Universidad Politécnica de Madrid

Escuela Técnica Superior de Ingenieros Aeronáuticos

Analysis and optimization of trajectories for Ballistic Missiles Interception

A thesis submitted for the degree of
Doctor of Philosophy in Aeronautical Engineering

Daniel Montero Yéboles.
Aeronautical Engineer

2015

Ciencia y Tecnología Aeroespaciales (130B)
Escuela Técnica Superior de Ingenieros Aeronáuticos
Universidad Politécnica de Madrid

Analysis and optimization of trajectories for Ballistic Missiles Interception
Author: Daniel Montero Yéboles, Aeronautical Engineer

Thesis Advisor: Dr. Pedro Sanz-Aránguez, PhD Aeronautical Engineer

Copyright © 2015 by Daniel Montero Yéboles
All rights reserved.

No part of the material protected by this copyright notice may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without the prior written permission of the author.

Typeset by the author with the L^AT_EX documentation system.

Universidad Politécnica de Madrid
Escuela Técnica Superior de Ingenieros Aeronáuticos

Analysis and optimization of trajectories for Ballistic Missiles Interception

A thesis submitted for the degree of
Doctor of Philosophy in Aeronautical Engineering

Daniel Montero Yéboles.
Aeronautical Engineer

2015

In loving memory of my father Ángel,
who always wanted for me to become an Engineer.

Acknowledgements

I would like to acknowledge many of the professors I had at the Escuela Técnica Superior de Ingenieros Aeronáuticos for the enthusiasm they showed teaching difficult subjects, which inspired me to love Physics and Mathematics. In general, I would like to acknowledge the education I received in this faculty, since I was given the main blocks of all the technical knowledge I have, and I was taught to always be demanding with myself.

In particular, I would like to acknowledge my Thesis Advisor, professor Pedro Sanz-Aránguez, who helped me focus this thesis in a realistic way and introduced me to the world of optimization. I wouldn't have started this thesis if he hadn't taught me that subject.

I would like to acknowledge the friends I made while studying at the ETSIA. Especially Mercedes Marzal Pitarch, Sonia Martínez Belinchón, Marta Pellicer Yagüe, Ana Pérez Marín, Rafael Marín Aguilar, Miguel González Cuadrado and Carlos Hernández Medel. It was my privilege to spend some years by their side.

I would like to thank many of the colleagues I had along the years in GMV. They taught me many things and they were a source of inspiration in many occasions to solve difficult technical problems. In particular, I would like to thank Pedro López-Adeva Fernández-Layos and Miguel Antonio Antón Díez since without the push they gave me I would have given up on finishing this thesis.

I also thank professor Manuel Pérez Cortés who has always been a role model for me. His compatibility of an academic dedication while keeping the highest responsibilities within GMV convinced me to complete this work.

I would like to acknowledge my friend Marco Antonio García Matatoros and his brother Pedro, who after so many years so far away are still there for me. I can still picture ourselves when we were only 10 years old.

I thank my family for the support they have always given me. I wouldn't be an Engineer if my father had not convinced me to be so, and I would be nothing without the love of my mother María del Carmen Yéboles, who has always been there for me.

I thank my sister Eva and my brother Raúl, who have become friends apart from sister and brother through the years.

Finally, I would like to thank my dear girlfriend Mihaela Ecaterina Gheorghiu, for all her support through the worst part of the development of this thesis.

This page intentionally left blank.

Contents

Abstract	xvii
-----------------	-------------

Notation notes	1
Notation for matrices	2
Notation for vectors	3
Notation for quaternions	4
Notation for the change of basis matrix	4

I ICBM interception	5
----------------------------	----------

1 Intercontinental Ballistic Missiles	7
1.1 Introduction	8
1.2 History of the ICBMs	9
1.2.1 First steps	9
1.2.2 Intercontinental Ballistic Missiles in the U.S. and the U.S.S.R.	11
1.2.2.1 Development	11
1.2.2.2 Reduction treaties	13
1.2.3 Intercontinental Ballistic Missiles in other countries	14
1.2.3.1 France	14
1.2.3.2 Israel	14
1.2.3.3 China	14
1.2.3.4 United Kingdom	15
1.2.3.5 India	15
1.2.3.6 North Korea	16
1.2.3.7 Iran	16
1.3 List of ICBMs	17
1.4 Characteristics of Intercontinental Ballistic Missiles	19
1.4.1 Shape	19
1.4.2 Propulsion system	19
1.4.3 Navigation system	19
1.4.4 Control system	20
1.4.5 Trajectory	21
Chapter 1 references	23

2	Missile Defence	25
2.1	Technical challenges of the problem	26
2.2	History	27
2.3	Ballistic Missile Defense System (BMDS)	29
2.3.1	Introduction	29
2.3.2	System components	30
2.3.2.1	Command and Control, Battle Management and Communications	30
2.3.2.2	Space Tracking and Surveillance System (STSS)	30
2.3.2.3	System sensors	31
2.3.2.4	Interception Systems	32
	Chapter 2 references	40
II	Simulation of the missiles	43
3	Equations of motion for the missiles	45
3.1	Equations of motion	46
3.1.1	Notation for the equations	46
3.1.2	Linear momentum	47
3.1.3	Angular momentum with respect to the center of mass of the missile	48
3.2	State vector	49
3.2.1	Introduction	49
3.2.2	State vector components	49
3.2.3	Obtaining data from the missile state vector	51
3.2.3.1	Position of the missile	51
3.2.3.2	Attitude of the missile	52
3.2.3.3	Velocity of the missile	52
3.2.3.4	Angular velocity of the missile	53
3.3	State Equations	54
3.3.1	Position	54
3.3.2	Velocity vector	54
3.3.3	Attitude	55
3.3.4	Angular velocity vector	55
3.3.5	Compilation of equations	57
3.3.6	Linearization of the equations	57
3.4	Characteristics of the equations to be solved	58
3.4.1	Problem to be solved	58
3.4.2	Behaviour of the problem	58
3.4.2.1	Well-Posedness	58
3.4.2.2	Stability	58
3.5	Numerical method for the integration of the equations	60
3.5.1	Characteristics of the numerical methods	60
3.5.1.1	Order of a numerical method	60
3.5.1.2	Stability of numerical methods	61

3.5.2	Considered numerical methods	63
3.5.3	Chosen numerical method	66
3.5.4	Selection of the time step	67
	Chapter 3 references	68
4	Forces and moments acting on the missile	69
4.1	Introduction	70
4.2	Gravity force and moment	72
4.2.1	Force of gravity	72
4.2.2	Gravity moment	74
4.3	Aerodynamic force and moment	75
4.3.1	Atmosphere model	75
4.3.2	Aerodynamic model	79
4.3.2.1	Coefficients	79
4.3.2.2	Considerations for spinning missiles	84
4.4	Thrust force and moment	86
4.4.1	Formula for the thrust force	86
4.4.2	Thrust force in a <i>de Laval</i> nozzle	87
4.4.2.1	General expression	87
4.4.2.2	Coefficients	88
4.4.3	Thrust force in the simulation	90
4.4.4	Thrust moment	90
4.5	Control forces	91
4.5.1	Control forces in the ICBM	92
4.5.1.1	Existing controls	92
4.5.1.2	Control equations	94
4.5.1.3	Control forces	96
4.5.2	Control forces in the interceptor missile	107
4.5.2.1	Existing controls	107
4.5.2.2	Control equations	108
4.5.2.3	Control forces	108
4.6	Considered noise in the external forces acting on the missile	113
	Chapter 4 references	115
5	Structure of the Simulator	117
5.1	Main characteristics of the simulator	118
5.1.1	Language and existing code packages used	118
5.1.2	Missiles implemented in the simulation	118
5.1.2.1	ICBM in the simulation	118
5.1.2.2	Interceptor missile in the simulation	119
5.2	High level program structure	120
5.2.1	Graphs notation	120
5.2.2	Numerical integration chart	121

5.2.3	Guidance-Control flow chart	122
5.2.4	Compute $\vec{F}$ chart	123
5.2.5	Stability analysis chart	125
5.2.6	ICBM simulation chart	127
5.2.7	ICBM-GBI simulation chart	129
5.3	Outputs of the simulation	131
	Chapter 5 references	132
6	Simulation examples and comparison with available data and other simulators	133
6.1	Simulation examples	134
6.1.1	ICBM simulation case	134
6.1.1.1	Trajectory	134
6.1.1.2	Geodetic position	135
6.1.1.3	Speed	136
6.1.1.4	Euler angles and flight path angles	138
6.1.1.5	Angular velocity	145
6.1.2	Interceptor trajectory	146
6.1.2.1	Trajectory	146
6.1.2.2	Geodetic position	147
6.1.2.3	Speed	148
6.1.2.4	Euler angles and flight path angles	150
6.1.2.5	Angular velocity	157
6.2	Comparison with available data	158
6.2.1	ICBM simulation	158
6.2.1.1	General behaviour of the simulator	158
6.2.1.2	Comparison with available data	158
6.2.2	GBI simulation	159
6.3	Comparison with other simulation platforms	159
6.3.1	Comparison with simple simulators	159
6.3.2	Comparison with the simulator in reference [3]	160
6.3.3	Comparison with generic simulation platforms	161
6.3.4	Comparison with Engineering simulators	161
6.3.5	Results of the comparison	161
	Chapter 6 references	162
III	Guidance algorithms	163
7	Guidance strategies and aiming	165
7.1	Guidance strategies	166
7.1.1	Atmospheric phase	166
7.1.1.1	Launch	166
7.1.1.2	Gravity turn	166
7.1.1.3	Yaw and roll control	168

7.1.2	Outer space phase	168
7.1.2.1	Final flight path angles	168
7.1.2.2	Boost flight termination	168
7.1.2.3	Reentry preparation (ICBM)	169
7.1.2.4	EKV guidance (GBI)	170
7.1.3	Reentry phase (ICBM)	170
7.2	The Lambert problem	171
7.2.1	Solutions of Lambert's problem	171
7.2.2	Solution of the Lambert problem used in the simulation	174
7.2.3	Errors because of using a solution of the Lambert problem for aiming	175
7.3	Aiming algorithms in the simulator	181
7.3.1	Initial aiming for the ICBM	181
7.3.1.1	Basic algorithm	181
7.3.1.2	Final State Transition Matrix	184
7.3.2	Initial aiming for the GBI missile	188
7.3.3	Aiming after launch	190
7.3.3.1	ICBM	190
7.3.3.2	GBI	190
	Chapter 7 references	191
8	Conventional ascent guidance	193
8.1	Introduction	194
8.2	Atmospheric ascent guidance	196
8.2.1	Conventional approaches	196
8.2.2	Atmospheric ascent guidance in the simulator	196
8.3	Exoatmospheric ascent guidance	198
8.3.1	Delta guidance	198
8.3.1.1	Formulation	198
8.3.1.2	Delta guidance in the missiles simulator	199
8.3.2	Path-Adaptive guidance	199
8.3.2.1	Formulation	199
8.3.2.2	Path-Adaptive guidance in the missiles simulator	199
8.3.3	Lambert guidance	200
8.3.3.1	Formulation	200
8.3.3.2	Lambert guidance in the missiles simulator	200
8.3.4	Q guidance	201
8.3.4.1	Formulation	201
8.3.4.2	Behaviour and problems of the Q guidance algorithm	202
8.3.4.3	Q guidance in the missiles simulator	203
8.3.5	Linear Tangent Guidance (LTG)	205
8.3.5.1	Iterative Guidance Mode (IGM)	205
8.3.5.2	Power Explicit Guidance (PEG)	209
8.3.6	Comparison of exoatmospheric ascent guidance algorithms	218

8.3.7	Algorithm to be used in the simulator for conventional ascent guidance . . .	221
	Chapter 8 references	222
9	Conventional terminal guidance	225
9.1	The EKV during the terminal guidance	226
9.2	EKV attitude control	226
9.3	EKV divert control	228
9.3.1	Proportional Navigation (PN)	229
9.3.2	Augmented Proportional Navigation and gravity compensation	230
9.3.3	Performance of Proportional Guidance	232
9.3.4	Predictive Guidance	233
9.3.5	Comparison of terminal guidance algorithms	234
9.3.6	Algorithm to be used in the simulator for conventional terminal guidance .	234
	Chapter 9 references	235
10	Optimal guidance	237
10.1	Optimal control theory	238
10.1.1	Historical note	238
10.1.2	Basic equations	238
10.1.2.1	Problem formulation	238
10.1.2.2	Euler-Lagrange theorem	239
10.1.2.3	About the transversality condition	240
10.1.2.4	Pontryagin's Minimum Principle	241
10.1.3	Linear systems	242
10.1.3.1	Linear quadratic regulator (LQR)	242
10.1.3.2	Linear quadratic tracking (LQT)	246
10.1.3.3	Fixed final state (LQ)	250
10.1.3.4	Constraints in the controls in quadratic regulators	252
10.1.3.5	Integration of the equations for the LQR and LQT	253
10.1.4	Linearization	255
10.2	Optimal guidance algorithms for the interception problem	256
10.2.1	General considerations	256
10.2.2	Overall description of the optimal guidance algorithms that have been im- plemented	259
10.2.3	Optimal terminal guidance	260
10.2.4	Global optimal interception guidance	263
10.2.4.1	In the ascent phase	263
10.2.4.2	In the terminal phase	265
10.2.5	Optimal tracking guidance	266
10.2.5.1	In the ascent phase	266
10.2.5.2	In the terminal phase	269
10.2.6	Global optimal guidance using an augmented state vector	270
10.2.6.1	In the ascent phase	272

10.2.6.2 In the terminal phase	274
Chapter 10 references	275
11 Comparison of guidance algorithms	277
11.1 Introduction	278
11.1.1 Parameters to be compared	278
11.1.2 Cases to be analysed	279
11.1.2.1 Geographical cases	279
11.1.2.2 Guidance algorithms to be compared	279
11.1.2.3 Number of executions	280
11.2 Results obtained with each guidance algorithm	280
11.2.1 Conventional ascent guidance with conventional terminal guidance	280
11.2.2 Conventional ascent guidance with conventional terminal guidance active only the last 1000 km	283
11.2.3 Conventional ascent guidance with optimal terminal guidance	285
11.2.4 Conventional ascent guidance with optimal terminal guidance active only the last 1000 km	287
11.2.5 Global optimal interception	289
11.2.6 Optimal tracking guidance	291
11.2.7 Global optimal guidance using an augmented state vector	293
11.3 Comparison of results	295
Chapter 11 references	300
IV Results and conclusions	301
12 Analysis of guidance algorithms	303
12.1 Behaviour of the guidance algorithms without noise nor delays in the target estimation	304
12.2 Behaviour of the guidance algorithms when noise is considered in the target state estimation	305
12.3 Behaviour of the guidance algorithms when delays are considered in the target state estimation	307
12.4 Feasibility of the Ballistic Missile Defense system in terms of guidance	309
Chapter 12 references	311
13 Achievements and conclusions	313
13.1 Achievements	314
13.2 Conclusions	316
13.3 Possible future research	318
Chapter 13 references	319

Appendices	321
A Frames of reference	323
A.1 ECI (Earth-Centered Inertial)	324
A.2 ECEF (Earth-Centered Earth-Fixed)	325
A.3 Conversion between ECI coordinates and ECEF coordinates	326
A.3.1 Conversion between CIS coordinates and Mean Earth-Centered Inertial of Date coordinates	327
A.3.2 Conversion between Mean Earth-Centered Inertial of Date coordinates and Mean True Earth-Centered Inertial of Date coordinates	328
A.3.3 Conversion between Mean True Earth-Centered Inertial of Date coordinates and True Earth-Centered Earth-Fixed coordinates	330
A.3.4 Conversion between True Earth-Centered Earth-Fixed coordinates and Mean Earth-Centered Earth-Fixed coordinates (CTS)	332
A.3.5 Summary of transformations ECI-ECEF	333
A.4 Geodetic coordinates	334
A.5 Conversion between geodetic coordinates and ECEF Cartesian coordinates	335
A.5.1 From geodetic coordinates to ECEF coordinates	335
A.5.2 From ECEF coordinates to geodetic coordinates	337
A.6 The Navigation frame	339
A.7 Conversion between ECEF coordinates and Navigation coordinates	340
A.8 The Body frame	342
A.9 Conversion between Navigation coordinates and Body coordinates	343
Appendix A references	344
B Change in the reference frame	345
B.1 Change of basis matrix	346
B.2 Rotation matrix	347
B.2.1 Rotation	347
B.2.2 Definition of rotation matrix	347
B.2.3 Rotation matrix and change of basis matrix	348
B.2.4 Givens rotation matrices	348
B.2.5 Composition of Givens rotation matrices	349
B.3 Euler angles	350
B.4 Rotation quaternions	352
B.4.1 Basic formula	352
B.4.2 Quaternions and the rotation matrix	354
B.4.3 Quaternions and the angular velocity vector	356
B.4.4 Quaternions and Euler angles	359
Appendix B references	361

C	Angular velocity vectors	363
C.1	Angular velocity of the ECEF frame w.r.t. the ECI frame	364
C.2	Angular velocity of the Navigation frame w.r.t. the ECEF frame	366
C.2.1	Derivatives of the geodetic coordinates	366
C.2.1.1	Curvature of a curve in a point	366
C.2.1.2	Derivative of the geodetic latitude	367
C.2.1.3	Derivative of the geodetic longitude	369
C.2.1.4	Derivative of the altitude	370
C.2.2	Expression for the angular velocity of the Navigation frame w.r.t. the ECEF frame	370
C.3	Angular velocity of the Body frame w.r.t. the Navigation frame	371
	Appendix C references	372
D	Equations of motion	373
D.1	Vectorial derivatives in moving frames	374
D.1.1	Coriolis theorem	374
D.1.2	Properties of the angular velocity vector	375
D.2	Relative motion	376
D.3	Equations of motion for a single particle	378
D.3.1	Linear momentum	378
D.3.2	Angular momentum	379
D.4	Equations of motion for a system of particles with constant mass	380
D.4.1	Linear momentum	380
D.4.2	Angular momentum	382
D.5	Equations of motion for a rigid body	384
D.5.1	Definition of rigid body and properties	384
D.5.1.1	Definition	384
D.5.1.2	Properties	384
D.5.2	Equations of motion	387
D.5.2.1	Linear momentum	387
D.5.2.2	Angular momentum	387
D.6	Equations of motion for the missile system	390
D.6.1	Considered system	390
D.6.2	Linear momentum of the missile system	392
D.6.2.1	Term related to the Coriolis acceleration	393
D.6.2.2	Term related to the relative acceleration	394
D.6.2.3	Final expression	395
D.6.3	Angular momentum of the missile system	396
D.6.3.1	Term related to the centrifugal acceleration	397
D.6.3.2	Term related to the angular acceleration	397
D.6.3.3	Term related to the Coriolis acceleration	398
D.6.3.4	Term related to the relative acceleration	402
D.6.3.5	Final expression	403

Appendix D references	404
E Gravitational potential	405
E.1 Gravitational potential	406
E.2 Gravity potential	412
Appendix E references	414
F Orbital Motion Problems	415
F.1 Introduction to orbital motion	416
F.1.1 Orbital elements	416
F.1.2 Basic equations	417
F.1.3 Orbit determination	421
F.2 Considered Orbital Motion Problems	422
F.2.1 Kepler's Problem	423
F.2.1.1 Formulation	423
F.2.1.2 Kepler's equation	424
F.2.1.3 Kepler's problem in this thesis	424
F.2.2 Lambert's Problem	425
F.2.2.1 Lambert theorem	425
F.2.2.2 Consequence of Lambert theorem	426
F.2.2.3 Solutions of Lambert's problem	427
F.2.2.4 Lambert's problem in this thesis	427
Appendix F references	428
G Missile parameters	429
G.1 ICBM parameters	430
G.1.1 Motors parameters	430
G.1.1.1 Stage 1	430
G.1.1.2 Stage 2	431
G.1.1.3 Stage 3	432
G.1.1.4 Propulsion System Rocket Engine	433
G.1.2 Control parameters	434
G.1.2.1 Control parameters in stage 1	434
G.1.2.2 Control parameters in stage 2	434
G.1.2.3 Control parameters in stage 3	435
G.1.2.4 Control parameters in the post-boost phase	435
G.1.2.5 Control parameters in the reentry vehicle	436
G.1.3 Missile components	436
G.1.3.1 Skirt section A6950	438
G.1.3.2 Stage 1 motor structure	438
G.1.3.3 Stage 1 motor fuel	439
G.1.3.4 INSTG I-II section A6750	441
G.1.3.5 Stage 2 motor structure	441
G.1.3.6 Stage 2 motor fuel	442

G.1.3.7	INSTG II-III section A6560	444
G.1.3.8	Stage 3 motor structure	444
G.1.3.9	Stage 3 motor fuel	445
G.1.3.10	Post-Boost Control System	447
G.1.3.11	Reentry Vehicle	448
G.1.3.12	Shroud assembly	449
G.1.4	Missile characteristics per stage	450
G.1.4.1	Missile with stage 1 active	450
G.1.4.2	Missile with stage 2 active	452
G.1.4.3	Missile with stage 3 active	453
G.1.4.4	Missile in the post-boost phase	454
G.1.4.5	Reentry vehicle	454
G.2	Interceptor parameters	455
G.2.1	Motors parameters	455
G.2.1.1	Stage 1	455
G.2.1.2	Stage 2	457
G.2.1.3	Stage 3	458
G.2.1.4	EKV	459
G.2.2	Control parameters	460
G.2.2.1	Control parameters in stage 1	460
G.2.2.2	Control parameters in stage 2	460
G.2.2.3	Control parameters in stage 3	460
G.2.2.4	Control parameters in the EKV	461
G.2.3	Missile components	461
G.2.3.1	Skirt section	463
G.2.3.2	Stage 1 motor structure	463
G.2.3.3	Stage 1 motor fuel	464
G.2.3.4	S1/S2 Interstage	466
G.2.3.5	Stage 2 motor structure	466
G.2.3.6	Stage 2 motor fuel	467
G.2.3.7	S2/S3 Interstage	469
G.2.3.8	Stage 3 motor structure	469
G.2.3.9	Stage 3 motor fuel	470
G.2.3.10	Guidance module and deployment module	472
G.2.3.11	EKV	473
G.2.3.12	Shroud assembly	474
G.2.4	Missile characteristics per stage	475
G.2.4.1	Missile with stage 1 active	475
G.2.4.2	Missile with stage 2 active	477
G.2.4.3	Missile with stage 3 active	478
G.2.4.4	Missile in the post-boost phase	479
G.2.4.5	EKV	479
	Appendix G references	480

Symbols	483
Definitions	489
Abbreviations	498

Abstract

Intercontinental Ballistic Missiles are capable of placing a nuclear warhead at more than 5,000 km away from its launching base. With the lethal power of a nuclear warhead a whole city could be wiped out by a single weapon causing millions of deaths. This means that the threat posed to any country from a single ICBM captured by a terrorist group or launched by a 'rogue' state is huge. This threat is increasing as more countries are achieving nuclear and advanced launcher capabilities.

In order to suppress or at least reduce this threat the United States created the National Missile Defense System which involved, among other systems, the development of long-range interceptors whose aim is to destroy incoming ballistic missiles in their midcourse phase.

The Ballistic Missile Defense is a high-profile topic that has been the focus of political controversy lately when the U.S. decided to expand the Ballistic Missile system to Europe, with the opposition of Russia. However the technical characteristics of this system are mostly unknown by the general public.

The Interception of an ICBM using a long range Interceptor Missile as intended within the Ground-Based Missile Defense System by the American National Missile Defense (NMD) implies a series of problems of incredible complexity:

- The incoming missile has to be detected almost immediately after launch.
- The incoming missile has to be tracked along its trajectory with a great accuracy.
- The Interceptor Missile has to implement a fast and accurate guidance algorithm in order to reach the incoming missile as soon as possible.
- The Kinetic Kill Vehicle deployed by the interceptor boost vehicle has to be able to detect the reentry vehicle once it has been deployed by ICBM, when it offers a very low infrared signature, in order to perform a final rendezvous manoeuvre.
- The Kinetic Kill Vehicle has to be able to discriminate the reentry vehicle from the surrounding debris and decoys.
- The Kinetic Kill Vehicle has to be able to implement an accurate guidance algorithm in order to perform a kinetic interception (direct collision) of the reentry vehicle, at relative speeds of more than 10 km/s.

All these problems are being dealt simultaneously by the Ground-Based Missile Defense System that is developing very complex and expensive sensors, communications and control centers and long-range interceptors (Ground-Based Interceptor Missile) including a Kinetic Kill Vehicle.

Among all the technical challenges involved in this interception scenario, this thesis focuses on the algorithms required for the guidance of the Interceptor Missile and the Kinetic Kill Vehicle in order to perform the direct collision with the ICBM.

The involved guidance algorithms are deeply analysed in this thesis in part [III](#) where conventional guidance strategies are reviewed and optimal guidance algorithms are developed for this interception problem.

The generation of a realistic simulation of the interception scenario between an ICBM and a Ground Based Interceptor designed to destroy it was considered as necessary in order to be able to compare different guidance strategies with meaningful results.

As a consequence, a highly representative simulator for an ICBM and a Kill Vehicle has been implemented, as detailed in part [II](#), and the generation of these simulators has also become one of the purposes of this thesis.

In summary, the main purposes of this thesis are:

- To develop a highly representative simulator of an interception scenario between an ICBM and a Kill Vehicle launched from a Ground Based Interceptor.
- To analyse the main existing guidance algorithms both for the ascent phase and the terminal phase of the missiles. Novel conclusions of these analyses are obtained.
- To develop original optimal guidance algorithms for the interception problem.
- To compare the results obtained using the different guidance strategies, assess the behaviour of the optimal guidance algorithms, and analyse the feasibility of the Ballistic Missile Defense system in terms of guidance (part [IV](#)).

As a secondary objective, a general overview of the state of the art in terms of ballistic missiles and anti-ballistic missile defence is provided (part [I](#)).

Notation notes

The notation used within this PhD thesis for vectors, matrices and quaternions is explained herein.

Notation for matrices

In general, uppercase single letters will represent matrices.

Rectangular matrices will be represented inside brackets:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Column matrices will be represented inside braces:

$$C = \begin{Bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{Bmatrix}$$

Row matrices will be represented as:

$$C = [C_{11} \quad C_{12} \quad C_{13}]$$

A superscript T means the transpose of the matrix and a superscript of -1 its inverse.

The identity matrix will be represented by the letter I , and it will have the appropriate dimensions for the operations to make sense, usually 3×3 :

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When it needs to be clarified, the dimensions will be given by subscripts like $A_{4 \times 3}$ or $I_3 = I_{3 \times 3}$.

We will write the determinant of a matrix A as $|A|$.

Notation for vectors

We will use an arrow on the top of a symbol to mean that the symbol represents a vector. For example:

$$\vec{v}$$

Several physical quantities, such as the velocity vector of a particle or the acceleration vector, vary depending on the reference frame from which these quantities are measured. When the used reference frame is not clear it will be noted in the following way:

$$(\vec{v})_S$$

where S is the reference frame from which these vectors are measured.

$\vec{v}^a$ represents the coordinates of vector $\vec{v}$ relative to a reference frame a , and actually stands for a column matrix of 3 elements in a 3 dimensional vector space:

$$\vec{v}^a = \begin{Bmatrix} v_x^a \\ v_y^a \\ v_z^a \end{Bmatrix}$$

The basis used to represent reference frame a is composed of the three versors: $\vec{i}_a, \vec{j}_a, \vec{k}_a$. By the very meaning of coordinates of a vector it must be that:

$$\vec{v} = v_x^a \cdot \vec{i}_a + v_y^a \cdot \vec{j}_a + v_z^a \cdot \vec{k}_a$$

This last equation will sometimes be abbreviated as:

$$\vec{v} = \begin{bmatrix} \vec{i}_a & \vec{j}_a & \vec{k}_a \end{bmatrix} \cdot \vec{v}^a$$

It has to be noted that a vector can be measured with respect to a reference frame S while their coordinates could be given in a different frame, for instance a . This will be denoted as:

$$(\vec{v}^a)_S$$

Notation for skew-symmetric matrix

Given a vector $\vec{v}$ its associated skew-symmetric matrix in the basis a will be denoted as:

$$\tilde{v}^a = \begin{bmatrix} 0 & -v_z^a & v_y^a \\ v_z^a & 0 & -v_x^a \\ -v_y^a & v_x^a & 0 \end{bmatrix}$$

Notation for quaternions

Quaternions will be written using any of the following equivalent notations:

$$q = q_0 + q_1 \cdot i + q_2 \cdot j + q_3 \cdot k$$

$$q = q_0 + \vec{q}$$

$$q = [q_0, \vec{q}]$$

$$q = [q_0, (q_1, q_2, q_3)]$$

q_0 is called the scalar part and $\vec{q} = (q_1, q_2, q_3)$ is called the vector part.

If a quaternion has no scalar part it will be called a pure quaternion.

Notation for the change of basis matrix

The matrix that allows changing from reference frame a to reference frame b will be written as C_a^b and it is the matrix that satisfies for every possible vector $\vec{v}$:

$$\vec{v}^b = C_a^b \cdot \vec{v}^a$$

All the frames of reference in this document will be orthonormal. As a consequence the change of basis matrices will be orthonormal too, that is, they will satisfy the equation:

$$C_a^b \cdot (C_a^b)^T = I$$

Part I

ICBM interception

This part of the thesis briefly describes the interception problem, indicating the main characteristics of existing ICBMs as well as the different layers of the Ballistic Missile Defence implemented by the United States of America.

This information will be later used in part [II](#) in order to simulate an ICBM and a kinetic interceptor in a realistic way.

This page intentionally left blank.

Chapter 1

Intercontinental Ballistic Missiles

This chapter reviews the history and characteristics of Intercontinental Ballistic Missiles.

1.1 Introduction

Ballistic missiles are missiles that after a powered phase follow a ballistic flightpath to its target.

They are usually divided according to their range (see reference [1]) as:

- Battlefield range ballistic missile (BRBM): Range less than 100 km
- Tactical ballistic missile: Range between 150 km and 300 km
- Theatre ballistic missile (TBM): Range between 300 km and 3,000 km
 - Short-range ballistic missile (SRBM): Range of 1,000 km or less
 - Medium-range ballistic missile (MRBM): Range between 1,000 km and 3,000 km
- Intermediate-range ballistic missile (IRBM): Range between 3,000 km and 5,500 km
- Intercontinental ballistic missile (ICBM): Range greater than 5,500 km

This latter group is the one of interest in this document.

The term ICBM is normally used when the missiles are launched from a ground site, while the term SLBM is used when they are launched from a submarine. However, it has to be noted that nowadays all the submarine-launched ballistic missiles (SLBMs) have a range of more than 5,500 km, so in fact SLBMs are all ICBMs.

In this document we will generally call them ICBMs paying attention only to their range.

ICBMs are very complex and expensive weapons, so they are only used to place nuclear warheads in a distant target. That is why having an enemy with ICBMs implies the biggest possible threat for any country.

1.2 History of the ICBMs

1.2.1 First steps

Modern rockets were born when Robert Goddard (1882-1945) built the first liquid-fuel rocket attaching a supersonic nozzle to a liquid-fueled combustion chamber in 1926.

The interest in rocketry spread in Austria, Britain, Czechoslovakia, France, Italy, Germany and Russia in the 1920s, with an special remark to the work of the GIRD, *Group for the Study of Reactive Motion* (Группа изучения реактивного движения), established in 1931 in Leningrad where over 100 experimental engines were built up to 1937. Sergey Korolev (1907-1966), the future leader of the Soviet space program, participated in the creation of GIRD.

The Reichswehr (latter to become the Wehrmacht) began to take an interest in rocketry in 1932 since the Treaty of Versailles limited Germany's access to long distance artillery. The Wehrmacht created a research team, joined by Wernher von Braun (1912-1977). This group developed the A series of rockets, among which the A-4, commonly known as V-2 was created.

The V-2 was the first operational ballistic missile. It had a weight at launch of 13,000 kg, a range of 300 km with a highest point of altitude of 90 km and an impact speed of about 1,100 m/s, and carried a 738 kg warhead.

The key technologies for the V-2 were large liquid-fuel rocket engines (1 stage rocket propelled by ethyl alcohol and liquid oxygen), and gyroscopic guidance.

The control was achieved using 4 rudders (taking into account supersonic aerodynamics) and 4 internal graphite vanes at the exit of the motor. The attitude of the missile was provided by 2 gimbaled gyroscopes.

Up to 6,048 V-2 rockets were built, among which 3,225 were launched, especially against the cities of London (1402) and Antwerp (1610). The attacks resulted in the deaths of about 9,000 civilians and military personnel.

Figure 1.1: Launch of a V2 Rocket
(picture from Wikimedia Commons)

1.2.2 Intercontinental Ballistic Missiles in the U.S. and the U.S.S.R.

1.2.2.1 Development

In the immediate post-WWII era, the U.S. and U.S.S.R. both started rocket research programs based on the German wartime designs, especially the V-2.

In the U.S.S.R., early development was focused on missiles able to attack European targets. This changed in 1953 when Sergei Korolev was directed to start the development of a true ICBM able to deliver the newly developed hydrogen bombs: the R-7. The first successful test was carried on 21 August 1957; the R-7 flew over 6,000 km (3,700 mi) and became the world's first ICBM.

In the U.S. the development of an ICBM was not initially a priority since the country had an overwhelming air superiority and truly intercontinental bombers. Things changed in 1953 with the Soviet testing of their first hydrogen bomb, but it was not until 1954 that the Atlas missile program was given the highest national priority.

The first successful flight of an Atlas missile to full range occurred on 28 November 1958. The first armed version of the Atlas, the Atlas D, had its first flight on 9 July 1959, and the missile was accepted for service on 1 September.

The R-7 and Atlas each required a large launch facility, making them vulnerable to attack, and could not be kept in a ready state continuously.

Failure rates were very high throughout the early years of ICBM technology.

Human spaceflight programs (Vostok, Mercury, Voskhod, Gemini, etc.) served as a highly visible means of demonstrating confidence in reliability, with successes translating directly to national defence systems.

