UNIVERSIDAD POLITÉCNICA DE MADRID
ESCUELA TÉCNICA SUPERIOR DE INGENIEROS
AERONÁUTICOS

PATTERN FORMATION OF PARAMETRICALLY
DRIVEN SURFACE WAVES INDUCED BY VIBRATION.

DOCTORAL THESIS

By:

Jose Miguel Perez de Gracia Delgado

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**DEPARTAMENTO DE MATEMÁTICA APLICADA A LA
INGENIERÍA AEROESPACIAL**

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Abstract

This thesis focus on the generation of subharmonic surface waves on fluids subject to forced vibration in the gravity-capillary regime with liquids of small viscosity. Three different problems have been considered: a rectangular container under horizontal vibration; the same geometry but under a combination of horizontal and vertical vibration; and a fully submerged vertically vibrated obstacle in a large container. An amplitude equation is derived from first principles that fairly precisely describes the subharmonic surfaces waves parametrically driven by vibration. That equation is two dimensional while the underlying problem is three-dimensional and permits spatially nonuniform forcing. Using this equation, the three systems have been analyzed, focusing on the calculation of the threshold amplitude, the pattern orientation, and the temporal character of the spatio-temporal patterns, which can be either strictly subharmonic or quasi-periodic, showing an additional modulation frequency. Dependence on the non-dimensional parameters is also considered. The theory is compared with the experiments available in the literature.

Resumen

Esta tesis se centra en la generación de ondas superficiales subarmónicas en fluidos sometidos a vibración forzada en el régimen gravitatorio capilar con líquidos de baja viscosidad. Tres problemas diferentes han sido estudiados: un contenedor rectangular con vibración horizontal, la misma geometría pero con una combinación de vibración vertical y horizontal y un obstáculo completamente sumergido vibrado verticalmente en un contenedor grande. Se deriva una ecuación de amplitud desde primeros principios para describir las ondas subarmónicas con forzamiento paramétrico inducido por la vibración. La ecuación es bidimensional mientras que el problema original es tridimensional y admite un forzamiento espacial no uniforme. Usando esta ecuación los tres sistemas han sido analizados, centrándose en calcular la amplitud crítica, la orientación de los patrones y el carácter temporal de los patrones espaciotemporales, que pueden ser estrictamente subarmónicos o cuasiperiodicos con una frecuencia de modulación temporal. La dependencia con los parámetros adimensionales también se considera. La teoría será comparada con los experimentos disponibles en la literatura.

1

Introduction

Pattern formation is quite an ubiquitous and broad field that ranges from fluid mechanics to chemistry and optics. Probably, the most studied case in fluid mechanics is the Rayleigh-Benard convection but Marangoni convection, Taylor-Couette flow and parametric waves (main topic of this thesis) have been the object of great interest and attention. In chemistry, reaction-diffusion systems have been of interest from the pattern formation point of view since the Turing (1952) seminal paper. Also solidification patterns are a main point of study as the properties of materials highly depend on this process. In optics, the presence of coherent structures are interesting, both scientific and technologically. We must not forget about the importance of pattern formation in biology that goes from the seemingly trivial formation of stripes on a zebra to the crucial process of cellular differentiation. A good point of study for all these could be the review by Cross and Hohenberg (1993), or the more recent books by Hoyle (2005) and Cross and Greenside (2009). These different fields, when studied from the pattern formation perspective, can be studied using similar methods.

In the fluid dynamics field, vibrated fluids have long attracted experimental and theoretical attention, especially so since the seminal work of Faraday (1831). Such interest reflects both their practical importance over a wide range of scientific and engineering problems such as liquid storage, mixing, structure formation, and aerodynamic design, and the rich variety of phenomena that can arise. Fluid properties, system size, timescales, symmetry properties, can vary enormously from one application to the following, and there are numerous possible effects to consider. Vertical vibration, for instance, can delay convection (Gresho & Sani 1970), suppress the Rayleigh-Taylor instability (Wolf 1969), induce mean flows (Vega, Knobloch & Martel 2001), and affect heat or mass transfer. Horizontal vibration can drive sloshing and other types of resonant and near-resonant mode dynamics at low frequencies (Miles 1984; Funakoshi & Inoue 1988; Feng 1997) and produce quasistatic structures called frozen waves (Talib *et al.* 2007) at intermediate frequencies on the interface between two immiscible fluids.

Theoretical efforts to explain this range of phenomena are necessarily broad in method and scope. Nonetheless, there are several simplifying assumptions that appear repeatedly in theoretical analyses. One simplification is in treating the forcing function as monochromatic with constant amplitude and direction. Except in precisely controlled experiments, this is an approximation because environmental vibrations are generally broad-band and vary in both magnitude and direction. However, may often be that the support structure surrounding the fluid, with its own mechanical resonances, transmits acceleration that is narrow-band and centered on specific frequencies and the monochromatic assumption can thus be regarded as a reasonable idealization. If multi-frequency vertical forcing is applied, on the other hand, complex surface wave states such as quasipatterns and superlattice patterns arise (Arbel & Fineberg 2002) and theoretical work (Topaz *et al.* 2004 and references therein) confirms the importance of the additional modes that come into play. The assumption of a constant axis of vibration is equally widespread and considerably simplifies the analysis. Recent work, however, illustrates the new phenomena that can arise with circular (Ivanova *et al.* 2001) or two-axis acceleration (Bestehorn *et al.* 2013).

The subharmonic instability is a robust transition that is observed easily in vertically forced Faraday waves, wavemaker experiments (Schuler 1933; Lin & Howard 1960; Barnard & Pritchard 1972; Lichter & Underhill 1987; Shemer & Lichter 1990; Underhill, Lichter & Bernoff 1991) and horizontally vibrated systems (Porter *et al.* 2012). These waves arise from parametric excitation (Benjamin and Ursell 1954, Garret 1970) and can exhibit complex behaviour such as modulations, mode interactions, and chaotic dynamics. Parametric excitation requires that the forcing acceleration exceeds a threshold value; the resulting waves can be periodic, with a frequency equal to one-half of the forcing frequency (as in the Faraday system), or quasi-periodic, exhibiting two frequencies whose sum coincides with the forcing frequency.

The case of a horizontal interface subject to purely *vertical vibrations* has become a paradigm of parametric excitation (Fauve 1995) because, in addition to experimental accessibility, it admits a fairly clean theoretical analysis under the assumption of a perfectly flat horizontal free surface undergoing perfectly vertical vibrations (Benjamin and Ursell 1954). The forcing is uniform and, in a comoving frame, the motionless flat surface solution persists for all vibration amplitudes, which greatly simplifies the analysis. Another common assumption in vibrated fluid systems is of uniform forcing. This is justified when an open container of fluid is vibrated in purely vertical fashion and capillary effects at the boundaries are ignored. The theoretical simplification of using a uniform parametric forcing mechanism helps

to explain why vertically forced Faraday waves have received more attention than horizontally forced systems. The assumption of uniform forcing is not valid, however, in more generic situations. Capillary effects in the Faraday problem, for example, generate harmonic waves at the boundaries that break the uniformity of the flat state. Faraday waves awake such large interest not only for its theoretical accessibility but also because the resulting patterns are very rich, finding stripes, squares and hexagons (Kudrolli & Gollub 1996), quasipatterns (Edwards & Fauve 1993), oscillons (Arbell Fineberg 2000). The problem continues to attract interest as shown in the recent review by Bush (2015) when considering the interaction of Faraday waves and drops.

Horizontally vibrated systems are more subtle because of the variety of wave patterns that appear already near threshold. Their structure strongly depends on the geometry, the gravity/capillary balance, and excitation details. Direct excitation produces *harmonic waves* (figure 1.1-bottom,right) exhibiting the forcing frequency at arbitrary small forcing amplitude. Experimental and theoretical analyses have concentrated in two distinguished limits: *sloshing modes* (illustrated in figure 1.1-top,right) are excited when the size of the container is comparable with the wavelength associated with the excitation frequency by the dispersion relation; and *cross waves* (see figure 1.1-top,left)) are purely subharmonic, modulated waves, with their crests perpendicular to the walls.

Interaction between both endwalls in horizontally vibrated containers produces *sloshing modes* that exhibit new dynamics (compared with cross waves) promoted by wave reflection at the sidewalls and endwalls. Both harmonic and subharmonic sloshing modes have been considered, among others, by Miles (1984), Funakoshi & Inoue (1988), Feng (1997), Hill (2003), and Faltinsen, Timokha, and collaborators (in a series of papers, see Faltinsen, Rognebakke & Timokha 2006, Hermann & Timokha 2008, and references therein) using modal expansions and considering fully nonlinear gravity waves under general excitation (not necessarily horizontal); Frandsen (2004) and Wu & Chen (2009), instead, addressed this problem using finite differences.

Cross waves (Barnard & Pritchard 1972) are strictly (2:1) subharmonic waves produced by a wavemaker consisting in a partially submerged horizontally vibrating plate. They were already identified by Faraday (1831) as a series of apparently permanent ridges projecting outward ‘like the teeth of a coarse comb’ (Miles and Henderson 1990). The parametric excitation of cross waves was uncovered by Garrett (1970) using the Havelock (1929) solution for the wavemaker problem in a semi-infinite rectangular tank.

The most influential theory of cross-waves was developed by Jones (1984),

Figure 1.1: Cross waves (top, left), sloshing modes (top, right), oblique subharmonic wavetrains (bottom, left), and harmonic wavetrains (bottom, right).

who derived a nonlinear Schrödinger (NLS) equation to describe their slow spatial and temporal evolution. Subsequent work included viscous effects (Lichter & Chen 1987; Bernoff, Kwon & Lichter 1989, Shemer & Kit 1989) and mode interactions (Ayanle *et al.* 1990). The case of propagation along an oblique angle was also considered in this context by Shemer & Chamesse (1999), who considered gravity-capillary waves and attributed obliqueness to side band instabilities. Becker & Miles (1991) considered radial cross waves in an annular container. Although these models were compared with some success to cross-wave experiments (Lichter & Bernoff 1988; Kit & Shemer 1989; Shemer & Kit 1989; Underhill, Lich & Bernoff 1991), it is important to note that these motivating experiments were all in the gravity wave regime and that the NLS models rely on assumptions appropriate to that limit. One such assumption is that cross-waves vary slowly in the downstream direction compared both to the crosswise wavelength and to the extent of the localized forcing. This permits a separation of lengthscales that places the forcing term in the homogeneous boundary condition of the NLS equation. Because bulk forcing is absent, this treatment is opposite in spirit to the uniform forcing assumed in systems like vertically forced Faraday waves, but it has a similar consequence: the spatial dependence of the forcing mechanism is again ignored.

Other vibrated systems that consider vertical and horizontal vibration

have not received much attention in connection with careful experimental measures of threshold amplitude and primary patterns. The available material in the literature concentrates in the fully nonlinear problem, either via numerical runs (Fradsen 1997; Chen & Nokes 2005; Liu & Lin 2008) or using model expansions.

Obliquely oriented subharmonic waves, whose crests are not perpendicular to the endwalls (as seen in figure 1.1-bottom, left), have been experimentally observed by Porter *et al.* (2012) in large horizontally vibrated containers, using driving frequencies for which capillary effects cannot be neglected. It is interesting that recent experiments on subharmonic capillary waves produced by oscillating a fully immersed wavemakers located far from the walls (Moisy *et al.* 2012) produced similar oblique waves, as did the experiments of Taneda (1994) using a half-submerged circular cylinder. Also, in the behaviour of mean flows described by Punzmann *et al.* (2014), oblique cross-waves seems to play a major role. This obliqueness was attributed to side-band effects by Shemer and Chamesse (1999). All these suggest that obliqueness could be unavoidable when capillary effects are present, but this is only a conjecture since no consistent, general theory exists, even at the linear level, that is able to predict both the subharmonic instability threshold in horizontally vibrated containers and the resulting patterns in the general capillary-gravity limit. Constructing such unifying general linear theory from first principles, not assuming particular orientations, is the main goal of this thesis.

The main difficulties associated with this goal came from the different sources of uncertainties in the modeling. One is contact line motion, which could contaminate quantitative comparison with experiments (or be blamed for disagreement). Empirical modeling, depending on tunable parameters, is not satisfactory in a theory derived from first principles. Here, instead, the contact line will be assumed to be pinned to the edge of the side-walls and endwalls, which can be experimentally performed by either filling up the container in a brimful way or using hydrophobic coatings. Fixing the contact line requires a careful analysis of wave reflection at the sidewalls and endwalls, which is not available in the literature (to my knowledge) and will be performed as a part of the analysis. Another source of uncertainties is the contamination of the free surface. The presence of surfactants is known to appreciably change the damping of the surface since Scott (1979). And it is also well established that tap water suffers of contamination as well, as distilled or filtered water do (to a lower extend). Thus, the only reliable option is to use (chemically produced) pure water as experimental fluid, but it must be noted that within hours of exposition the surface of pure water will be polluted as well (Henderson & Miles 1994). Since the

studies of Marangoni (1872), it has become usual to attach his name to the processes in which surface tension gradient (due to temperature differences or concentration of different species in a thin film) plays a role. Surface contamination can be modeled through Marangoni effect (Dorrestein 1951, Levich 1962, Nicolas & Vega 1999) and is responsible of producing a drift in Faraday waves (Martin, Martel & Vega 2002).

The various problems considered in this thesis show a fairly similar underlying physics. Because of this, the plan is to explain in details the physics for the first problem to be considered, and then concentrate only on the peculiarities of the remaining problems that will be addressed. These problems are:

- A brimful rectangular container driven by horizontal vibration aligned with one of the symmetry planes of the container (not the diagonal one).
- The same geometry as in the previous item, but combining horizontal and vertical forcing. Again the forcing is confined in one of the symmetry planes of the container.
- A large container with a fully immersed obstacle that is vibrated in the vertical direction.

In the first problem, parametric excitation is triggered by the harmonic oscillating flow produced by the vibrating end walls of the rectangular container. This harmonic field exhibits two distinguished components. Two *counterpropagating harmonic wavetrains* aligned along the vibrating endwalls of the container are present that propagate (and decay by viscous dissipation) inwards. An *oscillatory bulk flow* (OBF) is also present that is essentially different from the wavetrains because:

- It is slowly varying in space, while the wavetrains show an $O(1)$ (when using the non-dimensionalization of this thesis) wavelength.
- It is purely inviscid and affects a bulk region near the vibrating endwalls whose horizontal size is comparable to the depth of the container, while the wavetrains penetrate a distance determined by viscous dissipation.
- It exhibits a spatially constant pressure at the free surface (in spite of the presence of vertical pressure gradient), thus being independent of both gravity and capillary forces, while the wavetrains do depend on these.

This OBF produces a pressure field that is the counterpart of the spatially uniform, temporally periodic vertical-pressure-gradient field produced by inertial forces in the Faraday experiment. The essential difference is that the vertical pressure gradient produced by the OBF is not spatially uniform. Also, the second property (namely, that the OBF extends to a region comparable to the container depth) means that the OBF is the dominant forcing mechanism if the vibrating wall depth is at least somewhat large compared to the viscous-decaying length of the subharmonic wavetrains. This requires viscosity to be not too small, consistently with the experimentally accessible conditions with ordinary liquids when the container width and depth are comparable to each other and of the order of tens of cms and the forcing frequency is larger than 10 Hz. This will be the limit of interest on this thesis.

For the vibration of a rectangular container on the vertical $x - z$ plane, the mechanism of generation of harmonic wavetrains is the same as the one explained previously. It is different for the OBF, since now the effect of the bottom of the container has to be also considered. The effect of the bottom is, in fact, the one considered when studying Faraday waves, resulting in a pressure gradient that is constant in space. By the linearity of this flow, the independent effect of the lateral walls can be added to the one of the bottom. The result is a slowly varying pressure gradient with constant pressure at the free surface. This flow covers the whole domain, not just a region near the walls comparable with the depth of the container as in the case of a horizontally vibrated container. In any event, the same considerations about the differences between the harmonic wavetrains and the oscillatory bulk flow apply.

The last case considered is the vertical vibration of an immersed wave-maker. In this configuration, the harmonic wave field is produced by the upper face of the submerged obstacle. A couple of harmonic wavetrains moving in opposite directions is generated at the center of the container; they are parallel to the axis of the obstacle and move towards the end-walls. With a broad and deeply submerged obstacle, the OBF is similar to the one described. When any of these two conditions is not found, the situation is not that clear:

- For shallow and intermediate immersion depths, the OBF affects a region of size similar to the width of the obstacle, but this region opens wider with the depth of immersion.
- The vibration of shallow obstacles is such that both the OBF and the wavetrains are coupled, without a clear separation of them. It also increases the amplitude of the wavetrains (that could promote the sub-

harmonic instability).

- The pressure gradient resulting from the vibration of a slim obstacle can not be considered as slowly varying in space. But when the obstacle is very deeply submerged this slowly variation is to be expected.

The subharmonic instability is going to be studied when the conditions explained for the first problem are met (or close to it).

The OBF described in these three configurations was implicit already in early analyses of harmonic gravity waves produced by wavemakers (Havelock 1929) and thus also in subsequent works on cross waves that used the Havelock solution. This flow has been formulated in terms of a linear equation with well defined boundary conditions by Varas & Vega (2007) in the case of two- dimensional, horizontally vibrated containers.

As it happens in the Faraday system, selection of the (subharmonic) wave patterns at threshold is subtle. Here, such selection is determined by the OBF and the boundary conditions.

The approach in this thesis has the main ingredients for the analysis of subharmonic instability in nearly inviscid fluid systems with interfaces, provided that the vibration frequency is large enough (so that the excited wavelength is small compared to the system size).

1.1 Thesis plan

As already mentioned, this thesis studies the surface waves appearing in three different problems, none of which have received enough attention in the literature, focusing in the subharmonic instability. These three problems are the horizontal vibration of a rectangular container, the mixed horizontal/vertical vibration of a rectangular container and the vertical vibration of a fully immersed wavemaker in a large container. These problems will be considered in the realistic limit in which the spatial dependence of the parametric forcing mechanism producing subharmonic waves cannot be ignored. In other words, the goal of this thesis is to derive a model that is appropriate to precisely calculate the threshold of the subharmonic instability when the forcing is non-uniform. This model will take the form of a *general amplitude equation* and its appropriate boundary conditions. The general amplitude equation and boundary conditions provide both the subharmonic threshold amplitude and the critical eigenfunctions. Although the model will be obtained for the first problem, once it is obtained it will be adapted and applied to the mentioned problems.

The rest of the thesis is organized as follows.

The goal of chapter 2 is to obtain the above mentioned general amplitude equation and the boundary appropriate conditions that apply to the amplitude equation. These are the foundations of the theoretical analysis that is developed in the thesis and will be used, in particular, to analyze the subharmonic surface waves that are driven by horizontal vibration in a rectangular container. The usual approach in rectangular containers is to decompose the patterns into plane waves whose amplitudes evolve slowly according to an associated set of amplitude equations. Here, instead, a new approach is undertaken in which no particular pattern is assumed. Specifically, a *purely two-dimensional*, linear evolution equation for the free surface elevation is derived that has substantial interest in itself. It is a general amplitude equation since it is independent of the container geometry and allows for any possible spatially modulated pattern composed by waves in arbitrary orientations. Similar equations have been derived by phenomenological arguments for Faraday waves (Zhang & Viñals 1995). The boundary conditions account for wave reflection at the sidewalls and endwalls, and they are derived using the Fourier transform of the local solution in a region near the container walls. A linear stability analysis using normal mode expansions is applied to obtain the neutral stability curves. The minima of these curves select the critical amplitude and the orientation of the patterns. Weak mode interaction is also considered to study the secondary bifurcation. Then, the effect of varying the different non-dimensional parameters of the problem is studied. The model is validated with the experimental made by Porter *et al.* (2012).

Chapter 3 focusses on the surface waves appearing in a rectangular container that is subject to mixed vertical/horizontally vibration. To study this problem, the amplitude equation derived in chapter 2 is adapted (to include the effect of vertical forcing) and then the linear stability is studied (to obtain the neutral stability curves). The aim is to identify the effect of the two parameters controlling the forcing, namely the amplitude ratio and the phase between horizontal and vertical forcing. The differentiating features of the surface waves can be expressed in terms of the orientation, elucidating the quasiperiodic or subharmonic behaviour, and whether they are spread over the container or concentrated near the walls. The effect on the patterns of the two parameters controlling the forcing is then pursued.

In chapter 4, the effect on the free surface motion of the vertical vibration of an immersed wavemaker is studied. First, a theoretical framework is developed to calculate the two components of the harmonic flow, namely, the OBF and the wavetrains. The conditions of the flow far from the obstacle (the structure of the harmonic waves and a radiation condition) are used to obtain a solvability condition that provides both the OBF and the

harmonic wavetrains. Then, this theory is simplified to the case in which the wavemaker is wide and deeply submerged, which is precisely the case in which the OBF and the travelling waves are not coupled. The results of these two approximations are compared for different values of immersion. Under the assumption of a deeply submerged wavemaker, the amplitude equation is adapted for the produced by OBF wavemaker; the effect of the physical and geometrical parameters is studied. Surface contamination on water is also studied, considering its effect on the surface tension (whose gradient promotes an additional shear stress component) to derive the amplitude equation for this new situation; the effect of the contamination is tested for several wavemaker geometries/locations. Finally, a critical comparison between the results of the model and the experimental visualizations/measurements by Moisy *et al.* (2012) is made.

Chapter 5 closes the thesis with the main results and the conclusions extracted during the elaboration of this work, pointing towards some interesting possible continuations.

2

Horizontally vibrated container

2.1 Introduction

This chapter deals with the computation of the instability threshold (and associated patterns) for the primary subharmonic waves that are generated by forced horizontal vibration of a brimful rectangular container, vibrated along one of the primary symmetry planes of the container. The first part of this chapter deals with the derivation of a general amplitude equation, with appropriate boundary conditions (accounting for reflection of the waves at the endwalls and sidewalls) that will constitute the main theoretical framework for the second part of the chapter. The aim in this part is to calculate the acceleration, the orientation and the modulation frequency at the onset. A weakly nonlinear mode interaction analysis is also performed to elucidate the resulting patterns just above the threshold. With these in mind, this chapter is organized as follows. Section 2.2 is devoted to the problem formulation, which is simplified in the limit of small viscosity. The above mentioned amplitude equation, whose derivation and use are the main object of this thesis, is obtained in 2.3. The boundary conditions to be imposed to the previously obtained amplitude equation are deduced in section 2.4. Subsection 2.5.1 contains the linear stability analysis of the problem as performed via the amplitude equation formulation. Section 2.6 contains the neutral stability curves of the previous problem. The effect of varying the different parameters is computed in section 2.7. A comparison with experiments is made in section 2.8. The last section 2.9 contains a summary of the main results in the chapter.

2.2 Governing equations, quasi-potential approximation, and oscillatory bulk flow

This section is devoted to first formulate the problem in subsection 2.2.1 and then simplify the governing equations in subsection 2.2.2, where ad-

Figure 2.1: Sketch of the vibrating container.

vantage is taken of the assumptions that the viscosity is small and the container is large. Subsection 2.2.3 explains the different harmonic flow components appearing in the problem. The resulting formulation will be used in subsection 2.2.4 to calculate the OBF. This is a linear, harmonic, inviscid flow, responsible of triggering the subharmonic instability for containers of large depth and not too small fluid viscosity.

2.2.1 Formulation

A rectangular container of depth d^* and horizontal size $2L_1^* \times 2L_2^*$ (Fig.2.1) is considered, which is vibrating horizontally in the $2L_1^*$ -direction with an amplitude a^* and a frequency $2\omega^*$; the superscript $*$ denotes hereafter dimensional quantities. In order to account for the gravity and capillary limits in a smooth way, non-dimensionalization will be made using the characteristic time ω^{*-1} and length k^{*-1} defined as

$$d = k^* d^*, \quad L_1 = k^* L_1^*, \quad L_2 = k^* L_2^*, \quad a = k^* a^*, \quad (2.1)$$

where the wavenumber k^* is defined by the dispersion relation

$$\omega^{*2} = gk^* + \sigma k^{*3} / \rho. \quad (2.2)$$

Here, g is the gravitational acceleration, σ is the surface tension, and ρ is the density, all assumed constant. Using these, viscous and capillary-gravity effects are accounted for in terms of the modified Ohnesorge number and the gravity-capillary balance parameter,

$$C_g = \mu k^{*2} / (\rho \omega^*) \quad \text{and} \quad S = \sigma k^{*2} / (\sigma k^{*2} + \rho g), \quad (2.3)$$

respectively, where μ is the viscosity. Note that $0 \leq S \leq 1$, with the extreme values $S = 0$ and 1 corresponding to the *purely gravitational* and *purely capillary* limits, respectively.

In a static Cartesian coordinate system, with the xy plane including the unperturbed free surface, the x and y axes along the sidewalls and end-walls, respectively, and the z axis pointing upwards, the non-dimensional continuity and momentum equations are

$$\nabla \cdot \mathbf{u} + \partial_z w = 0, \quad (2.4)$$

$$\partial_t \mathbf{u} - w(\nabla w - \partial_z \mathbf{u}) + (\nabla \cdot \mathbf{u}^\perp) \mathbf{u}^\perp = -\nabla p + C_g(\Delta \mathbf{u} + \partial_{zz} \mathbf{u}), \quad (2.5)$$

$$\partial_t w + \mathbf{u} \cdot (\nabla w - \partial_z \mathbf{u}) = -\partial_z p + C_g(\Delta w + \partial_{zz} w). \quad (2.6)$$

Here, $\mathbf{u}$ and w are the horizontal and vertical velocity components, respectively, $p = \text{pressure} + (1 - S)z + (|\mathbf{u}|^2 + w^2)/2$ is the hydrostatic, stagnation pressure, and

$$\Delta = \partial_{xx} + \partial_{yy} \quad \text{and} \quad \nabla = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y \quad (2.7)$$

are the horizontal Laplacian and gradient operators, where ∂_t , ∂_z , ∂_{zz} , etc denote hereafter partial derivatives and $\mathbf{e}_x$ and $\mathbf{e}_y$ are the unit vectors in the x and y directions. In addition, $\mathbf{u}^\perp$ is the result of rotating 90° counterclockwise the vector $\mathbf{u}$, namely

$$\mathbf{u}^\perp = -v\mathbf{e}_x + u\mathbf{e}_y \quad \text{if} \quad \mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y. \quad (2.8)$$

Convective terms in (2.5)-(2.6) are the horizontal and vertical components of the nonpotential part of the three-dimensional convective terms, $-\mathbf{v} \times \nabla \times \mathbf{v}$ (with ∇ and $\mathbf{v}$ denoting hereafter the three-dimensional gradient operator and velocity vector, respectively).

Equations (2.4)-(2.6) apply in the domain occupied by the liquid, which is

$$-L_1 + a \cos 2t < x < L_1 + a \cos 2t, \quad -L_2 < y < L_2, \quad -d < z < f, \quad (2.9)$$

where $a > 0$ is the non-dimensional vibrating amplitude and f is the free surface elevation. The boundary conditions

$$\mathbf{u} = -2a\mathbf{e}_x \sin 2t, w = 0 \quad (2.10)$$

$$\text{at } x = \pm L_1 + a \cos 2t, \text{ at } y = \pm L_2, \text{ and at } z = -d,$$

$$f = 0 \text{ at } x = \pm L_1 + a \cos 2t \text{ and at } y = \pm L_2, \quad (2.11)$$

$$w = \partial_t f + \mathbf{u} \cdot \nabla f, \quad (2.12)$$

$$p - (|\mathbf{u}|^2 + w^2)/2 - (1 - S)f + S\nabla \cdot [\nabla f / (1 + |\nabla f|^2)^{1/2}] = \quad (2.13)$$

$$C_g\{2\partial_z w - 2(\partial_z \mathbf{u} + \nabla w) \cdot \nabla f + \nabla f \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) \cdot \nabla f\} / (1 + |\nabla f|^2),$$

$$\partial_z \mathbf{u} + \nabla w = \quad (2.14)$$

$$(\nabla \mathbf{u} + \nabla \mathbf{u}^\top) \cdot \nabla f - [2\partial_z w - (\partial_z \mathbf{u} + \nabla w) \cdot \nabla f] \nabla f \text{ at } z = f,$$

account for no-slip at the solid boundaries, kinematic compatibility, and equilibrium of normal and tangential stresses at the free surface. Here, the superscript $\top$ denotes the transpose. In the boundary condition (2.11) note that, as anticipated, the free surface is pinned to the upper edge of the container walls. The right-hand side of (2.13) is the viscous normal stress, $C_g[(\tilde{\nabla}\mathbf{v} + \tilde{\nabla}\mathbf{v}^\top) \cdot \mathbf{n}] \cdot \mathbf{n}$, where $\mathbf{n}$ is the upwards unit normal to the free surface. Similarly, the boundary condition (2.14) imposes zero tangential stress, namely $[(\tilde{\nabla}\mathbf{v} + \tilde{\nabla}\mathbf{v}^\top) \cdot \mathbf{n}] \times \mathbf{n} = \mathbf{0}$. The problem (2.4)-(2.6), (2.10)-(2.14) is invariant under reflection on the symmetry planes of the container, namely under the actions

$$x \rightarrow -x, \mathbf{u} \cdot \mathbf{e}_x \rightarrow -\mathbf{u} \cdot \mathbf{e}_x, t \rightarrow t + \pi/2 \quad \text{and} \quad y \rightarrow -y, \mathbf{u} \cdot \mathbf{e}_y \rightarrow -\mathbf{u} \cdot \mathbf{e}_y, \quad (2.15)$$

which will be referred to below as the *spatio-temporal reflection symmetry* and the *purely spatial reflection symmetry*, respectively. Invariance under the former symmetry implies periodicity, with a period π , which means in particular that quasi-periodic patterns cannot be invariant under this symmetry.

The main assumptions in this chapter are

$$L_1, L_2, d \gg 1, \quad a, C_g \ll 1, \quad C_g d \gg 1, \quad \text{and} \quad (\sqrt{C_g} + S)(L_1 + L_2) \gg 1. \quad (2.16)$$

The first assumption means that the size of the container is much larger than the surface wave wavelength (which is 2π , according to the nondimensionalization above). This is equivalent to the assumption that the forcing frequency is large compared to the frequency of the first capillary-gravity sloshing mode of the container. The assumptions that $a \ll 1$ and $C_g \ll 1$ imply that the steepness of the waves is small, namely

$$|\mathbf{u}| \ll 1, \quad |w| \ll 1, \quad |p| \ll 1, \quad |f| \ll 1, \quad (2.17)$$

and that the oscillatory motion is nearly inviscid, respectively. These allow for applying a quasi-potential approximation (Zhang & Viñals 1997), which is considered in the next subsection. The last two assumptions in (2.16) (further discussed in section 2.3 and 2.4) justify neglecting both nonlinear interaction with the counterpropagating, harmonic wavetrains and viscous dissipation in the Stokes layers attached to the sidewalls and endwalls.

Another important assumption is that the contact line is fixed at the brim of the container. This assumption (which can be fulfilled experimentally using either sharp edges or hydrophobic coatings) is important to avoid the uncertainty associated with a dynamic contact line, whose modelling is still not well understood (Snoeijer & Andreotti 2013).

