formed by two cylinders that are labeled using the same symbol (a triangle, a circle, etc). The circle corresponds to the solution of the system of Eq. (5.12). In this solution, the values of stretch of cylinder 1 are $\lambda_{z1} = \lambda_z^* = 1.1$ and $\lambda_{\theta1} = \lambda_\theta^* = 1.839$ while the values of stretch of cylinder 2 are $\lambda_{z2} = 1.307$ and $\lambda_{\theta2} = 2.133$. The other solutions (pairs of cylinders) obey the system of Eqs. (5.3) and (5.11).

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5.8 For $\lambda_z = 1.1$, the figure gives values of pressure of inflation vs values of hoop stretch for two points located at different altitudes of (i) a cylinder with $H/R = 0.05$ (dashed thick line) (ii) a cylinder with $H/R = 0.10$ (dashed thin line) and (iii) a thin-walled cylinder (solid line). All the cylinders are made of a Demiray material with $\alpha = 0.25$. There is only one line associated with the thin-walled cylinder (analytical results) because points A and B have the same hoop stretch. Whence, there is no bifurcation. The values associated with the other two cylinders have been obtained using numerical simulations. In each case, the figure shows two lines: one line is associated with the hoop stretch of point A while the other line is associated with the hoop stretch of point B. Both lines are very close. This means that both points have a non-distinguishing hoop stretch difference and that there is no bifurcation.

5.9 For $\lambda_z = 1.1$, the figure gives values of pressure of inflation vs values of hoop stretch for two points located at different altitudes of (i) a cylinder with $H = R = 0.05$ (dashed thick line) (ii) a cylinder with $H/R = 0.10$ (dashed thin line) and (iii) a thin-walled cylinder (solid line). All the cylinders are made of a neo-Hookean material. Analytical results (solid line) are just shown up to the onset of bifurcation to validate the results obtained numerically for the cylinders with different thicknesses (dashed lines). As loading is applied from the undeformed configuration (in the figure it corresponds with values in each curve for which $\lambda_0$ is close to 1), and prior to bifurcation, both points have a non-distinguishing hoop stretch difference. Under these conditions, the figure shows in each numerical simulation (each cylinder) two lines that are very close. When bifurcation occurs (close to maximum points in each curve), the structure is not able to support higher values of pressure. Furthermore, as pressure decreases, point B reverses the original path decreasing its hoop stretch while point A increases its hoop stretch. It follows that, in each case, the hoop stretch of point A is given by the dashed line which is at the right of the bifurcation point.
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5.11 For $\lambda_z = 1.1$, the plot gives values of pressure vs hoop stretch for two points (A and B) located at different altitudes of (i) a cylinder with $H/R = 0.05$ (dashed thick line), (ii) a cylinder with $H/R = 0.10$ (dashed thin line) and (iii) a thin-walled cylinder (solid line). All cylinders are made of the mixed material 5.1 with $k = 0.5$ and $\alpha = 0.25$. Analytical results (solid line) are just shown up to the onset of bifurcation to validate the results obtained numerically for the other two cylinders. For each cylinder there are two dashed lines. Each one of them gives values of pressure vs hoop stretch for one point (either point A or point B). The onset of bifurcation is identified when the hoop stretch of these two points differs. Under these circumstances, the two (dashed) lines start to diverge. Prior to bifurcation, both points have a non-distinguishing hoop stretch difference and the two (dashed) lines are very close. After the onset of bifurcation, for each case, the dashed line to the right gives the hoop stretch of point A (where the bulge appears) while the dashed line to the left gives the hoop stretch of point B. Axial propagation of bulging involves two periods: firstly, pressure remains essentially fixed during the ensuing propagation of the bulging instability mode beyond the onset of bifurcation until a suitable configuration is obtained; secondly, in subsequent motion, for further axial propagation of bulging pressure of inflation must be increased. This is clearly shown in the evolution of the hoop stretch of point A beyond the onset of bifurcation.

5.12 For the strain energy function (3.9), the figure shows parallel curves to the ones given in Fig. 5.11 for the mixed material model.

6.1 Stress-strain law for concrete in tension. The stress is plotted against the total strain, i.e. the sum of crack strain and concrete strain.[40]
A.1 Control parameters to be used within the input-file accompanied by the keyword STATIC, RIKS for the modified Riks method as implemented in Abaqus. The initial load proportionality factor (LPF) is computed as 
\[ \Delta \lambda_{in} = \Delta l_{in} / l_{period}, \] 
\[ [62] \text{ and } [65]. \]