For example it was the R-7 launch vehicle that placed the first artificial satellite in space, Sputnik, on 4 October 1957. The first human spaceflight in history was accomplished on a derivative of the R-7, Vostok, on 12 April 1961, by Soviet cosmonaut Yuri Gagarin.

In the U.S., the Atlas, Redstone, Titan, and Proton missiles were also the basis of space launch systems.

Figure 1.2: Launch preparation of an ATLAS-B missile
(picture from [2])

1.2.2.2 Reduction treaties

Up to 1970, the number of ICBMs and SLBMs was continuously increasing both in the U.S. and the U.S.S.R. For example in 1967 there were 1,054 ICBMs and 656 SLBMs in the U.S. The cost of this deployment was huge and the number of missiles was more than enough to ensure a mutual assured destruction. As a consequence some treaties were searched in order to reduce the number of deployed missiles. The end of the Cold War also help reaching agreements for their reduction.

The 1972 *Strategic Arms Limitation Talks* (SALT) treaty ([3]) froze the number of ICBM launchers of both the U.S. and the U.S.S.R. at existing levels, and allowed new submarine-based SLBM launchers only if an equal number of land-based ICBM launchers were dismantled.

Subsequent talks, called SALT II ([4]), were held from 1972 to 1979 and actually reduced the number of nuclear warheads held by the U.S. and the U.S.S.R.

Another treaty (START, *Strategic Arms Reduction Treaty*) was reached in 1991 between the U.S. and the U.S.S.R. and barred its signatories from deploying more than 6,000 nuclear warheads atop a total of 1,600 ICBMs, submarine-launched ballistic missiles, and bombers.

Its final implementation in late 2001 resulted in the removal of about 80 percent of all strategic nuclear weapons then in existence.

It was continued by the SORT treaty (*Strategic Offensive Reductions Treaty*) between the United States of America and the Russian Federation, that went into force in 2003. In this treaty both parties agreed to limit their nuclear arsenal to between 1700 and 2200 operationally deployed warheads each.

The most recent nuclear arms reduction treaty is the New START treaty (*Measures for the Further Reduction and Limitation of Strategic Offensive Arms*, [5]) between the United States of America and the Russian Federation , that entered into force in 2011. It limits the number of deployed strategic nuclear warheads to 1,550 and the number of deployed and non-deployed inter-continental ballistic missile launchers, submarine ballistic missile launchers, and heavy bombers equipped for nuclear armaments to 800. The number of deployed ICBMs, SLBMs, and heavy bombers equipped for nuclear armaments is limited to 700.

1.2.3 Intercontinental Ballistic Missiles in other countries

Many countries apart from the United States of America and Russia have developed ICBM capabilities since the 70s:

1.2.3.1 France

The French Centre of Spatial Studies (CNES, Centre National d'Études Spatiales) was formed in 1961. The CNES funded the development of a series of rockets named after precious stones (program Pierres précieuses) culminating with the Diamant (Diamond) rocket, the first French space launcher. In 1965, the Diamant rocket orbited the first French satellite, Astérix, following a successful launch from the Hammaguir test site in Algeria.

France then decided to develop the construction of the underwater-launched ballistic missile M1 (together with the development of the nuclear submarine "Le Redoutable"), and the development of the strategic ballistic surface-to-surface missile S2. Both systems became operational in 1971.

France focused afterwards on SLBMs and nowadays only has intercontinental missiles of this type (M45 and M51) in operation.

1.2.3.2 Israel

Jericho is a general designation given to the Israeli ballistic missiles. The name is taken from the first development contract for the Jericho I signed between Israel and Dassault in 1963.

The Jericho I was a short-range ballistic missile system publicly identified in 1971. It was continued by Jericho II, a solid fuel 2-stage long-range ballistic missile system with an estimated range of 7,800 km that was tested from 1987 to 1992.

The final version is the Jericho III which is supposed to have a payload capability of 1,000 kg and a range of more than 5,000 km.

1.2.3.3 China

After first testing a domestic-built nuclear weapon in 1964, China developed various warheads and missiles.

The Dong Feng 4 missile (DF-4) was the first ICBM operational in China. Its development was decided in 1965. It was deployed in 1975-76 and it is still operational. Its range is estimated to be between 5,500 and 7,000 km.

The DF-4 is to be substituted by the DF-31 missile. The latter was first tested in 1999 and its deployment started in 2009. This missile has a variant, DF-31A with possibly MIRVs (Multiple Independently targetable Reentry vehicles) capability. (It can hold 3 warheads in each missile and penetration and decoy aids to complicate missile defence efforts).

Beginning in the early 1970s, the liquid fuelled DF-5 ICBM was developed and used as a satellite launch vehicle in 1975. The DF-5, with a range of 10,000-12,000 km, was silo-deployed and entered into service in 1981. This missile was to be improved with the variant DF-5A, with a range increased to over 15,000 km and a more accurate guidance system, but there is no evidence that this system has been deployed yet.

China is now developing the Dong Feng-41 (DF-41) missile, with an estimated range between 12,000 km and 14,000 km, being then able to cover any position of the planet. This would make the DF-41 the world's longest ranged missile. The DF-41 was reported to have had its first flight test in 2012.

China is also developing the JL-2 SBLM, based on the DF-31 missile. This missile was first tested in 2012.

1.2.3.4 United Kingdom

In the early 1950s The United Kingdom had a nuclear deterrent capability based on the American GAM-87 Skybolt air-launched ballistic missile equipped with a nuclear warhead, and launched from bombers (V bombers). The V bomber fleet would become obsolete in 1965 and the United Kingdom wanted to have an independent British nuclear deterrent. With this aim the operational requirements for a medium range ballistic missile to be named Blue Streak were placed and the design was complete by 1957. However this missile was too expensive so the program was cancelled in 1960 and the missile was derived to space applications. It has to be noted that within this design the first missile silo was conceived.

The U.K. has not run a programme to develop an independent delivery system since the cancellation of the Blue Streak missile. Instead it has purchased U.S. delivery systems, fitting them with warheads designed and manufactured by the U.K.'s *Atomic Weapons Establishment* (AWE), the organization responsible for the design, manufacture and support of nuclear warheads in the U.K. American Polaris missiles were carried on British Royal Navy submarines between 1968 and the mid-1990s. They were replaced by the American missiles Trident II, that are now the only British nuclear weapon system in service. The U.K. currently has four Vanguard class submarines based in Scotland, armed with Trident II missiles.

1.2.3.5 India

By the start of 1980's, the *Defence Research and Development Laboratory* (DRDL) of India had developed competencies in the fields of propulsion, navigation and manufacture of materials.

This led to the birth of the *Integrated Guided Missile Development Programme* (IGMDP). The Agni missile series was initially conceived in the IGMDP as a technology demonstrator project in the form of a re-entry vehicle. The first missile of the series, Agni-I was tested in 1991. After its success, the Agni missile program was separated from the IGMDP upon realizing its strategic importance. It was designated as a special program in India's defence budget and provided adequate funds for subsequent development.

As of 2008, the Agni missile family comprises three deployed variants while two more variants are under testing and 1 is in development:

- Agni-I. Range: 700 - 1,250 km (Operational)
- Agni-II. Range: 2,000 - 3,000 km (Operational)
- Agni-III. Range: 3,500 - 5,000 km (Operational)
- Agni-IV. Range: 3,000 - 4,000 km (being tested)
- Agni-V. Range: 5,000 - 8,000 km (being tested)
- Agni-VI. Range: 8,000 - 10,000 km (Under development)

India successfully tested the missile Agni V in 2012, claiming entry into the ICBM club.

India is supposed to be developing a very long range ICBM called Surya (see [6]) but its development status is uncertain. India is also developing SLBMs for the Arihant class submarine (K-4 and K-5 missiles).

1.2.3.6 North Korea

In 1998 North Korea announced that they had used rocket Taepodong-1 to launch their first satellite "Kwangmyŏngsŏng-1". The satellite failed to achieve orbit, probably because of a failure in the third stage of the rocket.

This missile was believed by the U.S. Defense Intelligence Agency to be a technology demonstrator toward a longer-range missile development, namely the missile Taepodong-2.

The first Taepodong-2 test was conducted on July, 2006. The missile failed in mid-flight 35-40 seconds after launch. It is believed that this missile could have a range of up to 5,900 km, making it the first North Korean ICBM.

In 2009, North Korea announced that an Unha rocket would be used to launch the Kwangmyŏngsŏng-2 satellite. An analysis of the trajectory indicated that the first and second stage operated normally but the rocket's third stage failed to separate properly and no object was placed into orbit.

In 2012 the Unha-3 rocket was launched. The U.S. Northern Command confirmed that an object had entered into orbit.

The United States claimed that the launch was in fact a way to test the Taepodong-2 ICBM.

1.2.3.7 Iran

Iran started a long-range missile program with the development of the Shahab-1 missile (with a range of 1,000 km) between 1987 and 1994.

This missile was improved in the Shahab-2 version (with a range of up to 2,000 km) and first tested on 2006.

A collaboration between Iran and North Korea led to the development of the Shahab-3 missile (with a range of up to 1,280 km) based on the Nodong-1 North Korean missile.

There are alleged improvements of the Shahab missile in order to transform it into a long-range ICBM according to Israeli sources.

Also, it has been suggested in [7] that Iran has been developing independently of the Shahab family the Koussar missile, based on the Russian RD-216 engine and with a possible range of up to 5,000 km.

1.3 List of ICBMs

A list of ICBMs currently in operation or being tested or developed, according to references [1], [6] and [8] is provided herein:

Table 1.1: List of ICBMs (2015) (1)

	Country	Status	Launcher	First flight	Range (km)	Mass (kg)	MIRVs (number)
R-36M	U.S.S.R.	Active	Silo	1973	16,000	209,600	Yes (4-10)
R-36M2 Voevoda	U.S.S.R.	Active	Silo	1986	11,200	211,100	Yes (10)
UR-100N	U.S.S.R.	Active	Silo	1973	9,000	92,700	Yes (6)
RT-2PM Topol	U.S.S.R.	Active	Road-mobile	1985	10,500	45,100	No
RT-2UTTKh (Topol M)	Russia	Active	Silo, road-mobile	1994	10,500	47,200	Yes (4-6)
RS-24	Russia	Active	Silo, road-mobile	2007	10,500	49,000	Yes (3)
R-29R	Russia	Active	Submarine	1975	6,500	35,300	Yes (3)
R-29RK	Russia	Active	Submarine	?	6,500	34,400	Yes (?)
R-29RL	Russia	Active	Submarine	?	9,000	35,300	Yes (?)
R-29RM	Russia	Active	Submarine	1983	8,300	40,300	Yes (4)
R-29RMU Sineva	Russia	Active	Submarine	2004	11,500	40,300	Yes (6)
R-29RMU2 Layner	Russia	In dev.	Submarine	2011	>10,000?	40,300	Yes (12)
RSM-56 Bulava	Russia	Testing	Submarine	2004	8,300	36,800	Yes (6)
Minuteman III (LGM-30G)	U.S.	Active	Silo	1970	13,000	35,300	Yes (3)
Trident II (UGM-133)	U.S. & U.K.	Active	Submarine	1987	11,300	58,500	Yes (4)

Table 1.2: List of ICBMs (2015) (2)

	Country	Status	Launcher	First flight	Range (km)	Mass (kg)	MIRVs
M45	France	Active	Submarine	1986	6,000	35,000	Yes (6)
M51	France	Active	Submarine	2006	10,000	52,000	Yes (6)
DF-4	China	Active	Silo	1975	7,000	82,000	No
DF-31	China	Active	Silo	1999	8,000	42,000	No
DF-31A	China	In dev.	Silo	-	12,000	42,000	Yes (3)
DF-5	China	Active	Silo	1971	12,000	183,000	No
DF-5A	China	In dev.	Silo	1983	15,000	183,000	Yes (4)
DF-41	China	In dev.	Silo, road-mobile	2012	15,000	?	?
JL-2	China	Testing	Submarine	2012	14,000	42,000	?
Agni-V	India	Testing	Road/Rail mobile	2012	5,500	50,000	Yes (3)
Agni-VI	India	Testing	Road/Rail mobile	-	10,000	70,000	Yes (?)
Surya	India	In dev.	Road/Rail mobile	-	18,000	55,000	Yes (?)
K-4	India	Testing	Submarine	2014	3,500	?	?
K-5	India	In dev.	Submarine	-	?	?	?
Jericho-III	Israel	Active?	?	-	>5,000	30,000?	Yes (?)
Taepodong-2	North Korea	In dev.	Launch pad	2006	6,000	80,000	?
Koussar	Iran	In dev.	?	?	5,000	?	?

1.4 Characteristics of Intercontinental Ballistic Missiles

As it can be checked in tables 1.1 and 1.2 there is a great variety of ICBMs. However, some general characteristics are shared by all of them. These characteristics will be briefly indicated herein.

1.4.1 Shape

All the ICBMs summarized in section 1.3 are axisymmetric. These missiles are all very slender, with a high length/diameter ratio.

A difference can be noticed however between the submarine-launched missiles and the terrain-launched missiles. The submarine-launched missiles are usually smaller, with a length between 12 and 15 meters, and their length/diameter ratio is smaller, approximately between 5 and 8. The terrain-launched missiles are larger, with a length between 18 and 35 meters, and their length/diameter ratio is bigger, approximately between 9 and 12.

The only exception to these figures can be attributed to the Chinese ICBMs DF-31 and DF-41, which being silo-launched have a low length/diameter ratio.

In any case these missiles rarely have external control surfaces, showing basically a fuselage-body appearance.

1.4.2 Propulsion system

All the existing ICBMs have 2 or 3 stage liquid or solid rockets.

The use of only 2 stages can be associated with an older generation of missiles, being the case of the oldest (but still active) Russian and Chinese missiles, as well as the case of the oldest (already retired) American ICBMs. The newest generation of ICBMs is, in all countries, based on 3-stage rockets.

Something similar can be said about the propellant. Older missiles were all based on liquid propellants, more suitable for orbital launch vehicles, whereas the new generation is based on solid propellants. This allows lighter missiles.

Some exceptions can be highlighted: the new generation of Russian SLBMs is still based on liquid propellants.

1.4.3 Navigation system

The older generation of ICBMs was based only on inertial navigation, being the missiles completely autonomous.

In fact the need for appropriate accuracy for the inertial navigation in trajectories more than 5,500 km apart from the launching site resulted in a drastic improvement of inertial systems and guidance computers during the Cold War.

The purely inertial navigation was later on aided with star trackers, useful during the free-fall part of the trajectory. Most of the modern ICBMs incorporate these aiding navigation systems.

Finally the newest missiles are including Global Navigation Satellite Systems (GNSS) aids. GLONASS in the case of the Russian missiles and Compass in the case of the Chinese ones. The integration of GPS in the guidance system is now under testing in the case of the Trident II missiles, but it has not been deployed yet.

1.4.4 Control system

For the first stage, where the density of the atmosphere is still high and aerodynamic controls are useful, some ICBMs use aerodynamic surfaces both for control and stabilization.

They are usually trellised aerodynamic surfaces that are deployed after launch.

Figure 1.3: RT-2PM "Topol" missile after launch, with 8 deployed trellised aerodynamic surfaces (picture from [9])

However, the main control system that is used by ICBMs is thrust vectoring.

In normal conditions the thrust force is parallel to the missile axis and passes through the center of mass, generating a zero moment about this point. It is possible, however, to deflect this thrust force generating a moment that can create pitch and yaw in the missile. This can be done by several methods:

- Performing liquid injection in solid-propellant rockets.

In this case the rocket nozzle is fixed, but a fluid is introduced into the exhaust flow from injectors mounted around the aft end of the missile. This injection modifies the exhaust plume, resulting in a different thrust on that side and an asymmetric net force on the missile.

- Gimballing the rocket engine.

This often involves moving the entire combustion chamber and outer engine bell as on the Titan II's twin first stage motors, or even the entire engine assembly including the related fuel and oxidizer pumps. Such a system was used on the Saturn V and the Space Shuttle.

- Deflecting the rocket nozzle using electric servomechanisms or hydraulic cylinders.

The nozzle is attached to the missile via a ball joint with a hole in the center, or a flexible seal made of a thermally resistant material.

Roll control usually requires the use of two or more separately hinged nozzles, fins or vanes working together.

1.4.5 Trajectory

The flight path of an ICBM includes the following phases:

1. First stage flight

In this stage the pitch of the missile is changed from 90° to a lower value while trying to keep a low angle of attack in order to avoid lateral forces on the missile.

2. Second stage flight

In this stage the Kármán line that customarily represents the boundary between the Earth's atmosphere and the outer space is crossed. Once crossed, the aerodynamic forces on the missile can be neglected and a more aggressive control strategy can be used to ensure the required injection flight path angle is reached.

3. Third stage flight (if the missile has a third stage)

In this stage (or in stage 2 if the missile only has 2 stages) the final injection flight path angle is reached. The position when the third stage stops, together with the flight path angle and speed at this point determine the reentry point in the atmosphere and as a consequence the aiming point.

4. Post-boost phase

After the third stage has been separated, the missile is composed basically by a guidance module and the reentry vehicles. The guidance module sets the attitude of each reentry vehicle according to the reentry point in such a way that its axis is aligned with the flight path angle at reentry.

The guidance module sometimes has a small engine. This allows increasing the range and separating the targeting places of the reentry vehicles several hundred kilometers apart from each other using a single missile. This concept is known as Multiple Independently Targeted Reentry Vehicles (MIRVs).

The use of multiple warheads in each missile allows reducing the number of deployed ICBMs while maintaining the same number of warheads. This has 2 main advantages:

- It is less expensive since the deployment costs are directly related to the number of deployed ICBMs
- It allows fulfilling the terms of the reduction treaties (see section 1.2.2.2) that are more restrictive in terms of the number of launchers than in terms of nuclear warheads.

5. Reentry

Once a reentry vehicle (RV) is provided with the correct attitude for reentry, it is detached from the guidance module and provided with a certain spin. This spinning motion stabilizes the RV during the reentry phase towards its target.

The release of each RV is usually accompanied by the release of decoys in order to make the identification of the warhead by possible anti-missile systems more difficult.

The RVs follow individual ballistic trajectories, reenter the Earth's atmosphere, arm, and detonate according to the planned target profile (air burst or ground burst).

As an example a picture of the phases of the trajectory of the Minuteman III ICBM is shown:

Figure 1.4: Minuteman III flight path (picture from [10])

The deployment of Multiple Independently Targeted Reentry Vehicles is shown in the following picture:

Figure 1.5: Paths of MIRVs deployed by the Peacekeeper missile in a test (picture from Wikimedia Commons)

Chapter 1 references

- [1] National Air and Space Intelligence Center. *Ballistic & Cruise Missile Threat*. NASIC Public Affairs Office, Wright Patterson Air Force Base. Ohio, 2013. 8, 17
- [2] Embassy of the United States of America to Spain. Missiles: Atlas-B, Atlas Rocket, Atlas ICBM. <http://dspace.uah.es/dspace/handle/10017/1340>, 1957. [web page accessed on 22/09/2008]. 12
- [3] USA Department of State. STRATEGIC ARMS LIMITATION TALKS (SALT I). <http://www.state.gov/www/global/arms/treaties/salt1.html>, . [web page accessed on 13/09/2013]. 13
- [4] USA Department of State. Treaty between the United States of America and the Union of Soviet Socialist Republics on the Limitation of Strategic Offensive Arms. <http://www.state.gov/www/global/arms/treaties/salt2-2.html>, . [web page accessed on 13/09/2013]. 13
- [5] USA Department of State. Treaty between the United States of America and the Russian Federation on Measures for the Further Reduction and Limitation of Strategic Offensive Arms. <http://www.state.gov/documents/organization/140035.pdf>, April 2010. [web page accessed on 04/09/2015]. 13
- [6] George C. Marshall Institute and Claremont Institute. Missile Threat. <http://missilethreat.com/>. [web page accessed on 15/09/2013]. 16, 17
- [7] IHS Inc. Jane's defence weekly, September 2006. 16
- [8] Andrew Feickert. CRS Report for Congress. Missile Survey: Ballistic and Cruise Missiles of Foreign Countries. <http://www.au.af.mil/au/awc/awcgate/crs/rl30427.pdf>, March 2004. 17
- [9] FAS: Federation of American Scientists. RT-2PM - SS-25 SICKLE. <http://www.fas.org/nuke/guide/russia/icbm/rt-2pm.htm>, July 2000. [web page accessed on 10/09/2013]. 20
- [10] ICBM Prime Team, Prime - 19378, TRW Systems. *Minuteman Weapon System History and Description*. ICBM System Program Office, Hill Air Force Base Utah, second edition, July 2001. 22
- [11] T. D. Dungan. V2 rocket.com. <http://http://www.v2rocket.com>. [web page accessed on 10/09/2013].
- [12] James N. Gibson. *Nuclear Weapons of the United States. An Illustrated History*. Schiffer Publishing Ltd., Atglen, Pennsylvania, 1996.
- [13] Robert S. Norris and Thomas B. Cochran. *US - USSR/Russian Strategic Offensive Nuclear Forces 1945-1996*. Natural Resources Defense Council Inc., Washington DC, January 1997.
- [14] Michael J.H.Taylor. *Missiles of the World*. Charles Scribner's Sons, New York, third edition, 1980.

- [15] Office of the Under Secretary of Defense for Acquisition, Technology, and Logistics. Strategic Arms Reduction Treaty (START I). <http://www.acq.osd.mil/tc/treaties/start1/text.htm>, July 1991. [web page accessed on 04/09/2015].
- [16] The Nuclear Threat Initiative (NTI). Strategic Offensive Reductions Treaty (SORT). <http://www.nti.org/treaties-and-regimes/strategic-offensive-reductions-treaty-sort/>, May 2002. [web page accessed on 04/09/2015].
- [17] Pedro Sanz Aránguez. Resúmenes de Misiles y Vehículos Espaciales printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, 1996.

Chapter 2

Missile Defence

This chapter reviews the technical challenges of the ballistic missile defence problem and the components of the American Ballistic Missile Defense System.

It focuses later in the Ground-Based Midcourse Defense System, reviewing the characteristics of the Boost Vehicle and the Exo-atmospheric Kill Vehicle (EKV) that compose the Ground-Based Interceptor (GBI), since this is the missile that will be considered as the interceptor in this thesis.

2.1 Technical challenges of the problem

The problem of intercepting an ICBM using Anti-Ballistic Missiles (ABMs) is very challenging since the ABMs require a sophisticated guidance and control system to accomplish their mission.

The difficulty of the interception problem increased dramatically with the introduction of Multiple independently targetable reentry vehicle (MIRV) warheads (see section 1.4.5). Suddenly each launcher was throwing several warheads that would spread in space, requiring one interceptor for each warhead.

Also, when introducing MIRV warheads, decoy systems were added in order to fool the ABMs, and in many cases it was decided to detonate one nuclear warhead at high altitude, causing a Nuclear Electromagnetic Pulse (NEP) that would provoke blackouts in the radar detection systems on ground that should help the guidance of the ABMs.

This made ABMs economically ineffective.

The most convenient way to cope with MIRVs is destroying the ICBM in the boost phase, while it is still being powered and has a great IR signature, but since the powered phase of the ICBMs, especially when they are using solid propellants, is very short, this would only be possible if the ABMs are placed close to the launching site, which is not usually possible.

There are other problems related with the interception. For example the Soviet Union was developing in the 1960s the concept of Fractional Orbital Bombardment System (FOBS) (see [1]). With this concept a series of ICBMs would be placed in a low-Earth orbit and descend when required to its target. This system would provide global range for the ICBMs and would allow attacking the United States from the South, which is the opposite direction from which the NORAD (North American Aerospace Defense Command) early warning systems were oriented (the NORAD system is the one in charge of detecting missile attacks in the U.S.).

The FOBS was highly controversial and it was prohibited by the Outer Space Treaty (1967) that bars a state party to the treaty from placing nuclear weapons or any other weapons of mass destruction in the orbit of the Earth, installing them on the Moon or any other celestial body, or to otherwise station them in outer space.

Finally the effectiveness of ABMs is based on fixed trajectories for the ICBMs, which is usually the case, but it would be possible to develop ICBMs with a midcourse trajectory modification (see for example [2]). This is for instance the case of the RT-2UTTKh Topol M ICBM which according to Russia is capable of making evasive manoeuvres.

As a result of these technical difficulties there are many tactical ABM systems like:

- Arrow system (Israel)
- Patriot missile (United States)
- Hawk missile (United States)

but only 2 operational strategic ABM systems:

- The A-135 Russian system, placed around Moscow and based on the ABM-3 Gazelle and the ABM-4 Gorgon missiles
- The American National Missile Defense (NMD) system

China is developing now a strategic ABM system based on the SC-19 missile, but the system is still not operational. India is also developing a strategic ABM system: the Advanced Air Defence (AAD) missile system, but this development is not complete.

2.2 History

During the second World War Bell Labs was requested a study of the possibility of intercepting ballistic missiles like the recently introduced V2. The conclusion was that it was not possible at the time.

The situation changed in the 1950s with the introduction of high-speed computing systems and the U.S. started the development of the LIM-49 Nike Zeus missile in order to face Soviet ICBMs. The program, however, was canceled in 1963 because of its poor performance (it could only guide one missile each time and could not be used if there were high-altitude nuclear explosions because of NEPs). It was decided that it would be much simpler to build more nuclear warheads and guarantee mutually assured destruction.

Canada was also involved in anti-ballistic missiles in the 1950s. They created different technical innovations, like the use of terminal guidance to ensure interception, something difficult with the accuracy of the ground-based radar systems that decreases with the distance to the radar.

The Soviet Union started developing ABM systems in 1956 with the System A, based in the missile V-1000. This development went on until the A-35 anti-ballistic missile system designed to protect Moscow became operational in 1971. This system was designed for exo-atmospheric interception, but it was susceptible to the use of multiple warheads (MIRVs) and blackouts. The system was based on using nuclear warheads to destroy the incoming ICBMs, which in fact creates a problem itself since the detonation of the interceptor missile negatively affects the area it is protecting.

This system was upgraded in the 1980s to a 2-layer system, the A-135. The Gorgon missile (ABM-4) was used to intercept missiles outside the atmosphere, and the Gazelle (ABM-3) to intercept surviving missiles at a shorter range.

The United States also reinforced the ABM development at the late 1960s (project Nike-X). The idea was using radars capable of scanning much greater volumes of space and able to track many warheads and launch several missiles at once. Also the problem of the radar blackout had to be addressed using a new missile. The missile LIM-49 Spartan was to be used for long-range interception, while the missile Sprint was to be used for short-range interception. Once again this system was based on detonating nuclear warheads in the vicinity of the incoming ICBMs.

This system, called *Sentinel*, was abandoned due to its large cost and substituted by a minor version, the *Safeguard* system, deployed around the ICBMs silos in the Stanley R. Mickelsen Safeguard Complex, North Dakota.

In 1972 the *Anti-Ballistic Missile Treaty* was signed between the Soviet Union and the United States, restricting the use of strategic anti-ballistic missiles to 100 in order to protect a small area. Basically the treaty was allowing the already existing ABM systems, this is, the A-35 system around Moscow and the Safeguard system in North Dakota.

The Reagan administration started in the 1980s the Strategic Defense Initiative (SDI) often referred to as "Star Wars". The program aimed at providing a total shield against a massive Soviet ICBM attack:

What if free people could live secure in the knowledge that their security did not rest upon the threat of instant U.S. retaliation to deter a Soviet attack, that we could intercept and destroy strategic ballistic missiles before they reached our own soil or that of our allies?

(March 23, 1983, extract from President Reagan's Address to the Nation on Defense and National Security).

This initiative was extremely ambitious, not being ever operative. However it developed important technological innovations, like the concept of hit-to-kill vehicles or "Kinetic Kill Vehicles" (KKV) that would destroy the ICBMs by a direct hit, without using a nuclear warhead, thus avoiding the effects of a nuclear blast on the area the missile is supposed to protect. The first successful test of this concept was performed in 1984 when an RV from a Minuteman missile was intercepted at a height of 160 km.

The intended scope of this project was limited by president George H. W. Bush to a system based on interceptor missiles with the capacity of intercepting small numbers of ICBMs.

The SDI was renamed as Missile Defense Agency (MDA) in 2002. That year the United States abandoned the Anti-Ballistic Missile Treaty and started deploying ABM systems that were banned in the bilateral agreement. As a consequence the Russian Federation promptly dropped the START II agreement, that was pretending to completely ban MIRVs.

The whole program, called National Missile Defense (NMD) was initially conceived as a system based in long-range missile interceptors. In order to include other systems in the missile defense strategy the program was renamed as Ballistic Missile Defense System (BMDS) (although the old name is still used) and the part related to long-range missile interceptors was renamed as Ground-Based Midcourse Defense (GMD). The GMD is operative since 2006 with a limited but increasing capability intended for defending the United States of America against a missile attack by a rogue state.

The United States initially pretended to reinforce the GMD with a European Interceptor Site (EIS), that was to be a Ground-Based Midcourse Defense System with 10 silo-based 2-stage interceptors based in Poland, and a radar system placed in Brdy, Czech Republic.

The intention was to defend the United States of America against future long-range ICBMs from Iran.

However the system was opposed by a majority of the Polish population, and Russia threatened to place short-range nuclear missiles on its borders with NATO if the system was deployed (thus abandoning the *Intermediate-Range Nuclear Forces Treaty*, INF, of 1987 that banned the use of short and intermediate range nuclear weapons).

As a consequence the United States of America decided in 2009 to substitute the EIS with the Aegis system, both with Aegis ships to be placed on the Black Sea, and with ground based SM-3 missiles (Aegis ashore) to be placed in Redzikowo, Poland and Deveselu, Romania, in what is called European Phased Adaptive Approach (EPAA).

4 Aegis warships will be based at Rota, Spain, as part of the deployment of Aegis in Europe.

2.3 Ballistic Missile Defense System (BMDS)

2.3.1 Introduction

Ballistic missiles could be intercepted in 3 distinct phases of flight:

- Boost phase

In the boost phase, the ballistic missile's engine ignites and thrusts the missile into the atmosphere. This is the earliest point in the missile's flight path for an intercept. During the boost phase, the ballistic missile is easier to detect and track due to its bright, hot exhaust. The interception in the boost phase would be ideal since in this case any damage would be produced close to the launching site, and the detection of the missile is easier since it offers a huge infrared signature. For this reasons some studies encourage the focus in this interception (see [3]) criticizing the midcourse interception that involves huge detection problems (see [4]). However, the boost phase is very short and requires quick reaction in order to destroy the missile. This means that a very fast interception missile would be required, which implies a very heavy launcher (see reference [5]). Moreover, the interception in the boost phase requires the deployment of the interception systems close to the possible launching site, which is not always possible. As a consequence boost-phase defence systems are nowadays considered as unpractical (see pages 4-13 of [6]).

- Midcourse phase

Following the boost phase is the midcourse phase, in which the missile's booster burns out and the missile begins a ballistic trajectory. The midcourse phase is generally the longest phase of a ballistic missile's flight path, which offers the defender several opportunities to shoot down the ballistic missile. It is also in this phase that the missile may deploy a reentry vehicle (RV) as well as potential countermeasures that try to mask or disguise the RV. A successful midcourse intercept requires sensors that are able to find the RV amid the booster debris and countermeasures.

- Terminal phase

The terminal phase of a ballistic missile's flight is the last opportunity for intercept prior to detonation or impact. As the missile proceeds to the intended target, the RV has separated from the debris field due to atmospheric drag. A successful terminal intercept requires an advanced interceptor capable of countering any manoeuvres the RV may make as it closes in on its target.

All of these intercept scenarios - boost, midcourse, and terminal - require accurate missile tracking; quick reaction time; advanced interceptor missiles; reliable communications; and advanced sensors. All of them are considered by the BMDS which sets a layered protection system.

2.3.2 System components

The BMDS system is a system of systems focused on different strategies in order to intercept missiles of different ranges in different phases of flight. The most relevant components of the BMDS will be highlighted herein.

2.3.2.1 Command and Control, Battle Management and Communications

The Command and Control, Battle Management, and Communications (C2BMC) system is the integrating element of the Ballistic Missile Defense System (BMDS).

The C2BMC supports a layered missile defence capability that enables an optimized response to threats of all ranges in all phases of flight. It integrates and synchronizes the NMD sensor and weapon systems. It is placed in the Peterson Air Force Base in Colorado.

2.3.2.2 Space Tracking and Surveillance System (STSS)

This is a space-based sensor component with the purpose of detecting and tracking ballistic missiles in midcourse and during the boost phase by using low Earth orbit satellites with long and short wave infrared sensors.

This allows an earlier detection than what would be possible with other detection systems. The tracking data from the satellites would be sent to the Aegis ships (see section 2.3.2.4.4) and to the Kinetic Energy Interceptors (see section 2.3.2.4.5.1) minutes earlier than less accurate data provided by organic radars in the Aegis or THAAD (Theater High Altitude Area Defense). As of today 2 demonstration satellites are placed in orbit.

Also, the MDA has placed the satellite NFIRE (Near Field Infrared Experiment) in orbit on 24 April 2007. This satellite is designed to collect data for environmental background characterization (regional/seasonal atmospheric radiance variability, day-night, land-sea, clouds, auroral measurements, etc.) for Precision Tracking Space System sensors and for aiding the SM-3 IIB seeker development (see section 2.3.2.4.4). It also supports research of early launch detection and tracking capabilities in short wave infrared.

The information from the Space Tracking and Surveillance System (STSS) would be integrated in the NMD through the C2BMC.

2.3.2.3 System sensors

Different radars that estimate launch and impact points, detect sea-launched or intercontinental ballistic missiles and provide real-time ballistic missile tracking data to commit the launch of interceptors are integrated within the system:

- Early Warning Radars AN/FPS-132

Three Air Force Upgraded Early Warning Radars (UEWR), located in the Beale Air Force Base, California; the RAF base Fylingdales, United Kingdom; and the Thule Air Base, Greenland, are presently integrated into the Ballistic Missile Defense System (BMDS).

These are solid-state, phased-array, all-weather, long-range (up to 4800 km) radars that operate in the Ultra High Frequency Band. The radars in Beale and Thule have 2 radar faces while the radar in Fylingdales has 3 faces. Each face provides 120° coverage. The radar in Beale has an active aperture of 22.25 m of diameter; the ones in Fylingdales and Thule have an active aperture of 25.6 m of diameter.