2.2.2 Quasi-potential approximation

Ignoring viscous mean flows, which is consistent with the linear approximation that is the interest of this thesis (in Vega, Rüdiger & Viñals 2004 an extension of the quasi-potential approximation to include the mean flow can be found), the motion is inviscid in the bulk (outside the oscillatory boundary layers), where the vorticity and the Laplacian of the velocity are both zero. Thus, $\partial_z \mathbf{u} - \nabla w = \partial_{zz} \mathbf{u} + \Delta \mathbf{u} = \mathbf{0}$, $\nabla \cdot \mathbf{u}^\perp = \partial_{zz} w + \Delta w = 0$, and the conservation equations (2.4)-(2.6) simplify to

$$\nabla \cdot \mathbf{u} + \partial_z w = 0, \quad \partial_t \mathbf{u} = -\nabla p, \quad \partial_t w = -\partial_z p \quad \text{if } z < 0. \quad (2.18)$$

Eliminating the velocity in these equations yields

$$\Delta p + \partial_{zz} p = 0 \quad \text{if } z < 0, \quad (2.19)$$

and invoking (2.18), the boundary conditions (2.10)-(2.11) are linearized around $x = \pm L_1$ and rewritten as

$$\partial_x p + a \partial_{xx} p \cos 2t = 4a \cos 2t, \quad f + a \partial_x f \cos 2t = 0 \quad \text{at } x = \pm L_1, \quad (2.20)$$

$$\partial_y p = 0, \quad f = 0 \quad \text{at } y = \pm L_2, \quad \partial_z p = 0 \quad \text{at } z = -d. \quad (2.21)$$

On the other hand, using (2.16) and (2.17) and retaining only up to quadratic terms in the small quantities $\mathbf{u}$, w , p , f , a , and C_g , the boundary conditions (2.12)-(2.13) lead to

$$\partial_t f - w = -\nabla \cdot (f \mathbf{u}), \quad (2.22)$$

$$p - (1 - S)f + S \Delta f = 2C_g \partial_z w + (|\mathbf{u}|^2 + w^2)/2 - f \partial_z p \quad \text{at } z = 0. \quad (2.23)$$

The unknowns in this formulation are the pressure $p = p(x, y, z, t)$, the free surface deflection, $f = f(x, y, t)$, and the velocity components at the unperturbed free surface, $z = 0$, which do not depend on z and are given by $\mathbf{u} = \mathbf{u}(x, y, t)$ and $w = w(x, t)$. Determining these velocity components require two additional equations that, to the approximation relevant here, are

$$\partial_t \mathbf{u} = -\nabla p, \quad \partial_t w = -\partial_z p + 2C_g \Delta w \quad \text{at } z = 0. \quad (2.24)$$

The derivation of these equation is somewhat involved and is relegated to section 2.3.1. Note that the first of these equations coincides with the linear approximation of the inciscid horizontal momentum equation in (2.18). The second equation, instead, contains a viscous term that is not proportional to its counterpart in the viscous momentum equation, (2.6), because of the factor 2 that is not present in (2.6). This additional viscous contribution comes from viscous dissipation in the oscillatory boundary layer attached to

the free surface, which will be made clear in section 2.3.1. At the moment, however, we note that in the inviscid limit (the relevant limit in the next subsection) this term disappears and (2.24) coincide with (2.18).

The quasi-potential approximation (2.19)-(2.24) is the relevant one in the remaining of this chapter, to formulating the OBF (in the next subsection) and also to deriving both the general amplitude equation (in section 2.3) and the boundary conditions (in section 2.4) for the subharmonic waves. Even though the focus is in the linear approximation, quadratic terms have been retained in (2.23) because they are needed to obtain the parametric forcing terms (resulting from products of terms associated with the OBF and the subharmonic waves) in the general amplitude equation.

The structure of the flow under the above assumptions, which lead to a separation of scales, can be divided into distinct regions, illustrated in Fig. 2.2:

Figure 2.2: Structure of the flow identifying the distinguished regions.

1. Viscous boundary layers attached to the bottom of the container, the walls of the container and the free surface. These boundary layers have a thickness (non-dimensionalized by the wavenumber k^*) of the order of $\sqrt{C_g}$. Vorticity is also, in general, confined within these layers.
2. A layer beneath the free surface that is affected by the (propagation and reflection of) surface waves. This layer is thicker than the viscous boundary layer, of the order of the characteristic wavelength (i.e., $O(1)$ in non-dimensional units).
3. An interior region that excludes the above mentioned layers, referred to here as the bulk region.

2.2.3 Harmonic forcing terms

In comparison to the case of a vertically vibrated container, horizontal vibration produces a more complex harmonic (i.e., synchronous) flow. The

separation of scales associated with the limit considered in this chapter allows to distinguish two different harmonic components. These are (Fig. 2.3):

Figure 2.3: Forcing regions associated to harmonic surface waves and oscillatory bulk flow.

- A pair of counter-propagating wavetrains, travelling inwards from the opposing endwalls. These are surface waves generated by the movement of the (upper part of the) endwalls. They penetrate (from the endwalls) a distance determined by the damping, since the spatial decay rate is proportional to C_g . Their characteristic wavelength is given by the dispersion relation (2.2) with the forcing frequency $2\omega^*$ replacing the subharmonic frequency ω^* . The crests of these damped travelling waves are parallel to the vibrating endwalls.
- An oscillatory bulk flow (OBF). This flow is generated in the bulk region by the oscillation of the (entire surface of the) endwalls. This region extends inwards from the endwalls a distance comparable to the endwalls depth. The object of the next subsection will be to calculate this component of the harmonic forcing.

These two harmonic components provide a parametric forcing mechanism through quadratic terms in the boundary condition on the free surface (Miles & Henderson 1990; Higuera *et al.* 2014). Forcing from the oscillatory bulk flow (OBF) comes from the non-homogeneous (but slowly varying in space) pressure gradient at the free surface, which is the counterpart of the spatially constant pressure gradient associated to the vertical Faraday problem. Forcing from the harmonic wavetrains, instead, acts through resonant triad interactions (see Hammack & Henderson 1993). This is because since harmonic wavetrains exhibit a lengthscale comparable to the lengthscale of the (parametrically) excited subharmonic waves and thus resonance requires an additional condition on wavevectors.

In the general case, both forcing mechanisms will be present and will compete. The relative strength of each mechanism, namely the OBF and the surface waves, is determined by the container depth d and the magnitude of the damping C_g , respectively. Here, we shall consider the case in which the container depth is sufficiently large and forcing from the OBF dominates. This will be the relevant limit to explain the experimental available in the literature for this problem.

Both harmonic and (parametrically excited) subharmonic waves can promote a viscous mean flow, i.e., an overall circulation in the bulk region evolving on a slow timescale, induced by the shear stress associated with the propagation of gravity–capillary waves; the description of this was first made rigorous by Nicolas and Vega (1994) and further developed in a series of papers (see Higuera *et al.* 2014 for a review). The viscous mean flow can, in turn, modify the dynamics of surface waves. This mean flow can be incorporated in the (weakly nonlinear) analysis following the ideas in Vega, Rüdiger & Viñals (2004) but, since it appears in a third order term, it plays no role in the linear analysis considered in this thesis and will therefore be ignored.

2.2.4 The oscillatory bulk flow

The OBF is a linear harmonic flow produced by the vibrating endwalls. The specific property of this flow is that the flow variables (denoted hereafter with the superscript obf) are slowly varying in space. In particular,

$$|\nabla p^{\text{obf}}| \sim |\partial_z p^{\text{obf}}| \ll |p^{\text{obf}}|. \quad (2.25)$$

Thus, it is essentially different from the (also harmonic) counterpropagating wavetrains mentioned above, which show an $O(1)$ -wavevector and only affect an $O(1)$ -horizontal layer near the free surface. The OBF, instead, is slowly varying and penetrates deeper into the bulk. At leading order (as $a \rightarrow 0$), the associated pressure field is written as

$$p^{\text{obf}} = aP^{\text{obf}}(x, y, z) \cos 2t, \quad (2.26)$$

while the remaining flow variables are such that

$$|f^{\text{obf}}| \sim |w^{\text{obf}}| \sim |u^{\text{obf}}| \ll |p^{\text{obf}}|, \quad (2.27)$$

as is readily obtained, invoking (2.25), from the first boundary condition (2.23) and eq.(2.37). Using these, substituting (2.26) into eq.(2.19)-(2.21) and into the second boundary condition (2.23), and retaining only the leading

$O(a)$ -terms, yields

$$\Delta P^{\text{obf}} + \partial_{zz} P^{\text{obf}} = 0 \quad \text{in } |x| < L_1, |y| < L_2, -d < z < 0, \quad (2.28)$$

$$\partial_x P^{\text{obf}} = 4 \text{ at } x = \pm L_1, \quad \partial_y P^{\text{obf}} = 0 \text{ at } y = \pm L_2, \quad (2.29)$$

$$\partial_z P^{\text{obf}} = 0 \quad \text{at } z = -d, \quad P^{\text{obf}} = 0 \text{ at } z = 0. \quad (2.30)$$

Note the *zero-pressure boundary condition* at the unperturbed free surface, $z = 0$, which results from (2.27) and is a specific property of the OBF.

Because of the Neumann boundary conditions at the sidewalls, $y = \pm L_2$, the unique solution to (2.28)-(2.30) is independent of y and is readily solved in closed form,

$$P^{\text{obf}} = \frac{32d}{\pi^2} \sum_{m>0, \text{odd}} \frac{\sinh[m\pi x/(2d)] \cos[m\pi(z+d)/(2d)]}{m^2 \cosh[m\pi L_1/(2d)] \sin(m\pi/2)}, \quad (2.31)$$

where it has been taken into account that

$$\sum_{m>0, \text{odd}} \frac{4}{m\pi} \frac{\cos[m\pi(z+d)/(2d)]}{\sin(m\pi/2)} = 1 \quad \text{if } -d < z < 0. \quad (2.32)$$

The associated velocity components at $z = 0$ and the free surface elevation are readily calculated from equations (2.23) and (2.37) (2.29) (not used above), as

$$(\partial_z p^{\text{obf}}, f^{\text{obf}}, w^{\text{obf}}) = ag(x)(-4 \cos 2t, -\cos 2t, 2 \sin 2t), \quad \mathbf{u}^{\text{obf}} = \mathbf{0} \quad \text{at } z = 0, \quad (2.33)$$

where the forcing function g is given by

$$g(x) = -\partial_z P^{\text{obf}}/4 \equiv \frac{4}{\pi} \sum_{m>0, \text{odd}} \frac{\sinh[m\pi x/(2d)]}{m \cosh[m\pi L_1/(2d)]}. \quad (2.34)$$

It is noteworthy that $g(x)$ (and thus also $\partial_z p^{\text{obf}}$, w^{obf} , and f^{obf}) diverge logarithmically as $x \rightarrow \pm L_1$, but this weak singularity is consistent with the behavior of the harmonic flow in a $O(1)$ -region near the upper edge of the endwalls (namely, $|x \pm L_1| \sim |z| \sim 1$), as checked by Varas & Vega (2007). The function g is antisymmetric, namely

$$g(-x) \equiv -g(x), \quad (2.35)$$

which is due to the spatio-temporal reflection symmetry in (2.15). Also, if $d \ll L_1$, then $g(x)$ vanishes exponentially except in two $O(d)$ -regions near the endwalls. In particular, near $x = L_1$ (the approximation near $x = -L_1$ is obtained using (2.35)), g behaves as

$$g(x) = G(\zeta) \equiv \frac{4}{\pi} \sum_{m>0, \text{odd}} \frac{1}{m} \exp(-m\pi\zeta/2) \quad \text{with } \zeta = (x - L_1)/d. \quad (2.36)$$

Figure 2.4: The forcing function g vs. x/L_1 (left) and vs. $(x - L_1)/d$ (right) for $L_1/d = 0.1, 1, 2, 5, 10, 20$ and 50 ; arrows indicate increasing values of L_1/d .

For illustration, the function g is plotted in Fig.2.4 for several representative values of L_1/d . Note that when g is plotted vs. $(x - L_1)/d$ (right plot), only three curves are appreciated; the remaining curves coalesce to the approximation (2.36) as $L_1/d \geq 2$. This is because the error of the approximation is of the order of $e^{-\pi L_1/d}$.

2.3 The general amplitude equation for the linear evolution of subharmonic waves

This section is devoted to the derivation of the amplitude equation to study the subharmonic field triggered by the horizontal vibration of the container. Subsection 2.3.1 obtains a boundary condition for the velocity components just below the viscous boundary layer, which is the region where the amplitude equation is calculated. In subsection 2.3.2 the aim is to derive a *linear amplitude equation* for the free surface elevation, which provides the slow temporal evolution of the subharmonic waves.

2.3.1 Derivation of the evolution equations (2.24) for the velocity at the unperturbed free surface

Let us now derive eqs.(2.24), namely

$$\partial_t \mathbf{u} = -\nabla p, \quad \partial_t w = -\partial_z p + 2C_g \Delta w \quad \text{at } z = 0. \quad (2.37)$$

These equations coincide with their counterparts considered by Zhang & Viñals (1997), except for notation, since the quasi-potential approximation is written here in terms of the pressure, instead of the velocity potential. The derivation of (2.37) (and, in particular, ensuring that no additional quadratic terms resulting from convective terms are present) is subtle and is provided

here. There are two different ways of doing this, one is to solve the boundary layer using matched asymptotic expansions, which will be done for a related problem in section 4.5. Another, simpler (but more subtle) method will be the one used here.

In order to clarify the derivation, a special notation is introduced (just for this subsection), namely the superscripts ‘ufs’ and ‘ble’ denote the unperturbed free surface (where the velocity components will be calculated) and boundary layer edge (just below the oscillatory boundary layer that is attached to the free surface) where the boundary conditions (2.23) apply. It is important to mention that the quasi-potential variables used in subsection 2.2.2 would be ‘ble’, but the information from the boundary conditions at the free surface must be added.

To the approximation relevant here, the continuity equation in (2.4) and the boundary condition (2.14) are written as

$$\nabla \cdot \mathbf{u} + \partial_z w = 0 \text{ if } z \leq 0, \quad \partial_z \mathbf{u} + \nabla w = \mathbf{0} \text{ at } z = 0, \quad (2.38)$$

which imply that $\partial_z(\nabla \cdot \mathbf{u}) + \partial_{zz}w = 0$ and $\nabla \cdot (\partial_z \mathbf{u}) + \nabla \cdot (\nabla w) = 0$ at $z = 0$. Eliminating $\partial_z(\nabla \cdot \mathbf{u}) \equiv \nabla \cdot (\partial_z \mathbf{u})$ in these two equations yields

$$\partial_{zz}w = \Delta w \quad \text{at } z = 0. \quad (2.39)$$

This equation means that $\partial_{zz}w + \Delta w = 2\Delta w \neq 0$ at the free surface, while it vanishes in the bulk; thus, the above mentioned jump. Substituting (2.38)-(2.39) into the vertical momentum equation in (2.6) leads to

$$\partial_t w^{\text{ufs}} + 2\mathbf{u}^{\text{ufs}} \cdot \nabla w^{\text{ufs}} = -\partial_z p^{\text{ufs}} + 2C_g \Delta w^{\text{ufs}}. \quad (2.40)$$

On the other hand, applying the operators $\nabla \cdot$ and ∂_z to the momentum equations (2.5) and (2.6) respectively, and replacing (2.4) into the resulting equation leads to a standard Poisson equation for the pressure. Taking into account that $|\partial_z| \gg |\nabla|$ in the oscillatory boundary layer, the Poisson equation is written in first approximation as

$$\partial_{zz}p = \partial_z(\mathbf{u} \partial_z \mathbf{u}). \quad (2.41)$$

Integrating this equation across the oscillatory boundary layer yields

$$\partial_z p^{\text{ufs}} - \partial_z p^{\text{ble}} = \mathbf{u}^{\text{ufs}} \partial_z \mathbf{u}^{\text{ufs}} - \mathbf{u}^{\text{ble}} \partial_z \mathbf{u}^{\text{ble}}, \quad (2.42)$$

being the first order of $\mathbf{u}$ and ∇w continuous over the viscous layer and with vorticity equal to zero under the layer

$$\mathbf{u}^{\text{ble}} \partial_z \mathbf{u}^{\text{ble}} = \mathbf{u}^{\text{ble}} \nabla w^{\text{ble}} = \mathbf{u}^{\text{ufs}} \nabla w^{\text{ufs}}, \quad (2.43)$$

replacing $\nabla w = \partial_z \mathbf{u}$ on (2.42) the next relation is obtained:

$$\partial_z p^{\text{ufs}} - \partial_z p^{\text{ble}} = -2\mathbf{u}^{\text{ufs}} \cdot \nabla w^{\text{ufs}}. \quad (2.44)$$

The first equation (2.37) readily follows from the horizontal momentum equation (2.5), the pressure jump across the oscillatory boundary layer can be neglected in the approximation relevant here because

$$p^{\text{ufs}} - p^{\text{ble}} = O(\sqrt{C_g} |\mathbf{u}^{\text{ufs}} \cdot \nabla w^{\text{ufs}}|), \quad (2.45)$$

which is obtained from eq.(2.44), recalling that the thickness of the oscillatory boundary layer is $O(\sqrt{C_g})$. Substituting (2.44) into the horizontal momentum equation (2.5) and into (2.40) leads to

$$\partial_t \mathbf{u}^{\text{ufs}} = -\nabla p^{\text{ble}}, \quad \partial_t w^{\text{ufs}} = -\partial_z p^{\text{ble}} + 2C_g \Delta w^{\text{ufs}}, \quad (2.46)$$

which taking into account the notation in this section, coincides with the (2.37), which completes the derivation.

The expansion of the second equation in (2.37) does not coincide with the vertical momentum equation (2.6) due to a jump of Δw across the boundary layer. Both equations contain the same information but the terms on the expansion do not coincide on the viscous layer and outside of it.

2.3.2 The amplitude equation

The amplitude equation will be derived from the simplified equations (2.19)-(2.24) decomposing the flow variables appearing in these equations as

$$(p, f) = (p^{\text{obf}}, f^{\text{obf}}) + [e^{it}(P^s, F^s) + \text{c.c.}] + \dots, \quad (2.47)$$

$$(\mathbf{u}, w) = (\mathbf{u}^{\text{obf}}, w^{\text{obf}}) + [e^{it}(\mathbf{U}^s, W^s) + \text{c.c.}] + \dots, \quad (2.48)$$

where the harmonic OBF flow variables are as calculated in the last subsection, c.c. stands hereafter for the complex conjugate, and the superscript s refers to the subharmonic flow, which is such that

$$\begin{aligned} |\partial_t P^s| &\ll |P^s| \ll 1, & |\partial_t F^s| &\ll |F^s| \ll 1, \\ |\partial_t \mathbf{U}^s| &\ll |\mathbf{U}^s| \ll 1, & |\partial_t W^s| &\ll |W^s| \ll 1. \end{aligned} \quad (2.49)$$

These conditions implicitly imply a separation of temporal scales. Substituting (2.47)-(2.48) into (2.37) and setting to zero the coefficient of e^{it} lead to

$$\partial_t \mathbf{U}^s + i\mathbf{U}^s = -\nabla P^s, \quad \partial_t W^s + iW^s = -\partial_z P^s + 2C_g \Delta W^s, \quad (2.50)$$

which to the approximation relevant here (recall that ∂_t is small) yield

$$U^s = i\nabla P^s - \partial_t \nabla P^s, \quad W^s = i\partial_z P^s - \partial_{tz} P^s + 2C_g \partial_z \Delta P^s. \quad (2.51)$$

Substituting these and the expressions in (2.33) for the OBF into (2.48), then the resulting equation and (2.47) into (2.19)-(2.24), setting to zero the coefficient of e^{it} in the resulting equations (some of which will come from products of flow variables associated with the OBF and the subharmonic waves), and retaining only quadratic terms in the small quantities P^s , F^s , C_g , and a lead to the following formulation,

$$\Delta P^s + \partial_{zz} P^s = 0 \text{ in } -\infty < z < 0, \quad (2.52)$$

$$\partial_z P^s - F^s = -i\partial_t F^s - i\partial_{zt} P^s + 2iC_g \partial_z \Delta P^s + ag(x)\Delta \bar{P}^s/2, \quad (2.53)$$

$$P^s - (1 - S)F^s + S\Delta F^s = 2iC_g \partial_{zz} P^s + 3ag(x)\partial_z \bar{P}^s/2 \text{ at } z = 0,$$

$$\partial_z P^s \rightarrow 0 \text{ as } z \rightarrow -\infty. \quad (2.54)$$

Here, the function $g(x)$ is as given by (2.34), the overbar stands for the complex conjugate, and the boundary condition (2.54) results noting that the subharmonic surface waves only affect a horizontal layer of $O(1)$ -thickness attached to the free surface (the subharmonic flow being at rest below this layer).

Equations (2.52)-(2.54) provide the linear dynamics of the subharmonic surface waves, including the leading order effects of viscous damping and parametric forcing, accounted for in those terms proportional to $C_g \ll 1$ and $a \ll 1$, respectively. But this three-dimensional formulation can be reduced to a two-dimensional one by taking advantage of the fact that this problem is nearly singular. At leading order, setting $a = C_g = 0$ and neglecting time derivatives, equations (2.52)-(2.54) reduce to

$$\Delta P^s + \partial_{zz} P^s = 0 \text{ in } -\infty < z < 0, \quad (2.55)$$

$$\partial_z P^s - F^s = P^s - (1 - S)F^s + S\Delta F^s = 0 \text{ at } z = 0, \quad (2.56)$$

$$\partial_z P^s \rightarrow 0 \text{ as } z \rightarrow -\infty, \quad (2.57)$$

This unforced problem is singular, since it admits nontrivial, bounded solutions, namely

$$(P^s, F^s) = \sum_j A_j(x, y, t) (e^z, 1) e^{i(\nu_j x + \kappa_j y)}, \quad \text{with } \sqrt{\nu_j^2 + \kappa_j^2} = 1, \quad (2.58)$$

which invoking (2.47) represent combinations of plane travelling wavetrains. In fact, when the whole container (whose horizontal dimension is large) is considered, the complex amplitudes A_j and the unit wavevectors $k_j =$

$\nu_j e_x + \kappa_j e_y$ must be allowed to slowly vary in both space (horizontal directions) and time. This is the usual (quite involved) way to treat nearly inviscid surface waves in three space dimensions in the literature and generally requires considering a number of amplitude equations for the evolution of the amplitudes of the wavetrains, which are generally coupled, even in the linear approximation, because of reflection at the endwalls and sidewalls. The latter also implies that this approach depends in a critical way on geometry, meaning that the whole analysis must be customized for particular geometries. For instance, circular waves (instead of plane waves) should be considered in circular containers.

Here, instead, a simpler (but also somewhat more subtle) approach will be followed that provides a unique *general amplitude equation*. To this end, the first is to note that the wave fields (2.58) are such that

$$\partial_t F^s = 0, \quad \Delta P^s + P^s = 0, \quad \Delta F^s + F^s = 0, \quad (2.59)$$

which is equivalent to imposing that the complex amplitudes A_j and the wavenumbers $|\mathbf{k}_j| \equiv 1$ be both constant. Slow modulations due to viscous damping and parametric forcing promote slow spatial variations of both the complex amplitudes and the wavenumbers, which means that the exact equalities (2.59) must be replaced by

$$|\partial_t F^s| \ll 1, \quad |\Delta P^s + P^s| \ll |P^s|, \quad |\Delta F^s + F^s| \ll |F^s|. \quad (2.60)$$

These *resonance conditions* define slowly modulated resonant patterns, exhibiting wavenumbers that are close to one and implicitly imply a separation of scales in both space and time. Nonresonant patterns, not satisfying (2.60), decay to zero on the fast time scale $t \sim 1$ and can be ignored. As a second step, eq.(2.58) is written as

$$(P^s, F^s) = (e^z, 1) F^s(x, y, t), \quad (2.61)$$

where $F^s = \sum A_j e^{i(\nu_j x + \kappa_j y)}$ satisfies (2.60). The general amplitude equation for F^s will be obtained considering higher order terms (accounting for viscous damping, parametric forcing, and spatio-temporal modulations, neglected in (2.55)-(2.57)) in (2.61), namely

$$(P^s, F^s) = (e^z, 1) F^s(x, y, t) + (P_1^s, F_1^s) + \text{NRT}, \quad (2.62)$$

where NRT stand for nonresonant terms (not satisfying (2.60)). Instead, P_1^s and F_1^s are resonant and thus must be such that

$$|\partial_t P_1^s|, |\Delta P_1^s + P_1^s| \ll |P_1^s| \ll |P^s|, \quad |\partial_t F_1^s|, |\Delta F_1^s + F_1^s| \ll |F^s|. \quad (2.63)$$

Substituting (2.62) into (2.52)-(2.54) and invoking (2.60) and (2.63) yield

$$\partial_{zz}P_1^s - P_1^s = -(\Delta F^s + F^s)e^z \text{ in } -\infty < z < 0, \quad (2.64)$$

$$\begin{aligned} \partial_z P_1^s - F_1^s &= -2i\partial_t F^s + 2iC_g \Delta F^s - ag(x)\Delta \bar{F}^s/2, \\ P_1^s - F_1^s &= -S(\Delta F^s + F^s) + 2iC_g F^s + 3ag(x)\bar{F}^s/2 \text{ at } z = 0. \end{aligned} \quad (2.65)$$

$$\partial_z P_1^s = 0 \text{ at } z = -\infty, \quad (2.66)$$

This problem is singular (thus, it is not in general solvable) because its homogeneous part (setting to zero a , C_g , and F^s) exactly coincides with (2.55)-(2.57), which admits the nontrivial solutions (2.58). Thus, a *solvability condition* must be imposed and it is this solvability condition that provides the evolution equation for F^s . The solvability condition results multiplying (2.64) by e^z , integrating in $-\infty < z < 0$, integrating by parts, and imposing the boundary condition (2.66), which yields

$$\partial_z P_1^s - P_1^s = -(\Delta F^s + F^s)/2 \text{ at } z = 0. \quad (2.67)$$

Substituting here the boundary conditions (2.65) leads to the following linear evolution equation for F^s

$$\partial_t F^s + i(1 + 2S)(\Delta F^s + F^s)/4 = C_g(\Delta F^s - F^s) + iag(x)(3\bar{F}^s - \Delta \bar{F}^s)/4. \quad (2.68)$$

It is to be noted that since C_g , a , and ∂_t are all small, this equation contains terms that are not of the same order. In fact, at leading order $\Delta F^s + F^s = 0$, which precisely selects the allowable spatial patterns, and allows eq.(2.68) to be rewritten as

$$\partial_t F^s + i(1 + 2S)(\Delta F^s + F^s)/4 = -2C_g F^s + iag(x)\bar{F}^s. \quad (2.69)$$

This is precisely the above mentioned *general amplitude equation*, whose derivation was the main outcome of this subsection. Equation (2.69) applies in the bulk of the horizontal section of the container, outside four $O(1)$ -layers, near the endwalls and the sidewalls, where the waves are reflected. The approximation above involves a separation of spatial scales, an assumption that does not hold in these layers. The analysis of the wave reflection in these layers is given in section 2.4 and provides the boundary conditions. Equation (2.69) is linear and thus excludes wave-wave interactions, which could be enhanced by triad resonances (McGoldrick 1970) and facilitated at very small values of C_g (see below). These could mask the primary patterns calculated below, or even (when they occur at subcritical bifurcations) provide additional patterns at smaller forcing amplitudes than the linear threshold provided by (2.69)-(2.93). Interactions with the harmonic counterpropagating wavetrains can be ignored because of the last assumption (2.16), in which it has been taken into account that the OBF and the wavetrains penetrate a distance from the endwalls that scales with d and $1/C_g$, respectively.

2.4 Wave reflection near the endwalls and sidewalls

This analysis is in the spirit of Hocking (1987) and Nicolás, Rivas & Vega (1998); the reflection of waves aligned with the endwalls (such as the harmonic waves, ignored here) was already analyzed in the context of horizontal vibrations by Varas & Vega (2007).

Near the endwall $x = L_1$ (the endwall $x = -L_1$ and the sidewalls are treated similarly), the spatial variable x is shifted and the y -Fourier transforms of the pressure and the free surface deflection are considered, as

$$\hat{x} = x - L_1, \quad (P^s, F^s) = \int_{-1}^1 (q, \phi) e^{i\kappa y} d\kappa. \quad (2.70)$$

Substituting these into eqs.(2.55)-(2.57) (which are the relevant equations at leading order), the following equations and boundary conditions are obtained for the Fourier transforms q and ϕ

$$\partial_{\hat{x}\hat{x}}q + \partial_{zz}q - \kappa^2q = 0 \quad \text{in } 0 < \hat{x} < \infty, -\infty < z < 0, \quad (2.71)$$

$$\phi - \partial_zq = q - [(1 - S) + S\kappa^2]\phi + S\partial_{\hat{x}\hat{x}}\phi = 0 \quad \text{at } \hat{y} = 0, \quad (2.72)$$

$$\partial_{\hat{x}}q = \phi = 0 \quad \text{at } \hat{x} = 0, \quad \partial_zq \rightarrow 0 \quad \text{as } z \rightarrow -\infty. \quad (2.73)$$

If the boundary condition for ϕ at $\hat{x} = 0$ is ignored, then a simple solution exists, namely

$$(q_1, \phi_1) = (e^z, 1) \cos \nu x, \quad (2.74)$$

where the longitudinal wavenumber ν is defined as

$$\nu = \sqrt{1 - \kappa^2}. \quad (2.75)$$

In order to obtain a solution satisfying also the boundary condition for ϕ , a second solution is calculated that does not satisfy this boundary condition either; the solution satisfying the boundary condition will be obtained as a linear combination of both.

Ignoring again the boundary condition for ϕ at $\hat{x} = 0$, the equations and the remaining boundary conditions in (2.71)-(2.73) are solved via Fourier-cosine transform as

$$q_2 = \int_0^\infty B(\hat{\kappa}) e^{\sqrt{\kappa^2 + \hat{\kappa}^2} z} \cos(\hat{\kappa} \hat{x}) d\hat{\kappa}, \quad \phi_2 = \int_0^\infty B(\hat{\kappa}) \sqrt{\kappa^2 + \hat{\kappa}^2} \cos(\hat{\kappa} \hat{x}) d\hat{\kappa}, \quad (2.76)$$

where $B(\hat{\kappa})$ is calculated as follows. First, this expression for q_2 is introduced into the second boundary condition (2.72). Integrating the resulting equation for ϕ yields

$$\phi_2 = C e^{-\sqrt{(1-S\nu^2)/S} \hat{x}} + \int_0^\infty \frac{B(\hat{\kappa}) \cos(\hat{\kappa} \hat{x}) d\hat{\kappa}}{1 + S(\hat{\kappa} - \nu^2)}, \quad (2.77)$$

where C is an arbitrary constant and exponentially diverging behaviors of ϕ have been discarded. Identifying the two expressions for ϕ_2 in (2.76) and (2.77) and taking into account that

$$e^{-\sqrt{(1-S\nu^2)/S}\hat{x}} = \frac{2}{\pi} \int_0^\infty \frac{\sqrt{(1-S\nu^2)/S} \cos(\hat{\kappa}\hat{x}) d\hat{\kappa}}{\hat{\kappa}^2 + (1-S\nu^2)/S},$$

yields

$$B(\hat{k}) = \frac{2}{\pi} \frac{C \sqrt{S(1-S\nu^2)}}{\sqrt{1+\hat{\kappa}^2-\nu^2} [1+S(\hat{\kappa}-\nu^2)] - 1}, \quad (2.78)$$

where all integrals must be understood hereafter in the sense of the principal value. Substituting this into (2.76) and (2.77) (note that this expression for ϕ_2 exhibits better convergence properties than its counterpart in (2.76)) and setting $C = 1$ leads to

$$q_2 = \frac{2}{\pi} \int_0^\infty \frac{\sqrt{S(1-S\nu^2)} e^{\sqrt{\hat{\kappa}^2+\hat{\kappa}^2}z} \cos(\hat{\kappa}\hat{x}) d\hat{\kappa}}{\sqrt{1+\hat{\kappa}^2-\nu^2} [1+S(\hat{\kappa}-\nu^2)] - 1}, \quad (2.79)$$

$$\begin{aligned} \phi_2 = e^{-\sqrt{(1-S\nu^2)/S}\hat{x}} &+ \frac{2}{\pi} \int_0^\infty \frac{\sqrt{S(1-S\nu^2)} \cos(\hat{\kappa}\hat{x}) d\hat{\kappa}}{\sqrt{1+\hat{\kappa}^2-\nu^2} [1+S(\hat{\kappa}-\nu^2)]^2 - [1+S(\hat{\kappa}-\nu^2)]}. \end{aligned} \quad (2.80)$$

Now, once this second solution has been calculated, the general solution of (2.71)-(2.73), ignoring the boundary condition for ϕ at $\hat{x} = 0$, is written as

$$(q, \phi) = C_1(q_1, \phi_1) + C_2(q_2, \phi_2). \quad (2.81)$$

Invoking (2.74) and (2.80), and imposing the boundary condition that has been ignored so far (namely, $\phi = 0$ at $\hat{x} = 0$), it is seen that C_1 and C_2 must be such that

$$C_1 = -C_2 H_1(\nu^2, S), \quad (2.82)$$

where the scalar H_1 is given by

$$H_1 = 1 + \frac{2}{\pi} \int_0^\infty \frac{\sqrt{S(1-S\nu^2)} d\hat{\kappa}}{\sqrt{1+\hat{\kappa}^2-\nu^2} [1+S(\hat{\kappa}-\nu^2)]^2 - [1+S(\hat{\kappa}-\nu^2)]}. \quad (2.83)$$

Substituting (2.82) into (2.81) and setting, e.g., $C_2 = -1$, a solution of (2.71)-(2.73) is obtained that shows the asymptotic behavior (up to exponentially small terms)

$$\phi = H_1(\nu^2, S) \cos \nu \hat{x} + \frac{2\sqrt{S(1-S\nu^2)} \sin \nu \hat{x}}{(1+2S)\nu} \quad \text{as } \hat{x} \rightarrow \infty. \quad (2.84)$$

Figure 2.5: The scalar γ appearing in (2.88).

This behavior follows noting in (2.80) that $\int_0^\infty (\nu^2 - \hat{\kappa}^2)^{-1} \cos(\hat{\kappa} \hat{x}) d\hat{\kappa} = \pi \sin(\nu \hat{x}) / (2\nu)$ and that if $\hat{\kappa}^2 g(\hat{\kappa})$ is bounded, then $\int_0^\infty g(\hat{\kappa}) \cos(\hat{\kappa} x) d\hat{\kappa} \rightarrow 0$ exponentially as $\hat{x} \rightarrow \infty$. The limit (2.84) can also be written as

$$\phi = \sin[\nu \hat{x} - \alpha(\nu^2, S)] \quad \text{as } \hat{x} \rightarrow \infty, \quad (2.85)$$

where the phase shift α is given by

$$\alpha(\nu^2, S) = \tan^{-1} \frac{(1 + 2S)\nu H_1(S, \nu^2)}{2\sqrt{S(1 - S\nu^2)}}. \quad (2.86)$$

Still, the behavior (2.85) can also be written as

$$\partial_{\hat{x}} \phi - \gamma \phi = 0 \quad \text{at } \hat{x} = 0, \quad (2.87)$$

where (invoking also (2.75))

$$\gamma(\kappa^2, S) = \frac{\nu}{\tan \alpha} = \frac{2\sqrt{S(1 - S + S\kappa^2)}}{(1 + 2S)H_1(1 - \kappa^2, S)}. \quad (2.88)$$

For illustration, the scalar γ is plotted in Fig.2.5 vs. $|k_2|$ for various representative values of S , as indicated.

Matching with the outer solution, outside the layer analyzed in this section, and invoking (2.70) provides the following boundary condition for the solution in the bulk of the container

$$\partial_x F^s - \int_{-1}^1 \gamma(\kappa^2, S) \phi(\kappa) e^{i\kappa y} d\kappa = 0 \quad \text{at } x = -L_1, \quad (2.89)$$

where ϕ is the Fourier transform of $F(L_1, y)$, defined such that

$$F^s(L_1, y) = \int_{-1}^1 \phi(\kappa) e^{i\kappa y} d\kappa. \quad (2.90)$$

2.4. WAVE REFLECTION NEAR THE ENDWALLS AND SIDEWALLS

Note in Fig.2.5 that $\gamma(\nu) \equiv 0$ for $S = 0$, which means that the boundary condition (2.89) reduces to $\partial_{\hat{x}} F^s = 0$ in this limit. This is precisely the correct behavior of F^s at the wall in the gravity limit, which is fortunate because, in principle, the analysis above is singular at $S = 0$.

On the other hand, the strictly inviscid, unforced approximation above ignores both a Stokes boundary layer attached to the endwall and higher order terms resulting from products of $a \ll 1$ and the flow variables. If all these were taken into account, a fairly involved analysis (omitted here) would include higher order terms in the boundary condition (2.89), which would read

$$\partial_{\hat{x}} F^s - \int_{-1}^1 [(\gamma + \sqrt{iC_g} \gamma_1) \phi + a \gamma_2 \bar{\phi}] d\kappa = 0 \quad \text{at } \hat{x} = 0, \quad (2.91)$$

for some real coefficients γ_1 and γ_2 that are $O(1)$. Nevertheless, the $O(\sqrt{C_g})$ -coefficient γ_1 is expected to be somewhat small due to the fact that the contact line is pinned; this has been theoretically shown by Higuera, Nicolás & Vega (1994) and Martel, Nicolás & Vega (1998) for the first sloshing modes in various fluid configurations and experimentally checked by Howell et al. (2000). Of course, this cannot be true in the strict gravity limit, $S = 0$, but the additional term can still be neglected under the last assumption imposed in (2.16). Similarly, since the OBF extends a distance $d \gg 1$ in the bulk, the parametric forcing term in (2.91) can be neglected compared to the parametric forcing effect of the OBF. It is interesting to note that this term provides the parametric forcing effect of the vibrating endwalls acting as wave makers, which would be the only source for parametric forcing if the OBF were not present.

The analysis of the wave reflection provides the following boundary conditions, which are the counterparts of that for $x = -L_1$ in eq.(2.90), namely

$$\partial_x F^s \pm \int_{-1}^1 \gamma(\kappa^2, S) \phi_{\pm L_1}(\kappa) e^{i\kappa y} d\kappa = 0 \quad \text{at } x = \pm L_1, \quad (2.92)$$

$$\partial_y F^s \pm \int_{-1}^1 \gamma(\nu^2, S) \phi_{\pm L_2}(\nu) e^{i\nu x} d\nu = 0 \quad \text{at } y = \pm L_2, \quad (2.93)$$

where the real scalar γ is as given in eq.(2.88) (and plotted for various values of S in Fig.2.5) and the Fourier transforms $\phi_{\pm L_1}$ and $\phi_{\pm L_2}$ are defined such that

$$F^s(x, \pm L_2) = \int_{-1}^1 \phi_{\pm L_2}(\nu) e^{i\nu x} d\nu, \quad F^s(\pm L_1, y) = \int_{-1}^1 \phi_{\pm L_1}(\kappa) e^{i\kappa y} d\kappa. \quad (2.94)$$

Equations (2.69),(2.92)-(2.93) will be used in the following subsection to analyze the subharmonic instability. Note that this formulation is much simpler than current approximations since it is a scalar equation depending on the two horizontal coordinates only. Compared to the quasi-potential approximation, the pressure, the velocity, and dependence on both the vertical coordinate z and the fast timescale $t \sim 1$ are filtered out. Equations (2.69),(2.92)-(2.93) are invariant under the following reflection symmetries

$$(x, F^s) \rightarrow (-x, iF^s) \quad \text{and} \quad y \rightarrow -y, \quad (2.95)$$

which invoking (2.47) result from the symmetries (2.15). The solutions will not generally be invariant under the x -symmetry because invariant solutions would be such that $F^s(0) = iF^s(0)$, which implies the non-generic condition $F^s(0) = 0$. Non-invariance under this symmetry is consistent because the resulting patterns will generally be quasi-periodic.

2.5 Subharmonic instability threshold and patterns

This section is devoted to first calculating the subharmonic instability threshold, in subsection 2.5.1, and then to a discussion of the associated spatio-temporal patterns, in subsection 2.5.2. The starting point is the simplified formulation (2.69),(2.92)-(2.93).

2.5.1 Subharmonic instability

The instability threshold is calculated by imposing marginal instability, namely, requiring that the temporal behavior be purely oscillatory. Taking advantage of the fact that the coefficients of (2.69) do not depend on y , marginal instability can be analyzed decomposing the solution in normal modes in the y -direction, namely setting

$$F^s = \sum_{m=0}^{\infty} F_m(x, t) \cos(\kappa_m y + \delta_m), \quad (2.96)$$

where the transversal wavenumbers κ_m and the phase shifts δ_m must be such that

$$0 \leq \kappa_m \leq 1, \quad -\pi < \delta_m < \pi, \quad (2.97)$$

and are determined by the following equations

$$-\kappa_m \sin(\kappa_m L_2 + \delta_m) + \gamma(1 - \kappa_m^2, S) \cos(\kappa_m L_2 + \delta_m) = 0, \quad (2.98)$$

$$-\kappa_m \sin(-\kappa_m L_2 + \delta_m) - \gamma(1 - \kappa_m^2, S) \cos(-\kappa_m L_2 + \delta_m) = 0, \quad (2.99)$$

which result from substituting (2.96) into the boundary conditions at $y = \pm L_2$ in (2.93). After some algebra, eqs.(2.98)-(2.99) yield

$$\delta_m = m\pi/2 \pmod{2\pi}, \quad \tan(\kappa_m L_2 + \delta_m) = \gamma(1 - \kappa_m^2, S)/\kappa_m. \quad (2.100)$$

This equation defines the discrete values of the transversal wavenumber κ_m that are compatible with the boundary conditions at the sidewalls and provides a large number of values of κ_m (of the order of L_2/π) because L_2 is large, which can be sorted such that $0 < \kappa_1 < \kappa_2 < \dots$. Note, invoking (2.96), that even and odd values of m provide modes whose dependence on y is even and odd, respectively, which is consistent with the y -reflection symmetry in (2.95).

On the other hand, substitution of (2.96) into (2.69) and the boundary conditions in (2.92) at $x = \pm L_1$ leads to

$$\partial_t F_m + i(1 + 2S)[\partial_{xx} F_m + \nu_m^2 F_m]/4 = -2C_g F_m + iag(x)F_m, \quad (2.101)$$

$$\partial_x F_m \pm \gamma(1 - \nu_m^2, S)F_m = 0 \quad \text{at } x = \pm L_1 \quad (2.102)$$

where, for convenience, the longitudinal wavenumber, defined as

$$\nu_m = \sqrt{1 - \kappa_m^2} \geq 0, \quad (2.103)$$

will be used below instead of the transverse wavenumber κ_m . This is, in fact, a longitudinal wavenumber because, since $\partial_t, a, C_g \ll 1$, at leading order eq.(2.101) yields $F_m \sim e^{\pm i\nu_m x}$. Thus, the wavevector of the pattern is $\mathbf{k}_m = \nu_m \mathbf{e}_x + \kappa_m \mathbf{e}_y$, meaning that the angle of the wavevector with the endwalls is given by

$$\varphi = \cos^{-1} \nu_m. \quad (2.104)$$

Marginal instability is determined by requiring that the linear problem (2.101)-(2.102) exhibits purely oscillatory behaviors, which is done setting

$$F_m(x, t) = A_m F_m^+(x) e^{i\Omega t} + \bar{A}_m \bar{F}_m^-(x) e^{-i\Omega t}, \quad (2.105)$$

where the complex amplitude A_m is arbitrary in the present linear approximation, and the eigenfunctions $F_m^\pm$ are given by

$$i\Omega F_m^+ + i(1 + 2S)[\partial_{xx} F_m^+ + \nu_m^2 F_m^+]/4 = -2C_g F_m^+ + iag(x)F_m^-, \quad (2.106)$$

$$i\Omega F_m^- - i(1 + 2S)[\partial_{xx} F_m^- + \nu_m^2 F_m^-]/4 = -2C_g F_m^- - iag(x)F_m^+, \quad (2.107)$$

$$\partial_x F_m^+ \pm \gamma(1 - \nu_m^2, S)F_m^+ = \partial_x F_m^- \pm \gamma(1 - \nu_m^2, S)F_m^- = 0 \quad \text{at } x = \pm L_1, \quad (2.108)$$

as obtained by substituting (2.105) into (2.101)-(2.102). It is to be anticipated that

$$|\Omega| \ll 1, \quad (2.109)$$

which is compatible with the separation of scales assumption (2.60).

Equations (2.106)-(2.108) are invariant under the symmetries

$$(x, F_m^+, F_m^-) \rightarrow (-x, F_m^+, -F_m^-), \quad (\Omega, F_m^+, F_m^-) \rightarrow (-\Omega, \bar{F}_m^-, \bar{F}_m^+), \quad (2.110)$$

which mean, in particular, that the eigenvalues appear in complex conjugate pairs. Thus, the additional symmetry $(x, F_m^+, F_m^-) \rightarrow (-x, -F_m^+, F_m^-)$ is a consequence of these two and needs not be considered. Assuming simple eigenvalues (which are to be expected generically), the eigenspace must be one-dimensional in the complex domain, which invoking the first symmetry in (2.110) implies that

$$F_m^+(-x) \equiv F_m^+(x), \quad F_m^-(-x) \equiv -F_m^-(x) \quad (2.111)$$

and allows for integrating (2.106)-(2.108) only in $0 \leq x \leq L_1$. This is done numerically using two methods, whose results are used for mutual validation, as explained in Appendix A and B. In particular, for fixed values of S , C_g , L_1 , and d (which is implicit in the function g , see (2.34)), a marginal instability curve is obtained, namely

$$a = a(\nu_m), \quad \Omega = \Omega(\nu_m). \quad (2.112)$$

In fact, infinitely many such curves are obtained (with generally increasing values of a) because (2.106)-(2.108) involves a boundary value problem for a set of differential equations. Nevertheless, only that branch providing the smallest value of a (which will be referred to below as *the first branch*) needs to be considered; but see subsection 2.6 below for a discussion on the second branch for very small C_g . The minimum value of a in this first branch, a_c , corresponds to the instability threshold in the continuous case, as $L_2 \rightarrow \infty$. The associated minimum when only the discrete values of ν_m satisfying (2.100) and (2.103) are considered, will be denoted by a_c^d .

For illustration, a representative plot (which in fact corresponds to one of the experimental runs considered below, in section 2.8) of the functions (2.112) along the first branch is provided in Fig.2.6, where for convenience a and Ω are both rescaled with $(1 + 2S)/4$, and the allowable discrete values of ν_m are indicated with circles. In this case, $a_c^d = 0.04$, which corresponds to $m = 25$ and $\nu_m = 0.26$, meaning that the pattern is antisymmetric in y because m is odd. Also, the subharmonic waves are oblique because $\nu_m \neq 0$, and according to (2.104), the wavevector forms an angle $\varphi = \sin^{-1} 0.26 = 75^\circ$ with the endwalls. Note that (consistently with the large value of L_2) the discrete threshold is quite close to continuous threshold (the minimum of the $a = a(\kappa_m)$ curve), $a_c = 0.0399$, which corresponds to $\nu_m = 0.30$. Also, as anticipated, Ω is quite small (in fact, $\Omega \sim 10^{-6}$ at threshold), which is consistent with the large value of L_1 , see below.

2.5. SUBHARMONIC INSTABILITY THRESHOLD AND PATTERNS

Figure 2.6: Rescaled marginal instability threshold amplitude a (left) and rescaled frequency Ω (right) vs. the longitudinal wavenumber ν_m for $L_1 = 41.5$, $d = 46.1$, $C_g = 0.027$, and $S = 0.63$. The discrete values of ν_m for $L_2 = 41.5$ are plotted with circles.

Since Ω is small, the second symmetry (2.110) implies that the eigenfunctions can always be selected (requiring that $F_m^+(L_1) = -\bar{F}_m^-(L_1)$) such that

$$F_m^L = F_m^+ + \bar{F}_m^- \simeq 0 \text{ if } x > 0, \quad F_m^R = i(F_m^+ - \bar{F}_m^-) \simeq 0 \text{ if } x < 0, \quad (2.113)$$

which mean that these two functions are concentrated near the left and right endwalls, respectively. Also, invoking (2.111), these two functions are such that

$$F_m^L(-x) = iF_m^R(x). \quad (2.114)$$

In order to appreciate (2.113)-(2.114), the functions $F_m^\pm$, F_m^L , and F_m^R at threshold are plotted in Fig.2.7 for the case considered in Fig.2.6.

The reconstructed spatio-temporal patterns are obtained by substituting (2.105) into (2.96) (retaining only one mode), substituting this latter expression into (2.47), and concentrating only on subharmonic waves. The free surface elevation is given by

$$f = f_m(x, t) \cos(\kappa_m y + \delta_m), \quad (2.115)$$

where the longitudinal pattern f_m is written as

$$\begin{aligned} f_m = e^{it} |A_m| (e^{i\psi_m(t)} F_m^+ + e^{-i\psi_m(t)} \bar{F}_m^-) + \text{c.c.} = \\ e^{it} |A_m| [F_m^L \cos \psi_m(t) + iF_m^R \sin \psi_m(t)] + \text{c.c.}, \end{aligned} \quad (2.116)$$

in terms of the slowly varying phase ψ_m , defined as

$$\psi_m = \Omega t + \text{phase of } A_m. \quad (2.117)$$

Figure 2.7: Top: real (left) and imaginary (right) parts of the eigenmodes F^+ (solid lines) and F^- (dashed lines) vs. x/L_1 , at the instability threshold for the case considered in Fig. 2.6. Bottom: real (left) and imaginary (right) parts of $F^+ + \bar{F}^-$ (solid lines) and $i(F^+ - \bar{F}^-)$ (dashed lines) for the same eigenfunctions.

This represents a quasiperiodic pattern that exhibits the frequencies 1 and $\Omega \ll 1$. In order to elucidate how the patterns (2.115) are, the behavior of $f_m(x, t)$ on the fast timescale is considered, noting that it shows the symmetries

$$(\psi_m, f_m) \rightarrow (\psi_m + \pi, -f_m), \quad (t, f_m) \rightarrow (t + \pi, -f_m), \quad (2.118)$$

$$(x, \psi_m, t) \rightarrow (-x, \psi_m + \pi/2, t - \pi/2). \quad (2.119)$$

The symmetries (2.118) allow for restricting to a half of the period in both the slow and fast timescales, namely $0 \leq \psi_m(t) \leq \pi$ and $0 \leq t \leq \pi$, respectively. The following distinguished fast time behaviors in the interval $0 \leq t \leq \pi$ occur. (i) Near $\psi_m(t) = 0 \pmod{\pi}$, the longitudinal patterns is $f_m \simeq e^{it}(F^+ + \bar{F}^-) + \text{c.c.}$, which invoking the approximate symmetry (2.113) means that activity is concentrated near the left endwall; (ii) Near $\psi_m(t) = \pi/4 \pmod{\pi}$, instead, $f_m \simeq e^{it}[(1+i)F^+ + (1-i)\bar{F}^-]/\sqrt{2} + \text{c.c.}$, which invoking the approximate symmetry (2.113) means that activity is present near both endwalls. The fast time behavior (iii) near $\psi = \pi/2 \pmod{\pi}$ and near (iv) $\psi_m(t) = 3\pi/4 \pmod{\pi}$ are obtained from those near $\psi_m(t) = 0$ and $\pi/4$ using the symmetry (2.119).

Collecting the above, it follows that in the slow timescale, the system subsequently visits the four fast time behaviors described above, namely (i), (ii), (iii), (iv), (i), (ii),..., and so on, showing a slow, smooth transition between at intermediate values of $\psi_m(t)$. This can be seen in figure 2.8-left, where a space-time plot of the transverse root mean square of the free surface elevation:

$$\langle f \rangle_y = \sqrt{\frac{1}{2L_2} \int_{-L_2}^{L_2} f^2 dy}, \quad (2.120)$$

is given. Note that $\langle f \rangle_y \sim f_m$.

All these suggest that the (small but) nonzero values of the modulation frequency Ω are due to interaction between the dynamics near the two end-walls, meaning that $\Omega \rightarrow 0$ as $L_1 \rightarrow \infty$ (in fact, as L_1 is large compared to both d and C_g^{-1}), which will be confirmed in section 2.7 below.

It also should be mentioned that although this frequency is small it could lead to misleading conclusions, because depending on the initial conditions any of the four fast time behaviours could be seen during an experimental running. If this drift is not detected (which could easily happen with $\Omega \sim 10^{-6}$) the patterns could seem different under a repetition of the running.

2.5.2 Primary and secondary spatio-temporal patterns

In the formulation above, the subharmonic instability is associated with a Hopf bifurcation of the flat solution $F^s = 0$. In terms of the original formulation (2.4)-(2.6), (2.10)-(2.14), a system with periodic coefficients, the bifurcation is associated with a pair of complex Floquet exponents, $i(\pi \pm \Omega)$ (whose Floquet multipliers are both close to -1). Both approaches are equivalent. Here the Hopf bifurcation is considered, which is analyzed by adding cubic nonlinear terms in the general amplitude equation (2.69) and performing a weakly nonlinear analysis (Guckenheimer & Holmes 1983, Kuznetsov 1998). The complex amplitude A of the marginal mode satisfies a Landau equation,

$$A'(t) = \beta_1(a - a_c)A - \beta_2|A|^2A, \quad (2.121)$$

where the real part of β_1 is positive (because the instability sets in for $a > a_c$) and the real part of β_2 is also positive if the bifurcation is supercritical, as is assumed hereafter. As time increases, $A(t)$ converges to a periodic solution of the form

$$A = A^0 e^{i\Omega_1 t}, \quad (2.122)$$

where* $|A^0|^2 \sim |\Omega_1| \sim |a - a_c|$ as $a \rightarrow a_c$. Substituting (2.122) into (2.105) just provides a small correction of the frequency Ω in the temporally modulated

*There is a typo in Perez-Gracia et. al (2014) in this estimate

patterns described in the last subsection, which are the expected patterns when $|a - a_c|$ is sufficiently small and the bifurcation is supercritical.

On the other hand, since the transverse length L_2 is large, interaction with nearby modes is expected near threshold. Since the instability thresholds of the interacting modes are near the minimum of the curve $a = a(\kappa_m)$ defined in (2.112), eqs.(2.98)-(2.99), (2.103), (2.106)-(2.108) show that

$$\begin{aligned} |\kappa_m - \kappa_{m-1}| &\sim |\nu_m - \nu_{m-1}| \sim |F_m^\pm - F_{m-1}^\pm| \sim \\ |\Omega_m - \Omega_{m-1}| &\sim \sqrt{|a_{c,m} - a_{c,m-1}|} \sim 1/L_2^2 \ll 1. \end{aligned} \quad (2.123)$$

This is clear in Fig.2.6-left, where the mode $m = 25$ bifurcates first, but the $m - 1 = 24$ mode is nearly marginal. Interaction between the $(m - 1)$ -th and m -th modes is accounted for considering the free surface pattern

$$\begin{aligned} f = e^{it} [A_{m-1} F_{m-1}^+ e^{i\Omega t} + \bar{A}_{m-1} \bar{F}_{m-1}^- e^{-i\Omega t}] \sin(\kappa_{m-1} y + m\pi/2) + \\ e^{it} [A_m F_m^+ e^{i\Omega t} + \bar{A}_m \bar{F}_m^- e^{-i\Omega t}] \cos(\kappa_m y + m\pi/2) + \text{c.c.} + \dots, \end{aligned} \quad (2.124)$$

where the coupled evolution of A_{m-1} and A_m is given by the amplitude equations

$$A'_{m-1} \simeq [\beta_1(\varepsilon - \delta_1) - i\delta_2] A_{m-1} - (\beta_2 |A_{m-1}|^2 + \beta_3 |A_m|^2) A_{m-1} + \beta_4 \bar{A}_{m-1} A_m^2, \quad (2.125)$$

$$A'_m \simeq [\beta_1(\varepsilon + \delta_1) + i\delta_2] A_m - (\beta_2 |A_m|^2 + \beta_3 |A_{m-1}|^2) A_m + \beta_4 \bar{A}_m A_{m-1}^2. \quad (2.126)$$

These equations correspond to a double Hopf bifurcation with strong 1:1 resonance (Guckenheimer & Holmes 1983, Kuznetsov 1998), with the additional requirement that they are invariant under the actions $A_{m-1} \rightarrow -A_{m-1}$ and $A_{m-1} \leftrightarrow A_m$, which are associated with the transformations $y \rightarrow -(y + m\pi/\kappa_m)$ and $y \rightarrow \pi/(2\kappa_m) - y$, respectively. The slow frequency Ω appearing in (2.124) and the bifurcation parameter, the threshold splitting δ_1 , and the detuning δ_2 in (2.125)-(2.126) are defined in terms of the threshold amplitudes and eigenfrequencies of the interacting modes as

$$\begin{aligned} \Omega &= (\Omega_{m-1} + \Omega_m)/2, \quad \varepsilon = a - (a_{c,m-1} + a_{cm})/2, \\ \delta_1 &= (a_{c,m-1} - a_{c,m})/2, \quad \delta_2 = (\Omega_{m-1} - \Omega_m)/2. \end{aligned} \quad (2.127)$$

Equations (2.125)-(2.126) exhibit pure mode solutions, $(A_{m-1}, 0)$ and $(0, A_m)$. For these, eqs.(2.125)-(2.126) decouple and reduce to the Landau equation (2.121) considered above; pure modes are in the primary bifurcations at $a = a_{c,m-1}$ and $a = a_{c,m}$. There are also mixed mode solutions, with

Figure 2.8: Space-time plots of the transverse root mean square of the free surface elevation, $\langle f \rangle_y$ defined in eq.(2.120) of type I (left) and type II (right) patterns with $\Omega = 0.1$ (much larger than the typical experimental values to avoid a too large time interval) and $L_2 = 50$.

$A_{m-1} \neq 0$ and $A_m \neq 0$, that generally provide more complex temporally modulated patterns. In particular, if

$$|\delta_1|, |\delta_2| \ll \varepsilon, \quad (2.128)$$

then there are two types of simple mixed modes,

$$\text{type I: } (A_{m-1}, A_m) \simeq (\pm A, A), \quad \text{type II: } (A_{m-1}, A_m) \simeq (\pm iA, A), \quad (2.129)$$

where the + and - signs correspond to reflection in y and provide similar patterns.

Type I mixed modes produce patterns that are approximated by substituting $(A_{m-1}, A_m) = (A, A)$ into (2.124) then setting $\kappa_{m-1} = \kappa_m$ and $F_{m-1}^\pm = F_m^\pm$ (see (2.123)), to obtain

$$f = \sqrt{2}e^{it}|A|[F_m^L \cos \psi(t) + F_m^R \sin \psi(t)] \sin(\kappa_m y + m\pi/2 + \pi/4) + \text{c.c.}, \quad (2.130)$$

where $F_m^{L,R}$ are as defined in (2.113) and the slowly varying phase ψ is given by (2.117). Comparison with eqs.(2.115)-(2.117) shows that this pattern is quite similar to the primary pattern (except for the factor $\sqrt{2}$ and the $\pi/4$ phase shift in the y direction). Fig.2.8-left shows that activity alternates between the endwalls in the slow timescale.

The patterns associated with mixed modes of type II, instead, are different from the primary patterns. Substituting $(A_{m-1}, A_m) = (iA, A)$ into (2.124) and setting $\kappa_{m-1} = \kappa_m$ and $F_{m-1}^\pm = F_m^\pm$, as above, lead to

$$f = e^{it}|A| \left[F_m^L \cos[\kappa_m y + m\pi/2 + \psi(t)] + F_m^R \sin[\kappa_m y + m\pi/2 + \psi(t)] \right] + \text{c.c.} \quad (2.131)$$

Invoking (2.114), this pattern is invariant on the fast timescale under the spatiotemporal symmetry

$$t \rightarrow t + \pi/2, \quad x \rightarrow -x, \quad y \rightarrow (2\psi - m\pi + \pi/2)/\kappa_m - y. \quad (2.132)$$

consistent with the basic symmetries (2.15) in the present mode interaction context. Thus, activity cycles between the endwalls already in the fast timescale, as seen in Fig.2.8-right and also in Fig.2.9, where four representative snapshots of one period in the fast scale are considered. Both obliqueness and the symmetry (2.132) are also evident in Fig.2.9. The effect of the slow frequency does not lead here to any left-right modulation of the pattern; instead, it just provides a slow drift in the transverse direction, which must be combined with a transverse modulation, considered next.

The small difference between $(F_{m-1}^\pm, \kappa_{m-1})$ and $(F_m^\pm, \kappa_m)$, not accounted for above, produces small differences in longitudinal wavelength near both endwalls and, more importantly, a transverse modulation. This could be accounted for by using the full expression (2.124) to calculate the patterns (2.130) and (2.131), which would show that the amplitude of the pattern varies by an $O(1)$ quantity in the transverse direction. This transverse modulation is due to the interaction of just two modes, which locally cancel or reinforce each other. This effect is very sensitive to the wavelengths, amplitudes, and phases of the interacting modes. In other words, transverse modulations are very much affected by the neglected differences between the coefficients in the amplitude equations (2.125)-(2.126) and the symmetry-breaking parameters δ_1 and δ_2 . Analysis of these is well beyond the scope of the thesis.

Summarizing the above:

- (i) The patterns are generally oblique.
- (ii) The primary patterns at threshold are quasiperiodic, show two well-separated timescales, and show activity that cycle slowly between both endwalls; see Fig.2.8-left.
- (iii) Since $L_2 \gg 1$, interaction with modes exhibiting nearby wavenumbers is expected very soon after threshold and produces two types

Figure 2.9: Grey-scale contours of the free surface elevation (as given by (2.131) with $\psi(t) = \pi/4$) at $t = \psi$ (top, left), $\psi + \pi/4$ (top, right), $\psi + \pi/2$ (bottom, left), and $\psi + 3\pi/4$ (bottom, right), for the case in Figs.2.6-2.7, considering the interaction of the $m - 1 = 24$ -th and $m = 25$ -th modes.

of mixed-mode patterns. Type I patterns are qualitatively similar to the primary patterns. Type II patterns, instead, cycle between the two endwalls (see Figs.2.8-right and 2.9) already in the fast timescale. Both types of mixed-mode patterns may also show a significant transverse modulation, but this is affected by several small effects that have been neglected above.

- (iv) Since the second frequency Ω is small, experimental observation of the slow timescale may require very long experimental runs. In shorter runs, the portion of the slow oscillation that is visualized depends on initial conditions, and thus may change (in an apparently random fashion) from one run to another.
- (v) More complex modulated patterns resulting from, e.g., time-dependent solutions of (2.125)-(2.126), interaction between three or more modes, and cases in which the two bifurcations mentioned above are subcritical, are to be expected as $a - a_c$ increases further, but all these are again beyond the scope of the thesis.

2.6 The shape of the marginal instability curves

The validity of the theory developed above relies on the assumptions (2.16), which in practice require that $2L_1 \geq 12\pi$, $2L_2 \geq 12\pi$ (for at least six wavelengths to be present in the container), $e^{-d} \ll 1$, and $C_g \ll 1$. These can be generally considered to hold if

$$L_1, L_2 \geq 15, \quad d \geq 5, \quad \text{and} \quad C_g \leq 0.1. \quad (2.133)$$

Let us now elucidate how the non-dimensional instability threshold a_c and the associated values of the non-dimensional frequency shift Ω and longitudinal wavenumber ν_m depend on the various parameters of the problem, calculating a_c as explained in subsection 2.5.1. To this end, the attention is on the marginal instability curves associated with the first branch (2.112), which generally provides the smallest value of a . These curves are qualitatively similar to each other. Since L_2 is large, the threshold longitudinal wavenumber, ν_m , is roughly that corresponding to the minimum value of a_c in each resonance curve, denoted as ν_m^c .