- COBRA DANE radar

The COBRA DANE radar placed in the island of Shemya, Alaska, is also integrated into the Ballistic Missile Defense System (BMDS).

This radar operates in the L-band frequency. It has a face of 28.96 m in diameter with an overall height of 36.58 m. It has a range of 3200 km.

- Sea-Based X-Band radar

The SBX is an advanced X-Band radar mounted on a mobile, semi-submersible platform. It is 73.15 m wide and 118.9 m long and towers more than 85.34 m from keel to top.

Figure 2.1: Sea-Based X-Band radar
(picture from Wikimedia Commons)

Also, all the sensors included in the Aegis (SPY-1 radar) and Terminal High Altitude Area Defense (THAAD) missile systems (the AN/TPY-2 radar) would be integrated in the NMD through the C2BMC.

2.3.2.4 Interception Systems

2.3.2.4.1 National Missile Defense layers

The NMD is a system of systems and different elements are used to intercept ballistic missiles in different phases of flight.

The following picture depicts the Interception Subsystems included in the NMD, indicating in which phase of flight they can be used:

Figure 2.2: Elements of the NMD used against an incoming ballistic missile (picture from MDA)

Each of these subsystems will be briefly described hereafter, from shorter to larger range.

2.3.2.4.2 Patriot Advanced Capability-3 & Medium Extended Air Defense System

This is a land-based system based on the proven Patriot air and missile defence infrastructure, that was deployed in the Middle East in the Iraq War.

This system uses a new missile (PAC-3 or MIM-104F) that is more manoeuvrable than previous variants of the Patriot missile due to the use of 180 tiny pulse solid propellant rocket motors mounted in the forebody of the missile which serve to fine-align the missile trajectory with its target to achieve hit-to-kill capability.

The most significant upgrade to the PAC-3 missile is the addition of a Ka band active radar seeker. This allows the missile to drop its uplink to the system and acquire its target itself in the terminal phase of its intercept, which improves the reaction time of the missile against a fast-moving ballistic missile target.

The PAC-3 missile is accurate enough to select, target, and home in on the warhead portion of an inbound ballistic missile. The active radar gives a "hit-to-kill" capability that eliminates the need for a traditional proximity-fused warhead. However, the missile still has a small explosive warhead, called Lethality Enhancer, a directional warhead which launches a stream of low-speed steel fragments in the direction of the target in order to enhance the kill probability. This greatly increases the lethality against ballistic missiles of all types. It has also been added with an Upper-Tier Debris Mitigation capability to reduce missile waste caused by debris from upper-tier intercepts.

The PAC-3 upgrade has increased the system's lethality and effectiveness against ballistic missiles. It has also increased the scope of ballistic missiles that Patriot can engage, which now includes several intermediate range. This has been achieved reducing the capability of intercepting atmospheric aircraft and air-to-surface missiles, since it has now a smaller explosive warhead compared to older Patriot missiles.

This system is mounted on wheeled vehicles that carry several interceptors along with advanced radars that provide 360° coverage.

2.3.2.4.3 Terminal High Altitude Area Defense (THAAD)

This is a system designed to shoot down short, medium, and intermediate ballistic missiles in their terminal phase using a hit-to-kill approach. The missile carries no warhead but relies on the kinetic energy of the impact. THAAD was designed to hit Scuds and similar weapons, and has a limited capability against ICBMs. It is launched from wheeled vehicles.

This system is somehow similar to the Aegis BMD system (see section 2.3.2.4.4), and shares some components with the Aegis ashore system.

It uses the AN-TPY-2 radar (a transportable X-band, high resolution, phased-array radar designed specifically for ballistic missile defense). These radars play a vital role in the BMDS by acting as a forward based sensor. As of today 4 AN-TPY-2 radars are deployed, with a plan of deploying 12 radars.

Figure 2.3: Launch of a THAAD missile
(picture from MDA)

2.3.2.4.4 Aegis Ballistic Missile Defense (BMD) System

The Aegis BMD is designed to intercept ballistic missiles in the midcourse phase, prior to reentry, by expanding the Aegis Combat System with the addition of the AN/SPY-1 radar and the improvement of the *Standard* missile technologies.

There are many versions of the *Standard* missile: SM-2 missile was designed for endoatmospheric interceptions and equipped with a blast fragmentation warhead, while the SM-3 missile is designed for exoatmospheric interceptions in the midcourse phase and it is equipped with a "hit-to-kill" warhead (kinetic warhead) that is designed to destroy a ballistic missile's warhead by colliding with it. There are also several versions of the SM-3 missile, called "blocks", improving different aspects of the missile.

Future development of the Aegis BMD system includes Launch on Remote capability and upgraded *Standard* missile avionics and hardware. The Launch on Remote capability involves the use of off-board sensors to provide a targeting solution for the SM-3 launch. The ability to destroy ICBMs will be limited and only for the ships with version 5.1 of the Aegis BMD system (to be deployed in 2018) using the SM-6 increment 2 missile (see reference [7]).

This system is installed in U.S. Navy Ticonderoga class cruisers and Arleigh Burke class destroyers. The system has also been deployed in vessels of the Japan Maritime Self-Defence Force, with an intention of defending Japan against North Korean ballistic missiles. Australia is also installing Aegis BMD systems in one destroyer and the system could be deployed in vessels of other countries already operating the Aegis Combat System (Norway, Spain and South Korea).

A land-based component, Aegis Ashore, is also planned. This would consist of equipment which is commonly used by the Navy being deployed in land-based facilities (locations in Romania and Poland are being considered). This would include SPY-1 radars and a battery of SM-3 missiles.

2.3.2.4.5 Ground-Based Midcourse Defense

2.3.2.4.5.1 Description of the System

This is the element with the longest range of the NMD, and it is designed to destroy intermediate and long-range ballistic missile threats in the midcourse phase.

The system includes a Ground Support & Fire Control (GFC) System that consists of redundant fire control nodes, interceptor launch facilities, and a communications network. Data from satellites and ground based radar sources is used to task and support the intercept of target warheads using Ground-Based Interceptors.

The system is based on the Ground-Based Interceptor (GBI) missile made of 2 subsystems:

- Boost Vehicle (BV)

The Boost Vehicle is a three-stage solid fuel booster vehicle that carries the EKV as a payload toward the target's predicted location in space and then launches the EKV.

A 2 stage booster vehicle for shorter range defence was also developed and even tested, but it has not been deployed yet.

The initial concept for the booster vehicle came from Boeing and was called COTS (Commercial Off-the-Shelf) booster, because it used developed and commercially available rocket stages. The first flight test occurred on 31 August 2001. There were 2 flight tests of this booster vehicle (BV) and they were not successful so the GBI booster development program was restructured. An improvement of the BV was requested to Lockheed Martin Space Systems Company, which developed a version known as BV-Plus.

Additionally, Orbital Sciences Corp. (OSC) was awarded a contract to build an alternative booster (called OBV - Orbital Booster Vehicle) for the GBI. The OBV is based on the upper three stages of the company's Taurus XL commercial launch vehicle, modifying the payload fairing. After some tests, it was decided that the BVs would be contracted exclusively to OSC.

- Exoatmospheric Kill Vehicle (EKV)

The Exoatmospheric Kill Vehicle (EKV) is the intercept component of the Ground-Based Interceptor (GBI). Once released from the booster, the EKV mission is to engage and destroy incoming warheads using only the kinetic force of the direct collision to destroy the target warhead. There is no weapon or explosive element of the EKV.

There were previous prototypes from Lockheed Martin and Boeing, but Raytheon was selected in 1998 as prime contractor for the development of the EKV.

The Raytheon EKV has its own communication link to support target selection and intercept. The EKV is continuously updated with the latest information from the command and control center.

The EKV is also equipped with an infrared seeker, which is comprised of focal plane arrays and a cooling assembly attached to an optical telescope. The seeker software has to detect and track all incoming objects, and discriminate warheads from decoys and debris.

The EKV's manoeuvring system, known as DACS (Divert and Attitude Control System), has four rocket thrusters around the vehicle's body. The purpose is to steer the EKV to a head-on collision with a target at closing speeds of more than 25700 km/h.

The vehicle weighs approximately 64 kg, is 140 cm long and about 60 cm in diameter.

Figure 2.4: Exo-atmospheric Kill Vehicle
(picture from MDA)

There are two variants of the Exo-atmospheric Kill Vehicle (EKV) currently deployed. The first deployed version of the EKV is known as the Capability Enhancement-I, or CE-I, kill vehicle. The CE-I began deployment with the emplacement of the first GBI interceptor in July 2004. The last one was deployed in September 2007, at that time bringing the number of deployed GBIs to 24, all equipped with the CE-I.

The Missile Defense Agency (MDA) began the development of a new version of the EKV, the Capability Enhancement-II (CE-II), in 2005. The primary motivation for developing the new version was the obsolescence of parts in the original version, but a number of improvements were also made, like an upgraded infrared seeker with greater sensitivity and more accurate guidance instrumentation.

The Ground-Based Interceptors are placed at Fort Greely, Alaska and Vandenberg Air Force Base, California. A total of 30 interceptors were deployed at the end of 2010.

Boeing was selected by the NMD in 1998 as prime contractor for the interceptor missile and responsible for the integration of the booster rocket and the kill vehicle.

Figure 2.5: GBI missile being deployed in its silo
(picture from MDA)

2.3.2.4.5.2 GBI Tests

Each component of the Ground-Based Midcourse Defense System has been tested individually, but also some complete tests in an interception scenario against an ICBM with a mock warhead have been performed since October 1999 (17 tests up to June 2014). These tests have been executed in an iterative way, adding each time more complexity to the interception scenarios:

- The target missile has been improved. The first 4 tests used a modified Minuteman II missile by Lockheed Martin (MSLS) with an unarmed reentry vehicle (warhead). Tests 5-13 used the Target Launch Vehicle (TLV) or Minotaur II by Orbital. Tests 14, 15, 16 and 17 used the FTF LV-2 target by Lockheed Martin. The LV-2 uses Trident C4 Stage 1 and Stage 2 motors.
- The first 8 interception tests were performed with a booster by Lockheed Martin called PLV (Payload Launch Vehicle) made of the second and third stage of the Minuteman II missile (Aerojet SR19-AJ-1 and Hercules M57A1) with the designation NLGM-30F. The following tests were performed with the OBV booster.
- The first 5 tests used a single large decoy balloon with a much brighter IR signature than the warhead. The rest of the tests have used more realistic decoys.
- The sensor data used by the GBI have been improved since the first tests. The first 3 interception tests used existing radars not for GMD and a beacon in the mock warhead. The 4th test was the first one to include an X-Band Radar. Test 6 was the first one including data from the AN/SPY-1 Aegis tracking radar. Test 11 was the first one to use an operational sensor to support a GBI weapon task plan: The Beale UEWR radar was the primary sensor used for planning the intercept. Test 14 was the first one to include the sea-based X-band radar.
- The first 13 tests were performed using the CE-I version EKV. Tests 14 and 15 were executed with the CE-II version EKV. Since this version did not behave as expected test 16 was executed again using the CE-I version, while a study on the CE-II version performance was addressed (see [8]). Test 17 was performed with CE-II again.

Of these 17 interception tests, 9 have been successful. However, since more operational systems are included each time, the success of the first tests is less representative of the maturity of the system.

The system is deployed now, but the result of the tests indicates that it is still not completely operational. More upgrades and improvements are to be expected in the near future in order to achieve a better success rate.

Chapter 2 references

- [1] GlobalSecurity.org. R-36-O / SL-X-? FOBS. <http://www.globalsecurity.org/wmd/world/russia/r-36o.htm>, February 2013. [web page accessed on 01/11/2013]. 26
- [2] Mathew P. Gillis III. Optimal Mid-Course Modifications of Ballistic Missile Trajectories. Master's thesis, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, December 1975. 26
- [3] U.S.Army Center of Military History. History of Strategic Air and Ballistic Missile Defense, Volume II, 1956-1972. <http://www.history.army.mil/html/books/bmd/>, . [web page accessed on 01/11/2013]. 29
- [4] Theodore Postol. Explanation of Why the Sensor in the Exoatmospheric Kill Vehicle (EKV) Cannot Reliably Discriminate Decoys from Warheads. https://www.fas.org/spp/starwars/program/news00/postol_atta.pdf, May 2000. 29
- [5] American Physical Society. *Report of the APS Study Group on Boost-Phase Intercept Systems for National Missile Defense. Scientific and Technical Issues*. The National Academies Press, Washington DC, first edition, July 2003. 29
- [6] Committee on an Assessment of Concepts and Systems for U.S. Boost-Phase Missile Defense in Comparison to Other Alternatives. Division on Engineering and Physical Sciences. *Making Sense of Ballistic Missile Defense*. The National Academies Press, Washington DC, first edition, 2012. 29
- [7] Ronald O'Rourke. *Navy Aegis Ballistic Missile Defense (BMD) Program: Background and Issues for Congress*. Congressional Research Service, first edition, November 2014. 35
- [8] Randolph R. Stone. DODIG-2014-111, Exoatmospheric Kill Vehicle Quality Assurance and Reliability Assessment - Part A. <http://www.dodig.mil/pubs/documents/DODIG-2014-111.pdf>, September 2014. [web page accessed on 02/01/2015]. 39
- [9] Richard Matlock. Space & Missile Defense Symposium. http://www.mda.mil/global/documents/pdf/SMD_Aug13_DVbrief.pdf, August 2013.
- [10] Missile Defense Agency. U.S. Department of Defense, Missile Defense Agency. <http://www.mda.mil/>. [web page accessed on 01/11/2013].
- [11] Department of Defense. United States of America. Ballistic Missile Defense Review Report. http://archive.defense.gov/bmdr/docs/BMDR%20as%20of%2026JAN10%200630_for%20web.pdf, February 2010. [web page accessed on 01/11/2013].
- [12] U.S.Army Center of Military History. History of Strategic Air and Ballistic Missile Defense, Volume I, 1945-1955. <http://www.history.army.mil/html/books/bmd/>, . [web page accessed on 01/11/2013].
- [13] Congressional Budget Office. *Alternatives for Boost-Phase Missile Defense, A CBO Study*. Congressional Budget Office, The Congress of the United States, Washington DC, first edition, July 2004.

- [14] George C. Marshall Institute and Claremont Institute. Missile Threat. Defense Systems. <http://missilethreat.com/defense-systems/>. [web page accessed on 15/09/2013].
- [15] FAS: Federation of American Scientists. Ballistic Missile Defense. <http://fas.org/ssp/bmd/index.html>, February 2015. [web page accessed on 10/06/2015].
- [16] Missile Defense Advocacy Alliance. Ground Based Interceptor Exoatmospheric Kill Vehicle. <http://missiledefenseadvocacy.org/missile-defense-systems/u-s-deployed-intercept-systems/ground-based-midcourse-defense/ground-based-interceptor-exoatmospheric-kill-vehicle/>. [web page accessed on 08/11/2013].
- [17] Missile Defense Agency. Ballistic Missile Defense Intercept Flight Test Record. <http://www.mda.mil/global/documents/pdf/testrecord.pdf>, November 2014. [web page accessed on 08/01/2015].

This page intentionally left blank.

Part II

Simulation of the missiles

The development of a realistic simulation of an ICBM and its interceptor is necessary in order to obtain meaningful results when analysing the guidance of the interceptor.

This part of the thesis indicates the equations and models used for the simulation of these missiles and the general structure of the simulator. It also includes some analyses performed related to the simulator validation.

This page intentionally left blank.

Chapter 3

Equations of motion for the missiles

Appendix [D](#) indicates the general equations of motion applicable to the missile system. The notation to be used for these equations as well as their final form will be detailed in this chapter.

Also, the selected state vector will be detailed as well as the required equations to obtain from its variables all the useful information.

Finally, the numerical method to be used for the integration of the equations of motion is selected.

3.1 Equations of motion

3.1.1 Notation for the equations

The following notation will be used for the position of the center of mass of the missile (relative to the inertial reference frame, and with coordinates given in the inertial reference frame):

$$(\vec{r}_{CM}^i)_i = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad (3.1)$$

and for its velocity (relative to the inertial reference frame and with coordinates given in the body reference frame):

$$(\vec{v}_{CM}^b)_i = \begin{Bmatrix} v_x^b \\ v_y^b \\ v_z^b \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (3.2)$$

The following notation will be used for the rotation quaternion from the inertial to the body frame (see appendix B):

$$q_{bi} = \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} \quad (3.3)$$

and for the angular velocity vector of the missile:

$$\vec{\Omega}_{bi}^b = \begin{Bmatrix} \omega_x^b \\ \omega_y^b \\ \omega_z^b \end{Bmatrix} = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \quad (3.4)$$

The external forces acting on the particles p of the missile and the momentum of these external forces with respect to the center of mass will be represented as (with coordinates given in the body reference frame):

$$\sum_{\forall p} \vec{f}_{p\text{EXTERNAL}}^b = \begin{Bmatrix} F_x^b \\ F_y^b \\ F_z^b \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \quad (3.5)$$

$$\vec{\mathcal{M}}_{CM}^b = \sum_{\forall p} (\vec{r}_p)_b \wedge \vec{f}_{p\text{EXTERNAL}}^b \begin{Bmatrix} M_x^b \\ M_y^b \\ M_z^b \end{Bmatrix} = \begin{Bmatrix} \mathcal{L} \\ \mathcal{M} \\ \mathcal{N} \end{Bmatrix}$$

It has to be noted that the control forces and control moments are included within the external forces and moments.

The tensor of inertia in the b reference frame with respect to the center of mass of the missile will be denoted as:

$$I_{CM}^b(t) = \begin{bmatrix} I_x(t) & -J_{xy}(t) & -J_{xz}(t) \\ -J_{xy}(t) & I_y(t) & -J_{yz}(t) \\ -J_{xz}(t) & -J_{yz}(t) & I_z(t) \end{bmatrix} \quad (3.6)$$

Finally, the following notation will be used for the position and velocity of the mass flow centers:

$$\{(\vec{r}_e^b)_b\}_k = \begin{Bmatrix} x_e \\ y_e \\ z_e \end{Bmatrix}_k \quad (3.7)$$

$$\{(\vec{v}_e^b)_b\}_k = \begin{Bmatrix} u_e \\ v_e \\ w_e \end{Bmatrix}_k$$

where the subscript k indicates one of the nozzles (there could be up to N_{nz} nozzles)

3.1.2 Linear momentum

Since:

$$\left(\frac{d(\vec{v}_{CM})_i}{dt}\right)_i = \left(\frac{d(\vec{v}_{CM})_i}{dt}\right)_b + \vec{\Omega}_{bi} \wedge (\vec{v}_{CM})_i \quad (3.8)$$

equation D.106 can be re-written as:

$$\begin{aligned} \sum_{\forall p} \vec{f}_{p\text{EXTERNAL}} &= M(t) \cdot \left(\frac{d(\vec{v}_{CM})_i}{dt}\right)_b + \\ &+ M(t) \cdot \left(\vec{\Omega}_{bi} \wedge (\vec{v}_{CM})_i\right) + \sum_{k=1}^{N_{nz}} 2 \left(\vec{\Omega}_{bi} \wedge \dot{m}_k \cdot (\vec{r}_e)_b\right) \end{aligned} \quad (3.9)$$

Using the notation indicated in 3.1.1 the variation of the linear momentum is given by:

$$\begin{aligned} F_x &= M(t) (\dot{u} + q \cdot w - r \cdot v) + 2 \sum_{k=1}^{N_{nz}} \dot{m}_k (q \cdot z_{ek} - r \cdot y_{ek}) \\ F_y &= M(t) (\dot{v} + r \cdot u - p \cdot w) + 2 \sum_{k=1}^{N_{nz}} \dot{m}_k (r \cdot x_{ek} - p \cdot z_{ek}) \\ F_z &= M(t) (\dot{w} + p \cdot v - q \cdot u) + 2 \sum_{k=1}^{N_{nz}} \dot{m}_k (p \cdot y_{ek} - q \cdot x_{ek}) \end{aligned} \quad (3.10)$$

where k indicates a nozzle (up to N_{nz}) being $\dot{m}_k$ the mass flow through that nozzle.

3.1.3 Angular momentum with respect to the center of mass of the missile

Equation D.151 can be expressed as:

$$\begin{aligned} \vec{\mathcal{M}}_{CM}^b = & I_{CM}^b(t) \cdot \frac{d\vec{\Omega}_{bi}}{dt} + \vec{\Omega}_{bi} \wedge I_{CM}^b(t) \cdot \vec{\Omega}_{bi} + \frac{d(I_{CM}^b)}{dt} \cdot \vec{\Omega}_{bi} + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k \left\{ \{(\vec{r}_e)_b\}_k \wedge \left(\vec{\Omega}_{bi} \wedge \{(\vec{r}_e)_b\}_k \right) \right\} + \sum_{k=1}^{N_{nz}} \dot{m}_k [\{(\vec{r}_e)_b\}_k \wedge \{(\vec{v}_e)_b\}_k] \end{aligned} \quad (3.11)$$

where the subscript k indicates one of the nozzles (there could be up to N_{nz} nozzles)

As a consequence, using the notation indicated in 3.1.1 the variation of the angular momentum is given by:

$$\begin{aligned} \mathcal{L} = & I_x \cdot \dot{p} - J_{xz} \cdot \dot{r} + (I_z - I_y) \cdot q \cdot r - J_{xz} \cdot p \cdot q \\ & - J_{xy} \cdot \dot{q} - J_{yz} \cdot q^2 + J_{xy} \cdot r \cdot p + J_{yz} \cdot r^2 + \\ & + \dot{I}_x \cdot p - \dot{J}_{xy} \cdot q - \dot{J}_{xz} \cdot r + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k [p \{ (y_{ek})^2 + (z_{ek})^2 \} - qx_{ek}y_{ek} - rx_{ek}z_{ek}] + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k (y_{ek} \cdot w_{ek} - z_{ek} \cdot v_{ek}) \\ \\ \mathcal{M} = & I_y \cdot \dot{q} - (I_z - I_x) \cdot p \cdot r + J_{xz} \cdot (p^2 - r^2) \\ & - J_{xy} \cdot \dot{p} - J_{yz} \cdot \dot{r} - J_{xy} \cdot q \cdot r + J_{yz} \cdot p \cdot q + \\ & - \dot{J}_{xy} \cdot p + \dot{I}_y \cdot q - \dot{J}_{yz} \cdot r + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k [-px_{ek}y_{ek} + q \{ (x_{ek})^2 + (z_{ek})^2 \} - ry_{ek}z_{ek}] + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k (z_{ek} \cdot u_{ek} - x_{ek} \cdot w_{ek}) \\ \\ \mathcal{N} = & -J_{xz} \cdot \dot{p} + I_z \cdot \dot{r} - (I_x - I_y) \cdot p \cdot q + J_{xz} \cdot q \cdot r \\ & - J_{yz} \cdot \dot{q} - J_{xy} \cdot p^2 - J_{yz} \cdot p \cdot r + J_{xy} \cdot q^2 + \\ & - \dot{J}_{xz} \cdot p - \dot{J}_{yz} \cdot q + \dot{I}_z \cdot r + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k [-px_{ek}z_{ek} - qy_{ek}z_{ek} + r \{ (x_{ek})^2 + (y_{ek})^2 \}] + \\ & + \sum_{k=1}^{N_{nz}} \dot{m}_k (x_{ek} \cdot v_{ek} - y_{ek} \cdot u_{ek}) \end{aligned} \quad (3.12)$$

3.2 State vector

3.2.1 Introduction

The state vector of a system is a vector with a set of components (state variables) that completely determine the state of the system for each instant.

The state variables have to be such that once the state vector is known for a certain time t the future state of the system is known if the control vector is known. This is, the state vector has to be chosen so that it satisfies an equation for a Cauchy problem:

$$\dot{\vec{X}}(t) = \vec{f}(\vec{X}(t), \vec{u}(t)) \quad (3.13)$$

As a consequence the state vector has to be chosen carefully so that it represents completely the state of the missile for each instant t , and in order to ensure that the state of the system in the next instant $(t + dt)$ can be obtained directly from it, taking the control vector into account.

3.2.2 State vector components

In order to completely set the state of the missile for each instant we need to include the following entities in the state vector:

- Position vector of the center of mass of the missile.
- Velocity vector of the center of mass of the missile.
- Attitude of the missile.
- Angular velocity of the missile.

Having this information, the state of the missile is completely determined. We could obtain the position and velocity of any point of the missile in any instant with respect to any reference frame. This is shown in detail in section 3.2.3.

Note that the mass and the inertia tensor variables are not included in the state vector. These properties will be considered only dependent on the time since liftoff because the missiles have solid propellants for their boost phase, and this fuel is consumed in the same manner every time. This also allows obtaining every time the position of the center of mass of the missile with respect to the base of the missile.

This is not true after the third stage since in this case the missiles use liquid fuel for the post-boost phase (in the case of the ICBM) and for the EKV (in the case of the interceptor missile) so in this case the consumed fuel depends on the forces requested for the thrusters and the mass of the missile will depend on the guidance and control strategy. In this case the mass and inertia tensors will be computed each time taken into account the already consumed liquid fuel.

The chosen state vector for the missile is:

$$\vec{X}(t) = \{x \ y \ z \ u \ v \ w \ q_0 \ q_1 \ q_2 \ q_3 \ p \ q \ r\}^T(t) \quad (3.14)$$

where:

$$\bullet \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = (\vec{r}_{CM}^i)_i$$

= Position vector of the center of mass of the missile measured from the inertial reference frame in coordinates of the inertial reference frame.

$$\bullet \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = (\vec{v}_{CM}^b)_i$$

= Velocity vector of the center of mass of the missile measured from the inertial reference frame in coordinates of a body-fixed frame.

Taking into account that the equation of forces on the missile (equation 3.10) provides the velocity of the missile measured from an inertial reference frame $((\vec{v}_{CM}^b)_i)$ it seems sensible to choose the position vector also with respect to the inertial reference frame $((\vec{r}_{CM}^i)_i)$ as the representation of the position of the missile within the state vector, since this position can be obtained by a direct integration of the velocity vector.

$$\bullet \begin{Bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{Bmatrix} = q_{bi} = \text{rotation quaternion from the inertial frame to the body frame.}$$

It provides the attitude of the missile.

$$\bullet \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \vec{\Omega}_{bi}^b$$

= angular velocity vector of the missile with respect to the inertial reference frame in coordinates of a body-fixed frame.

Since the equation for the angular momentum (equation 3.12) provides the angular velocity of the missile with respect to the inertial reference frame $(\vec{\Omega}_{bi}^b)$ it will be easier to consider for the attitude of the missile the rotation quaternion from the inertial to the body frame, since it can be obtained by a direct integration using equation B.69.

3.2.3 Obtaining data from the missile state vector

The chosen state vector allows obtaining all the information about the state of the missile in any reference frame. We will show in this section how this information can be obtained.

3.2.3.1 Position of the missile

The state vector includes the position of the missile measured from the inertial reference frame $((\vec{r}_{CM}^i)_i)$.

The transformation matrix from the inertial reference frame to the ECEF reference frame can be obtained according to equation A.18 as just a function of time:

$$C_i^e = [A \cdot B \cdot C \cdot D] = f(t) \quad (3.15)$$

where all the required expressions for matrices A , B , C and D are provided in chapter A.

Having this transformation matrix and $\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = (\vec{r}_{CM}^i)_i$ it is easy to obtain the position vector of the center of mass of the missile in the ECEF reference frame as:

$$(\vec{r}_{CM}^e)_i = C_i^e \cdot (\vec{r}_{CM}^i)_i \quad (3.16)$$

where $(\vec{r}_{CM}^e)_i$ is in fact $(\vec{r}_{CM}^e)_e$ since both reference frames share the same origin.

Having $\begin{Bmatrix} x^e \\ y^e \\ z^e \end{Bmatrix} = (\vec{r}_{CM}^e)_e$ we can obtain the geodetic latitude and longitude of the center of mass of the missile applying equations A.32 and A.33 that provide $[\lambda, \varphi] = f(x^e, y^e, z^e)$.

The geodetic coordinates allow obtaining the transformation matrix C_e^n using equation A.35 ($C_e^n = f(\lambda, \varphi)$).

3.2.3.2 Attitude of the missile

The state vector includes the rotation quaternion from the inertial reference frame to the body frame $\{q_0, q_1, q_2, q_3\}^T$.

This quaternion allows obtaining directly the transformation matrix from the body reference frame to the inertial reference frame applying equation B.74 (in this case the expression is used based on a rotation quaternion from the inertial to the body reference frame q_{bi} instead of a rotation quaternion from the navigation to the body reference frame q_{bn}):

$$C_b^i = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \quad (3.17)$$

Since at this point we also have C_e^n and C_i^e we can apply the following relationship and obtain C_b^n :

$$C_b^n = (C_e^n) \cdot (C_i^e) \cdot (C_b^i) \quad (3.18)$$

Once we have C_b^n we have the rotation matrix that rotates from the navigation frame to the body frame ($R = C_b^n$). Having this transformation matrix we can apply equation B.52 and obtain the rotation quaternion from the navigation frame to the body frame (q_{bn}).

This latter quaternion is the one to be used in order to obtain the Euler angles of the missile applying equation B.76, since these angles only have sense measured from the navigation frame.

3.2.3.3 Velocity of the missile

The state vector includes the velocity vector of the center of mass as measured from the inertial reference frame ($(\vec{v}_{CM}^b)_i$).

We can obtain the velocity vector as measured from the ECEF reference frame applying equation A.19:

$$(\vec{v}_{CM}^e)_e = [A \cdot B \cdot C \cdot D] \cdot (\vec{v}_{CM}^i)_i + [A \cdot \dot{B} \cdot C \cdot D] \cdot (\vec{r}^i)_i \quad (3.19)$$

where of course we can obtain $(\vec{v}_{CM}^i)_i$ from $(\vec{v}_{CM}^b)_i$ by a simple change of coordinates:

$$(\vec{v}_{CM}^i)_i = C_b^i \cdot (\vec{v}_{CM}^b)_i \quad (3.20)$$

Once we have $(\vec{v}_{CM}^e)_e$ we can project this vector in the navigation frame obtaining the velocity vector in North-East-Down coordinates $\{v_n, v_e, v_d\}$:

$$(\vec{v}_{CM}^n)_e = C_e^n \cdot (\vec{v}_{CM}^e)_e = \begin{Bmatrix} v_n \\ v_e \\ v_d \end{Bmatrix} \quad (3.21)$$

3.2.3.4 Angular velocity of the missile

The state vector includes the angular velocity of the missile with respect to the inertial reference frame ($\vec{\Omega}_{bi}^b$).

$\vec{\omega}_{ei}^e$ can be obtained using equation C.9 as a function of time ($\vec{\omega}_{ei}^e = f(t)$).

$\vec{\omega}_{ne}^n$ can be obtained easily having $\{v_n, v_e, v_d\}$ and $[\lambda, \varphi, h]$ from equation C.33 ($\vec{\omega}_{ne}^n = f(v_e, v_n, \varphi, h)$).

Having these angular velocity vectors it is easy to obtain all the angular velocity vectors of the different reference frames with respect to each other. For example for $\vec{\omega}_{bn}^b$:

$$\begin{aligned} \vec{\Omega}_{bi}^b &= \vec{\omega}_{bn}^b + \vec{\omega}_{ne}^b + \vec{\omega}_{ei}^b \implies \vec{\omega}_{bn}^b = \vec{\Omega}_{bi}^b - \vec{\omega}_{ne}^b - \vec{\omega}_{ei}^b = \\ &= \vec{\Omega}_{bi}^b - C_n^b \cdot \vec{\omega}_{ne}^n - C_e^b \cdot \vec{\omega}_{ei}^e = \vec{\Omega}_{bi}^b - C_n^b \cdot \vec{\omega}_{ne}^n - C_n^b \cdot C_e^n \cdot \vec{\omega}_{ei}^e \end{aligned} \quad (3.22)$$

3.3 State Equations

As indicated in equation 3.13 the state vector has to be such that we can express an equation for each component in order to obtain its derivative:

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1, x_2, \dots, x_{13}; u_1, u_2, u_3) \\ \dot{x}_2(t) &= f_2(x_1, x_2, \dots, x_{13}; u_1, u_2, u_3) \\ &\vdots \\ \dot{x}_{13}(t) &= f_{13}(x_1, x_2, \dots, x_{13}; u_1, u_2, u_3)\end{aligned}\tag{3.23}$$

where $\{x_1, x_2, \dots, x_{13}\}^T = \vec{X}(t)$ is the state vector of the missile described in section 3.2.2 and $\{u_1, u_2, u_3\}^T = \vec{U}(t)$ is the control vector of the missile that will be detailed in chapter 4.

We will identify all these equations here.

3.3.1 Position

With the chosen components for the state vector the position derivative is very easy to obtain since:

$$\left(\dot{\vec{r}}_{CM}^i\right)_i = C_b^i \cdot (\vec{v}_{CM}^b)_i \tag{3.24}$$

where C_b^i is given by equation 3.17:

$$C_b^i = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \tag{3.25}$$

This is:

$$\begin{aligned}\dot{x} &= \{1 - 2(q_2^2 + q_3^2)\} \cdot u + 2(-q_0q_3 + q_1q_2) \cdot v + 2(q_0q_2 + q_1q_3) \cdot w \\ \dot{y} &= 2(q_0q_3 + q_1q_2) \cdot u + \{1 - 2(q_1^2 + q_3^2)\} \cdot v + 2(-q_0q_1 + q_2q_3) \cdot w \\ \dot{z} &= 2(-q_0q_2 + q_1q_3) \cdot u + 2(q_0q_1 + q_2q_3) \cdot v + \{1 - 2(q_1^2 + q_2^2)\} \cdot w\end{aligned}\tag{3.26}$$

3.3.2 Velocity vector

The derivatives of the velocity vector are directly given by the equation of forces on the missile (equation 3.10):

$$\begin{aligned}\dot{u} &= \frac{F_x}{M(t)} + r \cdot v - q \cdot w - \frac{2}{M(t)} \sum_{k=1}^{N_{nz}} \dot{m}_k (q \cdot z_{ek} - r \cdot y_{ek}) \\ \dot{v} &= \frac{F_y}{M(t)} - r \cdot u + p \cdot w - \frac{2}{M(t)} \sum_{k=1}^{N_{nz}} \dot{m}_k (r \cdot x_{ek} - p \cdot z_{ek}) \\ \dot{w} &= \frac{F_z}{M(t)} + q \cdot u - p \cdot v - \frac{2}{M(t)} \sum_{k=1}^{N_{nz}} \dot{m}_k (p \cdot y_{ek} - q \cdot x_{ek})\end{aligned}\tag{3.27}$$

3.3.3 Attitude

It has to be noticed that the chosen quaternion provides the rotation from the inertial to the body reference frame.

Since the angular velocity of the body frame with respect to the inertial frame (in coordinates of the body frame) is part of the state vector ($\{ p, q, r \}^T = \vec{\Omega}_{bi}^b$) we can obtain the derivatives of this quaternion using equation B.69:

$$\begin{aligned}\dot{q}_0 &= \frac{1}{2}(-p \cdot q_1 - q \cdot q_2 - r \cdot q_3) \\ \dot{q}_1 &= \frac{1}{2}(p \cdot q_0 + r \cdot q_2 - q \cdot q_3) \\ \dot{q}_2 &= \frac{1}{2}(q \cdot q_0 - r \cdot q_1 + p \cdot q_3) \\ \dot{q}_3 &= \frac{1}{2}(r \cdot q_0 + q \cdot q_1 - p \cdot q_2)\end{aligned}\tag{3.28}$$

3.3.4 Angular velocity vector

The derivatives of the angular velocity vector will be provided by the equation for the angular momentum (equation 3.12).

However, since in this equation the values of $\dot{p}$, $\dot{q}$ and $\dot{r}$ are mixed some operations are needed in order to isolate these variables.

Equation 3.12 can be re-written as:

$$\begin{aligned}I_x \cdot \dot{p} - J_{xy} \cdot \dot{q} - J_{xz} \cdot \dot{r} &= \mathcal{L} - (I_z - I_y) \cdot q \cdot r + J_{xz} \cdot p \cdot q \\ &+ J_{yz} \cdot q^2 - J_{xy} \cdot r \cdot p - J_{yz} \cdot r^2 - \dot{I}_x \cdot p + \dot{J}_{xy} \cdot q + \dot{J}_{xz} \cdot r + \\ &- \sum_{k=1}^{N_{nz}} \dot{m}_k [p \{ (y_{ek})^2 + (z_{ek})^2 \} - qx_{ek}y_{ek} - rx_{ek}z_{ek}] + \\ &- \sum_{k=1}^{N_{nz}} \dot{m}_k (y_{ek} \cdot w_{ek} - z_{ek} \cdot v_{ek}) = A_1 \\ -J_{xy} \cdot \dot{p} + I_y \cdot \dot{q} - J_{yz} \cdot \dot{r} &= \mathcal{M} + (I_z - I_x) \cdot p \cdot r - J_{xz} \cdot (p^2 - r^2) \\ &+ J_{xy} \cdot q \cdot r - J_{yz} \cdot p \cdot q + \dot{J}_{xy} \cdot p - \dot{I}_y \cdot q + \dot{J}_{yz} \cdot r + \\ &- \sum_{k=1}^{N_{nz}} \dot{m}_k [-px_{ek}y_{ek} + q(\{x_{ek}\}^2 + \{z_{ek}\}^2) - ry_{ek}z_{ek}] + \\ &- \sum_{k=1}^{N_{nz}} \dot{m}_k (z_{ek} \cdot u_{ek} - x_{ek} \cdot w_{ek}) = A_2 \\ -J_{xz} \cdot \dot{p} - J_{yz} \cdot \dot{q} + I_z \cdot \dot{r} &= \mathcal{N} + (I_x - I_y) \cdot p \cdot q - J_{xz} \cdot q \cdot r \\ &+ J_{xy} \cdot p^2 + J_{yz} \cdot p \cdot r - J_{xy} \cdot q^2 + \dot{J}_{xz} \cdot p + \dot{J}_{yz} \cdot q - \dot{I}_z \cdot r + \\ &- \sum_{k=1}^{N_{nz}} \dot{m}_k [-px_{ek}z_{ek} - qy_{ek}z_{ek} + r \{ (x_{ek})^2 + (y_{ek})^2 \}] + \\ &- \sum_{k=1}^{N_{nz}} \dot{m}_k (x_{ek} \cdot v_{ek} - y_{ek} \cdot u_{ek}) = A_3\end{aligned}\tag{3.29}$$

This is:

$$\begin{bmatrix} I_x & -J_{xy} & -J_{xz} \\ -J_{xy} & I_y & -J_{yz} \\ -J_{xz} & -J_{yz} & I_z \end{bmatrix} \cdot \begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \quad (3.30)$$

As a consequence the derivatives of the angular velocity vector will be provided by:

$$\begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = (I_{CM}^b)^{-1} \cdot \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \quad (3.31)$$

with:

$$\begin{aligned} A_1 &= \mathcal{L} - (I_z - I_y) \cdot q \cdot r + J_{xz} \cdot p \cdot q + J_{yz} \cdot q^2 - J_{xy} \cdot r \cdot p - J_{yz} \cdot r^2 + \\ &\quad - \dot{I}_x \cdot p + \dot{J}_{xy} \cdot q + \dot{J}_{xz} \cdot r + \\ &\quad - \sum_{k=1}^{N_{nz}} \dot{m}_k [p \{ (y_{ek})^2 + (z_{ek})^2 \} - qx_{ek}y_{ek} - rx_{ek}z_{ek}] - \sum_{k=1}^{N_{nz}} \dot{m}_k (y_{ek} \cdot w_{ek} - z_{ek} \cdot v_{ek}) \\ A_2 &= \mathcal{M} + (I_z - I_x) \cdot p \cdot r - J_{xz} \cdot (p^2 - r^2) + J_{xy} \cdot q \cdot r - J_{yz} \cdot p \cdot q + \\ &\quad + \dot{J}_{xy} \cdot p - \dot{I}_y \cdot q + \dot{J}_{yz} \cdot r + \\ &\quad - \sum_{k=1}^{N_{nz}} \dot{m}_k [-px_{ek}y_{ek} + q \{ (x_{ek})^2 + (z_{ek})^2 \} - ry_{ek}z_{ek}] - \sum_{k=1}^{N_{nz}} \dot{m}_k (z_{ek} \cdot u_{ek} - x_{ek} \cdot w_{ek}) \\ A_3 &= \mathcal{N} + (I_x - I_y) \cdot p \cdot q - J_{xz} \cdot q \cdot r + J_{xy} \cdot p^2 + J_{yz} \cdot p \cdot r - J_{xy} \cdot q^2 + \\ &\quad + \dot{J}_{xz} \cdot p + \dot{J}_{yz} \cdot q - \dot{I}_z \cdot r + \\ &\quad - \sum_{k=1}^{N_{nz}} \dot{m}_k [-px_{ek}z_{ek} - qy_{ek}z_{ek} + r \{ (x_{ek})^2 + (y_{ek})^2 \}] - \sum_{k=1}^{N_{nz}} \dot{m}_k (x_{ek} \cdot v_{ek} - y_{ek} \cdot u_{ek}) \end{aligned} \quad (3.32)$$

3.3.5 Compilation of equations

We can collect all these equations together to get the system state equations. The non-linearity of the equations is easy to observe:

$$\begin{aligned}
\dot{x} &= \{1 - 2(q_2^2 + q_3^2)\} \cdot u + 2(-q_0q_3 + q_1q_2) \cdot v + 2(q_0q_2 + q_1q_3) \cdot w \\
\dot{y} &= 2(q_0q_3 + q_1q_2) \cdot u + \{1 - 2(q_1^2 + q_3^2)\} \cdot v + 2(-q_0q_1 + q_2q_3) \cdot w \\
\dot{z} &= 2(-q_0q_2 + q_1q_3) \cdot u + 2(q_0q_1 + q_2q_3) \cdot v + \{1 - 2(q_1^2 + q_2^2)\} \cdot w \\
\dot{u} &= \frac{F_x}{M(t)} + r \cdot v - q \cdot w - \frac{2}{M(t)} \sum_{k=1}^{N_{nz}} \dot{m}_k (q \cdot z_{ek} - r \cdot y_{ek}) \\
\dot{v} &= \frac{F_y}{M(t)} - r \cdot u + p \cdot w - \frac{2}{M(t)} \sum_{k=1}^{N_{nz}} \dot{m}_k (r \cdot x_{ek} - p \cdot z_{ek}) \\
\dot{w} &= \frac{F_z}{M(t)} + q \cdot u - p \cdot v - \frac{2}{M(t)} \sum_{k=1}^{N_{nz}} \dot{m}_k (p \cdot y_{ek} - q \cdot x_{ek}) \\
\dot{q}_0 &= \frac{1}{2} (-p \cdot q_1 - q \cdot q_2 - r \cdot q_3) \\
\dot{q}_1 &= \frac{1}{2} (p \cdot q_0 + r \cdot q_2 - q \cdot q_3) \\
\dot{q}_2 &= \frac{1}{2} (q \cdot q_0 - r \cdot q_1 + p \cdot q_3) \\
\dot{q}_3 &= \frac{1}{2} (r \cdot q_0 + q \cdot q_1 - p \cdot q_2) \\
\dot{p} &= \left\{ (I_{CM}^b)^{-1} \right\}_{11} \cdot A_1 + \left\{ (I_{CM}^b)^{-1} \right\}_{12} \cdot A_2 + \left\{ (I_{CM}^b)^{-1} \right\}_{13} \cdot A_3 \\
\dot{q} &= \left\{ (I_{CM}^b)^{-1} \right\}_{21} \cdot A_1 + \left\{ (I_{CM}^b)^{-1} \right\}_{22} \cdot A_2 + \left\{ (I_{CM}^b)^{-1} \right\}_{23} \cdot A_3 \\
\dot{r} &= \left\{ (I_{CM}^b)^{-1} \right\}_{31} \cdot A_1 + \left\{ (I_{CM}^b)^{-1} \right\}_{32} \cdot A_2 + \left\{ (I_{CM}^b)^{-1} \right\}_{33} \cdot A_3
\end{aligned} \tag{3.33}$$

3.3.6 Linearization of the equations

A linearization of the equations 3.33 will be necessary for several purposes:

- To analyse possible divergence problems (see section 3.4.2.2.3)
- To be able to use linear methods on the system (see chapter 10)

The procedure basically consists in performing a differentiation of the state equation with respect to the state and control variables in the present position:

$$\begin{aligned}
\dot{\vec{X}}(t) &= \vec{f}(\vec{X}(t), \vec{u}(t)) \implies \\
\frac{d(\vec{X} + \overrightarrow{\Delta X})}{dt}(t) &= \vec{f}(\vec{X}(t), \vec{u}(t)) \Big|_{\vec{X}, \vec{u}} + \frac{\partial \vec{f}}{\partial \vec{X}} \Big|_{\vec{X}, \vec{u}} \cdot \overrightarrow{\Delta X}(t) + \frac{\partial \vec{f}}{\partial \vec{u}} \Big|_{\vec{X}, \vec{u}} \cdot \overrightarrow{\Delta u}(t) \implies \\
\overrightarrow{\Delta \dot{X}}(t) &= A \cdot \overrightarrow{\Delta X}(t) + B \cdot \overrightarrow{\Delta u}(t)
\end{aligned} \tag{3.34}$$

with:

$$A = \frac{\partial \vec{f}}{\partial \vec{X}} \Big|_{\vec{X}, \vec{u}} \quad B = \frac{\partial \vec{f}}{\partial \vec{u}} \Big|_{\vec{X}, \vec{u}} \tag{3.35}$$

3.4 Characteristics of the equations to be solved

3.4.1 Problem to be solved

Equation 3.33 is a system of ordinary differential equations with an initial value which can be expressed as:

$$\begin{aligned} \frac{d\vec{X}}{dt} &= \vec{f}(\vec{X}; t) \\ \vec{X} &= \vec{X}_0 \text{ in } t=t_0 \end{aligned} \quad (3.36)$$

This initial value problem (IVP) of ordinary differential equations (ODE) is usually known as a Cauchy problem.

3.4.2 Behaviour of the problem

3.4.2.1 Well-Posedness

A Cauchy problem is well posed ("correctly set") in the sense defined by Jacques Hadamard (see [1]) if:

- A solution for the problem exists
- The solution is unique
- The solution's behavior changes in a continuous way when the initial conditions vary.

The existence and unicity of a solution in the problem 3.36 is ensured if the function $\vec{f}(\vec{X}; t)$ fulfills the global condition of Lipschitz according to the Picard-Lindelöf theorem (see §3.2 in [2]). This happens (see again §3.2 in [2]) if the function $\vec{f}(\vec{X}; t)$ is continuous and has a continuous derivative ($\vec{f}(\vec{X}; t) \in \mathcal{C}^1$).

It can be easily observed that the system of equations to be solved (3.33) is continuous and with a continuous derivative ($\vec{f}(\vec{X}; t) \in \mathcal{C}^1$), which consequently ensures that it is a well posed problem.

3.4.2.2 Stability

3.4.2.2.1 Definition of stability of ordinary differential equations

The solution $\vec{X}(t)$ of a well posed Cauchy problem defined in $[t_0, \infty)$ is stable according to the Lyapunov definition (see [3] (A.M.Lyapunov, 1992)) if $\forall \epsilon > 0$ exists a parameter $\delta > 0$ such that the solution of the Cauchy problem:

$$\begin{aligned} \frac{d\vec{X}^*}{dt} &= \vec{f}(\vec{X}^*, t) \\ \vec{X}^*(t_0) &= \vec{X}_0^* \text{ with } \|\vec{X}_0 - \vec{X}_0^*\| < \delta \end{aligned}$$

exists, it is defined in $[t_0, \infty)$ and fulfills that $\|\vec{X}(t) - \vec{X}^*(t)\| < \epsilon \forall t \geq t_0$

If $\lim_{t \rightarrow \infty} \|\vec{X}(t) - \vec{X}^*(t)\| \rightarrow 0$ with $t \rightarrow \infty$ then $\vec{X}(t)$ is said to be asymptotically stable.

3.4.2.2.2 Stability in a linearized system of ODE

If we linearize the system of ordinary differential equations 3.36 about a solution $\vec{X}_s(t)$:

$$\begin{aligned}\vec{X}(t) &= \vec{X}_s(t) + \overrightarrow{\Delta X}(t) \\ \Rightarrow \frac{d\overrightarrow{\Delta X}}{dt}(t) &= \frac{\partial \vec{f}}{\partial \vec{X}}(\vec{X}_s(t), t) \cdot \overrightarrow{\Delta X} + \vec{h}(\overrightarrow{\Delta X}, t)\end{aligned}\quad (3.37)$$

where $\frac{\partial \vec{f}}{\partial \vec{X}}(\vec{X}_s(t), t)$ is the Jacobian of $\vec{f}(\vec{X}, t)$ with respect to $\vec{X}$ particularized in $\vec{X}_s(t)$ and $\vec{h}(\overrightarrow{\Delta X}, t)$ contains the non linear part of the problem, the stability of $\vec{X}_s(t)$ in 3.37 will be given by the stability of the solution $\overrightarrow{\Delta X}(t) = \vec{0}$.

If we consider that the Jacobian of $\vec{f}(\vec{X}, t)$ is locally constant ($\frac{\partial \vec{f}}{\partial \vec{X}}(\vec{X}_s(t), t) = L$), it can be proven (see §3.3 in [2]) that:

1. If all the eigenvalues λ_k of L have a negative real part, then $\|\overrightarrow{\Delta X}(t)\| \rightarrow 0$ in the problem 3.37 with $t \rightarrow \infty$. This is, the system is asymptotically stable.
2. If all the eigenvalues λ_k of L have a negative or null real part, and those with null real part have a geometric multiplicity equal to the algebraic multiplicity, then the system is stable.
3. In any other case, the system is unstable.

3.4.2.2.3 Stability of the missile equations

The system of equations to be solved (3.33) is not stable for all the system variables since the control forces are not always applied (see section 4.5) and even when they are applied not all the variables are controlled. As a consequence the system 3.33 has null eigenvalues and eigenvalues with a positive real part.

In order to be able to analyse the stability of the system we will linearize the system of equations (3.33) each time step about the present solution and obtain the eigenvalues of the Jacobian matrix.

Note that the system of equations 3.33 includes control forces ($\vec{u}$) that also depend on the state variables through the applied control equations (see section 4.5). As a consequence and when the control forces are active the linearization of the system of equations with respect to the state variables $\vec{X}$ will be expressed in the following way:

$$\dot{\vec{X}} = \vec{f}(\vec{X}, \vec{u}) \Rightarrow \Delta \dot{\vec{X}} = \frac{\partial \vec{f}}{\partial \vec{X}} \cdot \Delta \vec{X} + \frac{\partial \vec{f}}{\partial \vec{u}} \cdot \Delta \vec{u} = \frac{\partial \vec{f}}{\partial \vec{X}} \cdot \Delta \vec{X} + \frac{\partial \vec{f}}{\partial \vec{u}} \cdot \frac{\partial \vec{u}}{\partial \vec{X}} \cdot \Delta \vec{X} \quad (3.38)$$

Since the matrices A , B have been defined as (see section 3.3.6):

$$A = \frac{\partial \vec{f}}{\partial \vec{X}} \quad B = \frac{\partial \vec{f}}{\partial \vec{u}} \quad (3.39)$$

the eigenvalues that we need to compute are the ones related to the matrices:

$$\begin{aligned}A &\text{ when the control forces are not applied} \\ A + B \cdot \frac{\partial \vec{u}}{\partial \vec{X}} &\text{ when the control forces are applied}\end{aligned}\quad (3.40)$$

These expressions will be computed each time step during the simulation.

3.5 Numerical method for the integration of the equations

This section focuses on the selection of the numerical method to be used for the numerical integration of the equations of motion of the missiles.

A special attention has been given to this issue since the involved times in this problem are very large (the time of flight is around half an hour) so we need to choose a method that allows using a big time step (Δt) while maintaining good stability properties.

3.5.1 Characteristics of the numerical methods

In order to properly choose the most adequate numerical method for the problem we have to identify the characteristics of these methods. These characteristics will then be compared among the different methods.

3.5.1.1 Order of a numerical method

The following general expression can be used for most of the numerical methods, linear and non linear:

$$\sum_{j=0}^p \alpha_j \bar{u}^{n+1-j} = \Delta t H(\bar{u}^{n+1}, \dots, \bar{u}^{n+1-p}, t_n; \Delta t) \quad (3.41)$$

where:

- p is the number of steps involved in the method.
- Δt is the considered time step
- H is a function that depends on $\bar{u}^{n+1}, \dots, \bar{u}^{n+1-p}$ through the values of $\vec{F}(\bar{u})$ in these time steps.

The local truncation error in the step n is defined as the residue of the general expression 3.41 when instead of the approximation $\bar{u}^{n+1-j}$ the exact value $\bar{u}(t_{n+1-j})$ is used:

$$T^{n+1} = \sum_{j=0}^p \alpha_j \bar{u}(t_{n+1-j}) - \Delta t H(\bar{u}(t_{n+1}), \dots, \bar{u}(t_{n+1-p}), t_n; \Delta t) \quad (3.42)$$

If we develop $\bar{u}$ as a Taylor series and substitute it in equation 3.42 we will reach an expression like:

$$T^{n+1} = \mathcal{O}(\Delta t^{q+1}) \quad (3.43)$$

q is defined as the order of the numerical method, and indicates the local error in each step. Obviously and since Δt is small (< 1), the greater the order of the numerical method, the smaller the local truncation error.

3.5.1.2 Stability of numerical methods

Some general stability properties that can be required to a numerical method, such as consistency (see §3.6 in [2]), zero-stability (see also §3.6 in [2]), or convergence (ensured if the 2 previous conditions are met as shown in [4]), are already fulfilled if we choose a well-known conventional numerical method, so these requirements will not be used to select the integration method.

However, the stability characteristics of each numerical method can be compared in terms of its absolute stability, which will be different for each of them. As a consequence this will be the characteristic to analyse.

3.5.1.2.1 Absolute stability

A numerical method is said to be *absolutely stable* (see [4]) if when applied to the linear problem:

$$\begin{aligned} \frac{d\vec{u}}{dt} &= A\vec{u} \\ \vec{u} &= \vec{u}_0 \text{ in } t=t_0 \end{aligned} \quad (3.44)$$

it provides a solution such that:

$$\|\vec{u}^n\| \rightarrow \vec{0} \text{ when } n \rightarrow \infty \quad (3.45)$$

for any initial condition and time step Δt , where A is an $m \times m$ non singular matrix with eigenvalues with a negative real part.

The requirement for the eigenvalues of A is necessary in order for the original problem to tend to $\vec{0}$ (see section 3.4.2.2.2).

If we consider that A is diagonalizable then there exists a non singular matrix Q such that:

$$Q^{-1} \cdot A \cdot Q = \Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ & \dots & \\ 0 & \dots & \lambda_m \end{bmatrix} \quad (3.46)$$

and we can diagonalize the original problem with the change of basis $\vec{u} = Q \cdot \vec{v}$ obtaining:

$$\frac{d\vec{v}}{dt} = \Lambda \vec{v} \quad (3.47)$$

For this latter problem and considering the general equation of a linear method (see §1.4 in [2]) we get:

$$\sum_{j=0}^k \alpha_j \vec{v}^{n+1-j} = \Delta t \Lambda \sum_{j=0}^p \beta_j \vec{v}^{n+1-j} \quad (3.48)$$

It can be proven (see [2]) that a similar equation applies to the Runge Kutta and Predictor Corrector non linear methods).

Equation 3.48 can be expressed as m linear scalar equations that would result each of them from a scalar original problem ($\frac{dv}{dt} = \lambda_i v$ with $i = 1 \dots m$):

$$\sum_{j=0}^p \alpha_j v^{n+1-j} = \Delta t \lambda_i \sum_{j=0}^p \beta_j v^{n+1-j} \Rightarrow \sum_{j=0}^p \alpha_j v^{n+1-j} = \omega \sum_{j=0}^p \beta_j v^{n+1-j} \quad (3.49)$$

with $\omega = \Delta t \lambda_i$

The stability polynomial of the numerical system is defined as the polynomial obtained when solutions of the form $v^n = C \cdot r^n$ are used in equation 3.49:

$$\pi(r) = \sum_{j=0}^p (\alpha_j - \omega \beta_j) r^{p-j} \quad (3.50)$$

The roots of $\pi(r)$ will then provide the complete solution for v^n :

$$v^n = \sum_{\forall \alpha} \left(\sum_{k=1}^{s_\alpha} C_{\alpha k} \cdot n^{k-1} r_\alpha^n \right) + \sum_{\forall \beta} C_\beta r_\beta^n \quad (3.51)$$

where r_α is a root of $\pi(r) = 0$ with algebraic multiplicity $s_\alpha > 1$ and r_β is a root of $\pi(r) = 0$ with algebraic multiplicity 1.

From the definition of absolute stability it can be concluded observing equation 3.51 that a numerical method will be absolutely stable for a certain Δt if all the roots of the stability polynomial have a modulus smaller than one:

$$\begin{aligned} & \text{if } \forall r \text{ such that } \pi(r) = 0 \quad \|r\| < 1 \\ & \Rightarrow \text{numerical method is absolutely stable} \end{aligned} \quad (3.52)$$

There are not many numerical methods that are absolutely stable for all time steps Δt and for all eigenvalues λ with negative real part, and for this reason regions of absolute stability have to be considered (regions in the bidimensional space of $\omega = \Delta t \lambda$ where the condition 3.52 is fulfilled).

3.5.1.2.2 Regions of absolute stability

A *region of absolute stability* $\mathcal{R}_A$ is defined as the area in the bidimensional space of $\omega = \lambda\Delta t$ ($Re(\omega) \perp Im(\omega)$) where the numerical method provides a solution for equation 3.49 such that $v^n \rightarrow 0$ when $n \rightarrow \infty$ for any initial condition.

The limits of the region are given by the curve where the solutions are bounded ($\|r\| = 1$) which can be obtained using $r = e^{i\theta}$ in equation 3.50:

$$\sum_{j=0}^p \alpha_j e^{i(p-j)\theta} = \omega \sum_{j=0}^p \beta_j e^{i(p-j)\theta} \Rightarrow \omega = \frac{\sum_{j=0}^p \alpha_j e^{i(p-j)\theta}}{\sum_{j=0}^p \beta_j e^{i(p-j)\theta}} \quad (3.53)$$

This drawing provides the limits of the area in which all the eigenvalues of the matrix A multiplied by Δt have to be in order for the numerical method to be absolutely stable.

The regions of absolute stability $\mathcal{R}_A$ of the different numerical methods can be compared in order to decide whether they are adequate for solving the original problem 3.33. The greater the region and the closer it gets to the axis $Re(\lambda) = 0$, the better.

3.5.2 Considered numerical methods

The considered numerical methods will be briefly detailed herein indicating their formulation and their order for comparison.

The formulation will be given for a scalar problem for the sake of simplicity. The adaptation to a vectorial problem is straightforward.

The considered numerical methods are:

1. Adams Bashforth 2

This is a linear explicit numerical method involving 2 time steps:

$$\vec{u}^{n+1} = \vec{u}^n + \frac{\Delta t}{2} [3\vec{F}^n - \vec{F}^{n-1}] \quad (3.54)$$

The order of this method is 2: $T_{AdamsBashforth2}^{n+1} = \mathcal{O}(\Delta t^3)$.

2. Adams Bashforth 3

This is a linear explicit numerical method involving 3 time steps:

$$u^{n+1} = u^n + \frac{\Delta t}{12} [23F^n - 16F^{n-1} + 5F^{n-2}] \quad (3.55)$$

The order of this method is 3: $T_{AdamsBashforth3}^{n+1} = \mathcal{O}(\Delta t^4)$.

3. Adams Moulton 2

This method is also called Crank Nicolson or trapezoidal.

It is a linear implicit numerical method with a single step:

$$u^{n+1} = u^n + \frac{\Delta t}{2} [F^{n+1} + F^n] \quad (3.56)$$

The order of this method is 2: $T_{AdamsMoulton2}^{n+1} = \mathcal{O}(\Delta t^3)$.

4. Adams Moulton 3

This is a linear implicit numerical method involving 2 time steps:

$$u^{n+1} = u^n + \frac{\Delta t}{12} [5F^{n+1} + 8F^n - F^{n-1}] \quad (3.57)$$

The order of this method is 3: $T_{AdamsMoulton3}^{n+1} = \mathcal{O}(\Delta t^4)$.

5. Runge Kutta 2

This is a nonlinear explicit numerical method involving 2 time steps. There is an infinite number of possible Runge Kutta 2 methods, depending on the used intermediate point. We will consider here the method that uses the midpoint:

$$\begin{aligned} u^{n+1} &= u^n + \frac{\Delta t}{2} [k_1 + k_2] \\ k_1 &= F^n \\ k_2 &= F(u^n + \frac{\Delta t}{2} k_1; t_n + \frac{\Delta t}{2}) \end{aligned} \quad (3.58)$$

The order of this method is 2: $T_{RK2}^{n+1} = \mathcal{O}(\Delta t^3)$.

6. Runge Kutta 3

This is a nonlinear explicit numerical method involving 3 time steps.

There is an infinite number of possible Runge Kutta 3 methods, depending on the used intermediate points. We will consider here the following formulation:

$$\begin{aligned} u^{n+1} &= u^n + \frac{\Delta t}{6} [k_1 + 4 \cdot k_2 + k_3] \\ k_1 &= F^n \\ k_2 &= F(u^n + \frac{\Delta t}{2} k_1; t_n + \frac{\Delta t}{2}) \\ k_3 &= F(u^n - \Delta t k_1 + 2\Delta t k_2; t_n + \Delta t) \end{aligned} \quad (3.59)$$

The order of this method is 3: $T_{RK3}^{n+1} = \mathcal{O}(\Delta t^4)$.

7. Runge Kutta 4

This is a nonlinear explicit numerical method involving 4 time steps.

There is an infinite number of possible Runge Kutta 4 methods, depending on the used intermediate points. We will consider here the following formulation (known as *classical* Runge Kutta method):

$$\begin{aligned} u^{n+1} &= u^n + \frac{\Delta t}{6} [k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4] \\ k_1 &= F^n \\ k_2 &= F(u^n + \frac{\Delta t}{2} k_1; t_n + \frac{\Delta t}{2}) \\ k_3 &= F(u^n + \frac{\Delta t}{2} k_2; t_n + \frac{\Delta t}{2}) \\ k_4 &= F(u^n + \Delta t k_3; t_n + \Delta t) \end{aligned} \quad (3.60)$$

The order of this method is 4: $T_{RK4}^{n+1} = \mathcal{O}(\Delta t^5)$.

8. Predictor-Corrector methods

There are many Predictor-Corrector methods. We will only consider 2 methods that have been recommended among the PECE methods (combination of Adams Bashforth (AB) as predictor and Adams Moulton (AM) as corrector).

(a) PECE: AB1-AM2

This is a linear implicit numerical method composed of an Euler (P: prediction) and a Crank Nicolson (C: correction):

$$\begin{aligned} P : u^{n+1} &= u^n + \Delta t \cdot F^n \\ C : u^{n+1} &= u^n + \frac{\Delta t}{2} [F^{n+1} + F^n] \end{aligned} \quad (3.61)$$

The order of this method is 2: $T_{AB1-AM2}^{n+1} = \mathcal{O}(\Delta t^3)$.

(b) PECE: AB2-AM3

This is a linear implicit numerical method composed of an Adams Bashforth 2 (P: prediction) and an Adams Moulton 3 (C: correction):

$$\begin{aligned} P : \vec{u}^{n+1} &= \vec{u}^n + \frac{\Delta t}{2} [3\vec{F}^n - \vec{F}^{n-1}] \\ C : u^{n+1} &= u^n + \frac{\Delta t}{12} [5F^{n+1} + 8F^n - F^{n-1}] \end{aligned} \quad (3.62)$$

The order of this method is 3: $T_{AB2-AM3}^{n+1} = \mathcal{O}(\Delta t^4)$.

3.5.3 Chosen numerical method

The used criteria for the selection of the numerical method have been:

1. High numerical order

We want a numerical method with a low local truncation error, this is, a high numerical order.

An order of 3 has been chosen as a minimum requirement.

This means that several methods have to be discarded: Crank Nicolson, Adams Bashforth 2, Runge Kutta 2 and the predictor corrector method PECE AB1-AM2.

2. Good stability

We want a numerical method with good stability properties. Based on this we will discard numerical methods whose absolute regions do not include any eigenvalues with a positive real part, even small, in order to ensure that the method does not lead to unstable solutions when the real case includes eigenvalues with a real negative part but close to 0.

This leaves out Adams Moulton 2, Adams Moulton 3, Adams Bashforth 2, Runge Kutta 2 and the predictor corrector method PECE AB1-AM2.

3. Simplicity

As much as possible, we will discard numerical methods whose implementation is too difficult. Because of this we will discard the Runge Kutta methods that require computation in intermediate points.

The previous criteria leave only 2 possibilities among the considered numerical methods: the Adams-Bashforth 3 method and the PECE AB2-AM3 method. We will choose the PECE AB2-AM3 method because it has a bigger absolute stability region, even though it is more complex to implement.

Figure 3.1: Region of absolute stability of the $AB2 - AM3$ method

Figure 3.2: Region of absolute stability of the Adams Bashforth 3 method

3.5.4 Selection of the time step

Since, as indicated in section 3.4.2.2.2, we will compute the eigenvalues of the linearized real problem at all times, we will check that the product of all the eigenvalues with a negative real part times the time step is always within the absolute stability region of the AB2-AM3 method (figure 3.1) and decrease Δt temporarily if that is not the case.

With this method we will be able to integrate the equations 3.33 with a reasonably big Δt that allows a simulation time not too large for the considered trajectory times (up to 1 hour) while maintaining the stability for those situations (especially related to the aerodynamic forces and turn manoeuvres) in which the stability properties of the original problem are not so good, by reducing only then Δt .

Chapter 3 references

- [1] Jacques Hadamard. *Lectures on Cauchy's Problem in Linear Partial Differential Equations*. Yale University Press, New Haven, 1923. 58
- [2] Juan A. Hernández. *Cálculo Numérico en Ecuaciones Diferenciales Ordinarias*. Aula Documental de Investigación, Madrid, first edition, 2001. 58, 59, 61
- [3] A.M.Lyapunov. The general problem of the stability of motion. *International Journal of Control*, 55:531–773, March 1992. doi: 10.1080/00207179208934253A.M.LYAPUNOV. 58
- [4] J.D. Lambert. *Numerical Methods for Ordinary Differential Systems. The Initial Value Problem*. John Wiley & Sons, Baffins Lane, Chichester, England, 1991. 61
- [5] M. Prieto-Alberca. Curso de Mecánica Racional, Dinámica printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, November 1991.
- [6] Miguel Ángel Gómez Tierno, Manuel Pérez Cortés, and César Puentes Márquez. *Mecánica del Vuelo*. Instituto Universitario de Microgravedad Ignacio da Riva (IDR/UPM), Madrid, first edition, 2009.
- [7] Katsuhiko Ogata. *Modern Control Engineering*. Prentice-Hall International Inc., Upper Saddle River, New Jersey, third edition, 1997.
- [8] Jorgen Sand and Ole Osterby. *Regions of Absolute Stability*. Aarhus University, Ny Munkegade, Aarhus, Denmark, September 1979.
- [9] Michelle L.Ghrist, Jonah A. Reeger, and Bengt Fornberg. Stability Ordinates of Adams Predictor-Corrector Methods. http://amath.colorado.edu/faculty/fornberg/Docs/2012_GRF_AB_PC_stability_BIT_submitted.pdf, October 2014. [web page accessed on 27/10/2014].
- [10] Won Y. Yang, Wenwu Cao, Tae-Sang Chung, and John Morris. *Applied Numerical Methods Using MATLAB*. John Wiley & Sons, Hoboken, New Jersey, 2005.

Chapter 4

Forces and moments acting on the missile

The different forces and moments acting on the missile will be detailed herein and models will be provided for their simulation.