Dependence of these curves on L_1 , S , and C_g for $d = 5$ is summarized in Fig.2.10; for other values of d , the marginal curves are qualitatively similar to those in Fig.2.10.

These plots show that the instability threshold increases monotonously with C_g (namely as viscosity is increased), which was obviously to be expected.

Figure 2.10: Marginal instability curves, a_c vs. ν_m , for $d = 5$ for $L_1 = 15$ (left) and 50 (right), $S = 0$ (top), 0.5 (middle), and 1 (bottom). Each plot contains curves for $C_g = 0.001, 0.005, 0.01, 0.05$ and 0.1 , with arrows indicating decreasing values of C_g . Circles indicate the approximate values of the minima of the curves calculated in subsection 2.6.1.

Also as expected, the marginal curves are fairly smooth for the larger values of C_g , but strongly oscillate, showing resonance tongues, for very small values of C_g , near critical values of ν_m that are fairly constant as $C_g \rightarrow 0$; and the oscillations are steeper as L_1 increases. This behavior for very small C_g is analyzed in the following subsection. As a general comment, ν_m^c is generally intermediate between 0 and 1, which provides strictly oblique subharmonic waves.

The modulation frequency for the cases considered in Fig.2.10 is provided in Fig.2.11, where it can be seen that $|\Omega|$ is maximum (with a fairly flat behavior) at the cusps of the tongues and that $|\Omega|$ increases as either C_g or L_1 decrease.

Even though instability is given by the first branch of eigenvalues of (2.106)-(2.108), the second branch is not far from the first branch, especially for very small C_g . Thus, the second branch must be taken into account when continuing along the first branch to avoid jumps between branches.

Figure 2.11: Modulation frequency Ω for the cases considered in Fig.2.10, with arrows indicating decreasing values of C_g . Circles indicate the approximate values of Ω at the minima of the curves in Fig.2.10, as calculated in subsection 2.6.1.

For illustration, Fig.2.12 shows the first three branches for the case $d = 10$, $L_1 = 10$, $S = 1$, and $C_g = 0.001$. Note that the second and third branches break into isolas in some regions. L_1 has been taken not large enough (cf eq.(2.133)) in this figure to avoid that the tongues be too close together and the isolas be too narrow to be appreciated.

2.6.1 The role of spatial resonances at very small viscosity

As $C_g \rightarrow 0$, the eigenvalue problem (2.106)-(2.108) show spatial resonances when the unforced, strictly inviscid problem obtained setting $a = C_g = 0$ in (2.106)-(2.108) exhibits nontrivial solutions. The resulting equations (2.106)-(2.107) are readily solved, to obtain

$$F_m^\pm = A_m^\pm \cos(\tilde{\nu}_m^\pm x + \delta_m^\pm), \quad \text{with } \tilde{\nu}_m^\pm = \sqrt{\nu_m^2 \pm 4\Omega/(1+S)}. \quad (2.134)$$

Here, $A_m^\pm$ are arbitrary constants, which will be related among each other below. Imposing the boundary conditions (2.108) to F_m^+ and F_m^- yield (cf.

(2.100))

$$\delta_m^\pm = 0 \text{ or } \pi/2, \quad \tan(\tilde{\nu}_m^\pm L_1 + \delta_m^\pm) = \gamma(1 - \nu_m^2, S)/\tilde{\nu}_m^\pm. \quad (2.135)$$

These two equations (for the + and - superscripts) must hold simultaneously and provide a set of discrete values of the (ν_m, Ω) ; note that, in principle, there are four combinations of the set of equations, for $(\delta_m^-, \delta_m^+) = (0, 0)$, $(0, \pi/2)$, $(\pi/2, 0)$, and $(\pi/2, \pi/2)$. As $L_1 \rightarrow \infty$, $\Omega \rightarrow 0$ and $\tilde{\nu}_m^-$ and $\tilde{\nu}_m^+$ must be close to each other. Note that $F_m^\pm$ is an even and odd function of x for $\delta_m^\pm = 0$ and $\pi/2$, respectively.

Now, the effect of both parametric forcing and viscous damping is accounted for adding higher order terms in (2.134), namely substituting these by

$$F_m^\pm = A^\pm \cos(\tilde{\nu}_m^\pm x + \delta_m^\pm) + \text{HOT}, \quad (2.136)$$

where HOT stands for higher order terms in the limit $C_g, a \rightarrow 0$. Substituting these into (2.106)-(2.108) and applying solvability conditions to the (singular) linear problems providing HOT, the following linear results at leading order to calculate the threshold amplitude a and a correction Ω_1 to the frequency Ω , namely

$$(i\Omega_1 \pm i\hat{\delta} + 2C_g)A^\pm = \pm ia\beta_0 A^\mp, \quad (2.137)$$

where $\hat{\delta}$ is a rescaled spatial detuning measuring the distance from the vertices of the tongues and the coefficient β_0 is given by

$$\beta_0 = \int_{-L_1}^{L_1} g(x) \cos(\tilde{\nu}_m^+ x + \delta_m^+) \cos(\tilde{\nu}_m^- x + \delta_m^-) dx, \quad (2.138)$$

Since the forcing function g is odd, β_0 is nonzero only if the two cosines appearing in (2.136) exhibit different parities, namely if either $(\delta^+, \delta^-) =$

Figure 2.12: Counterpart of Fig.2.10 for the case $L_1 = 15$, $S = 0.5$, and $C_g = 0.005$, considering the first three branches (solid lines) and the approximations of the minima of these curves calculated as explained in subsection 2.6.1 for the first (circles), second (squares), and third (triangles) branches.

$(0, \pi/2)$ or $(\pi/2, 0)$; both possibilities lead to the same solutions noting that (2.134) is invariant under the action $(\Omega, F_m^\pm) \rightarrow (-\Omega, F_m^\mp)$. For consistency with the analysis in section 2.5.1, the value selected is:

$$\delta_m^+ = 0, \quad \delta_m^- = \pi/2. \quad (2.139)$$

The associated solutions of the system (2.135) provide the values of ν_m at the vertices of the tongues in Fig.2.10 and the associated values of Ω in Fig.2.11. The latter are plotted with circles in Fig.2.11, where it can be seen that the approximation is quite good. The threshold amplitude is calculated requiring that the homogenous linear system (2.137) exhibits nonzero solutions, which yields $a = \sqrt{4C_g^2 + \hat{\delta}^2}/|\beta_0|$, $\Omega_1 = 0$. This curve provides the marginal instability curves near each of the minima of the tongues in Fig.2.10. The latter are attained at $\hat{\delta} = 0$, which yields

$$a_c = 2C_g/|\beta_0|. \quad (2.140)$$

These values are plotted with circles in Fig.2.10, where it can be seen that the approximation is quite good. It is to be noted that $\Omega_1 = 0$, meaning that the forcing frequency behaves as $\Omega = \Omega_0 + O(\hat{\delta}^4)$ near the minima of the tongues, which is clearly appreciated in the flat behavior near the maxima and minima in Fig.2.11.

The above is concerned with interaction between natural modes exhibiting neighboring values of ν_m^+ and ν_m^- and provides the minima of the marginal instability curve associated to the first branch in Fig.2.11. Interaction between natural modes associated with $\nu_m^\pm$ and $\nu_{m-1}^\pm$ and with $\nu_m^\pm$ and $\nu_{m-2}^\pm$ provide the minima of the marginal instability curves in the second and third branches, and are plotted with squares and triangles, respectively, in Fig.2.12, where it can be seen that the approximation is again quite good.

The analysis above also shows that $\Omega \neq 0$ generically, because if $\Omega = 0$, then $F_m^+ = F_m^-$ and the forcing coefficient β_0 identically vanishes. For Faraday waves instead g is constant in (2.138) and β_0 is nonzero with $\Omega = 0$, which is the fundamental difference between horizontal and vertical oscillations that has been already mentioned.

Even though the limit of very small C_g is beyond the scope of the thesis for the reasons given, it helps to understand the resonance tongues appearing in Figs.2.10 and 2.12, and also to explain the essential differences with Faraday waves in analytical terms.

2.7 Effect of the parameters on the threshold amplitude and orientation

In this section, the effect on marginal instability of varying the non-dimensional parameters of the problem (from a reference case) is explored. Specifically, the dependence of the threshold amplitude and the modulation frequency on the viscosity C_g , the gravity-capillary parameter S , the depth of the container, and the length of the container L_1 is studied in two limiting cases. These are the continuous (infinite container) and the discrete (assuming a finite, rectangular container) limits, in which the wavenumber takes continuous values and is quantized, respectively. For the continuous case, studied in subsection 2.7.1, the container is large enough and the interaction between the vibrating endwalls is neglected. In the discrete case, instead, the endwalls are not too far apart and some interaction between them must be accounted for, as explained in subsection 2.7.2.

2.7.1 Very large (and deep) containers

Apart from the assumptions made previously, the value of $L_1 C_g$ is assumed to be large enough for the subharmonic waves (excited by the OBF) to be localized near the endwalls. Thus, they do not interact among each other. In addition, for the sake of simplicity, it will be assumed that either $L_2 = \infty$ or $L_2 = L_1$. In the experiments by Porter *et al.* (2012), $L_1 C_g$ is 1.05 with 5 cSt silicone oil and 50 Hz forcing and 2.1 with 10 cSt oil at the same frequency. Although $L_1 C_g$ can clearly be made larger with larger containers, higher frequencies, or higher viscosities, these values are sufficient to avoid the interaction of the endwalls for most practical purposes. This is because of the exponential decay of the subharmonic waves determined by equations (2.106)-(2.107) is relatively strong.

To illustrate the response of the system, a representative case with $L_1 = L_2 = d = 50$ is considered and the effect of varying one-by-one the different parameters is investigated. Note that, in spite of the fact that the OBF extends into the interior of the container when $d = L_1$, the interaction between subharmonic surface waves generated by the OBF near each endwall can still be weak because the non-uniform forcing $g(x)$ is much smaller in the center of the container than at the endwalls (see Fig. 2.3).

Figure 2.13 illustrates the effect of damping. As will be seen, this is nearly the only sensitive parameter in this limit. The neutral stability curve $a(\nu)$ (including the effect of quantization by the lateral boundary condition) is shown the left part for the indicated values of C_g . The resulting threshold amplitude, defined by the minima of the neutral stability curve, and its as-

Figure 2.13: Left: neutral stability curve for C_g : $C_g = 5 \times 10^{-2}$, $C_g = 10^{-2}$, and $C_g = 5 \times 10^{-3}$ (as C_g increases the curve moves upwards), $L_1 = d = 50$, and $S = 0.5$. Discrete modes selected by the lateral boundary condition are indicated by (open) circles. Right: threshold amplitude (dashed, thick line; values to be read on the left axis) and associated orientation (continuous, thick line; values on the right axes) vs. C_g . Thin lines indicate the threshold amplitude and associated orientation for the discrete modes selected by the lateral boundary condition.

sociated orientation (defined by the longitudinal wavelength ν) are shown in the right part of this Figure. Both the threshold amplitude and orientation vary monotonically with the damping C_g , except at very low values of C_g (in this case $L_1 C_g$ is no longer sufficiently large and some interaction between subharmonic surface waves near both endwalls will occur, as considered in the following section). A smooth transition is observed from nearly pure crosswise orientation at low (but not too low) damping to more oblique (but still far from parallel) waves at larger values of damping. The threshold amplitude, as expected, increases with damping. The discussion above is concerned with the continuous case $L_2 = \infty$. When the finite value of L_2 is taken into account, a large number of allowed modes (marked by open circles) can be seen in Fig. 2.13 because $L_2 = L_1$ is large. However, since the minimum of the neutral stability curve is located in the regular (non-oscillatory) part of the curve (which corresponds to low values of ν in the case of small damping), the effect of quantization on the selection of both the threshold value of the amplitude a and the orientation ν is weak; for the sake of clarity, the discrete case will not be represented in the following figures.

The effect of the gravity-capillary balance S is shown in Fig. 2.14. Again, both the critical amplitude and orientation vary monotonically (and slowly) with S except at very small values near the pure gravity wave limit. In this limit, a small deviation from pure cross-waves is also observed.

The effects of L_1 and of d are both weak. Figure 2.15 shows that when L_1 is increased beyond about 40 the limiting case of an infinitely large con-

2.7. EFFECT OF THE PARAMETERS ON THE THRESHOLD

Figure 2.14: Left: neutral stability curve for $C_g = 5 \times 10^{-3}$ (three curves at the top) and $C_g = 5 \times 10^{-2}$ (three curves at the bottom) for three values of gravity–capillary balance S : 0, 0.5 and 1 (as S increases the curves move downwards). All the curves correspond to $L_1 = d = 50$. Right: Threshold amplitude (discontinuous lines, left axis) and orientation (continuous lines, right axis) vs. S for $L_1 = d = 50$ and damping $C_g = 5 \times 10^{-3}$ (thin lines) and $C_g = 5 \times 10^{-2}$ (thick lines).

Figure 2.15: Left: Variation of threshold amplitude (discontinuous lines, left axis) and orientation (continuous lines, right axis) with container length L_1 for $S = 0.5$, $d = 50$ and damping $C_g = 5 \times 10^{-3}$ (thin lines) and $C_g = 5 \times 10^{-2}$ (thick lines). Right: Variation of threshold amplitude and orientation with container depth d for $L_1 = 50$, $S = 0.5$ and $C_g = 5 \times 10^{-3}$ and $C_g = 5 \times 10^{-2}$ (same interpretation as in the left plot).

tainer soon applies. If L_1 is not large, then more complex dependence is observed. However, in this case $L_1 C_g$ is no longer large and interaction between subharmonic waves near both endwalls is expected (considered in the following section). Finally, the effect of d is only apparent with the

threshold amplitude and this dependence is very weak except for relatively small values of d . The orientation of the pattern is nearly independent of d (only damping has a large effect).

The threshold frequency Ω_c , which measures the deviation from 2 : 1 resonance, is shown in Fig. 2.16. Although it is generically nonzero (the

Figure 2.16: Variation of frequency at threshold Ω_c with L_1 for $d = 50$, $S = 0.5$. Two curves are plotted corresponding to $C_g = 5 \times 10^{-3}$ (thin line) and $C_g = 5 \times 10^{-2}$ (thick line).

threshold value Ω_c is exactly zero only for particular combinations of the parameters), its value is very small except when L_1 is small (which is again the case considered in the following section). Thus, although the instability is in general associated with a finite (Hopf) frequency and quasiperiodic motion, in this limit it will appear, to a high degree of accuracy, as a pure 2 : 1 resonance.

2.7.2 Moderately large (and deep) containers

As in the previous subsection, an additional assumption about the product $L_1 C_g$ is made, where the damping C_g is conveniently small, such that $L_1 C_g$ is moderate or even small. This results in a moderate or large interaction between the waves excited at each endwall. This limit could, for example, be reached experimentally with a 9 cm square container of 5 cSt oil by using very low frequencies (less than 20 Hz, say, where $L_1 C_g \simeq 0.124$). Alternatively, this limit could be reached in the same container by using (pure) water where $L_1 C_g < 0.18$ for frequencies less than 100 Hz. As an illustrative case for this section, a container of section $L_1 = L_2 = 25$ and $d = 10$ is used. Damping will be varied to achieve different levels of interaction between the waves generated at both endwalls.

The effect of damping C_g is shown in Fig. 2.17. Although the trend is broadly similar to Fig. 2.13 of the previous section, some important differences arise. The appearance of nearly parallel waves encountered in the

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Figure 2.17: Counterpart of Fig. 2.13 with $L_1 = L_2 = 25$ and $d = 10$.

previous section for very small C_g values (due to the interaction between the two endwalls) occurs in this case for a wider range of damping values. At still larger values of damping, the dependence is similar to before. Additionally, there is a strong effect of quantization in this case, which noticeably influences the orientation. In the case of very small damping this quantization effect is dramatic. With $C_g = 10^{-3}$, for instance, the orientation changes from $\nu \simeq 0.95$ in the infinite L_2 case to $\nu \simeq 0.4$ for $L_2 = L_1 = 25$. This is related to the appearance (for very small C_g) of a large number of tongues with very similar amplitude, which can be expected to interact near threshold.

The effect of the gravity-capillary balance S is shown in Fig. 2.18 for $C_g = 5 \times 10^{-3}$ and, in contrast with the case of the previous section (see

Figure 2.18: Left: neutral stability curve for $C_g = 5 \times 10^{-3}$ for three values of gravity-capillary balance S : 0, 0.5 and 1 for $L_1 = L_2 = 25$ and $d = 10$. Modes selected by the lateral boundary condition (with $L_2 = L_1$) are indicated by (open) circles. Right: Variation of threshold amplitude (discontinuous thick line, left axis) and orientation (continuous thick line, right axis) with gravity-capillary balance S for $L_1 = d = 50$ and damping $C_g = 5 \times 10^{-3}$. Plots with thin lines correspond to threshold amplitude and orientation obtained from modes selected by the boundary condition

Fig. 2.14), this effect is now strong. The dependence observed for the thresh-

old amplitude is not unsimilar to that seen in the previous section (although quantization effects lead to an irregular curve), but the dependence of threshold orientation on S is much different. In the case of infinite L_2 , waves rotate from (almost pure) cross-waves in the gravity wave limit to nearly parallel waves in the capillary wave limit, passing through oblique angles (with jumps associated to changes from one tongue to another). Quantization effects when $L_2 = L_1$ produce additional new features, including different behaviour in the gravity wave limit where oblique waves are observed instead of cross-waves. As shown in figures 2.19 and 2.20, this effect becomes

Figure 2.19: Counterpart of Fig. 2.18 for $C_g = 10^{-2}$.

Figure 2.20: Counterpart of Fig. 2.18 for $C_g = 5 \times 10^{-2}$.

weaker for larger values of damping (as interaction between waves at both endwalls decreases).

The effect of the depth d , which is shown in Fig. 2.21, is similar to the case of large containers (Fig. 2.15). The effect of L_1 , in contrast, can be quite different with small damping (when strong interaction between endwalls is expected). If L_1 is not large enough, the threshold amplitude will correspond to the closest tongue to $\nu = 1$ (see Figs. 2.17 and 2.18), which corresponds to a purely parallel wave. Jumps from one tongue to another are expected as L_1 is varied (these are clearly seen in Fig. 2.21). Quantization

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Figure 2.21: Counterpart of Fig. 2.15 for $C_g = 5 \times 10^{-3}$ and $C_g = 5 \times 10^{-2}$ with $L_1 = 25$ and $d = 10$.

effects (not shown in Fig. 2.21 for the sake of clarity) lead to a more complex dependence of the orientation on L_1 . It is worth noting again that, in such cases where the successive tongues have similar minimum amplitudes, one expects interaction between a number of modes near onset, both in the finite and infinite L_2 case.

2.7.3 Subharmonic or quasiperiodic instability?

The magnitude of the modulation frequency at threshold, Ω_c , which is generically non-zero, depends strongly on damping. The left plot in Fig. 2.22 shows the frequency Ω_c vs. the damping, C_g . It is clear that, when

Figure 2.22: Left: frequency at threshold Ω_c as a function of damping C_g for a container with $L_1 = L_2 = 25$, $d = 10$ and $S = 0.5$. Thick lines correspond to L_2 infinite and thin lines to $L_2 = L_1$ case. Right: frequency at threshold Ω_c as a function of S for $C_g = 5 \times 10^{-3}$. Thick lines correspond to L_2 infinite and thin lines to $L_2 = L_1$ case.

there is strong interaction between the two endwalls, Ω_c is not so small, and this fact is not affected by quantization, as shown in the same plot.

The right part of Fig. 2.22 shows the frequency Ω_c as a function of the gravity-capillary balance S for $C_g = 5 \times 10^{-3}$. In the gravity wave limit, the frequency Ω_c is close to zero (this is consistent with the focus of previous theoretical analyses on pure subharmonic cross-waves). As S increases and

the pattern orientation varies from nearly crosswise to nearly parallel (see Fig. 2.18), the frequency Ω_c also increases toward values that would provide more noticeable modulation, in a practical experimental context. As seen in the same figure, when L_2 is finite the results are not qualitatively different, except for the appearance of (pure) 2 : 1 resonance at two particular values of S ; when L_2 is infinite, these pure 2 : 1 resonances do not appear since threshold values occur at the minima of one of the tongues and the frequency Ω is always maximal at these points. In practice, at such small values of the damping, complicated behaviour will be expected near onset since, in addition to the competition with triad resonances, the interacting modes associated with different tongues possess not only different longitudinal wavelengths but different frequencies as well. Quantization effects due to finite L_2 would add still more complexity to the potential interactions.

For larger values of the damping C_g , a comparatively simpler scenario appears. Figure 2.23 shows the results for $C_g = 10^{-2}$ and $C_g = 5 \times 10^{-2}$.

Figure 2.23: Counterpart of the left part of Fig. 2.22 for $C_g = 10^{-2}$ (left) and $C_g = 5 \times 10^{-2}$ (right).

In these cases, except near the capillary limit for $C_g = 10^{-2}$, the threshold is associated with a small longitudinal wavelength (not to instability tongues of oblique waves) and the frequency varies smoothly and can be zero for particular values of S . Still, in the gravity limit, a zero frequency seems to be found when L_2 is taken to be infinite. Quantization due to finite L_2 introduces some (slight) quantitative changes leading, for instance, to a nonzero frequency in the gravity limit (the associated frequency is still very small).

2.7.4 Recapitulation of effects of nondimensional parameters

Here the general effect of varying the parameters is summarized:

- As expected, a_c increases as either C_g is increased (which enhances viscous effects) or L_1 or d is decreased (which decreases the strength

of the OBF). The latter variation occurs mainly in the ranges $15 \leq L_1 \leq 25$ and $5 \leq d \leq 40$, where L_1 and $d = 10$ are somewhat comparable; for larger values of L_1 and d the curves saturate. Dependence of a_c on S is not monotonic and peaks at some intermediate value of S , which depends on the remaining parameters.

- The orientation is generally oblique (namely, $0 < \nu_m^c < 1$) except in two limits, both for quite small C_g . (i) For $S = 0$, the orientation is perpendicular to the endwalls, which is consistent with the fact that gravitational subharmonic waves were found to be perpendicular to the wavemakers in most experiments previous to Porter *et al.* (2012), Moisy *et al.* (2012) and Taneda (1994); confirming the conjecture in the Introduction that capillarity promotes obliqueness. And (ii) for $S \geq 0.5$, the patterns are almost parallel to the vibrating endwalls, which was the limit already considered by Varas & Vega (2007).
- ν_m shows abrupt transitions at some particular values of the parameters C_g , S , and L_1 , which are due to phase slips in the x direction; predicted and measured orientations may show large discrepancies in these regions.

The frequency Ω of the slow modulation is always very small, in the range $10^{-2} - 10^{-10}$. Calculations clearly show that

$$\Omega \rightarrow 0 \quad \text{as } L_1 C_g \rightarrow \infty \text{ and } L_1/d \rightarrow \infty, \quad (2.141)$$

which occurs when the longitudinal length L_1 is large compared with both the viscous decaying length of the waves and the horizontal extension of the OBF. In other words, Ω vanishes as interaction between the endwalls is weak. In the opposite limit, Ω increases as L_1 decreases, which is consistent with the observation (see, e.g., Becker and Miles 1991) that strict 2:1 subharmonic oscillations ($\Omega = 0$) of individual sloshing modes ($L_1 \sim 1$) are not possible in rectangular containers under horizontal oscillation.

2.8 Comparison with experiments

The non-dimensionalization used in this thesis is very convenient for theoretical purposes, but makes all non-dimensional parameters d , L_1 , L_2 , a , C_g , and S (defined in eqs.(2.1) and (2.3)) dependent on the dimensional wavenumber k^* . This is calculated by solving the cubic equation (2.2) at each ω^* (=one half of the forcing frequency), which is varied in experimental runs. Then, the dimensional threshold amplitude $a^* = k^* a_c$, the angle φ defined in eq.(2.104) that provides the obliqueness of the patterns, and the

dimensional frequency associated with the slow modulations $\Omega^* = \Omega/\omega^*$ are readily calculated.

Comparison is made with the experiments by Porter *et al.* (2012) and some new sets of experiments provided by the group of Porter using the same experimental device. The rectangular containers are of depth $d^* = 5$ cm and cross sections $2L_1^* \times 2L_2^* = 9 \times 9$, 10×10 cm² and 15×15 cm², using two silicone oils, with viscosities $\nu = 5$ and 10 cSt (and $\sigma \sim 19.7$ and 20.1 dyn/cm and $\rho = 0.913$ and 0.933 g/cm³, respectively), in the frequency range 25-80 Hz. The non-dimensional parameters defined in this chapter move in the ranges $25 \leq L_1, L_2 \leq 100$, $30 \leq d \leq 70$, $0.01 \leq C_g \leq 0.08$, and $0.2 \leq S \leq 0.8$ (the variation of these parameters with $2w^*$ can be found on Appendix D), which can be considered as satisfying the assumptions (2.16). In the paper it is described how they take care to maintain the contact line fixed, which is the boundary condition assumed in the thesis.

Figure 2.24: The dimensional threshold acceleration $4a^*\omega^{*2}$ vs. the forcing frequency $2\omega^*$ resulting from for several experimental runs (symbols) and from the theory developed above for $\mu = 5$ cSt (lower curves) and 10 cSt (upper curves); arrows indicate increasing values of depth.

The threshold accelerations in the various experimental runs are compared in Fig.2.24 with the theory developed in section 2.5. Note that theory and experiments match quite well; in fact, the theory predicts the amplitude within the experimental uncertainty.

It is to be noted that having used the appropriate boundary conditions at the sidewalls and endwalls (namely, (2.108)) is essential quantitatively, as illustrated in Fig.2.25, where one of the marginal instability curves obtained above is compared with its counterparts using Dirichlet and Neumann boundary conditions, namely

$$F_m^\pm = 0 \quad \text{and} \quad (\partial_x F_m^\pm, \partial_y F_m^\pm) = (0, 0) \quad \text{at} \quad (x, y) = (\pm L_1, \pm L_2), \quad (2.142)$$

respectively. Figure 2.25 shows that using Neumann boundary conditions would yield a 30% error in this case. The difference with the Dirichlet case is

Figure 2.25: Comparison of the instability threshold calculated above (solid line) with what would have been obtained using Dirichlet (dashed line) and Neumann (dot-dashed line) boundary conditions for the case of 5 cSt Silicone oil in a 9×9 cm² container, in Fig.2.24.

not so large because the constant γ appearing in (2.108) is somewhat large, namely $\gamma \simeq 2$, see Fig.2.5. Also, this indicates that the instability threshold is highly affected by the way in which the contact line is either fixed (as assumed in this thesis) or allowed to move, which is consistent with our comments above on the unavoidable experimental uncertainties.

The theoretically computed values of the pattern angle φ and the modulation frequency are in the ranges $71^\circ \leq \varphi \leq 74^\circ$ and $10^{-10} \leq \Omega \leq 10^{-2}$, respectively. Comparison with their experimental counterparts is difficult because the patterns only show a few wavelengths in the longitudinal direction and experimental uncertainties prevent measuring such small values of Ω in reasonably long experimental runs. Also, the precision with which the instability threshold can be approached is limited, meaning that the slowly modulated patterns illustrated in Fig.2.8 cannot be generally identified. The alternating patterns resulting from interaction between nearby modes illustrated in Fig.2.9, instead, occur for not so small values of $a - a_c$ and are visualized in several runs. As an example, Fig.2.26 shows the experimental pattern that is visualized in the case $\mu^* = 5$ cS, $d^* = 5$ cm, $2\omega^* = 5$ Hz, which (yield $L_1 = L_2 = 41.5$, $d = 46.1$, $C_g = 0.027$, $S = 0.63$ and thus) is precisely the case considered for illustration in Figs.2.6-2.9. As can be seen, the experimental pattern compares fairly well with its theoretical counterpart in Fig.2.9, in spite of the fact that some contact line motion was unavoidable for the not-so-small values of the forcing amplitude that were necessary to get a good visualization. The experimental pattern also shows a transverse modulation not accounted for in Fig.2.9, which is due to the small difference between the wavelengths of the interacting modes, as further explained in section 2.5.2. Some of the remaining experimental patterns are similar to this one (and also show a good qualitative comparison with

Figure 2.26: Counterpart of Fig.2.9 for the experimental run corresponding to the non-dimensional parameter values considered in Fig.2.9.

theory), but some others exhibit a more complex structure, which may result either from subcritical bifurcations or from interaction of more than two nearby modes. All of them are oblique, showing a pattern angle similar to that in Fig.2.9. This is consistent with the theoretically calculated angles, which as explained in above are in the range $71^\circ \leq \varphi \leq 74^\circ$.

Concerning comparison with the theory developed in the paper, two different kinds of patterns that involve mode interaction (see Figs.2.8 and 2.9) where predicted in subsection 2.5.2 explained. Type II is the counterpart of the experimental alternating pattern in Fig.2.26. Type I, instead, is a slowly modulated pattern that is qualitatively similar to the primary pattern that appears at threshold, both predicted for the first time in the framework of this thesis. The slow modulation gives a drift of activity one endwall to the other, which had not been detected experimentally when the theory was developed. In fact, the drift is so slow that careful visualizations must be maintained for very long times. In short runs, instead, the visualized patterns would show activity either near the left or the right endwall, or distributed among both endwalls, depending on the (non controllable) initial condition, which could be seen as unpredictable/non repeatable. After the theoretical prediction in Perez-Gracia et al. (2014), the slowly modulated patterns were experimentally reported in Tinao *et al.* (2014).

Summarizing, the theoretical instability threshold compares quite well with its experimental counterpart and the calculated and visualized experimental patterns also compare well in those cases in which such comparison is possible.

2.9 Concluding remarks

A general theory has been derived to calculate nearly-inviscid subharmonic waves in horizontally vibrated containers. The horizontal dimensions of the container are large compared to the wavelength of the basic capillary-gravity waves, meaning that the expected patterns will show a slow spatial modulation. In this limit, a quasipotential approximation derived in section 2.2.2 applies that greatly simplifies the formulation. The parametric instability that triggers the waves has been seen to be promoted by an oscillatory bulk flow that is uniquely determined by an inviscid linear problem formulated in section 2.2.4, whose solution is two-dimensional and allows for calculating the parametric forcing term in close form. Using these, a three-dimensional problem was derived in section 2.3 that applies in a $O(1)$ -vicinity of the unperturbed free surface and provides the linear evolution of the subharmonic field. Taking advantage of the fact that this problem is singular, an appropriate solvability condition has been applied that led to a simpler two-dimensional equation for the free surface elevation of general subharmonic waves (consisting in arbitrary linear combinations of plane waves), with boundary conditions that account for wave reflection at the sidewalls and endwalls. Using this simpler two-dimensional formulation and decomposing into normal modes in the transverse direction, an eigenvalue problem was obtained in section 2.5.1. This formulation shows that the instability is not strictly 2:1 subharmonic, but exhibits a second (small but nonzero) frequency that makes the resulting patterns generally quasiperiodic. The eigenvalue problem provides the instability threshold, the additional frequency, and the wave orientation at threshold. The resulting, generally oblique, quasiperiodic patterns were described in section 2.5.2, together with some secondary patterns resulting from mode interaction with nearby transverse modes (which is facilitated because the transverse size of the container is large).

In addition, the theory developed here was used in section 2.6 to elucidate the dependence of the instability threshold and the patterns orientation on the various non-dimensional parameters. In particular, it was seen that, except for gravity waves, the patterns are oblique and, moreover, align with the endwalls in some limits.