4.1 Introduction

In a generic way, the forces and moments acting on a missile are due to:

- Gravity:
 - Earth’s gravity
 - Lunar Gravity
 - Solar Gravity
 - Gravity from other planets
(Especially Venus, Jupiter and Mars).

- Aerodynamic forces
- Thrust
- Control forces
- Solar radiation

Solar radiation pressure (SRP) is the force due to the interaction of incident solar electromagnetic radiation with the surface of the spacecraft.

- Thermal forces
 - Thermal re-radiation

Insolation of the missile surface is partially absorbed and converted to heat that can be emitted. If the emission is anisotropic over the surface of the missile there will be a net acceleration due to the reactive force of the heat leaving the structure.

- Albedo

Albedo is the proportion of insolation that is reflected back from the Earth and includes the thermal radiation emitted by the Earth. Its effect in terms of force is also due to thermal re-radiation

- Magnetic torques

Reference [1] analyses the effects of magnetic torques and induced magnetic torques on satellites and indicates an order of magnitude of $0.1 \text{ N} \cdot \text{m}$ (for the Explorer XI satellite). Taking this value into account and because of the difficulty to simulate this moment, the magnetic torques will be neglected.

Reference [2] compiles the effect of different forces acting on a GLONASS satellite in a certain time and position and we can observe (see table 4.1) that most of them are much smaller than the Earth gravity U2.0 term.

Table 4.1: Force on a GLONASS satellite according to [2]

Force	acceleration magnitude (m/s^2)
Earth gravity monopole	6.14e-01
Earth gravity U2,0 ^a	1e-04
Lunar gravity	3.89e-06
Solar gravity	1.04e-06
Solar radiation pressure	7.18e-08
Earth gravity U4,0 ^b	1.06e-08
Thermal re-radiation	1.43e-09
Albedo	1.50e-09
Solid Earth tide (Lunar)	1.34e-09
Solid Earth tide (Solar)	4.50e-10
Earth gravity U6,0 ^c	2.37e-10
Planetary gravity (Venus)	1.15e-10
Earth gravity U7,0 ^d	4.33e-11
Planetary gravity (Jupiter)	1.31e-11
Planetary gravity (Mars)	2.70e-12

^aRelated to the gravitational potential with $\ell = 2$, $m = 0$ (see appendix section E.1)

^bRelated to the gravitational potential with $\ell = 4$, $m = 0$ (see appendix section E.1)

^cRelated to the gravitational potential with $\ell = 6$, $m = 0$ (see appendix section E.1)

^dRelated to the gravitational potential with $\ell = 7$, $m = 0$ (see appendix section E.1)

Taking into account the orders of magnitude in table 4.1 we will only take into account the following contributions for the forces and moments acting on the missile:

- Earth's gravity
- Aerodynamic forces
- Thrust
- Control forces

Expressions for these forces and moments will be obtained within this chapter.

4.2 Gravity force and moment

4.2.1 Force of gravity

Ideally, the force of gravity should be computed from the gravitational potential V (see appendix chapter E.1) using as many harmonic terms as possible, this is, using equation E.33.

However, this equation is very demanding from a computational point of view. For the purpose of this thesis a simplification will be used and it will be considered that the force of gravity can be approximated by the normal gravity field (see equation E.39) imposing $\vec{\omega} = \vec{0}$:

$$\vec{g} \simeq \vec{\gamma}_{\vec{\omega}=\vec{0}} \quad (4.1)$$

It has to be noted that the gravity potential (see appendix chapter E.2) is intended for providing a static value for the gravity vector over a particle in a moving reference frame (ECEF), so that this value can be compared with gravimetry measurements, and for this reason it includes the centrifugal acceleration.

In our case we are only interested in the gravitational attraction force. This means that the centrifugal acceleration has to be neglected. For this reason it is required to impose $\vec{\omega} = \vec{0}$ in equation 4.1 since the normal gravity potential U from which $\vec{\gamma}$ is obtained includes the centrifugal acceleration.

Since the considered Earth related to the normal gravity is a theoretical ellipsoid with a perfectly known shape and a perfectly uniform mass distribution, the normal gravity vector that it produces can be computed with unlimited accuracy by closed expressions.

These expressions can be found in reference [3] in spherical (S), rectangular (ECEF) (R) and ellipsoidal (E) coordinates:

$$\vec{\gamma}^E = \begin{Bmatrix} \gamma_u \\ \gamma_\beta \\ 0 \end{Bmatrix} \quad (4.2)$$

$$\vec{\gamma}^R = \begin{Bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{Bmatrix} = R_1 \cdot \vec{\gamma}^E \quad (4.3)$$

$$\vec{\gamma}^S = \begin{Bmatrix} \gamma_r \\ \gamma_{\varphi'} \\ 0 \end{Bmatrix} = R_2 \cdot \vec{\gamma}^R \quad (4.4)$$

where:

$$\begin{aligned}
\gamma_u(u, \beta) &= -\frac{1}{w} \left[\frac{GM_0}{u^2 + E^2} + \frac{\omega^2 a^2 E}{u^2 + E^2} \cdot \frac{q'}{q_0} \cdot \left(\frac{1}{2} \sin^2 \beta - \frac{1}{6} \right) \right] + \frac{1}{w} \omega^2 \cdot u \cdot \cos^2 \beta \\
\gamma_\beta(u, \beta) &= \frac{1}{w} \frac{\omega^2 a^2}{\sqrt{u^2 + E^2}} \cdot \frac{q}{q_0} \sin \beta \cos \beta - \frac{1}{w} \omega^2 \sqrt{u^2 + E^2} \cdot \sin \beta \cos \beta \\
R_1 &= \begin{bmatrix} \frac{u}{w\sqrt{u^2+E^2}} \cos \beta \cos \lambda & -\frac{1}{w} \sin \beta \cos \lambda & -\sin \lambda \\ \frac{u}{w\sqrt{u^2+E^2}} \cos \beta \sin \lambda & -\frac{1}{w} \sin \beta \sin \lambda & \cos \lambda \\ \frac{1}{w} \sin \beta & \frac{u}{w\sqrt{u^2+E^2}} \cos \beta & 0 \end{bmatrix} \\
R_2 &= \begin{bmatrix} \cos \varphi' \cos \lambda & \cos \varphi' \sin \lambda & \sin \varphi' \\ -\sin \varphi' \cos \lambda & -\sin \varphi' \sin \lambda & \cos \varphi' \\ -\sin \lambda & \cos \lambda & 0 \end{bmatrix}
\end{aligned} \tag{4.5}$$

The parameters indicated in equations 4.5 are provided herein:

$$\begin{aligned}
\{x, y, z\} &= \text{ECEF coordinates of the point.} \\
E &= \sqrt{a^2 - b^2} \\
u &= \left[\frac{1}{2} (x^2 + y^2 + z^2 - E^2) \cdot \left\{ 1 + \sqrt{1 + \frac{4E^2 z^2}{(x^2 + y^2 + z^2 - E^2)^2}} \right\} \right]^{1/2} \\
\beta &= \arctan \left(\frac{z\sqrt{u^2 + E^2}}{u\sqrt{x^2 + y^2}} \right) \\
w &= \sqrt{\frac{u^2 + E^2 \sin^2 \beta}{u^2 + E^2}} \\
q &= \frac{1}{2} \left[\left(1 + 3 \frac{u^2}{E^2} \right) \arctan \left(\frac{E}{u} \right) - 3 \frac{u}{E} \right] \\
q_0 &= \frac{1}{2} \left[\left(1 + 3 \frac{b^2}{E^2} \right) \arctan \left(\frac{E}{b} \right) - 3 \frac{b}{E} \right] \\
q' &= 3 \left[1 + \frac{u^2}{E^2} \right] \cdot \left[1 - \frac{u}{E} \arctan \left(\frac{E}{u} \right) \right] - 1
\end{aligned} \tag{4.6}$$

The ECEF coordinates can be obtained from the geodetic coordinates using equations A.27.

All the required constants to obtain the parameters in 4.6 can be found in reference [3].

Since according to 4.1 the centrifugal acceleration has to be neglected, the equations will be computed using the parameters given by equations 4.5 and 4.6 with the constants provided by [3] but imposing $\omega = 0$.

Easier expressions for the value of the normal gravity vector can be found in reference [3], but these expressions are valid for the normal gravity vector on the surface of the ellipsoid, and the error they imply with a greater height is not clearly specified.

4.2.2 Gravity moment

Because the gravitational force has a null angular momentum in the center of gravity of the missile, and this center of gravity is not exactly placed in the center of masses (because there is a gravity gradient), an angular moment will result in the center of mass.

This angular moment is usually neglected, since it is very small in comparison with other angular moments and only affects in the long term. However, for the sake of completeness, it will be taken into account in the simulation.

The gravity gradient torque of a spacecraft about non-spherical bodies, such as the Earth, has been studied in reference [4], showing that the main term of the gradient torque is contributed by the central component of the gravity field, whereas the oblateness of the Earth makes a contribution to the gravity gradient torque about, at most, 1% of the total contribution.

As a consequence we will consider for this term the Earth with a spherically symmetric mass distribution. With this assumption the gravitational force on any particle of the missile would be given by:

$$\vec{F} = \left\{ -\frac{GMm_p}{\|\vec{r}_p\|^3} \right\} \vec{r}_p \quad (4.7)$$

where:

- G is the gravitational constant $\simeq 6.67 \cdot 10^{11} N(m/kg)^2$
- M is the mass of the Earth
- m_p is the mass of the missile particle
- $\vec{r}_p$ is the position vector from the center of the Earth to the missile particle

The angular momentum on the center of mass of the missile would be given by:

$$\vec{\mathcal{M}}_{CM} = \sum_{\forall p} \vec{r}_p' \times \left\{ -\frac{GMm_p}{\|\vec{r}_p\|^3} \right\} \vec{r}_p \quad (4.8)$$

where:

- $\vec{r}_p'$ is the position vector from the center of mass of the missile to the missile particle ($\vec{r}_p' = \vec{r}_{CM} + \vec{r}_p$)

Using the approximation:

$$\frac{1}{\|\vec{r}_p\|^3} = \frac{1}{\|\vec{r}_{CM} + \vec{r}_p'\|^3} \simeq \frac{1}{\|\vec{r}_{CM}\|^3} - 3 \frac{1}{\|\vec{r}_{CM}\|^5} \vec{r}_{CM} \cdot \vec{r}_p' \quad (4.9)$$

equation 4.8 can be re-written as:

$$\vec{\mathcal{M}}_{CM} \simeq 3 \frac{GM}{\|\vec{r}_{CM}\|^5} \sum_{\forall p} m_p \vec{r}_p' \times (\vec{r}_{CM} \cdot \vec{r}_p') \vec{r}_{CM} \quad (4.10)$$

Taken into account the definition of the inertia tensor (see equation D.57) the terms in 4.10 can be expressed as:

$$\vec{\mathcal{M}}_{CM} \simeq 3 \frac{GM}{\|\vec{r}_{CM}\|^5} \vec{r}_{CM} \times I_{CM}^b \cdot \vec{r}_{CM} \quad (4.11)$$

This will be the expression used for this moment in the simulation.

4.3 Aerodynamic force and moment

This section will provide the aerodynamic force and moment acting on the missiles while travelling in the part of the atmosphere where the air density is significant.

Since this force and moment depend on the air density, a model for the atmosphere will be also provided herein.

4.3.1 Atmosphere model

An atmospheric model is required in order to provide the density as a function of height in the aerodynamic forces and moments.

The model indicated in [5] will be used with this aim.

We will use this model instead of the International Standard Atmosphere (ISA) model (see [6] and [7]) since the ISA model focuses on the lower part of the atmosphere, whereas the missiles to be simulated reach much higher altitudes.

The model described in [5] considers the following constants:

- $R^* = 8314.32 \text{ J/kg}$ (Universal gas constant)
- $M_0 = 28.964 \text{ kg/mol}$ (Sea level molecular weight of the air)
- $R_E = 6.3781 \cdot 10^6 \text{ m}$ (Earth radius)
- $g_0 = 9.7803 \text{ m/s}^2$ (gravity force at $Z=0$)
- $\gamma = 1.4$ (adiabatic polytropic constant)

This model can be used for different initial values of temperature and pressure at sea level (T_0 and P_0), although within this thesis the ISA standard sea level values will be used:

- $P_0 = 101325 \text{ Pa}$
- $T_0 = 288.15 \text{ K}$

The model described in [5] uses the concepts of:

- T_M = Molecular temperature. It is an artificial temperature based upon the assumption that the molecular weight of the air remains constant at its sea-level value throughout the whole atmosphere. It is related to the kinetic temperature (T) with equation:

$$T_M = \frac{M_0}{M} \cdot T \quad (4.12)$$

where:

- M_0 is the molecular weight of the air at sea level
- M is the molecular weight of the air at a certain altitude.

In this model, the atmosphere is divided in layers and a linear variation of the molecular weight of the air is considered within each layer. This model also considers a linear variation of the molecular temperature within each layer. The starting value for each layer is provided in table 4.2.

- h = geopotential altitude. It is an altitude based upon the assumption that the gravitational acceleration is constant with altitude, rather than obeying the inverse-square law. The geometric (= geodetic) altitude (Z) includes the variation of gravitational acceleration with the square of the geocentric distance.

As a consequence they are both related with the following equation:

$$h = \frac{R_E}{R_E + Z} \cdot Z \quad (4.13)$$

or, in differential form:

$$dh = \left(\frac{R_E}{R_E + Z} \right)^2 \cdot dZ = \left(\frac{g}{g_0} \right) \cdot dZ \quad (4.14)$$

Atmospheric layers are usually expressed in terms of geopotential altitude, since it is the one provided by the altimeters. However, the model described in [5] defines the atmosphere layers in terms of geometric altitude.

The different atmospheric layers considered within the atmosphere model described in [5] are indicated hereafter, providing the initial values of geometric height, molecular temperature and molecular weight for each layer.

Table 4.2: Variables for each atmospheric layer in the model described in [5]

Layer	$Z_{initial}$ Geometric altitude km	$T_{M_{h_{initial}}}$ Molecular temperature K	$M_{initial}$ Molecular weight kg/mol
Troposphere	0.0	T_0	M_0
Tropopause	11.0191	216.65	28.964
Stratosphere 1	20.0631	216.65	28.964
Stratosphere 2	32.1619	228.65	28.964
Stratopause	47.3501	270.65	28.964
Mesosphere 1	51.4125	270.65	28.964
Mesosphere 2	71.8020	214.65	28.962
Mesopause	86.00	186.946	28.962
Exosphere 1	100.00	210.65	28.880
Exosphere 2	110.00	260.65	28.560
Exosphere 3	120.00	360.65	28.070
Exosphere 4	150.00	960.65	26.920
Exosphere 5	160.00	1110.60	26.660
Exosphere 6	170.00	1210.65	26.500
Exosphere 7	190.00	1350.65	25.850
Exosphere 8	230.00	1550.65	24.690
Exosphere 9	300.00	1830.65	22.660
Exosphere 10	400.00	2160.65	19.940
Exosphere 11	500.00	2420.65	17.940
Exosphere 12	600.00	2590.65	16.840
Exosphere 13	700.00	2700.0	16.170

The model described in [5] considers the following equations for determining air pressure, temperature and density:

1. The hydrostatic equilibrium equation:

$$dP = -\rho g dZ \quad (4.15)$$

2. A linear variation for the molecular temperature (T_M) with the geometric height (Z) for each atmosphere layer in table 4.2:

$$T_M = T_{M_i} + L_{Z_i} \cdot (Z - Z_i) \quad (4.16)$$

with:

- T_{M_i} = molecular temperature at the beginning of the i atmosphere layer
 - $L_{Z_i} = \frac{T_{M_{i+1}} - T_{M_i}}{Z_{i+1} - Z_i}$ = slope for the line $T - Z$ in the i atmosphere layer
 - Z_i = geometric height at the beginning of the i atmosphere layer
3. The hypothesis of ideal gas for the air so that it fulfills the ideal gas law:

$$P = \rho R^* T / M \quad (4.17)$$

Substituting the air density of equation 4.15 in equation 4.17 we obtain:

$$P = -\frac{1}{g} \frac{dP}{dZ} \frac{R^* T}{M} \quad (4.18)$$

which can be expressed using the definitions in 4.12 as:

$$\frac{dP}{P} = -g \frac{M_0}{R^* T_M} dZ \quad (4.19)$$

In order to obtain the air pressure for a certain Z we have to integrate equation 4.19 using an expression for $g = f(Z)$. As a simplification we will use:

$$g = g_0 \left(\frac{R_E^2}{(R_E + Z)^2} \right) \simeq g_0 (1 - bZ) \quad (b = \frac{2.0}{R_E}) \quad (4.20)$$

This latter expression together with equation 4.16 allow expressing dP/P from equation 4.19 as:

$$\frac{dP}{P} = -\frac{g_0 M_0}{R^*} \cdot \frac{(1 - bZ)}{T_{M_i} + L_{Z_i} \cdot (Z - Z_i)} dZ \quad (4.21)$$

Equation 4.21 is integrated differently depending on whether the atmospheric layer is isothermal ($L_{Z_i} = 0$) or not:

- Isothermal layer

The integration leads to the following expression:

$$P = P_i \cdot \exp \left\{ -\frac{g_0 M_0}{R^* T_{M_i}} \cdot \left((Z - Z_i) - \frac{b}{2} (Z^2 - Z_i^2) \right) \right\} \quad (4.22)$$

- Non isothermal layer

The integration leads to the following equation:

$$P = P_i \left\{ \left(\frac{L_{Z_i}}{T_{M_i}} \right) \cdot (Z - Z_i) + 1 \right\}^{-\left\{ \left(\frac{g_0 M_0}{R^* L_{Z_i}} \right) \left[1 + b \left(\frac{T_{M_i}}{L_{Z_i}} - Z_i \right) \right] \right\}} \cdot \exp \left\{ \left(\frac{g_0 M_0 b}{R^* L_{Z_i}} \right) (Z - Z_i) \right\} \quad (4.23)$$

Once the air pressure is known, the air density is easily obtained from the ideal gas expression:

$$\rho = \frac{M_0}{R^* T_M} \cdot P = \rho_i \cdot \left(\frac{P}{P_i} \right) \cdot \left(\frac{T_{M_i}}{T} \right) \quad (4.24)$$

where $\frac{P}{P_i}$ can be obtained from equation 4.22 or 4.23 and $\frac{T_{M_i}}{T}$ can be obtained from equation 4.16.

It has to be noted that important errors can be derived from the use of any atmospheric model. In this case, the used model does not consider the amount of water vapor in the air, the season of the year nor the time of the day in which the model is used (there are obvious differences among the different seasons in the atmosphere, especially with higher values of latitude, and significant differences between night and day). Also, this model does not consider any type of wind.

As a consequence the errors implied for using this model will depend on many variables but could be relevant.

4.3.2 Aerodynamic model

4.3.2.1 Coefficients

The coefficients for aerodynamic forces and torques can be obtained by different ways. In this thesis it has been decided to use the Missile DATCOM model (see reference [8]). This model has been especially developed for missiles so it seems sensible to use it instead of other methods that would be adapted from airplanes, or analytical methods like the expressions in [9] that would be too complex for systems with variable geometry and very different Mach number along the trajectory.

(Note: in the following equations S_{ref} is a reference cross surface of the missile: we will use $\pi \cdot r^2$ being r the maximum radius of the stage of the missile, c_{ref} is a reference of the longitudinal length of the missile: we will use the length of the stage of the missile, b_{ref} is a reference of the lateral length of the missile: we will use the maximum diameter of the stage of the missile, and $V = \|(\vec{v}^b)_e\|$).

The Missile DATCOM provides the following coefficients for different values of Mach number and angle of attack α :

- Coefficient related to the drag force:

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } D = \text{Drag force} \quad (4.25)$$

- Coefficients related to the lift force:

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } L = \text{Lift force} \quad (4.26)$$

$$C_L/C_D = \frac{L}{D} \quad \text{with } L = \text{Lift force, and } D = \text{Drag force}$$

- Coefficients related to the axial force (force in X_{body} axis):

$$C_A = \frac{A}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } A = \text{Axial force} \quad (4.27)$$

$$C_{A_{\hat{q}}} = \frac{\frac{dA}{d\hat{q}}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \hat{q} = \frac{(\vec{w}_{bn})_y}{\left(\frac{2 \cdot V}{c_{ref}}\right)}$$

- Coefficients related to the lateral force (force in Y_{body} axis):

$$\begin{aligned}
C_Y &= \frac{Y}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } Y = \text{Axial force} \\
C_{Y_\beta} &= \frac{\frac{dY}{d\beta}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \beta = \text{sideslip angle} \\
C_{Y_{\hat{p}}} &= \frac{\frac{dY}{d\hat{p}}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \hat{p} = \frac{(\vec{w}_{bn})_x}{\left(\frac{2 \cdot V}{b_{ref}}\right)} \\
C_{Y_{\hat{r}}} &= \frac{\frac{dY}{d\hat{r}}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \hat{r} = \frac{(\vec{w}_{bn})_z}{\left(\frac{2 \cdot V}{b_{ref}}\right)}
\end{aligned} \tag{4.28}$$

- Coefficients related to the normal force (force in Z_{body} axis):

$$\begin{aligned}
C_N &= \frac{N}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } N = \text{Normal force} \\
C_{N_\alpha} &= \frac{\frac{dN}{d\alpha}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \alpha = \text{angle of attack} \\
C_{N_{\dot{\alpha}}} &= \frac{\frac{dN}{d\dot{\alpha}}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \dot{\alpha} = \frac{d\alpha}{dt} \\
C_{N_{\hat{q}}} &= \frac{\frac{dN}{d\hat{q}}}{\frac{1}{2}\rho V^2 S_{ref}} \quad \text{with } \hat{q} = \frac{(\vec{w}_{bn})_y}{\left(\frac{2 \cdot V}{c_{ref}}\right)}
\end{aligned} \tag{4.29}$$

- Coefficients related to the rolling moment (moment in X_{body} axis):

$$\begin{aligned}
C_{LL} &= \frac{\mathcal{L}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \mathcal{L} = \text{Roll moment} \\
C_{LL_\beta} &= \frac{\frac{d\mathcal{L}}{d\beta}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \beta = \text{sideslip angle} \\
C_{LL_{\hat{p}}} &= \frac{\frac{d\mathcal{L}}{d\hat{p}}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \hat{p} = \frac{(\vec{w}_{bn})_x}{\left(\frac{2 \cdot V}{b_{ref}}\right)} \\
C_{LL_{\hat{r}}} &= \frac{\frac{d\mathcal{L}}{d\hat{r}}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \hat{r} = \frac{(\vec{w}_{bn})_z}{\left(\frac{2 \cdot V}{b_{ref}}\right)}
\end{aligned} \tag{4.30}$$

- Coefficients related to the pitching moment (moment in Y_{body} axis):

$$\begin{aligned}
C_M &= \frac{\mathcal{M}}{\frac{1}{2}\rho V^2 S_{ref} \cdot c_{ref}} \quad \text{with } \mathcal{M} = \text{Pitch moment} \\
C_{M_\alpha} &= \frac{\frac{d\mathcal{M}}{d\alpha}}{\frac{1}{2}\rho V^2 S_{ref} \cdot c_{ref}} \quad \text{with } \alpha = \text{angle of attack} \\
C_{M_{\dot{\alpha}}} &= \frac{\frac{d\mathcal{M}}{d\dot{\alpha}}}{\frac{1}{2}\rho V^2 S_{ref} \cdot c_{ref}} \quad \text{with } \dot{\alpha} = \frac{d\alpha}{dt} \\
C_{M_{\hat{q}}} &= \frac{\frac{d\mathcal{M}}{d\hat{q}}}{\frac{1}{2}\rho V^2 S_{ref} \cdot c_{ref}} \quad \text{with } \hat{q} = \frac{(\vec{w}_{bn})_y}{\left(\frac{2 \cdot V}{c_{ref}}\right)}
\end{aligned} \tag{4.31}$$

- Coefficients related to the yawing moment (moment in Z_{body} axis):

$$\begin{aligned}
C_{LN} &= \frac{\mathcal{N}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \mathcal{N} = \text{Yaw moment} \\
C_{LN_\beta} &= \frac{\frac{d\mathcal{N}}{d\beta}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \beta = \text{sideslip angle} \\
C_{LN_{\hat{p}}} &= \frac{\frac{d\mathcal{N}}{d\hat{p}}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \hat{p} = \frac{(\vec{w}_{bn})_x}{\left(\frac{2 \cdot V}{b_{ref}}\right)} \\
C_{LN_{\hat{r}}} &= \frac{\frac{d\mathcal{N}}{d\hat{r}}}{\frac{1}{2}\rho V^2 S_{ref} \cdot b_{ref}} \quad \text{with } \hat{r} = \frac{(\vec{w}_{bn})_z}{\left(\frac{2 \cdot V}{b_{ref}}\right)}
\end{aligned} \tag{4.32}$$

It also provides the position of the center of pressure from the base of the missile for several values of the Mach number and angle of attack:

$$(\vec{r}_{C.P.})_{\text{from base}} \implies (\vec{r}_{C.P.})_b = (\vec{r}_{C.P.})_{\text{from base}} - (\vec{r}_{C.M.})_{\text{from base}} \tag{4.33}$$

The center of pressure is the point in which the aerodynamic forces are applied. It allows computing the final aerodynamic moment with respect to the center of mass of the missile $((\vec{\mathcal{M}}_a)_b)$:

$$(\vec{\mathcal{M}}_a)_b = (\vec{r}_{C.P.})_b \times \vec{F}_a + \mathcal{L} \cdot \vec{i}_b + \mathcal{M} \cdot \vec{j}_b + \mathcal{N} \cdot \vec{k}_b \tag{4.34}$$

being $\vec{F}_a$ the global aerodynamic force on the missile (see equation 4.40).

We will not use all these coefficients. Taking the symmetry of the missiles into account (they have an axial symmetry) for the static forces we will use the lift and drag coefficients:

1. Having the values of C_L for different α values we will obtain C_{L_0} and C_{L_α} :

$$C_L = C_{L_0} + C_{L_\alpha} \cdot \alpha \quad (4.35)$$

2. Having the values of C_D and C_L for different angles of attack we will obtain C_{D_0} and C_{D_L} :

$$C_D = C_{D_0} + C_{D_L} \cdot (C_L)^2 \quad (4.36)$$

Having expressions for C_D and C_L it is possible to obtain the axial, lateral and normal force each time from the angles α_{XZ} and α_{XY} obtained as:

$$\begin{aligned} \alpha_{XZ} &= \text{atan2} \left((\vec{v}^b)_n \cdot \vec{k}_b, (\vec{v}^b)_n \cdot \vec{i}_b \right) \\ \alpha_{XY} &= \text{atan2} \left((\vec{v}^b)_n \cdot \vec{j}_b, (\vec{v}^b)_n \cdot \vec{i}_b \right) \end{aligned} \quad (4.37)$$

since we can express:

$$\begin{aligned} L_{XY} &= \frac{1}{2} \rho V^2 S_{ref} \cdot (C_{L_0} + C_{L_\alpha} \cdot \alpha_{XY}) \quad \text{with } L_{XY} = \text{lift in the } Y_b \text{ axis} \\ &\implies C_{L_{XY}} = \frac{L_{XY}}{\frac{1}{2} \rho V^2 S_{ref}} \implies (C_D)_{XY} = C_{D_L} \cdot (C_{L_{XY}})^2 \\ L_{XZ} &= \frac{1}{2} \rho V^2 S_{ref} \cdot (C_{L_0} + C_{L_\alpha} \cdot \alpha_{XZ}) \quad \text{with } L_{XZ} = \text{lift in the } Z_b \text{ axis} \\ &\implies C_{L_{XZ}} = \frac{L_{XZ}}{\frac{1}{2} \rho V^2 S_{ref}} \implies (C_D)_{XZ} = C_{D_L} \cdot (C_{L_{XZ}})^2 \end{aligned} \quad (4.38)$$

and from this:

$$\begin{aligned} C_{A_{static}} &= C_{L_{XY}} \cdot \sin(\alpha_{XY}) + C_{L_{XZ}} \cdot \sin(\alpha_{XZ}) \\ &\quad - C_{D_0} - (C_D)_{XY} \cdot \cos(\alpha_{XY}) - (C_D)_{XZ} \cdot \cos(\alpha_{XZ}) \\ C_{Y_{static}} &= -C_{L_{XY}} \cdot \sin(\alpha_{XY}) - (C_D)_{XY} \cdot \sin(\alpha_{XY}) \\ C_{N_{static}} &= -C_{L_{XZ}} \cdot \sin(\alpha_{XZ}) - (C_D)_{XZ} \cdot \sin(\alpha_{XZ}) \end{aligned} \quad (4.39)$$

so we will use these expressions (equations 4.39) instead of the coefficients C_A , C_Y and C_N from Missile DATCOM ([8]).

The coefficients C_{LL} , C_M and C_{LN} provided by the Missile DATCOM model ([8]) will not be used as well. Taking the axial symmetry of the missile into account, there will not be any aerodynamic moment in static conditions when α and β are null, so the coefficients $C_{LL\beta}$, $C_{M\alpha}$ and $C_{LN\beta}$ will provide the complete value for $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ in static conditions.

The final expressions to be used for the aerodynamic forces will be:

$$\begin{aligned}
 A &= \frac{1}{2}\rho V^2 S_{ref} \cdot \left(C_{A_{static}} + C_{A_{\dot{q}}} \cdot \frac{q}{c_{ref}} \right) \\
 Y &= \frac{1}{2}\rho V^2 S_{ref} \cdot \left(C_{Y_{static}} + C_{Y_{\dot{p}}} \cdot \frac{p}{\left(\frac{2 \cdot V}{b_{ref}}\right)} + C_{Y_{\dot{r}}} \cdot \frac{r}{\left(\frac{2 \cdot V}{b_{ref}}\right)} \right) \\
 N &= \frac{1}{2}\rho V^2 S_{ref} \cdot \left(C_{N_{static}} + C_{N_{\dot{\alpha}}} \cdot \dot{\alpha} + C_{N_{\dot{q}}} \cdot \frac{q}{c_{ref}} \right)
 \end{aligned} \tag{4.40}$$

and the final expressions to be used for the aerodynamic moments will be:

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2}\rho V^2 S_{ref} b_{ref} \cdot \left(C_{LL_{\beta}} \cdot \beta + C_{LL_{\dot{p}}} \cdot \frac{p}{\left(\frac{2 \cdot V}{b_{ref}}\right)} + C_{LL_{\dot{r}}} \cdot \frac{r}{\left(\frac{2 \cdot V}{b_{ref}}\right)} \right) \\
 \mathcal{M} &= \frac{1}{2}\rho V^2 S_{ref} c_{ref} \cdot \left(C_{M_{\alpha}} \cdot \alpha + C_{M_{\dot{\alpha}}} \cdot \dot{\alpha} + C_{M_{\dot{q}}} \cdot \frac{q}{c_{ref}} \right) \\
 \mathcal{N} &= \frac{1}{2}\rho V^2 S_{ref} b_{ref} \cdot \left(C_{LN_{\beta}} \cdot \beta + C_{LN_{\dot{p}}} \cdot \frac{p}{\left(\frac{2 \cdot V}{b_{ref}}\right)} + C_{LN_{\dot{r}}} \cdot \frac{r}{\left(\frac{2 \cdot V}{b_{ref}}\right)} \right)
 \end{aligned} \tag{4.41}$$

The Missile DATCOM model ([8]) does not provide aerodynamic coefficients related to the velocity component in the axis of the missile ($u = (\vec{v}^b)_n \cdot \vec{i}_b$):

$$\begin{aligned}
 C_{M_u} &= \frac{\frac{d\mathcal{M}}{du}}{\frac{1}{2}\rho V^2 S_{ref} \cdot c_{ref}} \\
 C_{A_u} &= \frac{\frac{dA}{du}}{\frac{1}{2}\rho V^2 S_{ref}} \\
 C_{N_u} &= \frac{\frac{dN}{du}}{\frac{1}{2}\rho V^2 S_{ref}}
 \end{aligned} \tag{4.42}$$

so they will be neglected. This can be justified taking into account that the missile velocity component in the X_{body} axis (u) is approximately constant for short periods of time in a rocket.

The Missile DATCOM model ([8]) requires a file with the relevant input data to be used. Since the missiles have several stages a different input file has to be created for each of them. Once executed, the Missile DATCOM model produces several output files.

All these files are very long and will not be included in the thesis for the sake of brevity.

4.3.2.2 Considerations for spinning missiles

The reentry vehicle of the ICBM (see section G.1.3.11) is provided with a high spin rate activated before reentry through a hot gas spin system located in the aft section.

If the missile reenters the atmosphere with the body axis not aligned with the flight path it will have a wobbling motion (precession): the tip of the missile draws a cone around the axis of movement, with the tip of the cone on the center of mass of the missile. Because of imperfections that imbalance the center of mass there could also appear a nutation movement.

In normal conditions the precession and nutation movements (the wobbling) are damped with time (the missile is dynamically stabilized) but in this process there is a change in the flight path angle towards the direction of the spin (see [10]): for a right hand (clockwise) direction of rotation this component will always be to the right; for a left hand (counterclockwise) direction of rotation this component will always be to the left. This effect is caused by a gyroscopic drift.

Because of the spinning there may appear another effect acting on the missile: the Magnus effect: a cylinder rotating in an incoming airstream has a circulation created by the mechanical rotation that becomes a lift (Kutta-Joukowski lift).

Figure 4.1: Magnus effect
(picture from Wikimedia Commons)

This effect will appear in the reentry vehicle if the reentry is performed while important lateral winds are acting on the missile.

The effect of the Magnus effect would greatly depend on the existing winds in the reentry that are different in each layer of the atmosphere, so its effect is very difficult to simulate successfully. Since no winds have been considered in the simulation of the missiles, the Magnus effect will be neglected.

The missile DATCOM model ([8]) allows indicating whether the missile is spinning in order to adapt the results for this case.

The effects of the spin in the reentry vehicle could be simulated by using the coefficients for spinning missiles from the missile DATCOM model by simply integrating the forces and moments in each time step.

However, this would be very demanding from the computational point of view since the spin rate is very high, so the only way to successfully simulate this motion would be using a very small time step.