The container length and the damping together control the strength of

the interaction between subharmonic waves from opposing endwalls and representative cases were considered from the two limits of weak and significant interaction. A central result from this is the importance of the interaction in producing quasiperiodic solutions at onset. This is a very general effect of interacting waves produced by out-of-phase forcing mechanisms that can be understood as a result of symmetry-breaking in the weakly interacting limit.

In the limit of large containers and/or moderate (not too small) damping, the product $L_1 C_g$ is large enough that interaction between the subharmonic surface waves at each endwall is negligible. The instability is thus always close to pure 2 : 1 resonance (the modulation frequency is extremely small). In this regime we find that parameters other than damping have little influence on the threshold amplitude and pattern orientation. In the gravity wave limit, the orientation of the surface waves is nearly crosswise, as observed in many wavemaker experiments (Miles & Henderson 1990 and references therein). The main effect of the lateral walls can be expected to come from mode interaction just above onset, which would lead to more complicated patterns.

In the limit of moderately sized containers and/or small values of damping, the product $L_1 C_g$ is small enough that interaction between the subharmonic surface waves at each endwall cannot be neglected. There is a noticeable deviation from pure 2 : 1 resonance in this case, particularly when capillary effects dominate, leading to quasiperiodic solutions at onset. In addition to damping, the gravity–capillary balance and the presence of lateral walls have an important effect on threshold amplitude and (especially) pattern orientation in this regime. In the gravity wave limit the orientation is again nearly crosswise, while in the capillary wave limit the waves prefer an orientation that is nearly parallel to the vibrating endwalls. In any case, the selected orientation is strongly affected by the presence of lateral walls (quantization). Complicated dynamics are anticipated near onset, particularly in the capillary limit and when strong interaction between subharmonic waves generated at the two opposing endwalls occurs, due to the nonlinear interaction of multiple modes with different orientation and different frequency.

Comparison with measurements and visualizations in several experimental campaigns in section 2.8 showed a very good quantitative agreement in connection with the instability threshold and good qualitative agreement in connection with the visualized patterns when such comparison was meaningful (reasonably close to threshold).

3

Subharmonic instability in rectangular containers subject to mixed horizontal-vertical vibrations

3.1 Introduction

This chapter deals with the surface waves produced in a horizontal rectangular container by simultaneous vertical and horizontal vibration. As in chapter 2, the vibrating acceleration is strictly contained in one of the symmetry planes of the container. The amplitude equation derived in chapter 2 for strictly horizontal vibration is adapted to include the effect of the vertical vibration. Comparing both situations, it is known that the case of pure vertical vibration exhibits pure subharmonic patterns at onset, while (as shown in subsection 2.5.2) strictly horizontal vibration produces quasiperiodic patterns. The transition between these different temporal behaviors is examined in terms of the vertical-horizontal vibration amplitude ratio and the phase between these two components of the forcing.

This chapter is organized as follows. The problem is formulated in section 3.2 relying on the theoretical framework developed in Chapter 2, section 2.3, which is straightforwardly extended to the present formulation by adapting the forcing in the amplitude equation. This equation is used to calculate the subharmonic instability threshold and the resulting primary patterns in section 3.3. The results for varying values of the non-dimensional parameters are presented and discussed in section 3.4. Finally, the main conclusions are summarized in section 3.6.

3.2 Formulation

This section explains the problem object of this chapter and introduces the non-dimensional parameters relevant in the rest of the chapter. Subsection 3.2.1 details the main assumptions that allows the quasi-potential ap-

proximation and subsection 3.2.2 explains the different symmetries of the problem.

Figure 3.1: Sketch of the vibrating container.

A rectangular container of depth d^* and horizontal size $2L_1^* \times 2L_2^*$ (figure 3.1) is considered, which is subject to a mixed horizontal/vertical rigid-body horizontal-vertical vibration in the vertical plane containing the $2L_1^*$ -direction. Using the superscript $*$ to denote dimensional quantities, the container is translated parallel to itself, with all points of the container following the trajectory

$$x^* \rightarrow x^* + a^* \alpha \cos(2\omega^* t^*), \quad z^* \rightarrow z^* + a^* (1 - \alpha) \cos(2\omega^* t^* - \theta), \quad (3.1)$$

where x^* and z^* (see figure 3.1) are the longitudinal and vertical coordinates in a Cartesian coordinate system with the $z^* = 0$ plane at the unperturbed free surface and the x^* and y^* axes contained in the (vertical) symmetry planes of the container. The forcing amplitude is denoted by a^* , the phase θ (with $0 \leq \theta \leq 2\pi$) is the time shift between the horizontal and vertical forcing, and α (with $0 \leq \alpha \leq 1$) is a measure of the relative intensity of the horizontal forcing ($\alpha = 0$ corresponds to a purely vertical vibration and $\alpha = 1$ to a purely horizontal vibration). The trajectories are illustrated in figure 3.2 where the orbits of the container's points are plotted for different values of the parameters α and θ . In general, the container follows ellipses but for $\theta = 0, \pi$ the ellipses degenerate into straight lines. Obviously, the movement for $\alpha = 0$ and $\alpha = 1$ are also lines, independently of the value of the phase, being $\alpha = 0$ superimposed to the x axis and $\alpha = 1$ to the z axis.

The non-dimensionalization used is the same that was introduced in chapter 2. Then, viscous and capillary-gravity effects are accounted for in

Figure 3.2: Trajectories of a point of the container for in-phase vibration $\theta = 0$ (left), one of the cases of general out-of-phase vibration $\theta = \pi/4$ (middle) and symmetric out-of-phase vibration $\theta = \pi/2$ (right) for $\alpha = 0, 0.25, 0.5, 0.75, 1$, the arrow indicates increasing values of α .

terms of the modified Ohnesorge number C_g and the gravity-capillary balance with the parameter S , which were defined in equation (2.3). Note that $0 \leq S \leq 1$, with the extreme values $S = 0$ and 1 corresponding to the *purely gravitational* and *purely capillary* limits, respectively. Thus, the problem depends on 7 non-dimensional parameters, namely θ , α , C_g , S , d , L_1 , and L_2 , the three last parameters were defined in (2.1).

3.2.1 Main assumptions and the two-dimensional amplitude equation

Here, we consider the limit in which (i) the container is large compared with the wavelength of the excited waves, (ii) viscous effects are weak, and (iii) the forcing amplitude is small. This limit implies that the steepness of the generated waves is also small, namely

$$L_1, L_2, d \gg 1, \quad C_g \ll 1, \quad a \ll 1, \quad |f| \ll 1. \quad (3.2)$$

Noting that the non-dimensional forcing frequency is 1, the non-dimensional free surface elevation can be written (at leading order) as the linear combination of three components,

$$f \simeq [F^{\text{tw}}(\mathbf{x}, z, t) + F^{\text{obf}}(\mathbf{x}, z, t)]e^{2it} + F^{\text{s}}(\mathbf{x}, z, t)e^{it} + \text{c.c.} + f^{\text{mf}}(\mathbf{x}, z, t), \quad (3.3)$$

where

$$\mathbf{x} = (x, y) \quad (3.4)$$

and

$$|\partial_t F^{\text{tw}}| \ll |F^{\text{tw}}|, \quad |\partial_t F^{\text{obf}}| \ll |F^{\text{obf}}|, \quad |\partial_t F^{\text{s}}| \ll |F^{\text{s}}|, \quad |\partial_t f^{\text{mf}}| \ll |f^{\text{mf}}|. \quad (3.5)$$

The factor of e^{2it} is the *harmonic flow*, which is proportional to the forcing amplitude and is present for arbitrarily small forcing amplitude. This flow consists in two components:

- A pair of counter-propagating *travelling waves* denoted with the superscript ‘tw’, which are produced by horizontal vibration. These are outgoing purely traveling waves, as generated by the upper part of the vibrating end-walls, which act as wavemakers. These waves decay by viscous dissipation as they travel toward the inside, meaning that they penetrate a non-dimensional horizontal distance $\sim 1/C_g$ from the vibrating end-walls.
- The *oscillatory bulk flow*, denoted with the superscript ‘obf’, is strictly inviscid at leading order and slowly varying in space, namely

$$|\nabla F^{\text{obf}}| \ll |F^{\text{obf}}| \ll 1, \quad (3.6)$$

and is forced by both (i) the whole horizontally vibrating endwalls and (ii) the vertical vibration. Since the depth of the endwalls is $d \gg 1$, part (i) of the OBF penetrates a horizontal distance ($\sim d$) that is large as well. Part (ii) of the OBF, instead, is spread along the whole container. For ordinary liquids (not too low viscosity), deep containers, and forcing frequencies in the capillary-gravity limit,

$$C_g d \gg 1, \quad (3.7)$$

as it is assumed hereafter. This condition ensures that the oscillatory bulk flow dominates the travelling waves in connection with subharmonic waves generation. Thus, the travelling waves can be ignored in the analysis.

The *subharmonic flow*, denoted with the superscript ‘s’ in eq.(3.3), is triggered by the oscillatory bulk flow through a parametric instability mechanism and is the main object of the present chapter. An additional *mean flow*, denoted with the superscript ‘mf’ in eq.(3.3), is generated by both the harmonic and subharmonic waves. But this mean flow is very weak near threshold and is ignored here.

The assumptions (3.2) permit a *quasi-potential approximation* (Zhang & Viñals 1997). It was used in chapter 2 to derive the two-dimensional amplitude equation (section 2.3.2) and boundary conditions (section 2.4) that are quoted below and are the starting point for the analysis in this chapter. Summarizing the outcome of this derivation, the subharmonic component F^s satisfies the following *two-dimensional amplitude equation*

$$\partial_t F^s + i(1 + 2S)(\Delta F^s + F^s)/4 = -2C_g F^s + ia[\alpha g(x) + (1 - \alpha)e^{i\theta}] \bar{F}^s. \quad (3.8)$$

The amplitude equation (3.8) applies in a two-dimensional domain that coincides with the cross-section of the container, $-L_1 < x < L_1, -L_2 < y < L_2$. The equation is subject to the following *boundary conditions*

$$\partial_x F^s \pm \int_{-1}^1 \gamma(\kappa, S) \phi_{\pm L_1}(\kappa) e^{i\kappa y} d\kappa = 0 \quad \text{at } x = \pm L_1, \quad (3.9)$$

$$\partial_y F^s \pm \int_{-1}^1 \gamma(\nu, S) \phi_{\pm L_2}(\nu) e^{i\nu x} d\nu = 0 \quad \text{at } y = \pm L_2, \quad (3.10)$$

where Fourier transforms $\phi_{\pm L_1}$ and $\phi_{\pm L_2}$ are defined as

$$F^s(x, \pm L_2) = \int_{-1}^1 \phi_{\pm L_2}(\nu) e^{i\nu x} d\nu, \quad F^s(\pm L_1, y) = \int_{-1}^1 \phi_{\pm L_1}(\kappa) e^{i\kappa y} d\kappa. \quad (3.11)$$

Let us now discuss the amplitude equation (3.8) and the boundary conditions (3.9)-(3.10), which will constitute the starting point for the analysis in this chapter.

The amplitude equation (3.8) strictly coincides with its counterpart in the last chapter if $\alpha = 1$, as expected. The parameters α and θ are as discussed above and defined in (3.1). The function g and the parameter γ were defined (and discussed) in subsection 2.2.4 and section 2.4 respectively, but here the main ideas are reviewed for the sake of presentation. The function g appearing in the parametric forcing term in eq.(3.1) accounts for horizontal forcing. This function (plotted in figure 3.3-left) is given by

Figure 3.3: Left: the forcing function g appearing in (3.8) vs. x/L_1 for $L_1/d = 0.1, 1, 2, 5, 10, 20$, and 50 ; the arrow indicates increasing values of L_1/d . Right: the constant γ appearing in the boundary conditions (3.9)-(3.10) vs. κ for the indicated values of S .

$$g(x) = -\partial_z P^{\text{obf}}/4 \equiv \frac{4}{\pi} \sum_{m>0, \text{odd}} \frac{\sinh[m\pi x/(2d)]}{m \cosh[m\pi L_1/(2d)]}, \quad (3.12)$$

Note that g is odd and diverges as $x \rightarrow \pm L_1$, namely

$$g(-x) = -g(x), \quad g \sim \mp \log(L_1 \pm x) \quad \text{as } x \rightarrow \pm L_1. \quad (3.13)$$

In fact, g penetrates a distance $\sim d$ from the end-walls and concentrates near the end-walls as L_1/d increases. Thus, as in the last chapter, the effect of the horizontal component of forcing is somewhat concentrated near the endwalls. However, now the horizontal component of forcing (namely that term proportional to $(1 - \alpha)a$ in eq.(3.1)) is spatially constant and thus spread along the whole domain, which will have qualitative consequences in the patterns.

The real scalar γ appearing in (3.9)-(3.10) is plotted in figure 3.3-right. Note that $\gamma = 0$ in the gravity limit ($S = 0$) and increases as either κ or S increase. It must be noted that, at least in the purely horizontally vibrating case considered in chapter 2, this parameter γ has an important quantitative effect in the subharmonic threshold acceleration. Careful observation of both the last term present on the right hand side of equation (3.8) and figure 3.3-left allows to obtain some insight into the results of the object problem. When $\theta = 0$ the addition of some vertical vibration has the effect of displacing the curves plotted in figure 3.3-left vertically. The consequence is that one side of the container will be easier to destabilize. Then the expected subharmonic patterns become more dominant over one side of the container than the other. The behaviour is similar for values of the phase in $0 < \theta < \pi/2$ as there is always some component of the forcing displacing the forcing $g(x)$ up. The case for $\theta = \pi/2$ is special due to the addition being orthogonal to $g(x)$. This is explained in terms of symmetries in subsection 3.2.2.

The remainder of the chapter will be based on the formulation (3.8)-(3.10). At the moment, several remarks are in order, it is noted that according to equations (3.2) and (3.5), $|\partial_t F^s| \ll |F^s| \ll 1$, $C_g \ll 1$, and $a \ll 1$. Thus, equation (3.8) shows that $|\Delta F^s + F^s| \ll 1$, which selects the wavenumber of the possible plane wave solutions, $F^s = Ae^{i(\kappa_1 x + \kappa_2 y)}$, such that $\sqrt{\kappa_1^2 + \kappa_2^2} \simeq 1$, consistently with the non-dimensionalization above. Thus, the possible solutions are linear combinations of plane waves of this type, from which the equation selects a correction to the unit wavenumber, the orientation, and the slow temporal evolution.

3.2.2 Symmetries

The symmetries of the two-dimensional equation will play an important role below to anticipate the qualitative shape of the patterns and are analyzed here in some detail.

The problem (3.8)-(3.10) is *always invariant* under the symmetries

$$y \rightarrow -y \quad \text{and} \quad (\theta, x, F^s) \rightarrow (\theta + \pi, -x, iF^s), \quad (3.14)$$

which allows for restricting the analysis to the interval $0 \leq \theta \leq \pi$.

In addition, for *purely vertical forcing* ($\alpha = 0$), the problem (3.8)-(3.10) is invariant under the symmetry

$$x \rightarrow -x. \quad (3.15)$$

For *purely horizontal forcing* ($\alpha = 1$), the problem (3.8)-(3.10) is invariant under the symmetry

$$(x, F^s) \rightarrow (-x, iF^s). \quad (3.16)$$

Finally, for mixed horizontal/vertical ($0 < \alpha < 1$), symmetrically out-of-phase ($\theta = \pi/2$), the forcing term in (3.8) is rewritten as

$$i[\alpha g(x) + i(1 - \alpha)] \equiv [i\alpha g(x) - (1 - \alpha)] = G(x)e^{i\phi(x)}, \quad (3.17)$$

where, invoking (3.13), the real functions $G > 0$ and ϕ are such that

$$G(-x) = G(x), \quad \phi(-x) = -\phi(x). \quad (3.18)$$

In the remaining cases ($0 < \alpha < 1$, $\theta \neq 0, \pi/2, \pi$) none of these latter symmetries apply, which will have consequences in the expected patterns. The symmetries (3.14)-(3.18) are consistent with the symmetries of the physical problem, taking the form of the solution (3.3) into account.

There is an additional symmetry, analyzed in the next section, that only applies to the threshold patterns and is more subtle.

3.3 Threshold amplitude and primary patterns

This section is devoted to explaining the different temporal (such as quasiperiodicity) and spatial (such as orientation or distribution of the patterns over the domain) features of the patterns at the threshold. The difficulty is that there are seven parameters to be considered, namely $C_g, S, d, L_1, L_2, \theta$, and α , which are too many to be thoroughly considered. Instead, a *baseline case* is considered for the first five parameters, namely

$$C_g = 0.01, \quad S = 0.5, \quad d = 10, \quad L_1 = 50, \quad L_2 = 50. \quad (3.19)$$

Dependence on the forcing parameters α and θ , instead, will be analyzed in detail. The rest of the section is organized as follows. Subsection 3.3.1 obtains the neutral stability curves for the baseline case. Subsection 3.3.2 discusses the mentioned spatio-temporal behaviour of the patterns. Subsections 3.3.3-3.3.7 explains and classifies the nature of the eigenfunctions and the patterns for the different regions of the problem in the $\alpha - \theta$ plane.

3.3.1 Instability threshold

The instability threshold is calculated by imposing marginal instability, namely, requiring that the temporal behavior be either stationary or purely oscillatory. In addition, the threshold patterns can be decomposed into normal modes in the y -direction, seeking solutions of (3.8)-(3.10) of the form

$$F^s(x, t) = [F^+(x)e^{i\Omega t} + \bar{F}^-(x)e^{-i\Omega t}] \cos(\kappa y + \delta). \quad (3.20)$$

Substitution of (3.20) into (3.8)-(3.9) leads to

$$\begin{aligned} i\Omega F^\pm + i(1 + 2S)[\partial_{xx} F^\pm + \nu^2 F^\pm]/4 = \\ -2C_g F^\pm + ia[\alpha g(x) + (1 - \alpha)e^{i\theta}]F^\mp, \end{aligned} \quad (3.21)$$

$$\partial_x F^\pm \pm \gamma(\nu, S)F^\pm = \partial_x F^\mp \pm \gamma(\nu, S)F^\mp = 0 \quad \text{at } x = \pm L_1, \quad (3.22)$$

This problem determines $F^\pm$ except for a common amplitude and phase that remain undetermined. The problem is obviously invariant under the action

$$(\Omega, F^+, F^-) \rightarrow (-\Omega, \bar{F}^-, \bar{F}^+), \quad (3.23)$$

which (assuming simple eigenvalues, which is generic) implies that

$$F^- = \bar{F}^+ \quad \text{if } \Omega = 0. \quad (3.24)$$

The boundary conditions (3.10) at $y = \pm L_2$ will be ignored, assuming infinite container in the y direction and allowing the wavenumber ν to take continuous values in the interval $0 \leq \nu \leq 1$. Marginal instability is analyzed requiring that the homogeneous linear problem (3.21)-(3.22) exhibits nontrivial solutions. This gives the *continuous marginal instability curves*,

$$a = a(\nu), \quad \Omega = \Omega(\nu). \quad (3.25)$$

The minimum of the curve $a = a(\nu)$, denoted as a_c , precisely provides the *subharmonic instability threshold* for this continuous case.

Let us now turn into the above mentioned additional symmetry exhibited by the linearized stability system (3.21)-(3.22) at threshold. This symmetry is relevant since it allows to just calculate the threshold in the interval $0 < \theta < \pi/2$. In order to derive this additional symmetry, we note that at the instability threshold, eqs.(3.21)-(3.22) exhibit nontrivial solutions. Thus, the adjoint problem (using the usual L_2 -inner product),

$$\begin{aligned} i\Omega F_*^\pm + i(1 + 2S)[\partial_{xx} F_*^\pm + \nu^2 F_*^\pm]/4 = \\ 2C_g F_*^\pm - ia[\alpha g(x) + (1 - \alpha)e^{i\theta}]F_*^\mp, \end{aligned} \quad (3.26)$$

$$\partial_x F_*^\pm \pm \gamma(\nu, S)F_*^\pm = \partial_x F_*^\mp \pm \gamma(\nu, S)F_*^\mp = 0 \quad \text{at } x = \pm L_1, \quad (3.27)$$

must also exhibit nontrivial solutions. This property is implicit in the solvability condition that was applied to obtain the marginal instability curves (3.25); see subsection 2.3.2.

Now, the sets of equation (3.21)-(3.21) and (3.26)-(3.27) are invariant under the symmetry

$$(\theta, F_*^+, F_*^-) \leftrightarrow (-\theta, -i\bar{F}_*^+, -i\bar{F}_*^-). \quad (3.28)$$

Invoking uniqueness of the null spaces (which is expected generically), the solutions must be invariant under this symmetry too. Using this, the threshold amplitude must be invariant under the action $\theta \rightarrow -\theta$. Note, however, that the patterns for θ and $-\theta$ do not exhibit a simple relation to each other, as they are related through the adjoint. This is illustrated in subsection 3.3.7 in figures 3.13 and 3.14. Where can be seen that although the patterns are different, they are qualitatively similar, exhibiting the same features.

For illustration of the threshold, the continuous marginal instability curves (3.25) are given in figure 3.4 for the baseline case and the indicated values of α and θ . As can be seen, the marginal curves consist in a series of tongues, and the phase shift θ plays an important role, namely both the instability threshold a_c and the orientation behave monotonously for in-phase forcing ($\theta = 0$), but such dependence is more involved when the horizontal and vertical components are out of phase ($\theta = \pi/2$).

Figure 3.4: Neutral instability curves, rescaled threshold amplitude a vs. the wavenumber along x direction ν for the reference case (3.19), $\alpha = 0, 0.2, 0.4, 0.6, 0.8$, and 1 , and $\theta = 0$ (left) and $\pi/2$ (right); the arrow in the left plot indicates increasing α . The threshold amplitude, a_c , is indicated with plain circles.

3.3.2 Pattern orientation and additional frequency: primary patterns

As in chapter 2 the additional frequency Ω , measuring the deviation from the pure subharmonic case, tends to zero as $L_1 C_g \rightarrow \infty$ (instead, for pure vertically forcing $\Omega = 0$). In general (for not too large $L_1 C_g$ and with

some horizontal forcing), the additional frequency, which is not considered in figure 3.4 but will be discussed below, vanishes for small ν but becomes nonzero at larger ν . For some intermediate values of α (clearly appreciated in figure 3.4-right in the curve for $\alpha = 0.6$), the neutral stability curves exhibit tongues from the horizontal forcing (with minima close to where the neutral stability curve for $\alpha = 1$ exhibits them) and tongues from the vertical forcing (with minima close to where the neutral stability curve for $\alpha = 0$ exhibits them). As a consequence, some of the threshold accelerations plotted in figure 3.4 correspond to $\Omega = 0$ and some others, to $\Omega \neq 0$ (the tongues associated with vertical and horizontal tongues respectively). These two cases correspond to two qualitatively different scenarios:

- A *stationary bifurcation* if $\Omega = 0$. Invoking (3.3), (3.20), and (3.24), the associated threshold patterns are

$$f = 2e^{it}F^+ \cos(\kappa y + \delta) + \text{c.c.} \quad (3.29)$$

Thus, the patterns are *strictly subharmonic* (as in Faraday waves) in this case. The main difference is that F^+ is not spatially constant and thus the pattern shows a longitudinal modulation that may either be spread along the container or concentrated near the end-walls; see below.

- A *Hopf bifurcation* if $\Omega \neq 0$. Invoking (3.3) and (3.20), the associated threshold patterns are quasi-periodic, of the form

$$f_t = e^{it}|A|[F^L \cos \Omega t + F^R \sin \Omega t] \cos(\kappa t + \delta) + \text{c.c.}, \quad (3.30)$$

with

$$F^L = F^+ + \bar{F}^-, \quad F^R = i(F^+ - \bar{F}^-). \quad (3.31)$$

Since $\Omega \ll 1$ the pattern oscillates up and down in the fast timescale, as in the former case. But now, the longitudinal modulation, $F^L \cos \Omega t + F^R \sin \Omega t$, slowly drifts between the longitudinal patterns F^L and F^R , which depending on the parameters of the problem, may be either spread in the container or concentrated near the end-walls; see below. Even though Ω can be quite small in some cases, this slow drift has a strong qualitative effect since depending on the initial conditions, the ‘initial’ longitudinal modulation can be F^L , F^R , or any combination of these two. This means that any of these longitudinal modulations can be observed initially in the short timescale, which might lead to the wrong conclusion that the patterns are not repeatable, unless a sufficiently large experimental run is performed to appreciate the slow drift. The behaviour of these patterns is similar to what was found in

chapter 2, where it was explained also in terms of the function F^L, F^R and the additional frequency Ω in subsection 2.5.1.

Figure 3.5: Close up near $\theta = \pi/2$ showing the occurrence of the stationary (o) and Hopf bifurcation (x).

Now, for the reference case (3.19), these two scenarios occur in the regions θ vs. α plane indicated in figure 3.5. The common boundary of the stationary and Hopf bifurcations in figure 3.5 corresponds to a Takens-Bogdanov (TB) bifurcation (Kuznetsov 1998) occurring at the minimum of the neutral stability curve. In the TB bifurcation, two purely imaginary eigenvalues coalesce in a pair of real eigenvalues, one giving the marginal state and another giving stable dynamics*. Note that this boundary is fairly close to the straight lines $\theta = \pi/2$ and $\alpha = 1$, which will be analyzed in the following subsections in some detail. The straight lines $\alpha = 0$ and $\theta = 0, \pi/2$ will also be specifically considered because some symmetries are present in these cases, as anticipated in subsection 3.2.2, that allow for predicting the qualitative shape of the longitudinal modulation in these cases. The pattern orientations ν are considered in figure 3.6-left, where it can be seen that the patterns tend to be more cross-wave like when decreasing the value of α or increasing the value of θ .

Now, the distribution of the patterns over the container is considered. The patterns found in the Faraday problem cover the whole domain, while

*TB is a codimension-two bifurcation, which generically requires, at least, two free parameters to occur at isolated points. In principle, we have four free parameters, a, α, θ , and ν , but imposing that TB occurs at the minimum of the instability curve reduces to three the number of free parameters and thus determines the mentioned common boundary as a curve $\theta = \theta(\alpha)$ in the $\alpha - \theta$ plane considered in Fig.3.5

Figure 3.6: Grey-scale maps of wavenumber along x (left) and spread ($\delta_x = 10$) (right) of the patterns on the space α - θ .

the patterns for pure horizontal vibration patterns (described in the previous chapter 2) are concentrated near the walls (depending on the value of L_1 and d). When varying α from 0 to 1, the patterns should move from one behaviour to the other. This can be seen in figure 3.6-right where the integral of the threshold pattern amplitude over a region near the center of the container cross section (where the pure horizontally vibrated patterns are expected to be small), namely

$$\langle f \rangle = \frac{1}{2T} \frac{1}{2L_1} \frac{1}{2L_2} \int_0^T \int_{-\delta_x}^{\delta_x} \int_{-L_2}^{L_2} \sqrt{f^2} dy dx dt. \quad (3.32)$$

is plotted. Thus, this plot quantifies how close the patterns are the those resulting from vertical vibration. In other words, large values of $\langle f \rangle$ mean that the pattern is more spread over the whole container. Smaller values mean that the patterns are more concentrated toward one wall. As can be seen, the spread increases with α . The effect of θ seems smaller and it decreases the spread when increasing the phase between the forcing. For both the orientation ν and the spread $\langle f \rangle$, the effect of θ seems to intensify for smaller α .

3.3.3 Purely vertical vibrations ($\alpha = 0$)

For vertical vibrations, $\alpha = 0$, eqs.(3.21)-(3.22) are solved in closed form, as

$$F^\pm = A^\pm \cos(\beta_n x), \quad (3.33)$$

$$\text{with } \frac{A^-}{A^+} = \frac{(1 + 2S)(\nu^2 - \beta_n^2) - 8C_g}{4a}, \quad \beta_n \tan \beta_n L_1 = -\gamma, \quad (3.34)$$

for $n = 0, 1, 2, \dots$, meaning the longitudinal modulations $F^{L,R}$ in (3.30) are both spatially constant. The marginal instability curve (3.25) is

$$\Omega = 0, \quad a = 2\sqrt{C_g^2 + (\nu^2 - \beta_n^2)/32}. \quad (3.35)$$

Note that the minima of these curves, $a = a(\nu)$, occur at $\nu^2 = \beta_n^2$ and are all on the horizontal straight line $a = 2C_g = 0.02$. Thus, the instability threshold, $a_c = 2C_g$ is associated with infinitely many pattern orientations ν (as already illustrated in fig.3.4), which give different candidate eigenfunctions. The actual orientation should be determined by either nonlinear terms or finite effects, both omitted here.

3.3.4 Purely horizontal vibrations ($\alpha = 1$)

This is precisely the case analyzed in detail in chapter 2, where further details may be found. Summarizing the results of that chapter, the additional frequency Ω is always nonzero, which means that these patterns are quasi-periodic. Also, because the forcing function g is odd, in addition to the symmetry (3.23), the problem (3.21)-(3.22) is invariant under the reflection symmetry

$$(x, F^+, F^-) \rightarrow (-x, F^-, -F^+), \quad (3.36)$$

which permits selecting F^+ and F^- such that they are even and odd functions of x , respectively. Also, the common phase of F^+ and F^- , which is undetermined in principle, can be selected such the functions F^L and F^R defined in equation (3.31) concentrate in the left and right part of the container, respectively. For illustration, these functions are given in figure 3.7 for the reference case (3.19). Thus, the threshold patterns (3.30) shows a

Figure 3.7: The real part (solid line) and imaginary part (dashed line) of $F^{L,R}$ for the reference case (3.19) and $\alpha = 1$.

very slow periodic drift between patterns concentrated in the left and right parts of the container. Although Ω can be quite small, it may have a significant effect on the observed patterns, which can be fairly misleading if this slow drift is not taken into account (see the comment in 3.3.2). In fact, as further explained in 2.7, the additional frequency appears due to interaction between the vibrating end-walls and thus, Ω decreases as either $C_g L_1$ increases or d decreases. Furthermore, $\Omega \rightarrow 0$ as $C_g L_1 \rightarrow \infty$ for fixed d .

3.3.5 In-phase vibrations ($\theta = 0, \pi$)

If $\theta = 0, \pi$ and $0 < \alpha < 1$, then equations (3.21)-(3.22) exhibit no symmetry (in addition to (3.23)). The patterns for $\theta = 0$ and $\theta = \pi$ are related by the spatio-temporal symmetry (3.14). The additional frequency Ω can either be nonzero, as $\alpha > \alpha_c^1$, or zero for smaller α , where $0.998 < \alpha_c^1 < 1$ for the reference case (3.19). As already explained, the transition between these two scenarios occurs at a Takens-Bogdanov (TB) bifurcation. Because $\Omega \ll 1$ at $\alpha = 1$, such bifurcation takes place at a value of α that is very close to 1. Also, such transition is affected by the loss of the symmetry (3.36). In other words, as $\alpha < 1$, the forcing term in equations (3.21), which is $\alpha g(x) + 1 - \alpha$ and $\alpha g(x) - (1 - \alpha)$ for $\theta = 0$ and π respectively, becomes slightly more intense in the right and left parts of the container as α decreases from 1, meaning that one of the longitudinal modulations in figure 3.7 is favored. This

Figure 3.8: Left: the TB bifurcation scenario for the reference case at $\theta = 0$, plotting the marginal instability curves (3.25) and the associated longitudinal modulations defined in (3.31), $F^{L,R}$. Thick lines: the case $\alpha = 1$, where $\Omega = 2.79 \cdot 10^{-6} \neq 0$; thin lines: the case $\alpha = 0.98$, where $\Omega = 0$. Right: blow up of the neutral stability curves showing the different Takens-Bogdanov bifurcations along the neutral stability curve.

is illustrated in fig.3.8-left, where the transition at $\theta = 0$ is considered. As anticipated, decreasing α from 1 enhances those patterns that concentrate at the right of the container. As can be seen in figure 3.8-left, the slight inclusion of the vertical vibration has dramatic consequences in the observed patterns. In other words, at $\alpha = 1$ activity slowly drifts from left to right, with a frequency $\Omega \ll 1$, as explained in the last subsection. At $\alpha = 0.98$, instead, only one strictly subharmonic pattern is observed in which activity is concentrated at the right of the container.

The blow up of the marginal instability curves in 3.8-right illustrates well the several TB bifurcations that are present in the marginal instability curve at $\alpha = 0.98$, which are those points where two thin lines coalesce into a single thick line. As α is increased from 0.98 to 1, the first of these TB

points move to the left and reaches the minimum of the marginal instability curve (namely, the instability threshold) at some α intermediate between 0.98 and 1. This value of α precisely gives the transition between the strictly subharmonic and quasi-periodic instabilities.

As for the pure horizontal vibrations (last subsection) the concentration of activity near the end-walls of the container, for α close to one is more and more intense as d and $C_g L_1$ are decreased. The counterpart of figure 3.8 for $\theta = \pi$ is qualitatively similar, except that now it is activity near $x = -L_1$ that is enhanced. On the other hand, as α is decreased from 1 to 0, x -symmetry is gained since, as explained in subsection 3.3.3, the subharmonic patterns are x -symmetric at $\alpha = 0$.

3.3.6 Symmetrically-out-of-phase vibrations ($\theta = \pi/2$)

If $\theta = \pi/2$, then, the forcing of the problem retains some symmetries that seem responsible for the appearance of a region with $\Omega \neq 0$. Now, the additional frequency Ω is nonzero for $\alpha > \alpha_c$ and zero otherwise, for some critical α_c that, for the reference case (3.19), is such that $0.3 < \alpha_c^2 < 0.4$, see figure 3.5. This yields two qualitatively different scenarios for $\alpha > \alpha_c$ and $\alpha < \alpha_c$, which are considered in Fig.3.9. The first scenario ($\alpha > \alpha_c$)

Figure 3.9: The real (solid line) and imaginary (dashed line) parts of the eigenfunctions associated with longitudinal modulations, F^L (left) and F^R (right) for $\alpha = 0.4$ (top) and $\alpha = 0.3$ (bottom).

is illustrated in the upper plots, where the eigenfunctions for $\alpha = 0.4$ are given. These correspond to threshold patterns that slowly oscillates (with

the frequency $\Omega \ll 1$) between two subharmonic patterns. The lower plots, instead, illustrate the case $\alpha < \alpha_c$, where the pattern is strictly subharmonic.

In the case of $\Omega \neq 0$, plotted in fig.3.9-top, the symmetry 2.114 (which relates F^L and F^R for $\alpha = 0$) is no longer present, meaning that the patterns are not reflection symmetric with respect $x = 0$.

3.3.7 Summary of the expected patterns

In this subsection the different patterns along the line $\theta = 127\pi/256 \simeq 1.5585$, see figure 3.5, are explained for different values of α . As it will be seen below, the use of the eigenfunctions F^L and F^R is quite convenient, allowing to appreciate the features of the patterns quite easily, namely to recognize the existence of one or two modes interacting, and the presence of fixed points when the real part of the remaining mode is zero. For the values of $\alpha = 0.3, 0.4, 0.6, 1$ the following behaviours are encountered:

- Non-drifting patterns covering the whole domain for $\alpha = 0.3$. The eigenfunctions of this case are plotted in Fig.3.10, where it can be seen that only there is one mode. As the real part of F^R is almost zero it results on a pattern with fixed points. The spatiotemporal plot can be seen in Fig.3.13.

Figure 3.10: The real (solid lines) and imaginary (dashed lines) parts of the associated longitudinal modulations, F^L (left) and F^R (right) for $\alpha = 0.3$ and $\theta = 127\pi/256$.

- Drifting patterns covering the whole domain for $\alpha = 0.4$. This is a pattern similar to the ones described for the horizontal vibration but with the breaking of spatio-temporal symmetry responsible of the symmetry between F^L and F^R . The small real part of the eigenfunction produces the appearance of fixed points that slowly evolves. The two eigenfunctions of this case are plotted in Fig.3.11 and the combination of them are responsible of the drift. This can be seen in the spatiotemporal plot in Fig.3.13.

3.3. THRESHOLD AMPLITUDE AND PRIMARY PATTERNS

Figure 3.11: The real (solid lines) and imaginary (dashed lines) parts of the associated longitudinal modulations, F^L (left) and F^R (right) for $\alpha = 0.4$ and $\theta = 127\pi/256$.

- Non-drifting pattern concentrated near one wall for $\alpha = 0.6$. The two complex eigenvalues have split giving as a result a pattern concentrated near one wall. The eigenfunctions of this case are plotted in figure 3.12, where it can be seen that there is only one mode. As both real and imaginary part of F^R contribute the resulting pattern has no fixed points but, again, because of the small real part they appear to evolve on the large scale (faster than in the previous case because the larger real part). It can be seen in the spatiotemporal plot in Fig.3.13.

Figure 3.12: The real (solid lines) and imaginary (dashed lines) parts of the associated longitudinal modulations, F^L (left) and F^R (right) for $\alpha = 0.6$ and $\theta = 127\pi/256$.

- The drifting behaviour described in section 2.5.2 is again recovered when α is close to the pure horizontal vibration. The eigenfunctions can be seen in Fig.3.7 and the spatiotemporal behaviour in Fig.3.13.

Figures 3.13 and 3.14 show the different patterns for $\theta = 127\pi/256$ and $\theta = -127\pi/256$, respectively. The patterns show the same temporal modulation, same wavenumber, and similar concentration over the container. Also, both patterns are more strong on the right side of the container (when

Figure 3.13: Spatio-temporal plots of the different patterns at $\theta = 127\pi/256$, and (from left to right) $\alpha = 0.3, 0.4, \alpha = 0.6, \alpha = 1$.

Figure 3.14: Counterpart of figure 3.13 for $\theta = -127\pi/256$.

the patterns are standing). But, in the end, they are different patterns and exhibit different modulation along the x direction. This is because these two families of patterns are related among each other through the symmetry (3.28), which involves the adjoint problem (3.26)-(3.27), whose solution is strictly different from that of the direct problem (3.21)-(3.22). However, since C_g is small, comparison of these two problems shows that they are somewhat close to each other, which explains the approximate symmetry that could be seen between these two families of patterns.

3.4 The effect of the various non-dimensional parameters

As in the last chapter, we now elucidate roughly the effect of the various non-dimensional parameters on the results. To this end, we consider a baseline set of parameter values, namely

$$L_1 = 50, \quad d = 50, \quad C_g = 0.01, \quad S = 0.5, \quad (3.37)$$

and vary just one parameter at a time. This is made in subsections 3.4.1-3.4.4.

3.4.1 Varying the depth d

The effect of the depth is very different from the one that is observed for strictly vertical or horizontal vibration. For vertical vibration, the threshold is independent of this parameter (at least in the limit in which the depth is large enough to not affect the dispersion relation). For the horizontal vibration, instead, the depth controls the forcing. Thus, there is a smooth change in the threshold amplitude until the amplitude locks for large depths, which can reduce the value to the half of the original value. The change in the orientation, instead, behaves more straightforwardly.

The case of moderate d (namely, $d = 10$) was already considered in Fig.3.4, where it is seen that adding a horizontal component to a pure vertical vibration delays the subharmonic instability. The case of a deeper container ($d = 50$) is considered in Fig.3.15, where it is seen increasing the horizontal forcing component enhances the instability.

Figure 3.15: Neutral stability curves at $\theta = 0$ (left) and $\theta = \pi/2$ (right) in the baseline case (3.37) for $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$; the arrow in the left plot indicates increasing α .

3.4.2 Varying the length L_1

The length of the container has two effects. The first of these is associated with the shape of the parametric forcing term (whose overall forcing effect scales with L_1/d) in the amplitude equation, and is similar to the one already mentioned in connection with varying d . The second effect is connected with the effect of L_1 on the number of resonances tongues that are admissible for that particular value of L_1 .

Figure 3.16 shows the stability curves for L_1 only moderately large. For

Figure 3.16: Neutral stability curves at $\theta = 0$ (left) and $\theta = \pi/2$ (right) for $L_1 = 15$, $d = 10$, $C_g = 0.01$, $S = 0.5$, and $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$; the arrow in the left plot indicates increasing α .

the considered value of the product $L_1 C_g$, there is interaction between both endwalls of the container. Also, the damping is small enough, such that marginal instability curves showing resonance tongues are present. In this case, the minimum for the horizontal vibration is close to the parallel waves but become perpendicular for smaller values of α . This transition occurs in between $0.8 < \alpha < 1$ for $\theta = 0$ and between $0.6 < \alpha < 0.8$ for $\theta = \pi/2$. As the minima is located in one of the tongues the transition have to come in the form of a jump between tongues, so quick changes on the orientation of the patterns are expected.

3.4.3 Varying the damping C_g

The marginal instability curves for the case of a larger damping than that considered above are presented in Fig.3.17. Note that the orientation for $\alpha = 1$ is close to 45° . Also, for the considered value of C_g , the change on the orientation of the patterns is larger than the one for smaller damping. This variation is even larger in the case of $\theta = \pi/2$.

3.4. THE EFFECT OF THE VARIOUS NON-DIMENSIONAL PARAMETERS

Figure 3.17: Neutral stability curves for $\theta = 0$ (left) and $\theta = \pi/2$ (right) with $L_1 = 50$, $d = 10$, $C_g = 0.1$, $S = 0.5$ and $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$; the arrow in the left plot indicates increasing α .

3.4.4 Varying the gravity-capillary parameter S

Figure 3.18 shows the marginal instability curves in the extreme cases $S = 1$ and $S = 0$ of the gravity-capillary parameter, namely for purely capillary and purely gravity waves. Note that there does not seem to be a significant effect on the orientation when varying this parameter. The dependence of the threshold amplitude on the vertical/horizontal forcing balance, α , instead, shows opposite trends for $S = 0$ and 1.

Figure 3.18: Neutral stability curves for $\theta = 0$ (left) and $\theta = \pi/2$ (right) with $L_1 = 50$, $d = 10$, $C_g = 0.01$, $S = 0$ (top) and 1 (bottom), and $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$; the arrow in the left plot indicates increasing α .

3.5 Experimental predictions

Let us now turn into some specific experimental predictions for a realistic case. As in the last chapter, the non-dimensionalization used in this thesis is quite appropriate for the analysis, but makes all parameters frequency dependent. This must be taken into account in specific experiments, in which the container is fixed but the frequency is varied, which changes at the same time all nondimensional parameters, except for those relating the horizontal and vertical components of forcing, namely θ and α . For illustration, the variation of the critical acceleration when varying the forcing frequency (in the range $20 \text{ Hz} < 2\omega^* < 90 \text{ Hz}$) is plotted in figure 3.19-left, with the pattern orientation plotted in figure 3.19-right. This figure corresponds to a container of section $9 \text{ cm} \times 9 \text{ cm}$, depth 5 cm , using a silicone

Figure 3.19: Critical acceleration (left) and longitudinal wavenumber (right) against the forcing frequency for the following values of $\alpha = 0, 0.2, 0.4, 0.6, 0.8, 1$ and $\theta = 0$. The arrow indicates increasing values of α . The curve for Faraday waves is not present on the right figure because the selection is made by finite effects.

oil of viscosity 5 cSt , density 0.95 g cm^{-3} and surface tension 19 dyn cm^{-1} . This is, in fact, one of the cases studied in chapter 2 (the variation of the non-dimensional parameters with $2\omega^*$ can be found on Appendix D). As can be seen in the figure, the threshold amplitude increases monotonically with the forcing frequency. The orientation also varies smoothly, except at low frequencies, where there are some oscillations. These irregularities are due to the fact that C_g is small and L_1 is only moderately large, meaning that interaction between the endwalls occurs, which enhance the effect of resonance tongues in the marginal instability curves.

3.6 Concluding remarks

This chapter has considered the effect on the subharmonic instability of a free surface produced by simultaneous vertical and horizontal vibration.

The balance between these components of forcing has been account for in terms of two parameter: α controls the relative intensity of vertical and horizontal forcing and θ the phase between them. Both parameters have effect over the temporal behaviour, the orientation and the spread of the patterns over the container.

Concerning the parameter α , it varies from the extreme cases of purely vertical ($\alpha = 0$) and horizontal ($\alpha = 1$) forcing. The resulting patterns in the classical case of Faraday excitation are well known. At threshold, the patterns result from the transient bifurcation of one mode that covers the whole container in form of stripes that are strictly subharmonic. Purely horizontal forcing was considered in this thesis, in chapter 2, where it was shown that the resulting patterns are quasiperiodic. The additional frequency can be interpreted as the result of the interaction between the two endwalls of the container. This is a Hopf bifurcation in which two eigenvalues become unstable simultaneously.

The relevant patterns can either drift between the two endwalls or not. For the latter non-drifting patterns, the phase θ controls on which side of the container the patterns are concentrated. For θ in the interval $(-\pi/2, \pi/2)$, non-drifting patterns are concentrated over the right part of the domain, while in the other half, $(-\pi/2, -\pi) \cup (\pi, \pi/2)$, these patterns show more activity over the left side. Drifting patterns, instead, show a (generally very slow) drift of activity between the left and right endwalls. However, such drift occurs in a long timescale, while in not too long runs, the pattern could show a preference for one endwall (depending on the initial condition), but this preference would be reversed for the other side of the container over the long scale.

The spread of the patterns over the container is the expected. While moving from Faraday to horizontal forcing, the patterns stop being spread over the whole container and become concentrated towards the endwalls. Very close to $\alpha = 1$, the patterns can be concentrated on both walls (depending which time on the large time scale is selected). The patterns are also more spread when increasing θ .

The transition between standing patterns and quasiperiodic patterns happens soon when the addition of vertical vibration breaks the reflection symmetry of the horizontal forcing. Thus, the Hopf bifurcation breaks and as result one of the complex eigenvalues is stabilized, the other one is destabilized, leading to the disappearance of the drifting frequency. As explained along this chapter, this splitting of the eigenvalues is a Takens-Bogdanov bifurcation, in which the two complex eigenvalues are collapsing on the real axis and splitting in different directions of the real axis. As a consequence of one mode dominating over the other, patterns concentrated in *just one*

of the walls of the domain are found. The appearance of drifting patterns in a region close to $\alpha = 1$ is expected (from chapter 2 results). But, additionally, it is found in this chapter that close to $\theta = \pi/2$ there is a vicinity in which the resulting patterns are also quasiperiodic. This new region is related with the symmetries of the forcing explained in subsection 3.2.1. All these are discussed now that we may collect all results in the chapter. Figure 3.20 shows the common boundary between the two regions associated with strictly subharmonic and quasiperiodic patterns. In order to help on the visual inspection and the understanding of the problem, that region contained the common boundary is magnified in Fig.3.20. As can be seen, the common boundary shows more penetration with θ just when the first

Figure 3.20: Representation of the Takens-Bogdanov bifurcation on the $\alpha - \theta$ plane.

transition occurs (in $\theta = \pi/2$ close to $\alpha = 0.3$) and then θ decreases and the curve goes closer to the line $\theta = \pi/2$. This could be, of course, explained in terms of the interaction between the endwalls. However, such explanation is omitted because it would be somewhat speculative in the absence of a more complete complete set of calculations over the parameter space, which is beyond the object of this thesis. And, moreover, it is not clear that these narrow transitions can be seen on experiments due to the limited experimental accuracy.

The effect of the horizontal/vertical forcing balance over the orientation discussed in this chapter is somehow an unexpected result. As the parameter α is decreased, the patterns become more perpendicular to the walls, until the limit $\alpha = 0$, in which pattern selection is not obtained by the means considered in this chapter. Increasing θ , on the other hand, produces patterns more perpendicular to the walls.

It is worth mentioning that the problem studied in this chapter was analysed by a very straightforward adaptation of the amplitude equation obtained in subsection 2.3.2. This shows the generality of the mentioned amplitude equation to study subhamonic waves.

4

Surface waves excited by immersed wavemakers

4.1 Introduction

This chapter studies the generation of surface waves by an immersed oscillating wavemaker. The use of a completely submerged solid is very interesting because it avoids the generation of menisci on the wavemakers. Such menisci usually take place when using traditional wavemakers, without fixing the contact line, whose motion (which involves great modeling difficulties) directly excites capillary waves. With this approach, instead, the uncertainty over the modelling of the contact line motion disappears.

As in the previous chapters, the obstacle vibration induces two different types of surface waves: harmonic and subharmonic. Harmonic waves are excited by direct forcing and will always be present. Subharmonic waves, instead, arise as a parametric instability and are only triggered for forcing acceleration beyond a certain threshold. Once again, the subharmonic waves can be excited through two mechanisms: the oscillatory bulk flow or the harmonic waves. However, in contrast with what happens in the previous chapters, it is not generally clear that these two forcing mechanisms become decoupled. In other words, decoupling is only expected if the obstacle width and depth from the free surface are large compared to the basic capillary gravity wavelength. These conditions are needed for the oscillatory bulk flow to be slowly varying in space, a property that distinguishes this flow from the harmonic waves. Assuming that this condition holds, the theoretical framework developed in chapter 2 is valid for this new configuration. But, keeping in mind the results in chapter 2, where very accurate predictions were obtained even for a not-so-slowly modulated forcing (see figure 2.4-left), the theory will be somehow stretched a bit out of these limits.

The easy access to water makes experiments with it common. In fact, the available experiments in the literature in connection with immersed wave-

Figure 4.1: Sketch of the vibrating wavemaker.

makers are performed with tap water. Unfortunately, because of the great dissolution power of water, the free surface is easily contaminated. Taking into account that surfactants are known to affect the properties of the free surface and thus the generation of the waves, a new amplitude equation is obtained for this situation.

With these ideas in mind, this chapter is organized as follows. Section 4.2 describes the problem, the equations that are applicable, the characteristic parameters, and the main assumptions. The harmonic flow generated by the obstacle is the object of study in section 4.3 and it is followed by a discussion of the conditions for the theory developed in chapter 2 to be applicable. The subharmonic instability is studied in section 4.4 as well as the effect of the different geometrical and physical parameters involved in the problem. The effect of modelling the contamination of the surface with the Marangoni effect is considered in section 4.5. Comparison with experiments is attempted in section 4.6. The conclusions and some final comments are in section 4.7.

4.2 Formulation

A vertically vibrated immersed wavemaker (Figure 4.1) is considered. This solid is oscillating with an amplitude a^* and a frequency $2\omega^*$, inside of a brimful rectangular container. The non-dimensional cross-section and depth of the wavemaker (see figure 4.2) are $2l_w \times d_w$ and h_w is the immersion depth of the obstacle from the free surface, the span of the wavemaker is $2L_2$. The container is of cross section $2L_1 \times d$, with transverse direction $2L_2$. Thus, the wavemaker covers all the container spanning from one wall to another. This configuration can be seen in the figures 4.1 and 4.2.

The non-dimensionalization used in this chapter is the same used in the previous chapters. Viscosity is accounted for using the modified Ohnesorge number C_g and the gravity-capillary balance with the parameter S . The formulation is the same as in chapter 2, but the boundary conditions 2.10-2.11 need to be adapted to the new geometry and forcing needs adaptation too, since it is produced by vibration of the wavemaker in the vertical direction.

The new boundary conditions are

$$\mathbf{u} = 0, w = -2a \sin 2t \text{ and at } |x| < l_w, z = -h_w + a \cos 2t \quad (4.1)$$

$$\text{and } z = -h_w - d_w + a \cos 2t$$

$$\mathbf{u} = 0, w = 0 \text{ at } x = \pm l_w, -h_w - d_w + a \cos 2t < z < -h_w + a \cos 2t \quad (4.2)$$

$$\mathbf{u} = 0, w = 0 \text{ at } x = \pm L_1, \text{ at } y = \pm L_2, \text{ and at } z = -d, \quad (4.3)$$

$$f = 0 \text{ at } x = \pm L_1 \text{ and at } y = \pm L_2, \quad (4.4)$$

Note that boundary condition (4.4) implies that the free surface is pinned to the upper edge of the container walls. The remaining boundary conditions apply at the free surface and are those in the previous chapters, given in eqs.(2.12), (2.13), and (2.14), which account for kinematic compatibility, equilibrium of tangential stress, and equilibrium of normal stress, respectively. As in the previous chapters, the equations to be solved are the continuity and momentum equations. The resulting problem is invariant under reflection on the symmetry planes of the container, namely under the actions

$$x \rightarrow -x, \mathbf{u} \cdot \mathbf{e}_x \rightarrow -\mathbf{u} \cdot \mathbf{e}_x, \quad \text{and} \quad y \rightarrow -y, \mathbf{u} \cdot \mathbf{e}_y \rightarrow -\mathbf{u} \cdot \mathbf{e}_y, \quad (4.5)$$

which will be referred as the *longitudinal reflection symmetry* and the *transverse reflection symmetry*, respectively.

The main assumptions in this chapter are again

$$a, C_g \ll 1, \quad (4.6)$$

and imply that the steepness of the waves is small, namely

$$|\mathbf{u}| \ll 1, \quad |w| \ll 1, \quad |p| \ll 1, \quad |f| \ll 1, \quad (4.7)$$

and that the oscillatory motion is nearly inviscid, which allows for applying a quasi-potential approximation (Zhang & Viñals 1997) as explained in 2.2.2. Because of the difficulties of modelling a free contact line, this it is assumed to be fixed (no motion). This is experimentally reasonable in the longitudinal direction if the endwalls are apart from the wavemaker because of the damping of the waves, which could make the vicinity of the endwalls motionless. In the transverse direction, instead, meeting this assumption

Figure 4.2: Sketch of the transverse cross section of the container and the wavemaker. The deformation of the free surface is added at the top of the container to illustrate the expected shape of the amplitude, the aspect ratio is highly increased for illustrative purposes.

would require either using a specific coating or pinning the contact line in a corner. The following subsection adapts the boundary conditions to the problem considered in this chapter and reviews the necessary details to understand the formulation.

4.2.1 Quasi-potential approximation

To the approximation relevant here (under the assumptions made above), the free surface elevation is a linear combination of the following components

$$f \simeq [F^{tw}(\mathbf{x}, z, t) + F^{obf}(\mathbf{x}, z, t)]e^{2it} + F^s(\mathbf{x}, z, t)e^{it} + \text{c.c.} + f^{mf}(\mathbf{x}, z, t). \quad (4.8)$$

Here, that term proportional to e^{2it} is the harmonic flow produced by the vibration of the obstacle through direct excitation. That term with the superscript tw includes two counterpropagating wavetrains, which spread from the center of the container to the walls. They are parallel to the direction of the wavemaker and decay within the viscous length. It is expected that for deeply submerged obstacles ($h_w \gg 1$), the travelling waves are weak enough to be neglected. A theoretical framework is developed in section 4.3 to calculate the harmonic wavetrains and check whether they are negligible or not. The components with the superscript obf is the linear, inviscid flow around the obstacle, which has been called oscillatory bulk flow in the previous chapters. This flow penetrates in a region of order comparable to the width of the wavemaker (l_w) for not too deeply submerged obstacles; when the immersion depth is increased, the OBF opens wider. A more detailed discussion about these flows and when the amplitude equation from chapter 2 is valid can be found in section 4.3. The component proportional

to e^{it} are the subharmonic waves triggered by the harmonic flow. They are subharmonic (although not necessarily strictly 2:1) and they are expected to be almost perpendicular to the direction of the wavemaker, but no explicit assumption is made over the orientation of the patterns (the orientation is obtained from the neutral stability of the amplitude equation). The flow with the associated superscript mf is the viscous mean flow, which is originated by the harmonic flow but when it appears it couples both harmonic and mean flow. As already mentioned this mean flow appears at third order, thus for the linear limit considered here it can be neglected.

In this approximation for very low damping, the motion is inviscid in the bulk (outside the oscillatory boundary layers), where the vorticity and the Laplacian of the velocity are both zero.

As anticipated, the theoretical framework coincides with that developed in the previous chapters, except for the following changes. The boundary conditions (2.20) and (2.21) from the formulation on chapter 2 must be replaced by

$$\partial_y p = 0, \quad f = 0 \text{ at } y = \pm L_2, \quad \partial_x p = 0, \quad f = 0 \text{ at } x = \pm L_1, \quad (4.9)$$

$$\partial_z p = 0 \text{ at } z = -d, \quad (4.10)$$

$$\partial_x p = 0 \text{ at } x = \pm l_w, -h_w - d_w < z < -h_w, \quad (4.11)$$

$$\partial_y p + a \partial_{yy} p \cos 2t = 4 \cos(2t) \text{ at } |x| < l_w, z = -h_w \text{ and at } z = -h_w - d_w, \quad (4.12)$$

which are obtained by linearizing the boundary conditions (4.1)-(4.4) around $z = -h_w$ and $z = -h_w - d_w$. The quasi-potential approximation (2.19),(2.22)-(2.23) with boundary conditions (4.9)-(4.12) is the relevant one in the remainder of the chapter, to calculate the harmonic waves in section 4.3, to formulate the OBF in section 4.4 and to obtain the amplitude equation for a polluted surface in section 4.5.

4.3 Harmonic flow

This section is devoted to calculate the harmonic flow associated to the vibration of the wavemaker. As previously stated, the flow has two components: one is a flow in the region close to the wavemaker (oscillatory bulk flow), the other is a couple of counterpropagating wavetrains. The situation is similar to the one presented in chapter 2 when considering the horizontal vibration of a rectangular container. In that chapter, because the simplicity of the flow (the penetration of the harmonic waves determined by the viscous decay and the strength of the OBF by the container aspect ratio), it was easier to know when the instability was expected to be triggered by the

OBF. The dependence of the harmonic flow components on the dimensions of the obstacle and the submersion depth is more complicated for an immersed wavemaker. Only when the wavemaker width $2l_w$ and immersion depth h_w are both large (with the non-dimensionalization used here, this means that they are large compared to the wavelength), the main forcing contribution would be due to the OBF. In this case, the theory developed in chapter 2 can be applied here. If any of these conditions fails, the situation is not evident anymore but for different reasons. When the obstacle is shallow, there is not a clear separation of scales between both harmonic waves and OBF, thus both components appear mixed. The strength of the traveling waves increases the shallower the obstacle is, this being the other way of promoting subharmonic waves. When the obstacle is not wide enough the assumption of a pressure gradient slowly varying in space is in peril. Both of these situations makes the study of the subharmonic instability *not* be a direct extension of the results of chapter 2. There is some interest in knowing the limits in which the developed theory is applicable for the case of not wide or deeply submerged obstacles. One reason is that the only known (to the author's knowledge) experimental results the theory can be contrasted with (Moisy *et al.* 2012) are for a case in which the assumptions are doubtful.

The above makes the analysis of the harmonic flow both interesting and necessary. The study for different configurations will be made trying to identify when (i) there is a clear scale separation between OBF and traveling waves, (ii) the OBF dominates over the harmonic wavetrains, and (iii) the OBF is slowly varying in space. Let us decompose the harmonic flow in these two components,

$$f(\mathbf{x}, t) = (F^{\text{obf}} + F^{\text{tw}})e^{2it} + \text{c.c.}, \quad (4.13)$$

$$p(\mathbf{x}, z, t) = (P^{\text{obf}} + P^{\text{tw}})e^{2it} + \text{c.c.}, \quad (4.14)$$

$$w(\mathbf{x}, z, t) = (W^{\text{obf}} + W^{\text{tw}})e^{2it} + \text{c.c.}, \quad (4.15)$$

and take into account the fact that, far from the obstacle and for very low viscosity, the behaviour of the flow is known when assuming the symmetry of the solution, namely

$$f_{\pm\infty} \simeq Ae^{2it \mp i\tilde{\kappa}x} + \text{c.c.} \quad \text{as } x \rightarrow \pm\infty, \quad (4.16)$$

$$p_{\pm\infty} \simeq \frac{4A}{\tilde{\kappa}} e^{2it \mp i\tilde{\kappa}x + \tilde{\kappa}z} + \text{c.c.} \quad \text{as } x \rightarrow \pm\infty, \quad (4.17)$$

$$w_{\pm\infty} \simeq -2Aie^{2it \mp i\tilde{\kappa}x + \tilde{\kappa}z} + \text{c.c.} \quad \text{as } x \rightarrow \pm\infty, \quad (4.18)$$

where the wavelength $\tilde{\kappa}$ is obtained from the harmonic dispersion relation,

$$4 = (1 - S)\tilde{\kappa} + S\tilde{\kappa}^3. \quad (4.19)$$

Using these, the harmonic wavetrains are determined but for a constant factor A . The flow can be written as

$$f(\mathbf{x}, t) \simeq (F^{\text{obf}} + Ae^{\mp i\tilde{\kappa}x})e^{2it} + \text{c.c.} \quad \text{for } \pm x > 0 \quad (4.20)$$

$$p(\mathbf{x}, z, t) \simeq (P^{\text{obf}} + \frac{4A}{\tilde{\kappa}}e^{\mp i\tilde{\kappa}x}e^{\tilde{\kappa}z})e^{2it} + \text{c.c.} \quad \text{for } \pm x > 0 \quad (4.21)$$

$$w(\mathbf{x}, z, t) \simeq (W^{\text{obf}} - 2Ae^{\mp i\tilde{\kappa}x}e^{\tilde{\kappa}z})e^{2it} + \text{c.c.} \quad \text{for } \pm x > 0 \quad (4.22)$$

This decomposition is somewhat arbitrary and introduces a singular component due to the generation of wavetrains travelling in opposite directions at the origin. If these expressions are replaced in the quasi-potential approximation, a problem for the OBF and the amplitude A is obtained. The problem in these variables is singular as can be seen after replacing the previous decompositions on the boundary conditions. For $x = 0$ the value of the OBF flow variables is different from the right and the left side. This singular behaviour came from the way of writing the counterpropagating wave on each side of the wavemaker. The singularity affect to both flows and the intensity of this singular behaviour is $O(A)$. The jump of the variables is,

$$[F_x^{\text{obf}}] = 2Ai\tilde{\kappa} \text{ at } x = 0, \quad z = 0, \quad (4.23)$$

$$[P_x^{\text{obf}}] = 8Aie^{\tilde{\kappa}z} \text{ at } x = 0 \quad -h_w < z < 0, -\infty < z < -h_w - d_w. \quad (4.24)$$

Although, as in the previous chapters, nonsymmetric solutions are possible, we assume for simplicity that only symmetric solutions are present and impose symmetry from the outset. Thus, we only consider a half of the domain and impose the following boundary conditions for the quasi potential flow at $x = 0$

$$f_x(0, t) = p_x(0, z, t) = 0. \quad (4.25)$$

Replacing the flow expansion on the boundary conditions (4.9)-(4.12), (4.25) and in the quasi-potential approximation (2.19), (2.22)-(2.23), working in $x > 0$ the following problem is obtained

$$\Delta P^{\text{obf}} + \partial_{zz}P^{\text{obf}} = 0, \quad (4.26)$$

with the boundary conditions

$$F_x^{\text{obf}} = Ai\tilde{\kappa} \text{ at } x = 0, \quad z = 0, \quad (4.27)$$

$$P_x^{\text{obf}} = 4Aie^{\tilde{\kappa}z} \text{ at } x = 0 \quad -h_w < z < 0, -\infty < z < -h_w - d_w, \quad (4.28)$$

$$P_z^{\text{obf}} = -4Ae^{-i\tilde{\kappa}x + \tilde{\kappa}z} - 2 \text{ at } 0 < x < l_w, z = -h_w, -h_w - d_w, \quad (4.29)$$

$$P_x^{\text{obf}} = 4Aie^{-i\tilde{\kappa}\frac{l_w}{2} + \tilde{\kappa}z} \text{ at } -d_w - h_w < z < -h_w; x = l_w, \quad (4.30)$$

with the additional boundary condition on the free surface

$$4P^{\text{obf}} - (1 - S)P_z^{\text{obf}} + SP_{zxx}^{\text{obf}} = 0 \text{ at } z = 0. \quad (4.31)$$

To obtain the oscillatory bulk flow, it is necessary to impose an additional equation compatible with the asymptotic behaviour (4.16)-(4.18), which is the radiation condition and leads to

$$4P^{\text{obf}} - (1 - S)P_z^{\text{obf}} = 0 \text{ at } x \rightarrow \infty \quad (4.32)$$

So far, the inviscid case has been considered. When the viscosity is considered, the decay of the harmonic waves should be important over distances of order $x \sim 1/C_g$. If that is the case, the behaviour of the flow far from the obstacle in (4.16)-(4.18) should be replaced by $e^{\pm i\tilde{\kappa}x \mp C_g x}$, and also a suitable radiation condition compatible with this viscous effect should replace (4.32).