As a simplification, in the simulation the spinning effect will be simulated by neglecting any lateral aerodynamic force or torques acting on the missile during the reentry while the missile is spinning, and forcing a smooth transition from the angle of attack and sideslip angle of the missile when crossing the Kármán line to null values.

This approximation means that the spin drift effect will not be simulated.

The guidance module will command a final attitude for the reentry vehicle such that its precession motion is minimized in the reentry. This also minimizes the spinning drift effect and the errors made because of neglecting this effect.

4.4 Thrust force and moment

4.4.1 Formula for the thrust force

The equation for the thrust force of the missile can be obtained from the term related to the relative acceleration of the particles in the missile system (equation D.105) since this term was not included in the final expression for the linear momentum of the missile (equation D.106), but considered an external force:

$$\vec{E}_{\text{relative motion}} = - \left(\frac{d[\dot{m} \cdot (\vec{r}_e)_b]}{dt} \right)_b - \dot{m}(\vec{v}_e)_b \quad (4.43)$$

where:

- $(\vec{r}_e)_b$ = mass flow center (see equation D.95)
- $(\vec{v}_e)_b$ = mean exhaust velocity (see equation D.104)

This equation is completed with the effect related to the difference between the internal pressure in the nozzle of the missile (p_e) and the atmospheric pressure (p_0), since a global force of:

$$\vec{E}_{\text{difference pressure}} = -(p_e - p_0) \cdot A_e \vec{n}_e \quad (4.44)$$

will be produced by this difference.

In equation 4.44:

- A_e = surface area at the exit of the nozzle
- $\vec{n}_e$ = normal vector at the nozzle pointing outwards

Combining both forces we have the following formula for the thrust:

$$\vec{E} = - \left(\frac{d[\dot{m} \cdot (\vec{r}_e)_b]}{dt} \right)_b - \dot{m}(\vec{v}_e)_b - (p_e - p_0) \cdot A_e \vec{n}_e \quad (4.45)$$

This formula can be simplified. On the one hand the term $-\left(\frac{d[\dot{m} \cdot (\vec{r}_e)_b]}{dt}\right)_b$ can be neglected since for solid fuel rockets, as it is the case, the variation of the fuel mass with time is very low once a stage is in normal operation, and being the variation of the position of the center of mass of the missile very slow with time (see hypothesis D.81 and its consequences in appendix section D.6.1), the mass flow center vector $((\vec{r}_e)_b)$ is also almost constant.

On the other hand the mean exhaust velocity $((\vec{v}_e)_b)$, taking the axial symmetry into account, has an almost constant direction equal to the normal vector at the nozzle:

$$(\vec{v}_e)_b \simeq \|(\vec{v}_e)_b\| \vec{n}_e = v_e \vec{n}_e \quad (4.46)$$

As a consequence the thrust force will be computed with the expression:

$$\vec{E} = [\dot{m}v_e + (p_e - p_0)A_e](-\vec{n}_e) \quad (4.47)$$

4.4.2 Thrust force in a *de Laval* nozzle

4.4.2.1 General expression

The nozzles in the rockets usually follow a *de Laval* nozzle design, being the gas first compressed and later on expanded:

Figure 4.2: de Laval nozzle

Supposing an ideal expansion in the nozzle, this is:

- Stationary regime
- Unidimensional flow (good approximation taking into account the axial symmetry of the rocket)
- Homogeneous flow maintaining chemical composition and properties ($c_p, R, \gamma = \text{constant}$) within the nozzle.
- Ideal gas behaviour for the fluid in the expansion ($\frac{P}{\rho} = RT$)
- Isentropic adiabatic process ($\frac{P}{\rho^\gamma} = \text{constant}$)

we get the following equation for the speed at the nozzle exit in terms of the pressure and temperature of the combustion chamber (see [11]):

$$v_e = \sqrt{\frac{2\gamma}{\gamma-1} \frac{R}{M} T_c \left(1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right)} \quad (4.48)$$

where:

- $\gamma = \frac{c_p}{c_v}$ = isentropic expansion factor of the fluid
- c_p = specific heat of the gas at constant pressure
- c_v = specific heat of the gas at constant volume
- R = Universal gas law constant = 8314.5 J/(kmol K)
- M = gas molecular mass (kg/kmol)
- T_c = temperature of the gas at the exit of the combustion chamber
- p_e = pressure of the gas at the exit of the nozzle
- p_c = pressure of the gas at the exit of the combustion chamber

The mass flow in the throat (g) of the nozzle is:

$$\dot{m} = \rho_g v_g A_g \quad (4.49)$$

with:

- ρ_g = gas density at the throat of the nozzle
- v_g = gas speed at the throat of the nozzle
- A_g = surface area at the throat of the nozzle

It is considered that the nozzle is ideal if the Mach number is 1 at the throat of the nozzle. With the same hypotheses used for equation 4.48 and supposing an ideal nozzle we get from equation 4.49 the following equation for the mass flow rate (see [11]):

$$(\dot{m})_{ideal} = \frac{p_c A_g}{\sqrt{\frac{R}{M} T_c}} \Gamma(\gamma) \quad (4.50)$$

with:

$$\bullet \Gamma(\gamma) = \sqrt{\gamma} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

Using equations 4.48 and 4.50 in the equation for the thrust (equation 4.47) for the ideal case we get:

$$E = p_c A_g \Gamma(\gamma) \sqrt{\frac{2\gamma}{\gamma-1} \left(1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right)} + A_e (p_e - p_0) \quad (4.51)$$

4.4.2.2 Coefficients

The thrust coefficient is defined as:

$$C_E := \frac{E}{p_c A_g} \quad (4.52)$$

In the case of an ideal nozzle:

$$(C_E)_{ideal} = \Gamma(\gamma) \sqrt{\frac{2\gamma}{\gamma-1} \left(1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right)} + \frac{A_e}{A_g} \left(\frac{p_e}{p_c} - \frac{p_0}{p_c} \right) \quad (4.53)$$

so it can be observed that in the ideal case C_E depends on:

$$(C_E)_{ideal} = f \left(\gamma, \frac{p_e}{p_c}, \frac{A_e}{A_g}, \frac{p_0}{p_c} \right) \quad (4.54)$$

The continuity equation between the nozzle throat and the nozzle exit allows obtaining (supposing an ideal nozzle) (see [11]):

$$\frac{A_g}{A_e} = \frac{1}{\Gamma(\gamma)} \left(\frac{p_e}{p_c} \right)^{\frac{1}{\gamma}} \sqrt{\frac{2\gamma}{\gamma-1} \left(1 - \left(\frac{p_e}{p_c} \right)^{\frac{\gamma-1}{\gamma}} \right)} \quad (4.55)$$

This latter equation means that for an ideal nozzle $\left(\frac{p_e}{p_c} \right)_{ideal} = f \left(\gamma, \frac{A_e}{A_g} \right)$ so we can conclude that:

$$(C_E)_{ideal} = f \left(\gamma, \frac{A_e}{A_g}, \frac{p_0}{p_c} \right) \quad (4.56)$$

The parameter of characteristic speed is defined as:

$$C^* := \frac{\sqrt{\frac{R}{M} T_c}}{\Gamma(\gamma)} \quad (4.57)$$

The parameter of characteristic speed only depends on chemical properties of the gas and the temperature of the combustion chamber, which can also be considered as a property of the fuel.

Once the parameter of characteristic speed is known, the mass flow rate in the ideal case will be given by (see equation 4.50):

$$(\dot{m})_{ideal} = \frac{p_c A_g}{C^*} \quad (4.58)$$

The specific impulse is defined as the thrust divided by the fuel consumption per unit of time:

$$I_{sp} := \frac{E}{\dot{m}} = \frac{\dot{m} v_e}{\dot{m}} + \frac{(p_e - p_0) A_e}{\dot{m}} = v_e + \frac{(p_e - p_0) A_e}{\dot{m}} \quad (4.59)$$

It provides the efficiency of a rocket engine, indicating how much force can be obtained per fuel consumption. When $p_e = p_0$ the nozzle is considered as adapted and the specific impulse equals the exit speed of the gas.

Using equation 4.52 and equation 4.58 in equation 4.59 the specific impulse in the ideal case can be written as:

$$(I_{sp})_{ideal} = \left(\frac{E}{\dot{m}} \right)_{ideal} = \frac{(C_E)_{ideal} \cdot p_c A_g}{\frac{p_c A_g}{(C^*)_{ideal}}} = (C_E)_{ideal} \cdot (C^*)_{ideal} \quad (4.60)$$

Taking into account the dependencies of $(C_E)_{ideal}$ and $(C^*)_{ideal}$, the specific impulse in the ideal case only depends on the chemical properties of the fuel and the ratios $\frac{A_e}{A_g}, \frac{p_0}{p_c}$.

4.4.3 Thrust force in the simulation

The thrust in the real case will be given by applying efficiency factors to the equations in the ideal cases such as:

$$\Sigma_e = \frac{C_E}{(C_E)_{ideal}} = \text{efficiency parameter in the expansion} \quad (4.61)$$

$$\Sigma_b = \frac{C^*}{(C^*)_{ideal}} = \text{efficiency parameter in the combustion} \quad (4.62)$$

This means that the real thrust depends on:

- Efficiency factors (Σ_e and Σ_b)
- Chemical properties of the fuel (γ , R/M , T_c)
- A_e/A_g
(note that A_e is variable for some nozzles)
- $p_0/p_c = f(h)$
 $p_0 = f(h)$ (see section 4.3.1)
 p_c depends on the disposition of the solid fuel within the combustion chamber, since it is proportional to the exposed surface of the fuel (A_b):

$$p_c \propto A_b = f(t) \quad (4.63)$$

where A_b is the exposed surface of the fuel in the combustion chamber.

As a consequence providing the thrust as a function of time for a certain height requires detailed information about the propellant, the nozzle geometry and the propellant distribution in the combustion chamber that is not available and cannot be accurately simulated.

In order to overcome this problem the vacuum thrust, provided as a constant for each stage of the Minuteman III ICBM in some references (see chapter G.1) and by the manufacturer in the case of the GBI missile (see chapter G.2) will be used, and the following simplification will be applied:

$$E(\text{simulation}) = E(\text{vacuum}) - A_e \cdot p_0 \quad (4.64)$$

where p_0 will be simulated according to 4.3.1 and A_e can be observed for the ICBM and the interceptor in appendix section G.1 and appendix section G.2.

This simplification assumes that the exposed surface of the fuel remains almost constant with time. This is mostly true since the propellant is usually placed inside the combustion chamber with a geometry such that the exposed area varies as less as possible with time.

4.4.4 Thrust moment

The moments associated to the ejection of mass in the missile, this is, moment related to the Coriolis term (equation D.137) including jet damping (equation D.134) and moment related to the relative acceleration (equation D.150), have been included in the general equation for the angular momentum with respect to the center of mass of the missile (equation D.151).

Taking this into account and supposing an axial symmetry, no other moment related to the thrust is to be considered.

4.5 Control forces

Because of their configuration (the considered missiles do not have wings nor fins), the center of pressure of the considered missiles is closer to the fore of the missile than the center of mass of the missile. As a consequence, any angle of incidence (angle of attack or sideslip angle) that appears in the missile will create an aerodynamic moment with respect to the center of mass of the missile that will tend to increase that very angle. This is, the considered missiles are unstable platforms while they are inside the atmosphere.

Figure 4.3: Instability of missiles within the atmosphere

This instability is very severe because the missiles move within the atmosphere with a very high speed (up to Mach 12 when leaving the atmosphere), so any small angle of incidence will create an enormous destabilizing moment.

Because of this, control forces have to be applied in the missiles with a special care, not only to ensure that the missiles reach their desired targets, but to ensure that their trajectories remain stable at all times.

It has to be noted that no controls are available for the reentry vehicle, which is still an unstable platform (center of mass behind the center of pressure) and reenters the atmosphere at even larger speeds (up to Mach 26). In this case the stabilization is achieved by the spinning motion: the spinning provides a gyroscopic effect that limits, or cancels, the destabilizing moments.

The control forces applied in the missiles for stabilization and control will be detailed within this section.

4.5.1 Control forces in the ICBM

The control forces applied in the ICBM depend on the phase of the trajectory, since in each of them different controls are possible and a different control strategy is applied.

4.5.1.1 Existing controls

As explained in chapter 5 the Minuteman III LGM-30G missile is considered as the ICBM in the simulator.

This missile has the following controls for each stage:

1. First stage

The Minuteman III missile has 4 movable exhaust nozzles in the first stage. Each of them is capable of pivoting through an angle of ± 8 degrees from null, in a line parallel to the motor centerline.

The pair of laterally opposed nozzles pivot up and down for pitch control. The vertically opposed pair pivot sideways for yaw control. All four nozzles are used to maintain roll stability (see [12]).

Figure 4.4: Considered numbering and disposition of the nozzles in stage 1

2. Second stage

The missile has only 1 fixed nozzle in this stage. The control in pitch and yaw is obtained by a liquid injection Thrust Vector Control (TVC) system: freon from a toroidal storage tank is selectively injected into the nozzle at 4 points 90° apart. This produces shock waves in the flow which shifts pressure distributions inside the nozzle. This provides a thrust offset vector that causes the missile to correct its pitch and yaw (see [13]).

Figure 4.5: Considered numbering and disposition of the ejection points in stages 2 and 3

Roll control is accomplished by releasing warm gas through two pairs of roll control nozzles on opposite sides of the missile. This warm gas is provided by a gas generator separated from the gas generator used for pressurizing the freon (see [13]).

3. Third stage

Like in the second stage the missile has in this case only 1 fixed nozzle. A liquid injection TVC system (similar to the one used on the second stage motor) is used. As a difference with stage 2, instead of using a gas generator that takes the exhaust gases from burning solid propellant to provide the necessary pressure for the injectant, a pressurized helium tank is used. In this case the injectant used is strontium perchlorate (see [13]).

Roll control is obtained with a gas generator and a roll control valve at the forward end of the motor. Upon command the gas generator is ignited with an explosive squib device and a valve vents the gas through two nozzles which extend through the motor skirt (see [13]).

4. Post-Boost stage

The Minuteman missile has a Propulsion System Rocket Engine (PSRE) that provides the thrust required for post-boost manoeuvring.

The system is composed by a main motor which is a restartable liquid fuel motor, used for increasing the range of the missile, and 10 attitude-control engines (6 pitch and yaw motors and 4 smaller roll motors, see [12] and [13]). The time the PSRE main motor is used for increasing the range is then another control of the missile for its final guidance.

5. Reentry

The last part of the trajectory happens when the Reentry Vehicle (RV) is detached from the missile. This happens when the vehicle is placed with the right attitude for the reentry. Since up to 3 RVs can be deployed by a single missile, a deployment manoeuvre from the Post-Boost Control System would be required for each of them.

A hot gas spin system, located in the aft section, stabilizes the RV in its correct reentry orientation after deployment (see [12]).

4.5.1.2 Control equations

The Minuteman III ICBM was designed in the 1960's so it is sensible to suppose that it implements control equations based on classical control theory.

Using this supposition, each control channel (roll/pitch/yaw) will be controlled independently.

The dynamics of the system in each channel can be considered to represent a second order system, this is, it will be given by the following system equation:

$$\ddot{x} + 2 \cdot \omega \cdot \xi \cdot \dot{x} \pm \omega^2 \cdot x = 0 \quad (4.65)$$

As a consequence the control problem consists in obtaining the control function $u(t)$ such that when applied in the equation:

$$\ddot{x} + 2 \cdot \omega \cdot \xi \cdot \dot{x} \pm \omega^2 \cdot x = A \cdot u(t) \quad (4.66)$$

it provides an output $x(t)$ that tends to a certain reference x_{ref} in a controlled way.

For achieving this result we will apply a control function obtained from the following operations in the Laplace domain:

Figure 4.6: Scheme for obtaining the control function in the Laplace domain

With the strategy in figure 4.6 the control function in the Laplace domain would be:

$$U(s) = K_{DL} \cdot X_{ref} + (X_{ref} - X(s)) \cdot \left(K_P + \frac{K_I}{s} \right) + K_D \cdot X(s) \cdot s \quad (4.67)$$

Expressing the original problem (equation 4.66) in the Laplace domain we have:

$$s^2 \cdot X(s) + 2 \cdot \omega \cdot \xi \cdot s \cdot X(s) \pm \omega^2 \cdot X(s) = A \cdot U(s) \Rightarrow X(s) = \frac{A}{s^2 + 2 \cdot \omega \cdot \xi \cdot s \pm \omega^2} \cdot U(s) \quad (4.68)$$

so using equation 4.67 we get:

$$X(s) = \frac{A}{s^2 + 2 \cdot \omega \cdot \xi \cdot s \pm \omega^2} \cdot \left[K_{DL} \cdot X_{ref} + (X_{ref} - X(s)) \cdot \left(K_P + \frac{K_I}{s} \right) + K_D \cdot X(s) \cdot s \right] \quad (4.69)$$

After some manipulations in equation 4.69 we get:

$$\frac{X(s)}{X_{ref}} = \frac{A \cdot [K_I + (K_{DL} + K_P) \cdot s]}{(s^3 + (2 \cdot \omega \xi - A \cdot K_D) \cdot s^2 + (\pm \omega^2 + A \cdot K_P) \cdot s + A \cdot K_I)} \quad (4.70)$$

We can obtain the different gains (K_I, K_{DL}, K_P, A) by imposing the following conditions:

1. Neglecting the integral part of the control:

$$K_I = 0 \quad (4.71)$$

This could add inaccuracies and adds a pole that could add instabilities to the system behaviour, but makes obtaining the gains much easier.

2. Forcing that the stationary solution tends to x_{ref} :

$$\lim_{s \rightarrow 0} \frac{X(s)}{X_{ref}} = 1 \Rightarrow \frac{A \cdot (K_{DL} + K_P)}{\pm \omega^2 + A \cdot K_P} = 1 \quad (4.72)$$

3. Imposing the desired stability properties to the extended system.

The extended system is obtained taking the control forces into account. This is, if we have:

$$\ddot{x} + 2 \cdot \omega \cdot \xi \cdot \dot{x} \pm \omega^2 \cdot x = A \cdot u(t) = A \cdot (K_{DL} + K_P) \cdot x_{ref} + A \cdot [K_D \cdot \dot{x}(t) - K_P \cdot x(t)] \quad (4.73)$$

the system becomes an extended system (') given by:

$$\begin{aligned} \ddot{x} + (2 \cdot \omega \cdot \xi - A \cdot K_D) \dot{x} + (\pm \omega^2 + A \cdot K_P) \cdot x(t) &= A \cdot (K_{DL} + K_P) \cdot X_{ref} \\ \Rightarrow 2 \cdot \omega \cdot \xi - A \cdot K_D &= 2 \cdot \omega' \cdot \xi' \\ \Rightarrow (\pm \omega^2 + A \cdot K_P) &= \omega'^2 \end{aligned} \quad (4.74)$$

for which we will choose the values ω' and ξ' so that the extended problem has the desired stability characteristics.

When $\omega \neq 0$ the values $\omega' = 2.0$, $\xi' = 0.7$ will be considered as typical desired characteristics for a channel for which a normal dynamics is required and $\omega' = 4.0$, $\xi' = 0.7$ will be considered for a channel for which a faster dynamics is requested, this is, a channel we want to prioritize. When $\omega = 0$, the control forces directly affect the missile motion without any opposite force, so we will request a slower system dynamics (lower ω') and a bigger damping for the extended system (ξ'). The chosen values are $\omega' = 0.5$, $\xi' = 2.0$.

Once the gains have been computed and taken equation 4.71 into account the control function is given in the time domain from 4.67 as:

$$u(t) = K_{DL} \cdot x_{ref} + K_P \cdot (x_{ref} - x(t)) + K_D \cdot \dot{x}(t) \quad (4.75)$$

4.5.1.3 Control forces

The guidance strategies (see chapter 7) lead to control forces that will be detailed herein.

4.5.1.3.1 Adaptation to the stage 1

We will detail herein the adaptations of the control equations indicated in section 4.5.1.2 to the control channels in the stage 1.

As indicated in section 4.5.1.1 in the first stage the control is achieved by rotating the nozzles in 2 planes (y_b and z_b). This creates thrust forces in the y_b and z_b axes.

4.5.1.3.1.1 Stage 1 pitch control

All the motion in the stage 1 happens in a vertical plane defined by the initial heading requested for the missile (which is a result of the guidance module and decided according to the target position).

This is achieved in the real case by adjusting the controls so that the pitch is commanded within this heading. In the simulation we will achieve the same result by considering that the missile is placed before launch so that the angle $\phi = 0^\circ$ is aligned with the desired initial heading and then keeping $\phi = 0^\circ$ throughout the trajectory. This allows not implementing some complex control transformations while keeping the same behaviour in the simulated missile as in the real one.

According to this simplification the control in this case has to be applied in the z_b axis to induce pitch changes in the missile motion.

There are 2 different variables we want to control in this case (pitch angle: θ , and angle of attack: α), which are provided by the following equations:

$$\ddot{\Delta\theta} - \frac{\mathcal{M}_\theta}{I_{yy}} \cdot \Delta\theta = \frac{L_{thrust}}{I_{yy}} \cdot E_{z_b} \quad (4.76)$$

$$\ddot{\Delta\alpha} - \frac{\mathcal{M}_\alpha}{I_{yy}} \cdot \Delta\alpha = \frac{L_{thrust}}{I_{yy}} \cdot E_{z_b} \quad (4.77)$$

where:

- $\Delta\theta$ = variation of the pitch angle with respect to the reference value
- $\Delta\alpha$ = variation of the angle of attack of the missile with respect to the reference value
- $\mathcal{M}_\theta$ = moment of the external forces on the missile in the y_b axis proportional to $\Delta\theta$, without taking the control forces into account
- $\mathcal{M}_\alpha$ = moment of the external forces on the missile in the y_b axis proportional to $\Delta\alpha$, without taking the control forces into account
- I_{yy} = moment of inertia of the missile with respect to the y_b axis
- L_{thrust} = arm between the thrust application point (mass flow center) and the center of mass of the missile (in the x_b axis)
- E_{z_b} = thrust in the z_b direction

Note that $\frac{L_{thrust}}{I_{yy}} = A$ according to the formulation used in equation 4.66.

It has to be noted that $\mathcal{M}_{\dot{\theta}}$ is neglected in equation 4.76 and $\mathcal{M}_{\dot{\alpha}}$ is neglected in equation 4.77. The obtained control without considering these terms behaves properly since they only add a damping in the real response with respect to the initially considered.

The moments proportional to $\Delta\theta$ or $\Delta\alpha$ are obtained from the derivatives of the moment of the aerodynamic forces (the moment of the gravitational forces is very small and can be neglected) and from the derivatives of the pitch angle and the angle of attack with respect to the present state vector:

$$\begin{aligned}\mathcal{M}_\theta &= \frac{\partial \mathcal{M}}{\partial \vec{X}} \cdot \frac{\partial \vec{X}}{\partial \theta} = \frac{\partial \mathcal{M}}{\partial \theta} \\ \mathcal{M}_\alpha &= \frac{\partial \mathcal{M}}{\partial \vec{X}} \cdot \frac{\partial \vec{X}}{\partial \alpha} = \frac{\partial \mathcal{M}}{\partial \alpha}\end{aligned}\tag{4.78}$$

The control forces are obtained with the following procedure:

1. We obtain the control E_{z_b} in equation 4.76 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 2.0$ rad/s, $\xi' = 0.7$
2. We obtain the control E_{z_b} in equation 4.77 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 4.0$ rad/s, $\xi' = 0.7$

Within this control equation, the used α_{ref} is always:

$$\alpha_{ref} = 0.0\tag{4.79}$$

3. We obtain the final E_{z_b} adding the results for channels θ and α .

As a simplification and trying to decouple the motions as much as possible only nozzles 2 and 4 (see figure 4.4) will be moved for pitch control. We will suppose a symmetric distribution of thrust among the nozzles. This is:

$$(E_{z_b})_2 = (E_{z_b})_4 = \frac{E_{z_b}}{2}\tag{4.80}$$

Basically what we are doing with this combination of controls is commanding a *zero lift* trajectory (see section 7.1.1.2) while trying to maintain a certain reference value for the pitch of the missile.

Since the *zero lift* trajectory is prioritized (it has a faster dynamics) the achieved trajectory will keep a low angle of attack, and after the *initial kick* the flight path angle of the missile in the vertical plane will decrease according to equation 7.1.

4.5.1.3.1.2 Stage 1 yaw control

As explained before the missile is launched in the simulation as a simplification with the roll angle aligned with the desired heading. As a consequence the required control in this case has to be applied in the y_b axis to induce yaw changes in the missile motion.

There are 2 different variables we want to control in this case (yaw: ψ , and sideslip angle: β), which are provided by the following equations:

$$\ddot{\Delta\psi} - \frac{\mathcal{M}_\psi}{I_{zz}} \cdot \Delta\psi = \frac{L_{thrust}}{I_{zz}} \cdot E_{y_b} \quad (4.81)$$

$$\ddot{\Delta\beta} - \frac{\mathcal{M}_\beta}{I_{zz}} \cdot \Delta\beta = \frac{L_{thrust}}{I_{zz}} \cdot E_{y_b} \quad (4.82)$$

where:

- $\Delta\psi$ = variation of the yaw angle with respect to the reference value
- $\Delta\beta$ = variation of the sideslip angle of the missile with respect to the reference value
- $\mathcal{M}_\psi$ = moment of the external forces on the missile in the z_b axis proportional to $\Delta\psi$, without taking the control forces into account
- $\mathcal{M}_\beta$ = moment of the external forces on the missile in the z_b axis proportional to $\Delta\beta$, without taking the control forces into account
- I_{zz} = moment of inertia of the missile with respect to the z_b axis
- L_{thrust} = arm between the thrust application point (mass flow center) and the center of mass of the missile (in the x_b axis)
- E_{y_b} = thrust in the y_b direction

Note that $\frac{L_{thrust}}{I_{zz}} = A$ according to the formulation used in equation 4.66.

It has to be noted that $\mathcal{M}_{\dot{\psi}}$ is neglected in equation 4.81 and $\mathcal{M}_{\dot{\beta}}$ is neglected in equation 4.82. The obtained control without considering these terms behaves properly since they only add a damping in the real response with respect to the initially considered.

The moments proportional to $\Delta\psi$ or $\Delta\beta$ are obtained from the derivatives of the moment of the aerodynamic forces (the moment of the gravitational forces is very small and can be neglected) and from the derivatives of the flight path angle in the horizontal plane and the sideslip angle with respect to the present state vector:

$$\begin{aligned} \mathcal{M}_\psi &= \frac{\partial \mathcal{M}}{\partial \vec{X}} \cdot \frac{\partial \vec{X}}{\partial \psi} = \frac{\partial \mathcal{M}}{\partial \psi} \\ \mathcal{M}_\beta &= \frac{\partial \mathcal{M}}{\partial \vec{X}} \cdot \frac{\partial \vec{X}}{\partial \beta} = \frac{\partial \mathcal{M}}{\partial \beta} \end{aligned} \quad (4.83)$$

The control forces are obtained with the following procedure:

1. We obtain the control E_{y_b} in equation 4.81 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 2.0$ rad/s, $\xi' = 0.7$
2. We obtain the control E_{y_b} in equation 4.82 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 4.0$ rad/s, $\xi' = 0.7$

Within this control equation, the used β_{ref} is always:

$$\beta_{ref} = 0.0 \quad (4.84)$$

3. We obtain the final E_{y_b} adding the results for channels ψ and β .

As a simplification and trying to decouple the motions as much as possible only nozzles 1 and 3 (see figure 4.4) will be moved for yaw control. We will suppose a symmetric distribution of thrust among the nozzles. This is:

$$(E_{y_b})_1 = (E_{y_b})_3 = \frac{E_{y_b}}{2} \quad (4.85)$$

Basically what we are doing with this combination of controls is requesting a null sideslip angle in order to avoid structural moments that could destroy the missile, while trying to follow the desired yaw.

Since the control in the sideslip channel is prioritized (it has a faster dynamics) the achieved trajectory will keep a low sideslip angle.

4.5.1.3.1.3 Stage 1 roll control

In this case the required control has to be applied in such a way that it creates a roll motion. Taking figure 4.4 into account this can be achieved by moving the nozzles in the following way:

Nozzle 1 $\Rightarrow$ motion of an angle δ in the y_b axis

Nozzle 2 $\Rightarrow$ motion of an angle δ in the z_b axis

Nozzle 3 $\Rightarrow$ motion of an angle $-\delta$ in the y_b axis

Nozzle 4 $\Rightarrow$ motion of an angle $-\delta$ in the z_b axis

The roll angle is provided by the following equation:

$$\ddot{\Delta\phi} - \frac{\mathcal{M}_\phi}{I_{xx}} \cdot \Delta\phi = \frac{L_{roll}}{I_{xx}} \cdot E_{roll} \quad (4.86)$$

where:

- $\Delta\phi$ = variation of the roll angle of the missile with respect to the reference value
- $\mathcal{M}_\phi$ = moment of the external forces on the missile in the z_b axis proportional to $\Delta\phi$, without taking the control forces into account
- I_{xx} = moment of inertia of the missile with respect to the x_b axis
- L_{roll} = arm between the mass flow center and the central point of the nozzles ($y_b = 0, z_b = 0$)
- E_{roll} = thrust in the each nozzle according to the δ angle

Note that $\frac{L_{roll}}{I_{xx}} = A$ according to the formulation used in equation 4.66.

The moment proportional to $\Delta\phi$ is obtained from the derivatives of the moment of the aerodynamic forces and from the derivatives of the roll angle with respect to the present state vector:

$$\mathcal{M}_\phi = \frac{\partial \mathcal{M}}{\partial \vec{X}} \cdot \frac{\partial \vec{X}}{\partial \phi} = \frac{\partial \mathcal{M}}{\partial \phi} \quad (4.87)$$

The control force E_{roll} in equation 4.86 is obtained using the expression 4.75 computing the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 2.0$ rad/s, $\xi' = 0.7$.

Within this control equation, the used ϕ_{ref} is always:

$$\phi_{ref} = 0.0 \quad (4.88)$$

4.5.1.3.1.4 Control saturation

Once all the different control functions have been computed we have for each nozzle a desired thrust in y_b :

$$\text{Nozzle 1} \Rightarrow (E_{y_b})_1 = (E_{y_b})_{1\psi} + (E_{y_b})_{1\phi}$$

$$\text{Nozzle 2} \Rightarrow (E_{y_b})_2 = 0.0$$

$$\text{Nozzle 3} \Rightarrow (E_{y_b})_3 = (E_{y_b})_{3\psi} + (E_{y_b})_{3\phi}$$

$$\text{Nozzle 4} \Rightarrow (E_{y_b})_4 = 0.0$$

and z_b :

$$\text{Nozzle 1} \Rightarrow (E_{z_b})_1 = 0.0$$

$$\text{Nozzle 2} \Rightarrow (E_{z_b})_2 = (E_{z_b})_{2\theta} + (E_{z_b})_{2\phi}$$

$$\text{Nozzle 3} \Rightarrow (E_{z_b})_3 = 0.0$$

$$\text{Nozzle 4} \Rightarrow (E_{z_b})_4 = (E_{z_b})_{4\theta} + (E_{z_b})_{4\phi}$$

The thrust in the axial direction in each nozzle i is obtained easily having as input the total thrust in the engine:

$$(E_{x_b})_i = \sqrt{(E_{nozzle})_i^2 - (E_{y_b})_i^2 - (E_{z_b})_i^2} \quad (4.89)$$

where:

$$(E_{nozzle})_i \simeq \frac{E_{engine}}{4} \quad (4.90)$$

supposing an equal distribution of thrust among the nozzles, being E_{engine} the global thrust of the stage 1 motor (see section G.1.1.1).

The angles that the nozzles have to be deflected to get these thrust values can then be computed as:

$$\lambda_i = \text{atan2}(\sqrt{(E_{y_b})_i^2 + (E_{z_b})_i^2}, (E_{x_b})_i) \quad (4.91)$$

$$\mu_i = \text{atan2}((E_{z_b})_i, (E_{y_b})_i) \quad (4.92)$$

Since the maximum possible deflection for each nozzle is 8° (see section 4.5.1.1), whenever a requested nozzle angle is over this value, we saturate it:

$$\text{if } \lambda_i > 8^\circ \Rightarrow \lambda_i = 8^\circ \quad (4.93)$$

4.5.1.3.2 Adaptation to the stage 2

We will detail herein the adaptations of the control equations indicated in section 4.5.1.2 to the control channels in the stage 2.

As indicated in section 4.5.1.1 in the second stage the control is achieved by ejecting freon in several points. This creates a moment of the thrust with respect to the center of mass of the missile in the y_b and z_b axes.

4.5.1.3.2.1 Stage 2 pitch control

Since the missile is launched in the simulation with the roll angle aligned with the desired heading, the control has to be applied in the z_b axis to induce pitch changes in the missile motion. As a consequence only ejection points 1 and 3 (see figure 4.5) will be used for pitch control.

The control forces in this case are obtained in the following way:

1. While the missile is inside the atmosphere:

There are 2 different variables we want to control in this case (pitch: θ , and angle of attack: α), which are provided by the following equations:

$$\ddot{\Delta\theta} - \frac{\mathcal{M}_\theta}{I_{yy}} \cdot \Delta\theta = \frac{L_z}{I_{yy}} \cdot \Delta E_{1-3} \quad (4.94)$$

$$\ddot{\Delta\alpha} - \frac{\mathcal{M}_\alpha}{I_{yy}} \cdot \Delta\alpha = \frac{L_z}{I_{yy}} \cdot \Delta E_{1-3} \quad (4.95)$$

where all the parameters are the same as in equations 4.76 and 4.77 but:

- L_z = distance between the ejection points 1 or 3 and the center of the ejection points section.
- ΔE_{1-3} = difference of thrust between ejection point 1 and ejection point 3

Note that $\frac{L_z}{I_{yy}} = A$ according to the formulation used in equation 4.66.

It has to be noted that $\mathcal{M}_{\dot{\theta}}$ is neglected in equation 4.94 and $\mathcal{M}_{\dot{\alpha}}$ is neglected in equation 4.95. The obtained control without considering these terms behaves properly since they only add a damping in the real response with respect to the initially considered.