A collocated Boundary Element Method technique (with $\mathbb{P}_1$ elements) is used to obtain the solution of equation (4.26) with boundary conditions (4.27)-(4.30) and (4.32). The boundary condition involving second derivatives of the gradient pressure in the x direction (4.31) will be discretized with a finite element method (with $\mathbb{P}_1$ elements). And finally a solvability condition will be used to compute the amplitude of the harmonic waves.

Figure 4.3 shows the gradient of the pressure P_z on the free surface for different values of the immersion depth h_w , the wavemaker length l_w and the gravity-capillary parameter S . As expected, the forcing decreases when the immersion depth h_w is increased. It can also be seen that the forcing is larger for the smaller values of S ; but the effect is weaker. The pressure gradient for a short wavemaker is in figure 4.3-left where it can be seen that P_z is more concentrated and singular when the obstacle is closer to the surface. The amplitude of the generated harmonic waves for different immersion depths of the wavemaker can be seen in figure 4.4.

Looking at the previous results, the discussion of the applicability of the theoretical framework developed in chapter 2 is possible. There are important questions to answer: (i) when to consider that the harmonic wave is negligible; and whether the pressure gradient is slowly varying in space. From figure 4.4, the value of the wave amplitude is less than 5% of the OBF if $h_w \gtrsim 1.5$. Then from this deep of immersion it seems reasonable to consider that the harmonic waves are much smaller than the OBF. From figure 4.3 it can be seen that obstacles for immersion depths $h_w = 3.76, 6.22, 8$ exhibit a slowly varying in space gradient, so it seems these cases fall under the assumptions of the theory previously developed. It could also seem

Figure 4.3: P_z (solution of 4.27-4.32) given for the following depths from the obstacle to the surface, $h_w = 0.5, 0.82, 1.37, 2.27, 3.76, 6.22, 8$ with arrows indicating decreasing values of h_w , for $2l_w = 3$ (left) and $2l_w = 9$ (right). The gravity-capillary parameter is $S = 0.25$ (top), $S = 0.5$ (middle), $S = 0.75$ (bottom) and $d_w/2l_w = 4l_w$.

reasonable to include the fourth one ($h_w = 2.27$) reminding the good results obtained in chapter 2 with a not-so-smooth forcing. The three more shallow obstacles in figure 4.3-left are not slowly varying in space and the three in figure 4.3-right show that the OBF and the harmonic waves are not separated (due to the used decomposition). These three obstacles should probably not produce good results when studied with the present theory.

In order to apply the theory developed in the previous chapter, in the rest of this chapter the following assumptions are made: the harmonic waves are negligible (when compared with the OBF) and the oscillatory bulk flow is slowly varying in space.

Figure 4.4: Variation of real part (solid line) and imaginary part (dashed line) of the harmonic wave amplitude A with the depth of the obstacle h_w , for $2l_w = 3$ (left) and $2l_w = 9$ (right). The gravity-capillary parameter is $S = 0.25, 0.5, 0.75$ with arrows indicating increasing values of S .

4.4 Subharmonic waves

This section adapts the amplitude equation derived in section 2.3.2 to the problem considered in this chapter. Then the threshold of the subharmonic instability is studied. In subsection 4.4.1 the harmonic flow around the obstacle is analyzed. The onset is studied in subsection 4.4.2 and the effect of the different parameters is explored in subsection 4.4.3

4.4.1 Oscillatory bulk flow

In order to have separation of scales between harmonic waves and OBF, and a slowly varying (in space) OBF the assumption that the obstacle is deeply immersed and wide is made, taking $h_w \gg 1$ and $l_w \gg 1$. As explained in section 4.3 this condition is obtained for relatively small immersion depths (even for not too large obstacles), and it could be assumed to be fulfilled for h_w as low as 2.27. In these cases the harmonic surface waves will be very small and as they provide much less forcing than the oscillatory bulk flow, they can be ignored. For a deeply immersed and wide obstacle the oscillatory bulk flow obtained is slowly varying in space implying:

$$|\nabla p^{\text{obf}}| \sim |\partial_z p^{\text{obf}}| \ll |p^{\text{obf}}|. \quad (4.33)$$

With this the dynamic condition on the free surface will simplify to $P^{\text{obf}} = 0$.

The OBF is a linear, inviscid flow generated by the vibration of the oscillating obstacle. With the previous assumptions the pressure can be written as

$$p^{\text{obf}} = aP^{\text{obf}}(x, y, z) \cos 2t, \quad (4.34)$$

after replacing it on the quasi-potential approximation, the relevant equations are:

$$\Delta P^{\text{obf}} + \partial_{zz} P^{\text{obf}} = 0 \quad \text{in } |x| < L_1, |y| < L_2, -d < z < 0, \quad (4.35)$$

$$\partial_x P^{\text{obf}} = 0 \quad \text{at } x = \pm L_1, \quad (4.36)$$

$$\partial_z P^{\text{obf}} = 0 \quad \text{at } z = -d, \quad (4.37)$$

$$P^{\text{obf}} = 0 \quad \text{at } z = 0, \quad (4.38)$$

$$\partial_z P^{\text{obf}} = 4 \quad \text{at } z = -h_w, -h_w - d_w, \quad -l_w < x < l_w, \quad (4.39)$$

$$\partial_x P^{\text{obf}} = 0 \quad \text{at } x = \pm l_w, \quad -h_w - d_w < z < -h_w. \quad (4.40)$$

This problem is solved numerically via a Boundary Element Method (more details about these methods are on Appendix C) obtaining both the pressure and the pressure gradient. As in chapter 2 the forcing term $g(x)$ is related to the pressure gradient in $z = 0$ through $g(x) = -\partial_z P^{\text{obf}}/4$.

The variation of the pressure gradient with the spatial coordinate x can be seen in figure 4.5 for different values of the immersion depth; this is the counterpart of figure 4.3. As expected, when decreasing the immersion depth the intensity of the forcing decreases. As the boundary condition (4.31) is replaced with (4.38) the dependence on S is lost. This is consistent with the results on previous figure 4.3, where it could be seen that for the three deeper wavemakers the results were almost identical for different S , and they are also identical to the results in figure 4.5. The difference on P_z for the three more shallow wavemaker ($h_w = 0.5, 0.82, 1.37$) is considerable for the two wavemakers studied. Also as expected, for a short l_w the OBF obtained is less concentrated with the deeply submerged obstacles assumption. When l_w is larger and h_w is not large, the approximation for the OBF does not show the oscillatory behaviour that appears in figure 4.3-right, instead it adopts an almost square shape. In any event, the developed theory is not applicable. For the fourth shallow immersion (for both wavemakers) there are some differences but they are small enough to be reasonable to consider also the application of the theory to this case.

The effect of changing the height of the wavemaker for a fixed immersion depth and for two different values of the length of the wavemaker is shown in figure 4.6. Although it affects the forcing, the effects are small and it is not considered a relevant parameter of the problem.

4.4.2 Critical amplitude

Under the assumptions of a subharmonic instability triggered by an OBF that is slowly modulated in space (that results from large immersion depth

Figure 4.5: Forcing of the problem for different values of the depth at which the obstacle is located $h_w = 0.5, 0.82, 1.37, 2.27, 3.76, 6.22, 8$, with arrows indicating decreasing values of h_w , for $2l_w = 3$ (left) and $2l_w = 9$ (right), with $d_w/2l_w = 4l_w$.

Figure 4.6: Forcing of the problem for different values of the height of the wavemaker $d_w/2l_w = 0.5, 1, 2, 4, 8$, with arrows indicating increasing values of $d_w/2l_w$, for $2l_w = 3$ (left), $2l_w = 9$ (right) and immersion depth $h_w = 2.27$.

of the wavemaker), the forcing obtained in subsection 4.4.1 can be introduced in the amplitude equation obtained in 2.3.2,

$$\partial_t F^s + i(1 + 2S)(\Delta F^s + F^s)/4 = -2C_g F^s + iag(x)\bar{F}^s. \quad (4.41)$$

Using the same modal expansions as in chapter 2:

$$F^s = \sum_{m=0}^{\infty} F_m(x, t) \cos(\kappa_m y + \delta_m), \quad (4.42)$$

$$F_m(x, t) = A_m F_m^+(x) e^{i\Omega t} + \bar{A}_m \bar{F}_m^-(x) e^{-i\Omega t}, \quad (4.43)$$

the following equations and boundary conditions are obtained:

$$i\Omega F_m^+ + i(1 + 2S)[\partial_{xx} F_m^+ + \nu_m^2 F_m^+]/4 = -2C_g F_m^+ + iag(x)F_m^-, \quad (4.44)$$

$$i\Omega F_m^- - i(1 + 2S)[\partial_{xx} F_m^- + \nu_m^2 F_m^-]/4 = -2C_g F_m^- - iag(x)F_m^+, \quad (4.45)$$

$$\partial_x F_m^+ \pm \gamma(1 - \nu_m^2, S) F_m^+ = \partial_x F_m^- \pm \gamma(1 - \nu_m^2, S) F_m^- = 0 \quad \text{at } x = \pm L_1, \quad (4.46)$$

Equations (4.44)-(4.46) are invariant under the symmetries

$$(x, F_m^+, F_m^-) \rightarrow (-x, \pm F_m^+, \pm F_m^-), \quad (\Omega, F_m^+, F_m^-) \rightarrow (-\Omega, \bar{F}_m^-, \bar{F}_m^+). \quad (4.47)$$

When the frequency is $\Omega = 0$, there is only one amplitude equation (because the second equation becomes the c.c. of the first one). The equation is invariant with respect to the change $F^\pm \rightarrow e^{ic} F^\pm$, and knowing that one equation is the c.c. of the other, the previous invariance implies that $e^{ic} F^- = e^{-ic} \bar{F}^+$, thus a phase c can be chosen to obtain eigenfunctions such that $F^- = \bar{F}^+$.

A representative neutral stability curve is plotted in figure 4.7; the non-dimensional parameters are for one of the experimental comparison in section 4.6. It can be seen that the selected value of the amplitude $a_c = 0.12$ is for $\nu = 0$ being the resulting pattern formed by cross-waves. It can also be seen that there is a transition from a strictly 2:1 pattern to a quasi-periodic pattern around $\nu = 0.16$.

Figure 4.7: Rescaled marginal instability threshold amplitude a (left) and rescaled frequency Ω (right) for the case of $L_1 = 167$, $C_g = 0.0025$, $S = 0.65$, $d = 111.3$, $d_w = 11.2$, $l_w = 2.8$ and $h_w = 2.23$.

The function F^+ at threshold is plotted in 4.8; as it involves a stationary bifurcation one eigenfunction is representative of the amplitudes (recalling the symmetry $F^\pm \rightarrow e^{ic} \bar{F}^\mp$).

4.4.3 Effect of parameters on the threshold

The theory adapted in section 4.4 is used in this subsection to explore the dependence of the threshold on the various non-dimensional parameters of the problem. The range on the variation of the parameters is $1 < h_w < 8$, $1 < l_w < 12$, $0.001 < C_g < 0.05$, $0.25 < S < 0.75$ with $L_1 = 200$. The lower limits of h_w and $2l_w$ are a bit out of the region in which the theory can be

Figure 4.8: Eigenfunctions for the case considered in figure 4.7 The real part of the eigenfunction is in solid lines while the imaginary part uses dashed lines.

applied. This variation of the parameters is consistent with the discussion made previously and it is easily accessible experimentally with common fluids like water or oil using vibration frequencies in the range of 10 – 100 Hz. The aspect ratio of the obstacle is locked to $d_w/2l_w = 4$ because of the limited effect of this parameter, as shown in figure 4.6. The transverse direction of the container L_2 is assumed to be infinite; in the case in which this dimension is finite it would introduce discretization on the neutral stability curve forcing ν to have values consistent with the boundary conditions on $y = \pm L_2$.

The effect of the immersion depth h_w and the wavemaker width l_w on the amplitude threshold and the orientation is plotted in figures 4.9 and 4.10, respectively. Figure 4.9 also shows the effect of C_g and S , and figure 4.10 also analyzes the effect of C_g .

The threshold amplitude increases when increasing the damping C_g or the depth h_w at which the obstacle is located. It also decreases when increasing the gravity-capillary parameter S but the effect is fairly weak. The angle φ with the obstacle axis (defined on (2.104)) decreases when increasing the damping or the gravity-capillary parameter; the patterns, then, lose the pure cross-waves behaviour. The angle φ also decreases for small dampings with the immersion depth. For the largest damping considered the angle φ first decreases and then slowly increases with the immersion depth. The viscosity is the most relevant parameter for the orientation and the dependence is weak on the geometrical parameters and the gravity-capillary parameter. With these values of the wavenumber ν , the patterns would exhibit an orientation angle between 90° and 82° . In the explored range the threshold patterns do not show modulation frequency, being $\Omega = 0$.

The figure 4.10 shows the dependence of the threshold amplitude and orientations when varying the length of the obstacle. As expected, the larger the obstacle, the stronger the forcing, needing a smaller amplitude to obtain

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Figure 4.9: Variation of the rescaled threshold amplitude (left) and orientation (right) with the distance of the obstacle to the surface h_w , for $2l_w = 3$ (continuous lines) and $2l_w = 9$ (discontinuous lines), for the following values of the damping $C_g = 0.001, 0.005, 0.01, 0.05$ with arrows indicating increasing values of C_g , the gravity-capillary parameter is $S = 0.25$ (top), $S = 0.5$ (middle), $S = 0.75$ (bottom).

subharmonic waves. Concerning the orientation, similarly to the effect of the immersion depth. If the damping is small, then angle φ decreases with the length of the obstacle. If the damping is large the angle φ first decreases and then slowly increases with the length of the wavemaker.

4.5 Amplitude equation for contaminated surface

The free surface of a fluid could be contaminated with the addition of surfactants. This process could happen naturally due to the solubility of the considered fluid or could be intentional with the idea of varying some property of the fluid, for example, increasing the damping. Considering the availability of water it is expected that some experiments would use

Figure 4.10: Variation of the rescaled threshold amplitude (left) and orientation (right) with the length of the obstacle $2l_w$, for $h_w = 2.27$ (continuous lines) and $h_w = 6.22$ (discontinuous lines), for the following values of the damping $C_g = 0.001, 0.005, 0.01, 0.05$ with arrows indicating increasing values of C_g , the gravity-capillary parameter is fixed to $S = 0.5$.

water as in Moisy *et al.* (2012) and knowing how easily it is contaminated it is sensible to try to include some contamination effects. This section is devoted to extend the developed theory to the case in which the surface is polluted; in the end the aim is to obtain an amplitude equation that accounts for the contamination effects.

It is well established that water suffers from surface contamination since Henderson & Miles (1994). A very good candidate for modelling the contamination is the Marangoni effect as shown in Nicolas & Vega (1999). Following the spirit of that work, the contamination will be included via the Marangoni effect, neglecting dilatational and shear surface viscosity, and surfactant solubility. So the aim of this section will be to adapt the amplitude equation obtained in section 2.3 for the case in which the free surface is contaminated. The next subsection 4.5.1 obtains the Marangoni tangent stress equation in terms of the flow variables. Subsection 4.5.2 solves the viscous boundary layer using the previous equation as boundary condition. The resonant terms of the dynamic and boundary conditions are obtained in subsection 4.5.3. The subsection 4.5.4 obtains the expressions for the velocity components in terms of the pressure to replace them on the boundary conditions. Subsection 4.5.5 imposes the resonance condition obtaining the amplitude equation for a contaminated surface. The subsection 4.5.6 studies the marginal instability of this problem and the effect of the contamination on the threshold.

4.5.1 Marangoni tangent stress equation

Here the tangent stress equation and the conservation of the surfactant on the free surface is used to obtain a boundary condition that includes

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the contamination in the formulation of the problem. The tangent stress equation in the free surface with the Marangoni effect is:

$$\sqrt{C_g}(\partial_z \mathbf{u} + \nabla w) = -\frac{\gamma}{\Gamma_0} \frac{S}{\sqrt{C_g}} \nabla c \quad \text{at } z = 0, \quad (4.48)$$

where c is the concentration of contaminant, Γ_0 is the mean concentration and $\gamma = \frac{\Gamma_0}{\sigma(\Gamma_0)} \frac{d\sigma}{d\Gamma_0}(\Gamma_0)$. Here it has been assumed that the surface tension varies only with the contamination, Γ , being independent on any other variable like temperature. In this equation it appears the variation of the surface tension with the concentration but other terms from the surface tension correction are not present. This is because with the small value of the steepness, consistent with the assumptions made, namely $a \ll 1$ and $|\nabla f| \ll 1$. The conservation equation reads:

$$\partial_t c + \nabla \cdot (c\mathbf{u}) = 0 \quad \text{at } z = 0, \quad (4.49)$$

Assuming that the contaminant concentration behave in time like $c = \Gamma_0 + Ce^{it}$ and $\mathbf{u} = \mathbf{U}e^{it}$, conservation eq.(4.49) can be written as:

$$iC + \Gamma_0 \nabla \cdot \mathbf{U} = 0. \quad (4.50)$$

From this it is obtained $C = i\Gamma_0 \nabla \cdot \mathbf{U}$, which after substitution on (4.48) yields

$$\sqrt{C_g}(\partial_z \mathbf{U} + \nabla W) = -i\gamma \frac{S}{\sqrt{C_g}} \nabla(\nabla \cdot \mathbf{U}) \quad \text{at } z = 0. \quad (4.51)$$

4.5.2 Solving boundary layer

To obtain an amplitude equation below the viscous boundary layer the correction of the variables when moving from the free surface to the region below the viscous boundary layer needs to be calculated. This is the counterpart to the procedure on 2.3.1. The continuity equation and momentum equations are

$$\partial_z w + \nabla \cdot \mathbf{u} = 0, \quad (4.52)$$

$$\partial_t \mathbf{u} = -\nabla p + C_g(\Delta \mathbf{u} + \partial_{zz} \mathbf{u}), \quad (4.53)$$

$$\partial_t w = -\partial_z p + C_g(\Delta w + \partial_{zz} w), \quad (4.54)$$

Taking into account the harmonic behaviour of the flow, variables can be written as:

$$(\mathbf{u}, w, p) = (\mathbf{U}, W, P)e^{it} + c.c., \quad (4.55)$$

and the resulting equations read

$$\nabla \cdot \mathbf{U} + \partial_z W = 0, \quad (4.56)$$

$$\mathbf{iU} = -\nabla P + C_g(\Delta \mathbf{U} + \partial_{zz} \mathbf{U}), \quad (4.57)$$

$$\mathbf{iW} = -\partial_z P + C_g(\Delta W + \partial_{zz} W), \quad (4.58)$$

with the tangent stress boundary condition at the free surface written as

$$\partial_z \mathbf{U} + \nabla W = -\mathbf{i}\gamma \frac{S}{C_g} \nabla(\nabla \cdot \mathbf{U}) \quad \text{at } z = 0. \quad (4.59)$$

Now the following stretched coordinate $\eta = z/\sqrt{C_g}$ is introduced,

$$\sqrt{C_g} \nabla \cdot \mathbf{U} + \partial_\eta W = 0, \quad (4.60)$$

$$\mathbf{iU} = -\nabla P + C_g \Delta \mathbf{U} + \partial_{\eta\eta} \mathbf{U}, \quad (4.61)$$

$$\sqrt{C_g} \mathbf{iW} = -\partial_\eta P + C_g^{3/2} \Delta W + \sqrt{C_g} \partial_{\eta\eta} W, \quad (4.62)$$

$$\partial_\eta \mathbf{U} + \sqrt{C_g} \nabla W = -\mathbf{i}\gamma \frac{S}{\sqrt{C_g}} \nabla(\nabla \cdot \mathbf{U}) \quad \text{at } z = 0. \quad (4.63)$$

Integrating the horizontal momentum equation 4.61 and imposing boundary conditions on $\eta = 0$ and $\eta \rightarrow \infty$, being C_g small, yields (at leading order):

$$\mathbf{U} = \mathbf{i}\nabla P + \mathbf{A}e^{\sqrt{\mathbf{i}}\eta} \quad (4.64)$$

where $\mathbf{A}$ is a vector and it represents the correction to the first order of the velocity that accounts for the small terms coming from the viscosity and the contamination. Introducing the expression for $\mathbf{U}$ in the continuity equation 4.56 and integrating yields W ,

$$W = -\sqrt{C_g} \mathbf{i}\Delta P \eta - \frac{\sqrt{C_g}}{\sqrt{\mathbf{i}}} \nabla \cdot \mathbf{A} e^{\sqrt{\mathbf{i}}\eta}. \quad (4.65)$$

4.5.3 Resonant terms on the kinematic and dynamic boundary conditions

In this subsection, the solution from the oscillatory bulk flow is introduced in the boundary conditions to identify subharmonic resonant terms; the nonresonant terms and the terms of order smaller than C_g will be neglected. In the quadratic terms the superscript s and obf are added to clarify the origin of each contribution. The kinematic boundary condition is:

$$\partial_t f = -\nabla f \cdot \mathbf{u} + w + f \partial_z w \quad \text{at } z = 0, \quad (4.66)$$

where:

$$f = \underbrace{-\frac{a}{2}g(x)e^{2\mathbf{i}t}}_{obf} + \underbrace{e^{\mathbf{i}t}F^s}_{s} + c.c.,$$

$$u = \underbrace{-iae^{2it} + c.c.}_{obf} + \underbrace{e^{it}U^s + e^{it}A + c.c.}_s,$$

$$w = \underbrace{-iag(x)e^{2it} + c.c.}_{obf} + \underbrace{e^{it}W^s - e^{it}\frac{\sqrt{C_g}}{\sqrt{i}}\varphi^s + c.c.}_s,$$

and $\varphi^s = \nabla \cdot \mathbf{A}$.

keeping the leading order terms and the first correction, the resonant terms in the kinematic condition are:

$$iF^s + \partial_t F^s = W^s - \frac{\sqrt{C_g}}{\sqrt{i}}\varphi^s - \frac{a}{2}g(x)(\partial_z \bar{W}^s - \bar{\varphi}^s) \quad \text{at } z = 0. \quad (4.67)$$

The normal component of the dynamic condition is

$$p + f\partial_z p - \frac{1}{2}\mathbf{u}^2 - \frac{1}{2}w^2 - (1-S)f + S\Delta f - 2C_g\partial_z w = 0 \quad \text{at } z = 0, \quad (4.68)$$

where the pressure and the gradient pressure are decomposed as:

$$p = \underbrace{-2axe^{2it} + c.c.}_{obf} + \underbrace{e^{it}P^s + c.c.}_s,$$

$$\partial_z p = \underbrace{-2ag(x)e^{2it} + c.c.}_{obf} + \underbrace{e^{it}\partial_z P^s + c.c.}_s.$$

If the leading order and the first correction terms are retained, the resonant terms are:

$$P^s - 2ag(x)\bar{F}^s - \frac{a}{2}g(x)\partial_z \bar{P}^s + iag(x)(\bar{W}^s - i\sqrt{-iC_g}\bar{\varphi}^s) - (1-S)F^s + \\ + S\Delta F^s - 2C_g(\partial_z W^s - \varphi^s) = 0 \quad \text{at } z = 0. \quad (4.69)$$

4.5.4 Replacing velocity components

The next step is to replace the velocity in the boundary conditions using the approximations from the momentum equation and the corrections from the viscous and Marangoni effects:

$$U^s = i\partial_x P^s - \partial_{xt} P^s - \partial_{tt} U^s \approx i\partial_x P^s - \partial_{xt} P^s, \quad (4.70)$$

$$W^s = i\partial_z P^s - \partial_{zt} P^s - \partial_{tt} W^s \approx i\partial_z P^s - \partial_{zt} P^s. \quad (4.71)$$

Substituting these expressions in the continuity equation gives,

$$\Delta P^s + \partial_{zz} P^s = 0. \quad (4.72)$$

The kinematic condition in terms of the pressure, the surface amplitude and the contamination reads,

$$iF^s + \partial_t F^s = i\partial_z P^s - \partial_{zt} P^s + i\sqrt{iC_g}\varphi^s - \frac{a}{2}g(x)(i\Delta\bar{P}^s - \bar{\varphi}^s) \text{ at } z = 0, \quad (4.73)$$

and the dynamic condition is

$$P^s - (1-S)F^s + S\Delta F^s = 2ag(x)\bar{F}^s + \frac{a}{2}g(x)\partial_z\bar{P}^s - ag(x)(\partial_z\bar{P}^s + \sqrt{-iC_g}\bar{\varphi}^s) + 2C_g(-i\Delta P^s - \varphi^s) = 0 \text{ at } z = 0. \quad (4.74)$$

It is also necessary to replace the velocity on the tangent stress boundary condition, that gives

$$\sqrt{i}\varphi + \sqrt{C_g}\Delta(2i\partial_z P + i\sqrt{i}\sqrt{C_g}\varphi) = -i\gamma\frac{S}{\sqrt{C_g}}\Delta(i\Delta P + \varphi) \text{ at } z = 0. \quad (4.75)$$

4.5.5 Solvability condition

The goal of this subsection is to obtain a solvability condition from the first order correction terms of the equations. It is the same procedure that was previously made on 2.3.2 and it is not repeated here. Decomposing the flow variables as,

$$P^s(x, y, z, t) = e^z F^s(x, y, t) + P_1^s(x, y, z, t) \quad (4.76)$$

$$F^s(x, y, t) = F^s(x, y, t) + F_1^s(x, y, t) \quad (4.77)$$

and substituting the values of $\partial_z P_1^s$ and P_1^s in the boundary conditions, the solvability condition yields the new amplitude equation for a contaminated surface,

$$F_t^s + i\left(\frac{1+2S}{4}\right)(\Delta F^s + F^s) = C_g(-F^s - i\varphi^s) + \frac{i\sqrt{iC_g}}{2}\varphi^s + iag(\bar{F}^s - \frac{i}{4}\bar{\varphi}^s), \quad (4.78)$$

$$\sqrt{i}\varphi^s - 2i\sqrt{C_g}F^s = \gamma\frac{S}{\sqrt{C_g}}F^s - i\gamma\frac{S}{\sqrt{C_g}}\Delta\varphi^s. \quad (4.79)$$

These equations have to be completed with the boundary conditions that are exactly like the ones used in the rest of the thesis:

$$\partial_x F^s \pm \int_{-1}^1 \gamma(\kappa^2, S)\phi_{\pm L_1}(\kappa)e^{i\kappa y} d\kappa = 0 \text{ at } x = \pm L_1, \quad (4.80)$$

$$\partial_x \varphi^s \pm \int_{-1}^1 \gamma(\kappa^2, S)\hat{\phi}_{\pm L_1}(\kappa)e^{i\kappa y} d\kappa = 0 \text{ at } x = \pm L_1, \quad (4.81)$$

The scalar γ is as given in equation (2.88) and the Fourier transform $\phi_{\pm L_1}$, $\hat{\phi}_{\pm L_1}$ are defined such that

$$F^s(\pm L_1, y) = \int_{-1}^1 \phi_{\pm L_1}(\kappa) e^{i\kappa y} d\kappa, \quad \varphi^s(\pm L_1, y) = \int_{-1}^1 \hat{\phi}_{\pm L_1}(\kappa) e^{i\kappa y} d\kappa. \quad (4.82)$$

4.5.6 Subharmonic instability

Using the same modal expansions as in chapter 2:

$$F^s = \sum_{m=0}^{\infty} F_m(x, t) \cos(\kappa_m y + \delta_m), \quad \varphi^s = \sum_{m=0}^{\infty} \varphi_m(x, t) \cos(\kappa_m y + \delta_m), \quad (4.83)$$

to study the marginal stability of the problem (4.78)-(4.81), stationary or pure oscillatory solutions are sought,

$$\begin{aligned} F_m(x, t) &= F_m^+(x) e^{i\Omega t} + \bar{F}_m^-(x) e^{-i\Omega t}, \\ \varphi_m(x, t) &= \varphi_m^+(x) e^{i\Omega t} + \bar{\varphi}_m^-(x) e^{-i\Omega t}, \end{aligned} \quad (4.84)$$

Thus, the following equations and boundary conditions are obtained:

$$\begin{aligned} i\Omega F_m^{\pm} \pm i\left(\frac{1+2S}{4}\right)(\partial_{xx} F_m^{\pm} + \nu^2 F_m^{\pm}) = \\ C_g(-F_m^{\pm} \pm i\varphi_m^{\pm}) \pm \frac{\pm i\sqrt{\pm i C_g}}{2} \varphi_m^{\pm} \pm iag(F_m^{\pm} \pm \frac{i}{4} \varphi_m^{\pm}) \end{aligned} \quad (4.85)$$

$$\sqrt{\pm i} \varphi_m^{\pm} \mp 2i\sqrt{C_g} F_m^{\pm} = \gamma \frac{S}{\sqrt{C_g}} F_m^{\pm} \mp i\gamma \frac{S}{\sqrt{C_g}} (\partial_{xx} \varphi_m^{\pm} - \kappa_m^2 \varphi_m^{\pm}) \quad (4.86)$$

$$\partial_x F_m^+ \pm \gamma(1 - \nu_m^2, S) F_m^+ = \partial_x F_m^- \pm \gamma(1 - \nu_m^2, S) F_m^- = 0 \quad \text{at } x = \pm L_1, \quad (4.87)$$

$$\partial_x \varphi_m^+ \pm \gamma(1 - \nu_m^2, S) \varphi_m^+ = \partial_x \varphi_m^- \pm \gamma(1 - \nu_m^2, S) \varphi_m^- = 0 \quad \text{at } x = \pm L_1, \quad (4.88)$$

A representative neutral stability curve is plotted in figure 4.11 for the same non-dimensional values used in figure 4.7. It can be seen that the selected value of the amplitude is $a_c = 0.22$ for $\nu = 0.15$, which means that with the addition of contamination the resulting waves are now oblique respect to the wavemaker. The transition from a strictly 2:1 pattern to a quasi-periodic pattern is delayed due to the contamination to the value $\nu = 0.30$.

The function F^+ at threshold is plotted in 4.12. As it involves a stationary bifurcation one eigenfunction is representative of the amplitudes (recalling the symmetry $F^{\pm} \rightarrow e^{ic} \bar{F}^{\mp}$).

Figure 4.11: Counterpart of figure 4.7 for a contaminated surface with $\gamma = 2.5$ showing the real part of F^+ (solid lines) and the imaginary part (dashed lines).

Figure 4.12: Counterpart of figure 4.8 for a polluted surface.