The moments proportional to $\Delta\theta$ or $\Delta\alpha$ are obtained as in equation 4.78.

The control forces are then obtained with the following procedure:

- (a) We obtain the control ΔE_{1-3} in equation 4.94 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 2.0$ rad/s, $\xi' = 0.7$
- (b) We obtain the control ΔE_{1-3} in equation 4.95 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 4.0$ rad/s, $\xi' = 0.7$

Within this control equation, the used α_{ref} is always:

$$\alpha_{ref} = 0.0 \quad (4.96)$$

- (c) We obtain the final ΔE_{1-3} adding the results for channels θ and α .

Basically what we are doing with this combination of controls is requesting a *zero lift* trajectory in order to avoid structural moments that could destroy the missile, while trying to follow the desired θ_{ref} .

Since the control in the angle of attack channel is prioritized (it has a faster dynamics) the achieved trajectory will keep a very low angle of attack and the flight path angle of the missile in the vertical plane will decrease according to equation 7.1.

2. Once the missile has left the atmosphere:

In this case we can forget about the control in the angle of attack and we can control directly in the pitch angle (θ) which is provided by the following equation:

$$\ddot{\Delta\theta} - \frac{\mathcal{M}_\theta}{I_{yy}} \cdot \Delta\theta = \frac{L_z}{I_{yy}} \cdot \Delta E_{1-3} \quad (4.97)$$

where all the parameters are the same as in equation 4.94.

Note that $\frac{L_z}{I_{yy}} = A$ according to the formulation used in equation 4.66.

Since we are outside the atmosphere the moment proportional to $\Delta\theta$ can be neglected:

$$\mathcal{M}_\theta = 0 \quad (4.98)$$

We obtain the control ΔE_{1-3} in equation 4.97 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 0.5$ rad/s, $\xi' = 2.0$.

We will suppose that only one of the ejection points is affected each time and the complete value of ΔE_{1-3} will be assigned to one ejection point (1 or 3) depending on whether we require a pitch down or a pitch up manoeuvre.

What we are doing in this case is focusing all the available control in achieving the pitch angle that allows reaching the desired injection point, once the missile has crossed the Kármán line.

4.5.1.3.2.2 Stage 2 yaw control

Since the missile is launched in the simulation with the roll angle aligned with the desired heading, the required control in this case has to be applied in the y_b axis to induce yaw changes in the missile motion.

As a consequence only ejection points 2 and 4 (see figure 4.5) will be used for yaw control.

The control forces in this case are obtained in the following way:

1. While the missile is inside the atmosphere:

There are 2 different variables we want to control in this case (yaw: ψ , and sideslip angle: β), which are provided by the following equations:

$$\ddot{\Delta\psi} - \frac{\mathcal{M}_\psi}{I_{zz}} \cdot \Delta\psi = \frac{L_y}{I_{zz}} \cdot \Delta E_{2-4} \quad (4.99)$$

$$\ddot{\Delta\beta} - \frac{\mathcal{M}_\beta}{I_{zz}} \cdot \Delta\beta = \frac{L_y}{I_{zz}} \cdot \Delta E_{2-4} \quad (4.100)$$

where all the parameters are the same as in equations 4.81 and 4.82 but:

- L_y = distance between the ejection points 2 or 4 and the center of the ejection points section.
- ΔE_{2-4} = difference of thrust between ejection point 2 and ejection point 4

Note that $\frac{L_y}{I_{zz}} = A$ according to the formulation used in equation 4.66.

It has to be noted that $\mathcal{M}_{\dot{\psi}}$ is neglected in equation 4.99 and $\mathcal{M}_{\dot{\beta}}$ is neglected in equation 4.100. The obtained control without considering these terms behaves properly since they only add a damping in the real response with respect to the initially considered.

The moments proportional to $\Delta\psi$ or $\Delta\beta$ are obtained as in equation 4.83.

The control forces are then obtained with the following procedure:

- (a) We obtain the control ΔE_{2-4} in equation 4.99 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 2.0$ rad/s, $\xi' = 0.7$
- (b) We obtain the control ΔE_{2-4} in equation 4.100 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 4.0$ rad/s, $\xi' = 0.7$

Within this control equation, the used β_{ref} is always:

$$\beta_{ref} = 0.0 \quad (4.101)$$

- (c) We obtain the final ΔE_{2-4} adding the results for channels ψ and β .

Basically what we are doing with this combination of controls is requesting a null sideslip angle in order to avoid structural moments that could destroy the missile, while trying to follow the desired ψ_{ref} .

Since the control in the sideslip channel is prioritized (it has a faster dynamics) the achieved trajectory will keep a low sideslip angle.

2. Once the missile has left the atmosphere:

In this case we can forget about the control in the sideslip angle and we can control directly in the yaw angle (ψ) which is provided by the following equation:

$$\ddot{\Delta\psi} - \frac{\mathcal{M}_\psi}{I_{zz}} \cdot \Delta\psi = \frac{L_y}{I_{zz}} \cdot \Delta E_{2-4} \quad (4.102)$$

where all the parameters are the same as in equation 4.99.

Since we are outside the atmosphere the moment proportional to $\Delta\psi$ can be neglected:

$$\mathcal{M}_\psi = 0 \quad (4.103)$$

We obtain the control ΔE_{2-4} in equation 4.102 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing $\omega' = 0.5$ rad/s, $\xi' = 2.0$.

We will suppose that only one of the ejection points is affected each time and the complete value of E_{2-4} will be assigned to the corresponding ejection point (2 or 4) depending on whether we require a turn left or a turn right manoeuvre.

What we are doing in this case is focusing all the available control in achieving the yaw angle that allows reaching the desired injection point, once the missile has crossed the Kármán line.

4.5.1.3.2.3 Stage 2 roll control

In this case as indicated in 4.5.1.1 the roll motion is achieved by releasing gas through two pairs of roll control nozzles on opposite sides of the missile.

The roll angle is provided by the following equation:

$$\ddot{\Delta\phi} - \frac{\mathcal{M}_\phi}{I_{xx}} \cdot \Delta\phi = \frac{L_{roll}}{I_{xx}} \cdot E_{roll} \quad (4.104)$$

where all the parameters are the same as in equation 4.86 but:

- L_{roll} = stage 2 diameter according to the indications in 4.5.1.1
- E_{roll} = thrust generated by all roll control nozzles

Note that $\frac{L_{roll}}{I_{xx}} = A$ according to the formulation used in equation 4.66.

The moment proportional to $\Delta\phi$ is obtained as in equation 4.87.

The control force E_{roll} in equation 4.104 is obtained using the expression 4.75 computing the gains with equations 4.71, 4.72 and 4.74 imposing:

1. While the missile is inside the atmosphere: $\omega' = 2.0$ rad/s, $\xi' = 0.7$
2. Once the missile has left the atmosphere: $\omega' = 0.5$ rad/s, $\xi' = 2.0$

Within this control equation, the used ϕ_{ref} is always:

$$\phi_{ref} = 0.0 \quad (4.105)$$

4.5.1.3.2.4 Control saturation

In stage 2 we have to limit the maximum thrust that could be subtracted by ejecting freon. As a supposition it has been considered that the maximum possible subtraction is a 20 %:

$$\Delta E_{maximum} = 0.2 \cdot \frac{E_{engine}}{4} \quad (4.106)$$

considering that the global thrust of the engine E_{engine} in stages 2 and 3 (as indicated in section G.1.1.2 and G.1.1.3) is equally distributed among the ejection points.

We also have to limit the maximum roll moment we consider as possible using the available control. As a supposition we set:

$$\text{Maximum Thrust}_{roll} = 1000 \text{ N} \quad (4.107)$$

4.5.1.3.3 Adaptation to the stage 3

The available controls in stage 3 are similar to the ones in stage 2: the ejection of liquid is used for controlling the missile in pitch and yaw (with the only difference that, instead of freon, strontium perchlorate is used) and the release of gas through pairs of roll control nozzles on opposite sides of the missile is used for roll control.

In this stage the missile is always outside the atmosphere. As a consequence the control is similar to the one for stage 2 once the missile has left the atmosphere.

In fact, the same values of ω' and ξ' used for computing the pitch, yaw and roll control in stage 2 outside the atmosphere are applied in this case.

Also the considered values for control saturation are the same ones as in section 4.5.1.3.2.4.

4.5.1.3.4 Adaptation to the post-boost phase

We will detail herein the adaptations of the control equations as indicated in section 4.5.1.2 to the control channels in the post-boost phase.

In this stage as detailed in G.1.2.4.1 we consider that the controls are directly the control moments $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_z$ so we will use values for ω' and ξ' with a slower dynamics and a bigger damping:

- For pitch control: $\omega' = 0.5 \text{ rad/s}$, $\xi' = 2.0$
- For yaw control: $\omega' = 0.5 \text{ rad/s}$, $\xi' = 2.0$
- For roll control: $\omega' = 0.5 \text{ rad/s}$, $\xi' = 2.0$

In this case the reference roll angle (ϕ_{ref}) is always null:

$$\phi_{ref} = 0.0 \quad (4.108)$$

As a simplification we will impose as saturation a supposed maximum value for these control moments:

$$\text{Maximum Moment} = 100 \text{ N} \cdot \text{m} \quad (4.109)$$

4.5.2 Control forces in the interceptor missile

4.5.2.1 Existing controls

As explained in chapter 5 the GBI missile is considered as the interceptor missile in the simulator. The GBI missile is a modern missile with controls much simpler than the ones used in the Minuteman III missile.

- First, second and third stages

All these stages have a single movable exhaust nozzle capable of pivoting through an angle of ± 5 degrees from null, in a line parallel to the motor centerline (see [14]).

The motion of the nozzle pivoting in the vertical plane of the body frame allows controlling in pitch and in the angle of attack. The motion of the nozzle in a horizontal plane allows controlling in yaw and in the sideslip angle.

No control is considered in roll. This is because having a fast computer it is possible to project at all times the required reference angles taking the roll angle into account, in the same way that having a fast computer allows using strap-down IMUs instead of gimballed ones. Also the missiles have axial symmetry so any roll moment on the missile will be low and the roll angle will be easy to compute without much errors.

- Exoatmospheric Kill Vehicle (EKV)

The EKV has a system called DACS (Divert and Attitude Control System) made of 4 divert thrusters and 2 attitude control systems using liquid propellant.

The attitude control systems are used to provide the adequate attitude for the EKV EO/IR sensor in order to track the ICBM. The divert thrusters allow providing the required lateral deviations to reach the incoming target.

Figure 4.7: Exoatmospheric Kill Vehicle
(picture from reference [15])

4.5.2.2 Control equations

The GBI missile is a modern missile so it is possible that it implements control equations from the modern control theory and controls simultaneously both horizontal and vertical channels.

However and for the sake of simplicity a similar approach as in the case of the Minuteman III missile will be used: a second order system represented by equation 4.66 will be considered for the dynamics of the system in each channel, and the control term will be computed according to equation 4.75 using equations 4.71, 4.72, and 4.74 to obtain the gains once a desired behaviour for each channel (setting desired values of ω' and ξ') is chosen.

When $\omega \neq 0$ the values $\omega' = 2.0$, $\xi' = 0.7$ will be considered as typical desired characteristics for a channel for which a normal dynamics is required and $\omega' = 4.0$, $\xi' = 0.7$ will be considered for a channel for which a faster dynamics is requested.

When $\omega = 0$, the control forces directly affect the missile motion without any opposite force, so we will request a slower system dynamics (lower ω') and a bigger damping for the extended system (ξ'). The chosen values are $\omega' = 0.5$, $\xi' = 2.0$.

4.5.2.3 Control forces

The guidance strategies (see chapter 7) lead to control forces that will be detailed herein.

4.5.2.3.1 Adaptation to the stages 1 and 2 inside the atmosphere

We will detail herein the adaptations of the control equations indicated in section 4.5.1.2 to the control channels in the stages 1, 2 and 3 while the missile is within the atmosphere.

As indicated in section 4.5.2.1 the first 3 stages of the GBI missile have the same controls: a unique nozzle pivots up to ± 5 degrees from null, in a line parallel to the motor centerline (see [14]). This creates thrust forces in the y_b and z_b axes.

As a simplification we will suppose that the missile is placed before launch so that the angle $\phi = 0^\circ$ is aligned with the desired initial heading. This is probably true since it is very easy to set the origin of the roll angle in the desired initial heading direction by the guidance computer once the target position has been input.

In any case, and since the roll is not controlled in this missile, the vertical plane in the navigation frame will not be the plane $y_b = 0$ and the horizontal plane in the navigation frame will not be the plane $z_b = 0$.

Because of this we need to project the θ and ψ errors in the y_b and z_b directions taking the roll angle into account:

$$\text{Projection in } z_b : \text{angle}_{z_b} = \theta \cdot \cos \phi + \psi \cdot \sin \phi \quad (4.110)$$

$$\text{Projection in } y_b : \text{angle}_{y_b} = -\theta \cdot \sin \phi + \psi \cdot \cos \phi \quad (4.111)$$

where:

θ = pitch

ψ = yaw

ϕ = roll angle of the missile

angle_{z_b} and angle_{y_b} will be the angles to control.

4.5.2.3.1.1 Control in z_b

There are 2 different variables we want to control in this case (angle $_{z_b}$ and angle of attack: α), which are provided by the following equations:

$$\frac{d^2 \Delta \text{angle}_{z_b}}{dt^2} - \frac{\mathcal{M}_{\text{angle}_{z_b}}}{I_{yy}} \cdot \Delta \text{angle}_{z_b} = \frac{L_{thrust}}{I_{yy}} \cdot E_{z_b} \quad (4.112)$$

$$\ddot{\Delta \alpha} - \frac{\mathcal{M}_{\alpha}}{I_{yy}} \cdot \Delta \alpha = \frac{L_{thrust}}{I_{yy}} \cdot E_{z_b} \quad (4.113)$$

where:

- $\Delta \text{angle}_{z_b}$ = variation of angle $_{z_b}$ with respect to the reference value
- $\Delta \alpha$ = variation of the angle of attack of the missile with respect to the reference value
- $\mathcal{M}_{\text{angle}_{z_b}}$ = moment of the external forces on the missile in the y_b axis proportional to $\Delta \text{angle}_{z_b}$, without taking the control forces into account.
- $\mathcal{M}_{\alpha}$ = moment of the external forces on the missile in the y_b axis proportional to $\Delta \alpha$, without taking the control forces into account.
- I_{yy} = moment of inertia of the missile with respect to the y_b axis
- L_{thrust} = arm between the thrust application point (mass flow center) and the center of mass of the missile (in the x_b axis)
- E_{z_b} = thrust in the z_b direction

Note that $\frac{L_{thrust}}{I_{yy}} = A$ according to the formulation used in equation 4.66.

It has to be noted that $\mathcal{M}_{\frac{d(\text{angle}_{z_b})}{dt}}$ is neglected in equation 4.112 and $\mathcal{M}_{\dot{\alpha}}$ is neglected in equation 4.113. The obtained control without considering these terms behaves properly since they only add a damping in the real response with respect to the initially considered.

The moments proportional to $\Delta \text{angle}_{z_b}$ and $\Delta \alpha$ are obtained from the derivatives of the moment of the aerodynamic forces (the moment of the gravitational forces is very small and can be neglected) and from the derivatives of angle $_{z_b}$ with respect to the present state vector.

The control forces are obtained with the following procedure:

1. We obtain the control E_{z_b} in equation 4.112 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing: $\omega' = 2.0$ rad/s, $\xi' = 0.7$
Within this control equation, the used reference for angle $_{z_b}$ is the projection according to the present roll (equation 4.110) of the θ_{ref} and ψ_{ref} reference angles.
2. We obtain the control E_{z_b} in equation 4.113 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing: $\omega' = 4.0$ rad/s, $\xi' = 0.7$
Within this control equation, the used α_{ref} is always:

$$\alpha_{ref} = 0.0 \quad (4.114)$$

3. We obtain the final E_{z_b} adding the results for channels angle $_{z_b}$ and α .

Basically what we are doing with this combination of controls is commanding a *zero lift* trajectory while trying to achieve the desired reference pitch angle in the vertical plane.

Since the *zero lift* trajectory is prioritized (it has a faster dynamics) after the *initial kick* the achieved trajectory will keep a low angle of attack, and the flight path angle of the missile in the vertical plane will decrease according to equation 7.1.

4.5.2.3.1.2 Control in y_b

There are 2 different variables we want to control in this case (angle $_{y_b}$ and sideslip angle: β), which are provided by the following equations:

$$\frac{d^2 \Delta \text{angle}_{y_b}}{dt^2} - \frac{\mathcal{M}_{\text{angle}_{y_b}}}{I_{zz}} \cdot \Delta \text{angle}_{y_b} = \frac{L_{thrust}}{I_{zz}} \cdot E_{y_b} \quad (4.115)$$

$$\ddot{\Delta \beta} - \frac{\mathcal{M}_{\beta}}{I_{zz}} \cdot \Delta \beta = \frac{L_{thrust}}{I_{zz}} \cdot E_{y_b} \quad (4.116)$$

where:

- $\Delta \text{angle}_{y_b}$ = variation of angle $_{y_b}$ with respect to the reference value
- $\Delta \beta$ = variation of the sideslip angle of the missile with respect to the reference value
- $\mathcal{M}_{\text{angle}_{y_b}}$ = moment of the external forces on the missile in the z_b axis proportional to $\Delta \text{angle}_{y_b}$, without taking the control forces into account.
- $\mathcal{M}_{\beta}$ = moment of the external forces on the missile in the z_b axis proportional to $\Delta \beta$, without taking the control forces into account.
- I_{zz} = moment of inertia of the missile with respect to the z_b axis
- L_{thrust} = arm between the thrust application point (mass flow center) and the center of mass of the missile (in the x_b axis)
- E_{y_b} = thrust in the y_b direction

Note that $\frac{L_{thrust}}{I_{zz}} = A$ according to the formulation used in equation 4.66.

It has to be noted that $\mathcal{M}_{\frac{d(\text{angle}_{y_b})}{dt}}$ is neglected in equation 4.115 and $\mathcal{M}_{\beta}$ is neglected in equation 4.116. The obtained control without considering these terms behaves properly since they only add a damping in the real response with respect to the initially considered.

The moments proportional to $\Delta \text{angle}_{y_b}$ and $\Delta \beta$ are obtained from the derivatives of the moment of the aerodynamic forces (the moment of the gravitational forces is very small and can be neglected) and from the derivatives of angle $_{y_b}$ with respect to the present state vector.

The control forces are obtained with the following procedure:

1. We obtain the control E_{y_b} in equation 4.115 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing: $\omega' = 2.0$ rad/s, $\xi' = 0.7$

Within this control equation, the used reference for angle $_{y_b}$ is the projection according to the present roll (equation 4.111) of the θ_{ref} and ψ_{ref} reference angles.

2. We obtain the control E_{y_b} in equation 4.116 with the expression 4.75 obtaining the gains with equations 4.71, 4.72 and 4.74 imposing: $\omega' = 4.0$ rad/s, $\xi' = 0.7$

Within this control equation, the used β_{ref} is always:

$$\beta_{ref} = 0.0 \quad (4.117)$$

3. We obtain the final E_{y_b} adding the results for channels angle $_{y_b}$ and β .

Basically what we are doing with this combination of controls is requesting a null sideslip angle in order to avoid structural moments that could destroy the missile, while trying to follow the desired $\psi_{y_{ref}}$.

Since the control in the sideslip channel is prioritized (it has a faster dynamics) the achieved trajectory will keep a low sideslip angle.

4.5.2.3.1.3 Control saturation

Once all the different control functions have been computed we have for the nozzle a desired thrust in y_b and z_b :

$$\begin{aligned} (E_{y_b}) &= (E_{y_b})_{\text{angle}_{y_b}} \\ (E_{z_b}) &= (E_{z_b})_{\text{angle}_{z_b}} \end{aligned}$$

The thrust in the axial direction is obtained having as input the total thrust (E) in the engine:

$$(E_{x_b}) = \sqrt{(E^2 - (E_{y_b})^2 - (E_{z_b})^2)} \quad (4.118)$$

The angles that the nozzle has to be deflected to get these thrust values can then be computed as:

$$\lambda = \text{atan2}(\sqrt{(E_{y_b})^2 + (E_{z_b})^2}, E_{x_b}) \quad (4.119)$$

$$\mu = \text{atan2}(E_{z_b}, E_{y_b}) \quad (4.120)$$

Since the maximum possible deflection is 5° (see section 4.5.2.1), whenever a requested angle is over this value, we saturate it:

$$\text{if } \lambda > 5^\circ \Rightarrow \lambda = 5^\circ \quad (4.121)$$

4.5.2.3.2 Adaptation to the stage 2 outside the atmosphere and stage 3

We will detail herein the adaptations of the control equations indicated in section 4.5.1.2 to the control channels in the stages 2 and 3 while the missile is outside the atmosphere.

Once the missile has left the atmosphere we can forget the control in the angle of attack and the sideslip angle and focus in achieving the pitch angle and the yaw angle that allow achieving the final flight path angles.

The equations are similar to the ones described in section 4.5.2.3.1. As a difference in this case we change the gains to be used, since we do not have in this case any opposite force to the control forces, and impose as desired values for both z_b and y_b channels: $\omega' = 0.5 \text{ rad/s}$, $\xi' = 2.0$. The considered values for control saturation are the same ones as in section 4.5.2.3.1.3.

4.5.2.3.3 Adaptation to the EKV

We will detail herein the adaptations of the control equations as indicated in section 4.5.1.2 to the control channels in the EKV.

In this stage the missile is always outside the atmosphere. As a difference with the previous stages, the EKV missile has a control in roll which allows stopping the roll rate that appears on the GBI missile in the previous stages. The values for ω' and ξ' will be the typical ones applied when the missile is outside the atmosphere: $\omega' = 0.5 \text{ rad/s}$, $\xi' = 2.0$ (for pitch, yaw and roll control).

As detailed in G.2.2.4.1 in this case we consider that the controls are directly the control moments $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_z$ so as a simplification we will impose as saturation a supposed maximum value for these control moments:

$$\text{Maximum Moment} = 100 \text{ N} \cdot \text{m} \quad (4.122)$$

4.6 Considered noise in the external forces acting on the missile

In order to successfully analyse the behaviour of a guidance algorithm within this interception environment it is necessary to consider a certain variability among the different simulations. This is, stochastic variables have to be considered.

The following stochastic variables have been considered for the simulation:

- Gravity force

Gaussian errors will be considered in the gravity force.

Reference [16] (Tenzer et al., 2009) provides a map with the gravity disturbances evaluated at the Earth's surface using the EGM2008 gravitational model ([17]):

Figure 4.8: Gravity disturbances evaluated at the Earth's surface

The statistics related to this map, computed at the 1×1 arc degree grid of points of the Earth's surface are also provided by reference [16] (Tenzer et al., 2009):

Table 4.3: Statistics for the Gravity disturbances at the Earth's surface

Minimum (mGal)	Maximum (mGal)	Average (mGal)	σ (mGal)
-303	293	-0.7	29

We will neglect as an approximation the mean value of the gravity disturbance in table 4.3, and apply the standard deviation to estimate the error in the gravitational attraction within the missile trajectory as if it were a Gaussian distribution.

This is, the average value for the gravity force will be computed as indicated in section 4.2.1 and a value of 29 mGals will be used for the standard deviation.

$$\vec{g}(\varphi, \lambda, h)_{\text{simulated}} \simeq \mathcal{N}(\vec{g}(\varphi, \lambda, h)_{\text{theoretical}}, \sigma^2) \quad (4.123)$$

with $\sigma = 29$ mGal.

- Thrust force

Taking into account the simplifications used in the expressions for the thrust force in section 4.4 it is very difficult to include stochastic variables within the model. As a simplification we will directly consider that the whole thrust force follows a Gaussian distribution with as average value the nominal one and considering a standard deviation given by a % of the nominal thrust:

$$E_{\text{simulated}} = \mathcal{N}(E_{\text{nominal}}, \sigma) \quad (4.124)$$

with $\sigma = \% \cdot E$

- Aerodynamic force and moments

No stochastic parameters have been considered in the aerodynamic forces and moments nor in the atmospheric model since the related parameters do not usually vary according to a Gaussian distribution. As a consequence it has been preferred to increase the % of variation of the thrust force to include within this variation the variability in the aerodynamic forces and moments and the atmospheric model.

The simulator manages these stochastic variables using boolean commands, so that it is possible to activate or deactivate the presence of stochastic variables in the simulation according to the case that is to be executed.

Chapter 4 references

- [1] Robert J. Naumann. *NASA Technical Report TR R-183: Observed Torque-Producing Forces Acting on Satellites*. George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, December 1963. 70
- [2] Marek Ziebart. *High Precision Analytical Solar Radiation Pressure Modelling for GNSS Spacecraft*. PhD thesis, University of East London, June 2001. 70, 71
- [3] National Imagery and Mapping Agency, Department of Defense. *Technical Report: NIMA TR8350.2: World Geodetic System 1984. Its Definition and Relationship with Local Geodetic Systems*. National Geospatial-Intelligence Agency, amendment 1, third edition, January 2000. URL <http://earth-info.nga.mil/GandG/publications/tr8350.2/wgs84fin.pdf>. 72, 73
- [4] Peter C. Hughes. *Spacecraft attitude Dynamics*. Dover Publications, Inc., Mineola, New York, second edition, 2004. 74
- [5] Frank J. Regan and Satya M. Anandakrishnan. *Dynamics of Atmospheric Re-Entry*. American Institute of Aeronautics and Astronautics, Inc., Washington, DC, first edition, 1993. 75, 76, 77
- [6] Engineering Sciences Data Unit (ESDU). *ESDU 77022: Equations for calculation of International Standard Atmosphere and associated off-standard atmosphere*. Information Handling Services Inc., Houndsditch, London, Amendment C edition, November 2008. 75
- [7] Engineering Sciences Data Unit (ESDU). *ESDU 77021: Properties of a standard atmosphere*. Information Handling Services Inc., Houndsditch, London, Amendment B edition, March 2005. 75
- [8] William B. Blake. *AFRL-VA-WP-TR-1998-3009. Missile DATCOM User's Manual - 1997 Fortran 90 Revision*. Dover Publications, Inc., Wright-Patterson Air Force Base, Ohio, first edition, February 1998. 79, 82, 83, 85
- [9] Jack N. Nielsen. *Missile Aerodynamics*. McGraw-Hill Book Company Inc., The Maple Press Company, York, PA, first edition, 1960. 79
- [10] C.L.Farrar and D.W. Leeming. *Battlefield Weapons Systems and Technology Series. Volume X: Military Ballistics. A Basic Manual*. Brassey's Publishers Limited, Pergamon Group, Oxford, England, 1983. 84
- [11] J. Salvá-Monfort. Apuntes de motores cohete printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, September 1995. 87, 88, 89
- [12] ICBM Prime Team, Prime - 19378, TRW Systems. *Minuteman Weapon System History and Description*. ICBM System Program Office, Hill Air Force Base Utah, second edition, July 2001. 92, 93

-
- [13] David P. Blanks, Anthony M. Logue, Stephen J. Skotte, Douglas M. Bruce, Ralph A. Sandfry, and Michael L. Zywiec. A Two-Stage Intercontinental Ballistic Missile (ICBM) Design Optimization Study and Life Cycle Cost Analysis. Master's thesis, Faculty of the School of Engineering of the Air Force Institute of Technology Air University, Wright-Patterson Air Force Base, Ohio, December 1992. 92, 93
- [14] ATK. ATK Space Propulsion Products Catalog. <http://cms.atk.com/SiteCollectionDocuments/ProductsAndServices/ATK-Motor-Catalog-2012.pdf>, September 2012. 107, 108
- [15] Missile Defense Advocacy Alliance. Ground Based Interceptor Exoatmospheric Kill Vehicle. <http://missiledefenseadvocacy.org/missile-defense-systems/u-s-deployed-intercept-systems/ground-based-midcourse-defense/ground-based-interceptor-exoatmospheric-kill-vehicle/>. [web page accessed on 08/11/2013]. 107
- [16] Robert Tenzer, K. Hamayun, and Peter Vajda. Global maps of the CRUST 2.0 crustal components stripped gravity disturbances. *Journal of geophysical research*, 114, May 2009. doi: 10.1029/2008JB006016. 113
- [17] Nikolaus K. Pavlis, Simon A. Holmes, Steve C. Kenyon, and John K. Factor. An Earth Gravitational Model to Degree 2160: EGM2008. *General Assembly of the European Geosciences Union*, April 2008. 113
- [18] Brian L. Stevens and Frank L. Lewis. *Aircraft Control and Simulation*. John Wiley & Sons, Inc., New York, 1992.
- [19] J. Peláez. Teoría del potencial y aplicaciones printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, September 1993.
- [20] C. Sánchez-Tarifa. Aerorreactores y turbinas de gas printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, November 1978.
- [21] ATK. ATK Space Propulsion Products Catalog. https://wiki.umn.edu/pub/AEM_Air_Launch_Team/LaunchTrajectoryDesign/Space_Products_Catalog_B.pdf, April 2007.

Chapter 5

Structure of the Simulator

This chapter provides the overall structure and characteristics of the developed simulator for the missiles.

5.1 Main characteristics of the simulator

5.1.1 Language and existing code packages used

The simulator of the missiles has been implemented in MATLAB[®]. This is a high-level language and interactive environment for numerical computation, visualization, and programming especially suited for analysing data and developing algorithms. This language has been chosen since it allows a fast implementation and a powerful verification, visualization and debugging capability, ideal for testing the implemented equations and algorithms.

MATLAB[®] allows using existing code packages for many physical and mathematical problems. However it has been decided to only use the basic features of this framework and implement ad hoc all the particular packages related to physical or mathematical problems required for the simulation of the missiles. This limits possible problems that could arise from using code not specifically designed for our purposes.

5.1.2 Missiles implemented in the simulation

5.1.2.1 ICBM in the simulation

This thesis is based on the interception of an ICBM using a long-range kill vehicle. Since the only nation with these kill vehicles is the United States of America, the scenario to simulate implies an attack to this country.

In this scenario a massive attack by a superpower like Russia has no sense since the American Missile Defense System is designed only to be able to neutralize the threat posed by a very small number of incoming missiles. The Ground-Based Midcourse Defense System (see chapter 2) is more suited for the interception of a single missile launched by a terrorist group that somehow acquires a missile system, or for the interception of a reduced number of ICBMs launched by a 'rogue' state.

We will consider the case of a single missile launched from a 'rogue' state. The scenarios related to such an event usually considered by the American agencies involve North Korea or Iran (see for example [1]). Among these possibilities, we will consider arbitrarily the base for the missile in Iran.

In this scenario the considered ICBM should be from Iran, whose missiles are based on Russian or North Korean developments. However it was impossible to obtain relevant data about Russian, Indian or Chinese ICBMs. Even more difficult is to obtain information about North Korean or Iranian missiles.

In order to overcome this problem it has been decided to simulate the Minuteman III missile in its current version (LGM-30G), since for this missile enough information is available, with the hope that it will not be too different to ICBMs from other countries, being as it is a very old missile (developed in the 1960s) that has been probably used as a base in the development of more recent missiles by other countries.

An extensive search of existing data for this missile has been undertaken within this thesis. The final missile data to be used for the simulation of the Minuteman III LGM-30G ICBM (geometry and inertia data as well as motor and control parameters) can be found in Appendix G.

As indicated in section 4.3.2, the geometry data of the missile indicated in Appendix G has been used as an input for the Missile DATCOM program (see reference [2]) in order to obtain the aerodynamic coefficients to be applied for the different stages of the ICBM.

The Minuteman III is able to deploy up to 3 Multiple Independently targetable Reentry Vehicles (MIRVs), in the same way as most of the ICBMs (see table 1.1). However for the sake of simplicity we will only consider 1 RV for the interception problem. This is also the present state of the art in the Missile Ballistic Missile Defense System, since all the tests involved are dealing only with a single target (see [3]).

The case of several RVs will be probably addressed by launching several GBIs almost at the same time, once the system behaves properly against a single target.

5.1.2.2 Interceptor missile in the simulation

The GBI missile in its present configuration (based on the OBV Orbital Booster Vehicle by Orbital Sciences) will be the interceptor used for the simulation within this thesis. This has been decided since this is the only interceptor system within the NMD with real capability of ICBM interception, and since because of its long range this will be the most challenging scenario in terms of guidance strategies.

As indicated in chapter 2 the Ground Based Interceptor missiles (GBIs) are based at Fort Greely, Alaska and Vandenberg Air Force Base, California. In this thesis the considered base is Fort Greely.

As in the case of the ICBM, an extensive search of existing data for this missile has been undertaken. The results in terms of geometry, inertia data, motor parameters and control parameters can be found in Appendix G.

These data has been used as an input for the Missile DATCOM model to generate the aerodynamic coefficients to be applied for the different stages of the missile.

5.2 High level program structure

The main structure of the simulator is indicated herein using structure charts.

5.2.1 Graphs notation

The following conventions will be used in the charts:

- Magenta squares indicate start point and end point of a program
- A small black circle indicates start point or end point of a computation block
- A white square indicates a module performing an action
The action to be performed is denoted inside the square. Relevant output data from the action could also be denoted inside.
- Pink squares indicate computation blocks for which a structure chart exists within this document
- An ellipse or a circle indicates initial data available from outside the chart
The most relevant data is denoted inside the ellipse or circle.
In order to simplify the charts it will be supposed that all the data available at previous execution steps is available (as it is the real case), so the required initial data is only shown once.
- Red arrows indicate data flow
- A diamond indicates a condition
Each of the possible outputs of the condition (yes or no) leads to a different flow path for the program.
- Black arrows indicate execution flow
This execution flow can be followed from the start point to the end point going through all the conditions
- Dotted squares indicate an iteration loop

5.2.2 Numerical integration chart

As explained in section 3.5, the Predictor-Corrector method PECE AB2-AM3 (Adams-Bashforth 2, Adams-Moulton 3) is used to integrate the missile system equations.

This method was chosen for its big absolute stability region and its high order, even though it is complex to implement since it requires executing all the equations to obtain $\vec{F}$ twice, where:

$$\dot{\vec{X}} = \vec{F}(\vec{X}, \vec{U}, t). \quad (5.1)$$

All the elements of $\vec{F}$ are given in equations 3.33.