The effect of increasing the value of the Marangoni parameter γ on both the orientation and the threshold amplitude is shown in figure 4.13 for two different obstacles: a small obstacle $2l_w = 3$ and a larger one $2l_w = 9$. Two different depths are considered $h_w = 2.2, 6.4$; also two values of the damping are used $C_g = 0.005, 0.05$. The gravity-capillary parameter S instead is fixed to $S = 0.5$. As in section 4.4.3 the container is relatively large, with $L_1 = 200$, and the height of the obstacle is $d_w/2l_w = 4$. The threshold amplitude increases with the addition of contaminant to the free surface. The effect on the orientation is more complicated and a detail of the neutral stability curves is plotted in figure 4.14. The Marangoni effect initially increases the obliqueness of the resulting patterns but if this parameter keeps increasing the patterns become perpendicular to the walls again. The pattern orientation observed in the range of the considered parameters considered ($2 < h_w < 6$, $3 < 2l_w < 9$, $0.005 < C_g < 0.05$, $S = 0.5$ and $0 < \gamma S / \sqrt{C_g} < 1$) is between 72.5° and 90° .

Figure 4.13: Variation of the rescaled threshold amplitude (left) and orientation (right) with the contamination $\gamma S / \sqrt{C_g}$, for $h_w = 2.27$ (continuous lines) and $h_w = 6.22$ (discontinuous lines), for two values of the damping $C_g = 0.005, 0.05$ with arrows indicating increasing values of C_g . The length of the obstacle is $2l_w = 3$ (top) and $2l_w = 9$ (bottom), the gravity-capillary parameter is fixed to $S = 0.5$.

Figure 4.14: Detail of the neutral stability curves showing the variation of the forcing amplitude with the wavenumber for $\gamma S / \sqrt{C_g} = 0.14, 0.28, 0.42, 0.57, 0.71$; $h_w = 6.22$, $2l_w = 9$, $d_w/2l_w = 4l_w$, $C_g = 0.05$, $S = 0.5$, $L_1 = 200$ and the arrows indicate increasing values of $\gamma S / \sqrt{C_g}$.

4.6 Comparison with experiments

The experiments performed by Moisy *et al.* 2012 are very interesting and the closest ones to the configuration considered in this chapter. Thus the theoretical results of the model are going to be compared against these experimental results, although it must be noted that there are some important

differences.

The experimental setup used on the experiments can be seen in figure 4.15. The different dimensions used are $2L_1^* = 60$ cm, $2L_2^* = 30$ cm, $2l_w^* = 0.5$ cm, $d_w^* = 2$ cm. The values of the immersion depth are $h_w^* = 0.2, 0.4, 0.6, 0.8, 1$ cm. The span of the wavemaker does not cover the whole container; wavemakers of 12 and 24 cm are used. Thus this wavemaker is different to what the model assumes: a wavemaker spanning through the whole container (in the y direction). The working fluid is tap water, so contamination of the free surface is to be expected. Also no special care was taken to obtain a fixed contact line; this is not important for the boundary conditions on $x = \pm L_1$ but could be of importance for the transverse walls. In these conditions, the resulting patterns are reported to be standing and the angle φ they form with the wavemaker is around 70° .

Figure 4.15: Experimental setup used during the realization of the experiments (picture from Moisy *et al.* 2012).

With the non-dimensionalization used in this thesis the value of the physical and geometrical parameters moves when changing the frequency. With a frequency in the range $20 \text{ Hz} < 2\omega^* < 60 \text{ Hz}$ the non-dimensional parameters vary over the ranges $82 < L_1 < 232$, $1.37 < l_w < 3.9$, $0.55 < h_w < 8.33$, $0.31 < S < 0.78$, $0.0011 < C_g < 0.0031$ (the variation of these parameters with $2\omega^*$ can be found on Appendix D). Looking at the results of subsection 4.4.3 (where no contamination effect were considered) it is expected that the resulting patterns are strictly perpendicular to the walls. Some values of the immersion depth are a bit small to consider that the

obstacle is deeply submerged in the whole range of the experiments. The theory should be applicable for $h_w > 2$; these condition on the immersion exclude the most shallow obstacle (0.2 cm) but is fulfilled by the second more shallow obstacle from 40 Hz onwards. The rest of the wavemakers fall under the assumptions of the theory.

Both the experimental values and the values obtained from the amplitude eqs.(4.44)-(4.46) for the threshold amplitude in the experiments of Moisy *et al.* (2012) are plotted in figure 4.16 for the different immersion depths of the wavemaker. In the predictions obtained from eqs.(4.44)-(4.46) the modulation frequency is zero involving a 2:1 bifurcation and the orientation of the patterns is perpendicular to the wavemaker. The threshold amplitudes obtained with the model are around 40% – 60% smaller than the experimental ones. This result is no surprising as the experiments involves the use of tap water, which is easily polluted by surfactants. In both harmonic and subharmonic waves the paper of Moisy *et al.* (2012) reports that the decay rate of the waves is larger than the corresponding decay for pure water and they estimate a value of a 2 times larger decay.

Figure 4.16: Curves comparing the critical acceleration of the experiments (solid lines) with the ones obtained in the theory (dashed lines), for immersion depths $h_w^* = 0.2, 0.4, 0.6, 0.8, 1$ cm. The value of the amplitude increases with the immersion depth.

Using the amplitude equation derived in section 4.5 the threshold amplitude for the experiments of Moisy *et al.* (2012) is computed again with the addition of the contamination effect on the free surface. To choose a suitable value of the Marangoni parameter γ the decay of the longitudinal waves reported in Moisy *et al.* (2012) is used. These values are compared with the decay that a wave would exhibit theoretically when considering Marangoni effect. Values in the range $2 < \gamma < 3$ are consistent with this estimation. The forcing amplitude for the different depths of immersion of the wavemaker is plotted in figure 4.17. The modulation frequency obtained in the predictions is again zero involving a 2:1 bifurcation. The orientation of

the patterns is now oblique to the walls, obtaining an angle between 78° – 82° .

The effect of considering the discretization coming from the selection of the lateral walls (as done in chapter 2 with boundary conditions (2.98)–(2.99)) can be seen in figure 4.17-right. The presence of jumps (meaning the transition from one of the discrete κ_m to another) indicates that, although L_2 is large, the conditions on the lateral walls are important and should not be ignored.

Figure 4.17: Counterpart of figure 4.16 for a polluted surface (left) with $\gamma = 2.5$ and threshold amplitude considering the length of the container in the transverse direction (right).

Although the predictions (taking into account surface contamination) get closer to the experimental results, there is still a noticeable discrepancy between both curves. It is to be noted here that there are several differences between the experiments and the theory developed in this chapter:

- The wavemaker in the experiments is finite and there is a gap between the lateral walls and the wavemaker. In our theory the wavemaker goes from wall to wall.
- In the gap between the wavemaker and the lateral walls there are two rods that work as a guide for the shaker.
- The experiments are performed with a free contact line in the lateral walls. The theory considers that it is a fixed contact line.

All these differences can be relevant. The effect of the boundary conditions has been proved to be important, as can be seen in figure 4.17-right in this section and in figure 2.25 in chapter 2. There are also contact lines on the rods, which are closer to the subharmonic field than the walls (they can produce some reflection and dispersion), that are not taken into account in the model.

Also, there is some uncertainty in the contamination modelling. In the thesis the effect of tap water is modeled through Marangoni elasticity. Although there are also other possible effects such as dilatational and shear surface viscosities or surfactant solubility their effect is not expected to be that important.

Thus the differences in the threshold amplitude and orientation could be due to a combination of all these effects. One important and interesting question raised during the analysis of this problem is the origin of the wave obliqueness. Under the assumptions of the models the origin of the obliqueness can be summarized:

- With a clean surface an infinite wavemaker does not produce oblique waves due to the small viscosity of the water. The obliqueness only can come from the pattern selection of the transverse walls but this effect is probably weak
- With a polluted surface instead there are also oblique waves with an infinite wavemaker.

4.7 Concluding remarks

This chapter have analyzed the generation of surface waves due to a vibrating obstacle fully immersed in a fluid. A theoretical framework based on the quasi-potential approximation was developed to study the harmonic flows and then the amplitude equation obtained in chapter 2 was adapted to the new configuration with and without the effect of the contamination in the free surface.

The oscillation of the wavemaker promotes a couple of counterpropagating wavetrains and an oscillatory bulk flow around the obstacle for any oscillation amplitude; these flows have been studied in section 4.3. There it was checked that if the immersion depth and the obstacle width are large enough then the harmonic waves are small when compared to the oscillatory bulk flow and the OBF is slowly modulated. These facts where used in the section 4.4 to consider geometries where the harmonic waves are negligible and the OBF is slowly varying in space. With this, the study of the excitation of subharmonic waves can be made by adapting the general amplitude equation obtained in chapter 2 to the forcing produced by the oscillatory bulk flow generated by an immersed obstacle. For the case of very small damping the patterns are strictly perpendicular to the walls (in the range of parameters explored). When increasing the viscosity, which seems to play an essential role on the angle that the waves exhibit, the angle with the wavemaker is no longer perpendicular. The immersion depth of the

wavemaker also decreases this angle, obtaining angles with the walls ranging between 80° and 90° .

Section 4.5 includes the effect of surfactant contamination through the Marangoni effect. The tangent stress boundary condition, with surface tension varying with the surfactant concentration, is used to recalculate the amplitude equation. With this new source of damping the amplitude threshold and the orientation of the patterns are studied at the subharmonic onset. The addition of this effect increases the critical amplitude due to the additional damping introduced and also decreases the angle of the patterns. With the inclusion of Marangoni effect the angle of the waves with respect to the wavemaker is (in the explored range) 72.5° to 90° .

Section 4.6 compares the results provided by this theory with the closest experiments using the configuration considered in this chapter. The theoretical threshold amplitude obtained without contamination is approximately half of the experimental results by Moisy *et al.* (2012). The patterns predicted by the theory for this small viscosity are perpendicular to the walls. The decay of the waves in the experiments is consistent with the contamination of the free surface, which provides an additional source of damping. This is to be expected due to the use of tap water, which is easily polluted. When considering the Marangoni effect the order of the threshold amplitude of the experiments can be recovered but the slope is not quite correct. Also, it is very important to note that when contamination is taken into account, the new patterns are now oblique, being the angle around $78^\circ - 82^\circ$. There are several differences between the theory and the experiments that can explain the discrepancies when comparing both: such as the presence of contact lines, the gaps between the wavemaker and the lateral walls or the lateral tubes assisting in the transmission of the shaker vibration.

5

Conclusions

This thesis has studied three different problems of high interest in the field of parametrically driven surface waves, with the goal of understanding the generation of subharmonic instabilities in situations with non-uniform forcing. All of the developed work was made with the idea of possible comparison with experiments, in spite of its scarcity in the literature.

First, a theory has been developed to study the subharmonic instability for the horizontal vibration of a rectangular container triggered by the oscillatory bulk flow (OBF) generated by the container walls. One of the main guidelines during the development of the theory was to obtain it from first principles without any kind of phenomenological parameter adjustment. Also, the formulation of the theory is general enough to be easily extended to other configurations. This is possible because (i) the theory makes no assumption over the orientation of the patterns, (ii) the forcing is generic, and (iii) the spatial fast scales are not filtered. Not assuming any orientation of the patterns has the additional advantage of making the study of the obliqueness of the patterns possible. The model was validated with the experiments by Porter *et al.* (2012). Then the theory has been extended to two different related problems. First, the study of combined horizontal and vertical vibration of a rectangular container, which is a relatively ignored problem and the results can be seen as predictions. The second extension was to study the problem of an immersed wavemaker. The model helps to discuss the question of which is the origin of wave obliqueness on the set of very interesting experiments by Moisy *et al.* (2012).

The problem of a rectangular container horizontally vibrated was studied on chapter 2. The neutral stability analysis of the amplitude equation found that the spatio-temporal threshold is related with a pair of complex Floquet multipliers close to -1. The resulting patterns are oblique, slowly modulated (quasi-periodic) waves that are nearly standing in the short timescale and slowly drift back and forth between the two endwalls. Assuming supercritical bifurcations, a weakly nonlinear mode interaction analysis was made. The nearby modes are nearly marginal when the length

of the container is large leading to a secondary bifurcation very soon after threshold. The structure of the resulting oblique patterns alternates between both endwalls already in the short timescale. The quasi-periodicity of the primary patterns (and the subsequent mode interaction giving qualitatively different patterns) is an essential qualitative difference with Faraday waves. This drifting frequency is important also from the experimental point of view, as it could lead to a false lack of repeatability. The calculated threshold amplitude compares very well quantitatively with the experimental observations by Porter *et al.* (2012), the resulting patterns also agree well with the experiments qualitatively. The theory allowed to predict a different kind of pattern from the originally reported in Porter *et al.* (2012) that was confirmed independently in Tinao *et al.* (2014). Studying the variation of the threshold for the different parameters it was found that the obliqueness is enhanced by capillary effects and viscosity. It was also found that larger viscosities and container lengths lead to smaller drifting frequencies, making harder to detect it experimentally.

Chapter 3 considered the case of combined vertical and horizontal forcing on a rectangular basin. The amplitude equation derived in chapter 2 was adapted to include the effect of the new vertical vibration. It is well known that in the Faraday problem patterns at threshold are subharmonic. With pure horizontal forcing, instead, the analysis in chapter 2 has shown that patterns are quasiperiodic. This transition from $\Omega = 0$ to $\Omega \neq 0$ takes the form of a Takens-Bogdanov bifurcation occurring at the minimum of the neutral stability curve, in which a pair of complex eigenvalues merge into the real axis to split into a pair of two real eigenvalues, resulting into the stabilization of one of them while the other become the critical eigenvalue. The region where this bifurcation appears in the plane of the parameters corresponding to the relative strength and the phase between the two components of the forcing is identified. Quasi-periodic patterns are found in the regions close to pure horizontal vibration and phase $\theta = \pi/2$. The addition of vertical forcing seems to promote cross-wave patterns and moving the phase of the forcing from 0 to $\pi/2$ has a similar but weaker effect. Also, the different patterns kinds of patterns predicted by the theory are described, including patterns covering the whole container and patterns concentrated near one wall or drifting between both walls, as well as the region in the parameters plane where these patterns are to be found. All the results in this chapter should be seen purely as predictions due to the lack (to the author's knowledge) of experimental literature on this problem.

Chapter 4 studied the excitation of subharmonic waves by a vertically vibrating immersed obstacle. First a theory to calculate the harmonic flow was developed; from the results of this theoretical analysis the conditions

that produce an oscillatory bulk flow that dominates harmonic wavetrains as the triggering mechanism of subharmonic instability and leads to an OBF that is slowly modulated in space were identified. It is found that these conditions are satisfied for not too slim wavemakers and immersion depths large enough. With these ideas a theory corresponding to the assumption of a deep wavemaker, in which the OBF triggers the instability, is developed. Then the amplitude equation developed in chapter 2 was adapted for the OBF from an oscillating obstacle. The threshold amplitude for different wavemakers and viscosities were computed, finding that for very small viscosities the patterns are cross-waves but for larger viscosities the obliqueness of the patterns appears. The easy availability of water make experiments using this fluid common and knowing how easy it is to contaminate the surface of water the amplitude equation was adapted to take into account this situation, finding that the contamination enhances the obliqueness of the waves. The theory is compared with the experiments by Moisy *et al.* (2012), finding that if pure water is considered the predicted patterns are cross-waves, strictly subharmonic and that the threshold amplitude is like half of the experimental one. When the contamination is considered, oblique patterns and threshold amplitudes comparable with the experimental were obtained but the match is not complete. The discrepancies could be because of several differences between the experiments and the developed theory like the gap between the wavemaker and the walls, and the presence of rods supporting the shaker (that present contact lines).

The theory developed in chapter 2 for the case of the horizontally vibrated container is general enough to be easily extended to other problems. This has been done in the thesis for the case of horizontal and vertical vibration (on a symmetry plane) of a container and for the vertical vibration of an immersed obstacle. Other extensions for general vibrating systems with interfaces are possible, provided that the counterparts of the assumptions (2.16) hold, which occurs if the forcing frequency is large compared to the first sloshing mode and the vibrating acceleration and viscous effects are appropriately small.

For instance in this thesis the vibration of containers with certain symmetries has been studied. A more general case in which there are not symmetries could also be considered: such as studying the inclusion of horizontal forcing outside of the symmetry $x - z$ plane for a rectangular container, with or without vertical forcing. The same could be done for an immersed wavemaker: first the horizontal vibration of a submerged solid could be studied, and then the combination of both vertical and horizontal forcing.

Another interesting example is to consider *more general vibrating liquid systems with interfaces*, such as vibrating drops. The oscillatory bulk flow can

be still obtained in a similar way, providing the forcing pressure gradient at the free surface that must be used to obtain (using the suitable quasipotential approximation) the parametric forcing. The resulting problem applies in a $O(1)$ -region near the unperturbed free surface and can be always reduced to a general, two-dimensional amplitude equation using curvilinear coordinates along the unperturbed free surface when this is non-flat, as in, e.g., vibrating drops.

Other more delicate extensions are possible like excitation from resonant triads. For a very low viscosity, competition between harmonic waves and the OBF is expected, being the subharmonic field triggered by the dominant component of the harmonic flow or by a combination of both forcing mechanisms. To account for these effects the amplitude equation should incorporate some terms that were neglected, when the assumptions of slowly variation in space of the harmonic flow was made, which can no longer be done when the harmonic waves are present.

The inclusion of weakly nonlinear terms in a way consistent with the thesis, without phenomenological assumptions but from first principles, is also a possibility and it should represent a great challenge. With this model the integration of the equations for amplitudes beyond the threshold would be possible, capturing not just the obliqueness of the patterns but different features of them as, for example, the curvature. Within this model the effects of the mean flow should be incorporated.

In any event, the theory developed in the thesis is expected to be a step further to uncover the mechanisms for promoting subharmonic waves in general vibrating systems with interfaces, identifying the parametric forcing mechanisms and allowing the calculation of both the threshold acceleration and the resulting subharmonic patterns in a systematic way.

A Spectral method

Spectral methods have the advantage of providing high accuracy with relative few modes and the disadvantage of having problems dealing with complex geometries. As the geometry of the problems considered in the thesis is very simple, they are a very good choice.

A *spectral method* resulting from a truncated expansion of $F_m^\pm$ into harmonic x -functions satisfying the boundary conditions (2.108). This yields a homogeneous, linear system of algebraic equations for the complex mode amplitudes that must exhibit nontrivial solutions. This latter condition provides a complex solvability condition that allows for simultaneously calculating the real unknowns Ω and a .

In the spectral method F_m^+ and F_m^- are expanded using a Fourier series

$$F_m^\pm = \sum_j B_{mj}^\pm \sin(\delta_j^s x) + \sum_j C_{mj}^\pm \cos(\delta_j^c x), \quad (A.1)$$

where (odd and even) frequencies are selected by the boundary condition (2.108) through

$$\gamma(\kappa_m^2, S) \tan(\delta_j^s L_1) + \delta_j^s = 0 \quad (A.2)$$

$$-\delta_j^c \tan(\delta_j^c L_1) + \gamma(\kappa_m^2, S) = 0 \quad (A.3)$$

The algebraic eigenvalue problem resulting from the spectral problem associated with equations (2.106)-(2.108) (with $i\Omega$ replaced by a complex eigenvalue) is then solved to determine the eigenvalues with largest real part (which can be done using techniques such as Arnoldi's method; in practice, however, since a relatively small number of modes will be used in the spectral method, the whole set of eigenvalues could be instead computed).

Since the wavenumber of subharmonic waves must be close to one (see scaling in section 2) a *window* of modes around this frequency can be used in the spectral method. Of course, due to the weak singularity of the solution, the convergence of the method is not exponentially fast and a *tail* of high frequencies must be considered in this *window*. Anyway, given the weakly character of the singularity a high number of modes associated to high frequencies is not needed in practice.

It should be mentioned that the above observed symmetries are not imposed in the spectral method. Instead, the computation using the spectral method is used to (easily, since the method uses odd and even modes) confirm that any solution of the equations (2.106)-(2.108) contributing to the neutral stability curve preserves one of these symmetries. Although the

symmetry of the problem is not explicitly imposed after the Galerkin projection the algebraic structure of the problem couples the symmetric part of one mode with the antisymmetric part of the other mode.

The previous considerations were for the problem of chapter 2 and the differences in the treatment of the two other problems are explained now. The problem of a mixed vertically and horizontally vibration followed exactly the same procedure as the former problem but with the difference that the Galerkin projection did not couple the modes in any special way. The problem of the vibration of an immersed wavemaker needed more preparation as the forcing is obtained numerically, so the Galerkin projection has to take this into account. The interpolating coefficients of the forcing are obtained via splines and in each interval the analytical integration (of the Galerkin projection) of the interpolating polynomial is performed. From the contribution of each interval is obtained the integral on the whole domain, obtaining the coefficients for each mode. Because the forcing function of the problem is even, the algebraic structure of the eigenvalue problem couples the symmetric part of one mode with the symmetric part of the other, the same happens with the antisymmetric parts.

B Shooting method

A shooting method has been developed for each considered problem both for (mutual) validation and for its usage in calculating the neutral stability curves. For the problem of horizontal vibration of a rectangular container considered in chapter 2:

The problem (2.106)-(2.108) is solved numerically using a shooting method. Some remarks concerning the practical implementation of this technique are in order:

- Multiple shooting (from both endpoints, $x = \pm L_1$) is done to avoid ill-conditioning of the resulting algorithm when single shooting (either from an interior point or from one endpoint) is used; remark that $F_m^\pm \rightarrow 0$ far from the endpoints $x = \pm L_1$ when $L_1 \gg 1$. A system of algebraic equations on (free) values of $F_m^\pm$ and $\partial_x F_m^\pm$ at the endpoints, a and Ω is then derived by the imposition of continuity conditions at $x = 0$. In practice, the symmetry of the problem (2.106)-(2.108) (see below) can be exploited to separately compute each class of solutions. Shooting is then formulated on $(0, L_1)$.
- Since $g(x)$ is odd, the problem (2.106)-(2.108) exhibits the following symmetries

$$(x, F_m^+, F_m^-) \rightarrow (-x, -F_m^+, F_m^-) \quad (A.4)$$

$$(x, F_m^+, F_m^-) \rightarrow (-x, F_m^+, -F_m^-) \quad (A.5)$$

Every solution of the problem (2.106)-(2.108) relevant to the neutral stability curve (as it can be easily checked using a spectral method, see below) keeps one of these symmetries. This fact is used to compute separately both groups of solutions using the shooting method.

- Although equations (2.106)-(2.107) are singular at $x = \pm L_1$, the singularity is weak and the differential equation can be directly integrated provided that suitable adaptive step control and initial step predictions are implemented (an effective adaptive step control is also an important ingredient to avoid a high computational cost when $L_1 \gg 1$). Thus, no explicit separation of the singular part is required in order to integrate the differential equations.

For the problem of a mix of vertical-horizontal vibration of a rectangular container considered in chapter 3:

- The problem is formulated for general values of α and θ , because of this there is no special symmetry that helps to simplify the domain. Then the problem have to be formulated on the whole domain $(-L_1, L_1)$, multiple shooting from $x = \pm L_1$ is done and the continuity of the solution on $x = 0$ is used to obtain the set of algebraic equation that provide the values of a and Ω .
- The singularity of the equations is treated as in the previous case

For the problem of the vertical vibration of an immersed wavemaker considered in chapter 4:

- As in the problem of chapter, the symmetry of the problem (4.44)–(4.46) (see below) can again be exploited to compute the solutions. Shooting is then formulated on $(0, L_1)$.
- Since $g(x)$ is even, the problem (4.44)–(4.46) exhibits the following symmetries

$$(x, F_m^+, F_m^-) \rightarrow (-x, F_m^+, F_m^-) \quad (A.6)$$

$$(x, F_m^+, F_m^-) \rightarrow (-x, -F_m^+, -F_m^-) \quad (A.7)$$

Both of these solutions have been observed, but the second one appears well beyond the threshold and, in general, the computations (when using the shooting method) are made with the first symmetry.

- The forcing function $g(x)$ for this problem is obtained numerically, then for the shooting method evaluation of $g(x)$ in any point on $(0, L_1)$ could be needed. The value of the function $g(x)$ on these points is not necessarily available, to solve this numerical interpolation (splines) is used.

The linearity of the differential equations could be exploited when deriving the algebraic system to be solved (as a result of the implementation of the shooting algorithm). More precisely, using this linearity the resulting system can be reduced to a system of algebraic equations in a and Ω (it can be further manipulated to also eliminate Ω). Nevertheless, taking into account that continuation methods will be used to obtain neutral stability curves as some parameters are varied (see again below), the original problem arising from multiple shooting (a system of equations in a , Ω , as well as the initial conditions at the shooting points on $F_m^\pm$ and $\partial_x F_m^\pm$) can be solved since only a few iterations of a quasi-Newton method (a Broyden technique) are needed and then both the cost and robustness of each alternative are rather similar.

The problem (2.106)-(2.108) (and analogues problems in chapter 3 and 4) is solved with the (y component of the) wavenumber κ_m varying (continuously) in $[0, 1]$. This is done using continuation with a pseudo arclength algorithm with a linear prediction (Allgower & Georg 2003). It must be observed that a simple continuation in the κ_m would not be able to capture backward components of the neutral stability curve. It should be observed that, despite the appearance of some of the neutral stability curves in figure 2.10, they corresponds to a unique regular branch (and not composed by several crossing branches). Maybe the more clear need for a continuation method is the presence of isolas in figure 2.12.

Continuation methods are also used in section 2.7.4 to obtain neutral stability curves as the different parameters (C_g , d , L and S) are varied. In this case, local minima of the curves in the (κ_m, a_c) plane must be followed. In practice (to easily detect the appearance and disappearance of minima), this is done by computing some portions of the curves in the (κ_m, a_c) plane (associated with the tongues in the case of small damping), corresponding to some intervals in κ_m that are adapted along the continuation process. Local minima are computed using an adaptive interpolation scheme (of a_c as a function of κ_m) with error control.

Since, along this thesis, continuation techniques are (widely) used to compute the neutral stability curve following a selected number of branches (i.e. following a certain eigenvalue as some equation parameters, κ_m , d , S , L or C_g , are varied), care must be taken to check that ignored branches are not relevant to the computation of the threshold amplitude. Thus, in

order to check that no other eigenvalue contributes to the neutral stability curve, a (somehow large) set of eigenvalues (corresponding to those with the largest real part) associated to the problem (2.106)-(2.108) is computed using a spectral code for some sets of parameters (in practice, this is also used to verify the implementation of the shooting technique).

C Boundary element method

The boundary element method (BEM from now on) is an efficient method that allows to obtain a reduced version of the problem. By transforming a problem that involves the whole domain to a new one that only involves the boundaries, the dimension of the problem is reduced by one. It also has some disadvantages, the matrix obtained after applying this technique is a full matrix, is not sparse as the ones provided by finite element or finite differences. It can be applied to a reduce number of problems, as a fundamental solution (a Green function) must be known previously.

The Laplacian in some arbitrary domain is:

$$\Delta v = 0, \text{ on } C \in \mathbb{R}^2. \quad (C.8)$$

Now, the fundamental solution for the Laplacian in a two dimensional domain will be obtained. The fundamental solutions are important in the sense that they will be used as weighting functions when deriving the integral formulation of the problem.

The associated fundamental problem is the problem singular in the point (x_1, y_1) ,

$$\Delta w + \delta(x_1 - x, y_1 - y) = 0, \text{ on } C \in \mathbb{R}^2. \quad (C.9)$$

As the solution is expected to be symmetric with respect to the singularity, adopting a local polar coordinate system and because of the symmetry $\partial_{\theta\theta} = 0$, the problem yields:

$$\frac{1}{r} \partial_r (r \partial_r w) = 0, \quad (C.10)$$

this can be integrated, obtaining:

$$w = A \log r + B. \quad (C.11)$$

Where A and B are constants of integration, B is usually set to zero. From the definition of δ and integrating over a disk of radius ϵ the value of $A = -\frac{1}{2\pi}$ is obtained.

Multiplying the Laplacian for the fundamental solution, integrating over the domain and applying (twice) the Green theorems:

$$\int_C \Delta v w = \int_{\partial C} \frac{\partial u}{\partial n} w \, dl - \int_C \nabla v \nabla w \, dC = 0, \quad (C.12)$$

$$\int_{\partial C} \frac{\partial v}{\partial n} w \, dl - \int_{\partial C} v \frac{\partial w}{\partial n} \, dl + \int_C v \Delta w \, dC = 0. \quad (C.13)$$

This, together with the property of the delta function,

$$\int_C v \Delta w \, dC = - \int_C v \delta(x - x_1, y - y_1) = -v(x_1, y_1) \quad (C.14)$$

yield the following integral equation:

$$v(x_1, y_1) - \int_{\partial C} \frac{\partial v}{\partial n} w \, dl + \int_{\partial C} v \frac{\partial w}{\partial n} \, dl = 0 \quad (x_1, y_1) \in C \quad (C.15)$$

and for a point on the boundary (more about BEM can be found on Kythe 1995),

$$k(x_1, y_1)v(x_1, y_1) - \int_{\partial C} \frac{\partial v}{\partial n} w \, dl + \int_{\partial C} v \frac{\partial w}{\partial n} \, dl = 0, \quad (C.16)$$

with $k = 1, 1/2, \alpha/2\pi$ for an internal point, a regular point on the boundary and a point with two different tangents on the boundary respectively (α is the internal angle formed by the surface).

Here there are some details about the implementation of this numerical method. The interpolation between adjacent nodes was made with a linear approximation for both v and $\partial v/\partial n$. Then the integrals appearing on C.16 are solved analytically, as they (only) involves products of $\log(r)$ and $1/r$ by the linear interpolating functions. When considering the influence of a node over its own element, a singularity appears but it vanishes in the sense of the principal value of the integral, cancelling contributions over adjacent elements.

For nonsmooth points on the boundary, two different $\partial v/\partial n$ has been considered (one for each tangent). No additional equation is necessary while doing this, as always it is known one value of v and $\partial v/\partial n$ or the two values of $\partial v/\partial n$.

For the method to solve the problem that allows to obtain the harmonic flow in section 4.3 it is needed to know how the flow decay when $x \rightarrow \infty$ and $z \rightarrow -\infty$. It was found that $P_z \rightarrow 1/x^2$, $P \rightarrow 1/x^2$ when $x \rightarrow \infty$ and $P \rightarrow 1/z$ when $z \rightarrow \infty$.

D Curves of non-dimensional parameters

In this appendix the curves of variation of the non-dimensional parameters for the experimental runs can be found.

Here are the curves for the experiments in chapter 2 and 3 using silicone oil of $\sigma^* = 19.7 \text{ dyn cm}^{-1}$, $\rho^* = 0.913 \text{ g cm}^{-3}$ and $\nu^* = 5 \text{ cSt}$. The dimensions of the container used were $L_1 = 9 \text{ cm}$, $d = 5 \text{ cm}$. The variation of these parameters with the forcing frequency and with the non-dimensionalization considered in this thesis are below.

Figure D.1: Variation of L_1 (left) and d (right) with the forcing frequency $2\omega^*$ for the subharmonic waves.

Figure D.2: Variation of C_g (left) and S (right) with the forcing frequency $2\omega^*$ for the subharmonic waves.

The experiments in chapter 4 were made using water of $\sigma^* = 60 \text{ dyn cm}^{-1}$, $\rho^* = 1 \text{ g cm}^{-3}$ and $\nu^* = 1 \text{ cSt}$. The geometrical dimensions were $2l_w^* = 0.5 \text{ cm}$, $h_w^* = 0.2, 0.4, 0.6, 0.8, 1 \text{ cm}$ and $L_1^* = 60 \text{ cm}$. The variation of these parameters with the forcing frequency and with the non-dimensionalization considered in this thesis are below.

Figure D.3: Variation of $2l_w$ (top-left), h_w (top-right) with the arrow showing increasing values of immersion depths and L_1 (bottom) with the forcing frequency $2\omega^*$ for the subharmonic waves.

Figure D.4: Variation of C_g (left) and S (right) with the forcing frequency $2\omega^*$ for the subharmonic waves.

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