The algorithm is depicted in figure 5.1:

Figure 5.1: Flow chart of the numerical integration loop

5.2.3 Guidance-Control flow chart

The guidance algorithm is depicted together with the control algorithm since they are coupled.

The process starts from the present state of the missile. With this information and taking the target position (and velocity vector in the case of the GBI) into account the selected guidance algorithm (detailed in chapter 8, chapter 9 or chapter 10) provides the required attitude for the missile to reach the target.

As explained in chapter 7 the initially computed time to target is used in the guidance algorithm (for conventional ascent guidance algorithms, based in the solution of the Lambert problem) since that increases the robustness of the guidance scheme.

The output of the guidance algorithm is then used as reference values for θ_{ref} and ψ_{ref} that the control system has to achieve and maintain, providing a final control vector $\vec{U}$ according to the indications in section 4.5.

The control system takes into account the active control now in the missile, and the indication of whether aerodynamic forces can be neglected.

The computation flow is depicted in figure 5.2:

Figure 5.2: Flow chart of the guidance and control algorithms

5.2.4 Compute $\vec{F}$ chart

The system equations (equations 3.33) are integrated as detailed in section 5.2.2 using $\vec{F}$, which has to be computed repeatedly throughout the simulation.

The process starts with the computation of the air density, which is checked in order to indicate whether the atmosphere can be neglected. When this happens no aerodynamic forces and moments are computed and a null value is provided for all of them. This saves substantial computational time since the evaluation of the aerodynamic terms uses coefficients from the Missile DATCOM (see reference [2]) that are interpolated in a time-consuming algorithm.

The gravity and aerodynamic forces and moments are computed according to the expressions given in sections 4.2 and 4.3.

The time since liftoff is also checked to analyse whether a spinning moment should be activated. As indicated in section 1.4.5, soon after its release, the RV is provided with a spin motion that provides stability in the reentry process. This spinning moment is obtained within the simulation as an aerodynamic moment in the Ox_b axis that lasts several seconds.

The spinning of the missile is taken into account in the computation of aerodynamic forces and moments after the reentry according to the indications given in section 4.3.2.2.

The next step is the computation of the thrust forces. This is done using the control vector $\vec{U}$, previously calculated as indicated in section 5.2.3, according to the expressions given in section 4.5. The result of this computation is not only the global force and moment because of the thrust, but also the thrust force through each nozzle, when there is more than one. This is required since the mean exhaust velocity through each nozzle (proportional to the thrust vector through the nozzle) is used in the computation of the $I \cdot \frac{d\Omega_{bi}^b}{dt}$ term according to the expressions in 3.32.

The final steps are the computation of the global force and global moment on the missile, and the computation of the $I \cdot \frac{d\Omega_{bi}^b}{dt}$ term, which is the $\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix}$ vector in expression 3.31.

With all the previous information all the components of $\vec{F}$ can be computed using equations 3.33.

The computation flow is depicted in figure 5.3:

Figure 5.3: Flow chart of the computation of $\vec{F}$

5.2.5 Stability analysis chart

As indicated in section 3.5, a special attention has been given to the analysis of the stability of the numerical system, since the simulation times are large (about half an hour), which requests a big time step, but there are several situations (especially within the atmosphere and when performing some rotations) when a smaller time step has to be used to avoid that the numerical scheme becomes unstable.

In order to cope with this, the procedure indicated in section 3.5.4 will be followed:

1. The A matrix ($A = \frac{\delta \vec{f}}{\delta \vec{X}}$ as explained in section 3.3.6) is computed using as input the present missile state vector and the derivatives of the gravity and aerodynamic forces and moments with respect to the state vector, previously computed as shown in figure 5.3.
2. The B matrix ($B = \frac{\delta \vec{f}}{\delta \vec{u}}$ as explained in section 3.3.6) is computed using as input the active control vector $\vec{U}$, previously computed as indicated in section 5.2.3.
3. The global applicable system transition matrix (the one that relates $\vec{X}(t)$ with $\vec{X}(t + dt)$, this is $A + B \cdot \frac{d\vec{U}}{d\vec{X}}$) is computed ($\frac{d\vec{U}}{d\vec{X}}$ is also provided by the control algorithm as shown in figure 5.2).
4. The eigenvalues of the global system transition matrix are computed.
5. We check whether any of the eigenvalues with negative real part is outside the absolute stability region of the implemented numerical scheme (PECE AB2-AM3 as indicated in section 3.5).

If that is the case, we reduce the simulation time step to:

$$dt_{\text{next step}} = dt_{\text{present step}} \cdot 0.1 \quad (5.2)$$

6. The program maintains the low simulation step 2 seconds and then tries again with a bigger one, in order to speed up the simulation.

If the simulation step required to make a system stable is below a certain margin, we stop the simulation and indicate that a problem exists.

The computation flow is depicted in figure 5.4:

Figure 5.4: Flow chart of the stability analysis algorithm

5.2.6 ICBM simulation chart

It is possible with the simulator to only perform the simulation of the ICBM trajectory as a stand-alone program. The structure of the program in this case is shown in this section.

The simulation starts with the initialization of several data and the generation of the initial ICBM state vector.

With this aim astronomical data (required to obtain the C_e^i transformation matrix as indicated in Appendix section A.3), WGS84 data (required to obtain several transformation matrices and the gravity forces), specific geometric data and control parameters (see Appendix G), global constants, aerodynamics data as well as missile performance tables are used.

Also the simulation case data is input. These data includes the requested target position and the guidance type to be used.

Then the simulation loop is started, increasing the previous time an amount dt each cycle, recomputing then the transformation matrices C_e^i and C_{VECF}^{VECI} (which is $[A \cdot \dot{B} \cdot C \cdot D]^T$ according to equation A.19) and updating the missile inertial properties.

From this step, computation blocks already detailed are executed sequentially within the loop:

1. The guidance algorithm and the control algorithm are executed obtaining the required control vector $\vec{U}$
2. The vector $\vec{F}$ is computed as explained in section 5.2.4
3. The stability algorithm is executed in order to set the time step to be used in the following simulation step
4. The integration block is executed obtaining the state of the system in $t + dt$: $\vec{X}(t + dt)$

The loop is executed until the computed height is below the target height. In this moment the loop is stopped and the obtained accuracy ($\|\vec{X}_{target} - \vec{X}_{ICBM}\|_{final}$) is computed.

An external algorithm is included, as detailed in section 7.3.1.2, in order to compute the sensitivity matrix of the obtained final position with respect to the requested one, and compute variations to be applied as constant offsets to the indications provided by the guidance algorithm. This allows finishing the complete simulation only when the accuracy is below a desired margin.

The computation flow is depicted in figure 5.5:

Figure 5.5: Flow chart of the execution cycle of the ICBM simulation

5.2.7 ICBM-GBI simulation chart

In this case the simulation starts as in the case of the stand-alone ICBM simulator described in section 5.2.6.

Under certain circumstances, the GBI is launched towards the ICBM. In the simulator, as a simplification, this launch is usually performed when the ICBM has finished its boost phase. It is uncertain what the real condition is, since on the one hand the GBI has to be launched as soon as possible, but on the other hand the launch of an incoming ICBM has to be confirmed and its trajectory initially tracked so that the GBI launch is successfully aimed.

Once that the GBI is launched, each time step the state of both missiles is updated sequentially. The only interdependence is the guidance algorithm of the GBI, which requires information of the state of the ICBM (position and sometimes also velocity vector, depending on the used guidance algorithm).

This information would be given in the initial states of the interception by an observation algorithm (typically a Kalman Filter) that combines the information from the different ground based radars and tracking satellites of the Ballistic Missile Defense System (see section 2.3.2). Once the EKV is deployed from the GBI this information would be combined with information from the infrared sensor of the EKV. This observation problem is complex to simulate since it depends on many systems which are unknown. Since the problems involved in the tracking of the ICBM are out of the scope of this thesis, a simple Kalman Filter with the assumption of constant velocity vector is used within the simulator to provide the ICBM state data to be used by the guidance algorithm.

The end condition is different in this case to the stand alone case. We will check the distance ICBM-GBI at all times and stop the simulation when this distance is not decreasing any longer.

Note that in this case no repetition of the simulation cycle is done in order to refine the accuracy of the ICBM. The difference between the initial trajectory of the ICBM and the 'accurate one' will be low, and does not justify the repetition in terms of representativity of the interception problem.

The computation flow is depicted in figure 5.6:

Figure 5.6: Flow chart of the execution cycle of the ICBM-GBI interception simulator

5.3 Outputs of the simulation

Since the simulator is written in MATLAB[®] it is possible to store the complete data structure computed within each simulation with debugging purposes.

Apart from that, several graphics have been prepared so that they are available after each execution:

- Evolution of the geodetic coordinates (latitude, longitude and height) of the missile with time.
- Evolution of the distance of the missile with respect to the launching site with time.
- Evolution of the Mach number of the missile with time.
- Evolution of the components of the velocity vector of the missile (in the navigation frame) with time: V_{North} , V_{East} , V_{Down} and global speed.
- Evolution of the Euler angles of the missile with time.
The reference Euler angles from the guidance module, when applicable, are included in this representation
- Evolution of the angle of attack (AoA, α) and sideslip angle (AoS, β) of the missile with time.
- Variables related to the control in the vertical channel:
 - Evolution of the pitch angle of the missile (θ) with time.
The reference pitch angle (θ_{ref}) from the guidance module, when applicable, is included in this representation.
 - Evolution of the angle of attack (AoA, α) of the missile with time.
 - Evolution of the vertical flight path angle of the missile with time.
The reference vertical flight path angle from the guidance module, when applicable, is included in this representation.
- Variables related to the control in the horizontal channel:
 - Evolution of the yaw angle of the missile (ψ) with time.
The reference yaw angle (ψ_{ref}) from the guidance module, when applicable, is included in this representation.
 - Evolution of the angle of sideslip (AoS, β) of the missile with time.
 - Evolution of the horizontal flight path angle of the missile with time.
The reference horizontal flight path angle from the guidance module, when applicable, is included in this representation.
- Evolution of the angular velocity vector of the missile with time.
- Evolution of the simulation time with respect to the simulation cycle with time.
This is a measure of the evolution of the used time step (dt) with time.
- Evolution of the global force requested from the divert thrusters with time (only for the EKV).

Chapter 5 references

- [1] National Air and Space Intelligence Center. *Ballistic & Cruise Missile Threat*. NASIC Public Affairs Office, Wright Patterson Air Force Base. Ohio, 2013. 118
- [2] William B. Blake. *AFRL-VA-WP-TR-1998-3009. Missile DATCOM User's Manual - 1997 Fortran 90 Revision*. Dover Publications, Inc., Wright-Patterson Air Force Base, Ohio, first edition, February 1998. 119, 123
- [3] Missile Defense Agency. Ballistic Missile Defense Intercept Flight Test Record. <http://www.mda.mil/global/documents/pdf/testrecord.pdf>, November 2014. [web page accessed on 08/01/2015]. 119

Chapter 6

Simulation examples and comparison with available data and other simulators

This chapter provides an example of a simulation case for the trajectory of the ICBM and another example of a complete interception case, both computed with the simulator described in this part [II](#) and using guidance algorithms that will be later on described in part [III](#).

A comparison with known data will be performed in order to validate the simulator to the extent possible.

Finally, the characteristics of the simulator will be compared with other existing simulation platforms.

6.1 Simulation examples

6.1.1 ICBM simulation case

As an example, the case of a Minuteman III ICBM launched from Shahrud, Iran ($36^{\circ}25'05''\text{N}$ $54^{\circ}58'35''\text{E}$, $h = 1,345$ m) and aimed at New York ($40^{\circ}44'54.36''\text{N}$ $73^{\circ}59'8.36''\text{E}$, $h = 100$ m) is shown.

In order to successfully execute the simulation a guidance algorithm has to be used. Different guidance algorithms will be developed for the missiles simulation as explained in part III.

In particular, the example case shown herein was executed considering Q guidance for the ascent phase. The details of this guidance algorithm are explained in section 8.3.4.

In this example the gravity moments are neglected, so that the proper alignment of the missile for the reentry (as explained in chapter 7.1.2.3 and easily noticed in figure 7.2) is easier to observe.

6.1.1.1 Trajectory

The trajectory obtained for the ICBM in this simulation case is shown in the following figure:

Figure 6.1: ICBM trajectory from Iran to NYC (3D view)
(picture obtained with Google EarthTM mapping service)

6.1.1.2 Geodetic position

The variation of latitude, longitude and height with time along the trajectory is shown in the following picture:

Figure 6.2: ICBM trajectory from Iran to NYC (position)

6.1.1.3 Speed

The variation of the velocity vector of the missile with respect to the ECEF frame expressed in the navigation frame (North, East, Down coordinates) is shown in the following picture:

Figure 6.3: ICBM trajectory from Iran to NYC (velocity)

It can be seen that there is an initial profile in which the speed is increasing due to the thrust of the different stages of the missile, and a final profile in which the speed decreases abruptly due to the drag forces in the reentry. Between these 2 profiles the trajectory is not boosted and the changes in the speed are related to an orbital motion outside the atmosphere. A maximum speed of about 7 km/s is achieved by the missile.

A close-up of the boosted phase of the trajectory is shown in the following picture:

Figure 6.4: ICBM trajectory from Iran to NYC (velocity in the boosted phase)

The three stages of the missile can be easily observed in figure 6.4 since there is a change in the speed profile whenever there is a transition from one stage to another.

The Mach number related to the previously shown speed is shown in the following picture. The stars represent the moments when the Kármán line is crossed

Figure 6.5: ICBM trajectory from Iran to NYC (Mach number)

It can be observed that Mach numbers up to almost 10.0 are achieved when leaving the atmosphere (crossing the Kármán line after launch). Since the missile continues accelerating after crossing this line, the achieved Mach number in the reentry is much higher (approximately 26.5).

The oscillations observed in the graphic of the Mach number with time close to crossing the Kármán line are not due to oscillations in the speed (there are no oscillations in the speed as it can be observed in figure 6.3) but to changes in the considered temperatures in the different atmosphere layers, according to table 4.2.

6.1.1.4 Euler angles and flight path angles

The Euler angles during the trajectory are shown in the following picture (reference angles -when existing- in green, real angles in blue):

Figure 6.6: ICBM trajectory from Iran to NYC (Euler angles)

It can be observed that there is a sharp change in the reference angles after cutoff. This is because the attitude is changed then in order to align the missile with the speed in the reentry point. This will minimize lateral forces and moments when reentering.

The flight path angles during the trajectory are shown in the following picture (reference angles -when existing- in green, real angles in blue):

Figure 6.7: ICBM trajectory from Iran to NYC (flight path angles)

The control strategies are implemented in order to achieve these flight path angles with the highest possible accuracy. A close-up of the flight path angles at the end of the boosted phase is provided in order to appreciate how the achieved flight path angles are very close to the reference values:

Figure 6.8: ICBM trajectory from Iran to NYC (flight path angles in the boosted phase)

6.1.1.4.1 Pitch angle, vertical flight path angle, and angle of attack

The relevant part in terms of pitch and vertical flight path angle appears in the boosted phase of the missile, as it can be observed herein (reference angle -when existing- in green, real angles in blue):

Figure 6.9: ICBM trajectory from Iran to NYC (pitch angle and γ_z of the boosted phase)

The *gravity turn* manoeuvre (see section 7.1.1.2) can be observed in figure 6.9 including the *initial kick* (the sudden change in the reference pitch during 20 seconds).

While inside the atmosphere we can observe how, after the initial kick, the pitch angle falls according to a gravity turn manoeuvre.

Once the Kármán line is crossed we forget about controlling the angle of attack and the requested pitch angle, an output from the guidance module, is set. This ensures that the correct flight path angle is achieved at the end of the boosted phase.

Once the boosted phase finishes the correct attitude for reentry is commanded and achieved by the Post-Boost Control System (PBCS). Since this attitude change is done once the boosted phase has finished this only affects the attitude of the missile without changing its path angle.

After the boosted phase the attitude of the missile remains almost fixed along the trajectory (only small forces and moments as the gravity torque affect it) so the changes in pitch are due to the change of the navigation frame along the trajectory.

The achieved angle of attack in the boosted phase of the missile can be observed in the following picture:

Figure 6.10: ICBM trajectory from Iran to NYC (AOA of the boosted phase)

The effect of the gravity turn on the angle of attack can be observed: it increases during the *initial kick* and then decreases until it becomes almost null.

Once the missile crosses the Kármán line the control in angle of attack is stopped so the angle of attack varies according to the chosen pitch angle.

The angle of attach of the missile along the whole trajectory can be observed in the following picture:

Figure 6.11: ICBM trajectory from Iran to NYC (AOA)

Since the reentry vehicle is given at the injection point the pitch angle required for reentering (see section 7.1.2.3), there is an angle of attack at the injection point. This means that, when the spinning motion of the reentry vehicle starts, the angle of attack spins as well. The amplitude of this motion decreases as the missile gets closer to its reentry point, since the pitch angle at the injection point was selected in order to ensure that the axis of the missile is aligned with the velocity vector at the reentry.

In this case, as indicated before, the gravity moment has been neglected so that a perfect match ($\alpha = 0$) when reentering can be observed.

6.1.1.4.2 Yaw angle, horizontal flight path angle, and sideslip angle

The relevant part in terms of yaw and horizontal flight path angle appears in the boosted phase of the missile, as it can be observed herein (reference angle -when existing- in green, real angles in blue):

Figure 6.12: ICBM trajectory from Iran to NYC (yaw angle and γ_y of the boosted phase)

Within the *gravity turn* manoeuvre the priority is given to maintaining a low sideslip angle, so only small differences appear between the achieved yaw angle and the reference value.

Once the Kármán line is crossed we forget about controlling the sideslip angle and the requested yaw angle, an output from the guidance module, is set. This ensures that the correct flight path angle is achieved at the end of the boosted phase.

Once the boosted phase finishes the correct attitude for reentry is commanded and achieved by the Post-Boost Control System (PBCS). Since this attitude change is done once the boosted phase has finished this only affects the attitude of the missile without changing its path angle.

After the boosted phase the attitude of the missile remains almost fixed along the trajectory (only small forces and moments as the gravity torque affect it) so the changes in yaw are due to the change of the navigation frame along the trajectory.

The achieved sideslip angle in the boosted phase of the missile can be observed in the following picture:

Figure 6.13: ICBM trajectory from Iran to NYC (AOS of the boosted phase)

The sideslip angle is maintained as almost null while the missile is inside the atmosphere.

Once the missile crosses the Kármán line the control in sideslip angle is stopped so the sideslip angle varies according to the chosen yaw angle.

The sideslip angle of the missile along the whole trajectory can be observed in the following picture:

Figure 6.14: ICBM trajectory from Iran to NYC (AOS)

Since the reentry vehicle is given at the injection point the yaw angle required for reentering (see section 7.1.2.3), there is a sideslip angle at the injection point. This means that, when the spinning motion of the reentry vehicle starts, the sideslip angle spins as well. The amplitude of this motion decreases as the missile gets closer to its reentry point, since the yaw angle at the injection point was selected in order to ensure that the axis of the missile is aligned with the velocity vector at the reentry.

In this case, as indicated before, the gravity moment has been neglected so that a perfect match ($\beta = 0$) when reentering can be observed.

6.1.1.4.3 Roll angle

The reference roll angle is always zero along the trajectory and apart from several perturbations it is kept like that in the boosted phase of the trajectory:

Figure 6.15: ICBM trajectory from Iran to NYC (roll angle in the boosted phase)

However a spin motion is commanded in the reentry vehicle after it is detached from the deployment module. This is commanded in order to stabilize the reentry vehicle (RV) during the reentry (see section 1.4.5).

A close-up of the start of the spinning motion can be observed herein:

Figure 6.16: ICBM trajectory from Iran to NYC (start of spin)

6.1.1.5 Angular velocity

The angular velocity of the missile along the trajectory can be observed in the following figure:

Figure 6.17: ICBM trajectory from Iran to NYC (angular velocity)

As it can be observed in figure 6.17, apart from several perturbations (when a certain angular velocity is necessary to change the missile attitude) the angular velocity is null most of the time.

The only exception is the spinning motion, which is commanded for the reentry vehicle after it is detached from the deployment module in order to stabilize it during the reentry (see section 1.4.5).

Once this moment is stopped the roll rate keeps almost constant along the trajectory with the only exception of the effect on this roll rate of the gravity torque, which is neglected in this simulation case.

6.1.2 Interceptor trajectory

As an example, a trajectory of a GBI missile launched from Fort Greely, Alaska ($63^{\circ}54'18''\text{N}$ $145^{\circ}33'16''\text{W}$, $h = 478.5$ m) and aimed at an ICBM launched from Shahrud, Iran and aimed at New York is shown.

In order to successfully execute the simulation a guidance algorithm has to be used. Different guidance algorithms will be developed for the missiles simulation as explained in part III.

In particular, the example case shown herein was executed considering Q guidance for the ascent phase of the GBI missile and for the ascent phase of the ICBM. The details of this guidance algorithm are explained in section 8.3.4.

For the terminal phase of the EKV, proportional navigation with gravity compensation is used in this example. The details of this guidance algorithms are explained in section 9.3.2.

6.1.2.1 Trajectory

The trajectory obtained for the missiles in this simulation case is shown in the following figure:

Figure 6.18: GBI trajectory (3D view)
(picture obtained with Google EarthTM mapping service)

6.1.2.2 Geodetic position

The variation of latitude, longitude and height of the interceptor missile with time along the trajectory is shown in the following picture:

Figure 6.19: GBI trajectory (position)

6.1.2.3 Speed

The variation of the velocity vector of the interceptor missile with respect to the ECEF frame expressed in the navigation frame (North, East, Down coordinates) is shown in the following picture:

Figure 6.20: GBI trajectory (velocity)

It can be seen that there is an initial profile in which the speed is increasing due to the thrust of the different stages of the missile. After that profile the trajectory is not boosted (only the divert thrusters of the EKV are providing some control force) and the changes in the speed are related to an orbital motion outside the atmosphere.

A close-up of the boosted phase of the trajectory is shown in the following picture:

Figure 6.21: GBI trajectory (velocity in the boosted phase)

The three stages of the missile can be easily observed in figure 6.21 since there is an abrupt change in the speed profile whenever there is a transition from one stage to another.

The Mach number related to the previously shown speed is shown in the following picture. The star represents the moment when the Kármán line is crossed

Figure 6.22: GBI trajectory (Mach number)

It can be observed that Mach numbers close to 10.5 are achieved when leaving the atmosphere (crossing the Kármán line after launch).

The oscillations observed in the graphic of the Mach number with time close to crossing the Kármán line are not due to oscillations in the speed (there are no oscillations in the speed as it can be observed in figure 6.20) but to changes in the considered temperatures in the different atmosphere layers, according to table 4.2.

6.1.2.4 Euler angles and flight path angles

The Euler angles during the trajectory are shown in the following picture (reference angles in green, real angles in blue):

Figure 6.23: GBI trajectory (Euler angles)

The flight path angles during the trajectory are shown in the following picture (reference angles -when existing- in green, real angles in blue):

Figure 6.24: GBI trajectory (flight path angles)

The control strategies are implemented in order to achieve these flight path angles with the highest possible accuracy. A close-up of the flight path angles at the end of the boosted phase is provided in order to appreciate how the achieved flight path angles are very close to the reference values:

Figure 6.25: GBI trajectory (flight path angles in the boosted phase)

6.1.2.4.1 Pitch angle, vertical flight path angle, and angle of attack

The relevant part in terms of pitch and vertical flight path angle appears in the boosted phase of the missile, as it can be observed herein (reference angle -when existing- in green, real angles in blue):

Figure 6.26: GBI trajectory (pitch angle and γ_z of the boosted phase)

The *gravity turn* manoeuvre (see section 7.1.1.2) can be observed in figure 6.26 including the *initial kick* (the sudden change in the reference pitch during 20 seconds).

While inside the atmosphere we can observe how, after the initial kick, the pitch angle falls according to a gravity turn manoeuvre.

Once the Kármán line is crossed we forget about controlling the angle of attack and the requested pitch angle, an output from the guidance module, is set. This ensures that the correct flight path angle is achieved at the end of the boosted phase.

Once the boosted phase finishes the correct attitude for interception is commanded and achieved by the Exoatmospheric Kill Vehicle (EKV) by aligning the Ox_b axis with the line of sight to the target.

The achieved angle of attack in the boosted phase of the missile can be observed in the following picture:

Figure 6.27: GBI trajectory (AOA of the boosted phase)

The effect of the gravity turn on the angle of attack can be observed: it increases during the *initial kick* and then decreases until it becomes almost null.

Once the missile crosses the Kármán line the control in angle of attack is stopped so the value of the angle of attack varies according to the chosen pitch angle.

The angle of attack of the missile along the whole trajectory can be observed in the following picture:

Figure 6.28: GBI trajectory (AOA)

In this case no spinning motion is commanded in the EKV since it does not have to reenter the atmosphere and it is important that it remains stable in order to track the incoming ICBM and distinguish the warhead from any possible decoy or debris. The angle of attack varies with time according to the variation of the requested pitch angle.

6.1.2.4.2 Yaw angle, horizontal flight path angle, and sideslip angle

The relevant part in terms of yaw and horizontal flight path angle appears in the boosted phase of the missile, as it can be observed herein (reference angles -when existing- in green, real angles in blue):

Figure 6.29: GBI trajectory (yaw angle and γ_y of the boosted phase)

Within the *gravity turn* manoeuvre the priority is given to maintaining a low sideslip angle, so small differences appear between the achieved yaw angle and the reference value.

Once the Kármán line is crossed we forget about controlling the sideslip angle and the requested yaw angle, an output from the guidance module, is set. This ensures that the correct flight path angle is achieved and the end of the boosted phase.

Once the boosted phase finishes the correct attitude for interception is commanded and achieved by the Exoatmospheric Kill Vehicle (EKV) by aligning the Ox_b axis with the line of sight to the target.

The achieved sideslip angle in the boosted phase of the missile can be observed in the following picture:

Figure 6.30: GBI trajectory (AOS of the boosted phase)

Apart from some initial instabilities, the sideslip angle is maintained as almost null while the missile is inside the atmosphere.

Once the missile crosses the Kármán line the control in sideslip angle is stopped so the value of the sideslip angle varies according to the chosen yaw angle.

The sideslip angle of the missile along the whole trajectory can be observed in the following picture:

Figure 6.31: GBI trajectory (AOS)

In this case no spinning motion is commanded in the EKV since it does not have to reenter the atmosphere and it is important that it remains stable in order to track the incoming ICBM and distinguish the warhead from any possible decoy or debris. The sideslip angle varies with time according to the variation of the requested yaw angle.

6.1.2.4.3 Roll angle

The roll angle is not controlled along the boosted part of the trajectory, so even though because of the axial symmetry of the missile there are only small perturbations that generate a roll moment, a certain roll angle appears on the missile:

Figure 6.32: GBI trajectory (roll angle in the boosted phase)

This roll angle is controlled by the Exoatmospheric Kill Vehicle (EKV) that achieves a null roll angle shortly after it is released as shown in the following picture. It has to be noted that it is important that the EKV remains stable in order to track the incoming ICBM and distinguish the warhead from any possible decoy or debris.

Figure 6.33: GBI trajectory (EKV controlling roll angle)

6.1.2.5 Angular velocity

The angular velocity of the missile along the trajectory can be observed in the following figure:

Figure 6.34: GBI trajectory (angular velocity)

As it can be observed in figure 6.34, apart from several perturbations (when a certain angular velocity is necessary to change the missile attitude) the angular velocity is null most of the time.

6.2 Comparison with available data

6.2.1 ICBM simulation

The validation of the simulator will be done in terms of an analysis of the obtained parameters and by comparing data with available data for the real system.

6.2.1.1 General behaviour of the simulator

An extensive battery of cases has been executed with the simulator obtaining in all cases reasonable data. The simulator behaves as expected in all cases and provides the foreseen values for all the variables in terms of overall behaviour.

6.2.1.2 Comparison with available data

The exact performance of the LGM-30G Minuteman III missile is not disclosed and it is difficult to validate the simulator. However, some performance data is available and can be compared with the obtained data in the simulation.

Some indications about the behaviour of the Minuteman III missile are provided in reference [1] and these indications will be compared with the obtained data in a trajectory from Iran to NYC forcing constant values of the vertical flight path angles at injection to ($\gamma_{zref} = 30^\circ$ and $\gamma_{zref} = 20^\circ$):

Table 6.1: Comparison between data in [1] and data from the simulator

Parameter	Value in [1]	Value in the simulator $\gamma_{zref} = 20^\circ$	Value in the simulator $\gamma_{zref} = 30^\circ$
Mach at T = 19 s	1	0.96	0.96
Speed at T = 19 s	335 m/s	310.5 m/s	310 m/s
Δ Height at T = 19 s	2530 m	2596 m	2639 m
Mach at T = 39 s	3	2.8	2.77
Δ Height at T = 39 s	11,580 m	10,764 m	11,438 m
Δ Height at T = 45 s	15,240 m	14,460 m	15,540 m
Δ Height at T = 62 s	30,480 m	28,886 m	31,981 m
Distance traveled at T = 62 s	33,336 m	31,273 m	27,660 m
Δ Height at T = 121 s	96,012 m	96,566 m	114,155 m
Δ Height at T = 123 s	73,152 m	99,225 m	117,595 m
Distance traveled at T = 123 s	166,680 m	162,978 m	146,227 m

It can be seen in table 6.1 that the values in the simulator for $\gamma_{zref} = 20^\circ$ match relatively well the data in [1], being the simulated missile a bit slower at the beginning (difference of 6.7% in the Mach number in T = 39s), which becomes in slightly shorter distance traveled up to T = 62s (difference of 5%).

The data in [1] seems to be wrong for the indicated height at T = 123s, but the achieved height for T = 121s and the distance traveled at T = 123s are very similar to the ones achieved with the simulator setting $\gamma_{zref} = 20^\circ$ (difference of 2% in traveled distance).

It can be concluded that the first stage of the trajectory is a bit slower than the real one (maybe the simulator has a thrust for the first stage a bit smaller than the real value, or maybe the drag coefficient used in the simulator is overestimated), but apart from that the values provided by the simulator (for $\gamma_{zref} = 20^\circ$) seem to have acceptable errors in comparison with the data in [1], with the only exception of the indicated heights at $T = 123s$ that seems to be wrongful data.

6.2.2 GBI simulation

It has not been possible to obtain performance data related to the GBI. As a consequence the simulation of this missile could only be validated in terms of an analysis of the obtained parameters.

An extensive battery of cases has been executed with the simulator obtaining in all cases reasonable data. The simulator behaves as expected in all cases and provides the foreseen values for all the variables in terms of overall behaviour.

6.3 Comparison with other simulation platforms

A comparison between the simulator developed within this thesis and several other simulators available or indicated in the literature is provided in this section.

6.3.1 Comparison with simple simulators

Most of the papers dealing with ballistic missiles algorithms include simulations based on simplified simulators that are only useful to show general concepts. This leads to cases where sometimes developed algorithms are not directly applicable to the real case.

As an example [2] (W.J.Harlin and D.A.Cicci, 2007) proposes an algorithm for targeting (see section 7.3.1.2) based on modifications of the initial quaternions at launch that cannot in fact be changed since these missiles all have a vertical launch.

In other cases the algorithms indicated are applicable, but their benefit is reduced when applied in a realistic simulator (see for example the analysis in 7.2.3).

This is why we decided to implement a highly representative simulator to host the guidance algorithms.

6.3.2 Comparison with the simulator in reference [3]

There are not many simulation platforms used in a research environment with the level of detail included in the the one developed within this thesis.

As an exception, the simulator indicated by reference [3] can be noted. This simulator is a suite of FORTRAN codes used to model the flight of a three-stage solid fueled rocket. This simulator includes models for the aerodynamics, mass properties, and propulsion and includes a 7-8th order Runge-Kutta integration routine (RK78) for the integration of the equations of motion.

This simulator was validated using the Minotaur as a baseline for comparison. Also several of its building blocks have been validated, as the solid propellant propulsion model and the aerodynamics model, for which an industry tool (*Aerodsn*) is used.

The three-stage solid-fuel orbital flight model was validated using data for the Minuteman-III ICBM in [3].

The simulator developed within this thesis can be compared with this simulator:

- The simulator developed within this thesis includes a very detailed model for the gravity force, based on WGS84 (see section 4.2.1).
No reference is given about the gravity model used in [3], so it is to be expected that a more simplified model is used there.
- The simulator developed for this thesis uses the Missile DATCOM (see reference [4]) for the aerodynamic model. It is to be expected that this model, specifically designed for missiles, should at least be as reliable as the commercial tool *Aerodsn* used in [3].
- The propulsion model used in the simulator developed within this thesis is very simplified, so the model used in [3] is probably better.
- Validation of data

The validation performed in reference [3] is based on geometrical data and burnout times. A comparison of burnout altitudes and burnout speed for a single case is also provided (see table 7 in [3]) but the considered case is not indicated, the value obtained for the burnout altitude of the stage 2 was very different to the reference value, and of course the obtained burnout altitude and speed will completely depend on the considered launch scenario (particularly on the selected target).

The simulator developed within this thesis is based on geometric, fuel and control data obtained from all the available sources found (see Appendix G) and can be considered as the most extensive search and analysis of disclosed data ever made for the LGM-30G Minuteman III ICBM and the GBI missile. Also, as indicated in section 6.2.1, it has also been successfully compared with the limited available performance data of the Minutemant III ICBM.

- Scope

The simulator used in [3] focuses on the ascent phase and does not include within the same simulation 2 missiles in order to complete the simulation of an interception scenario.

It can be concluded that the simulator developed within this thesis is at least as good as the one indicated in [3] in terms of gravity and aerodynamic models, probably worse in terms of propulsive model, and presumably better in terms of configuration data.

As a main advantage, the simulator developed within this thesis is focused in the interception problem.

6.3.3 Comparison with generic simulation platforms

Generic simulation frameworks could also be considered. In this sense, the following simulation platforms were found suitable for the simulation of ballistic missiles:

- The CADAC environment (see reference [5] (Peter H. Zipfel, 2012)).
CADAC (Computer Aided Design of Aerospace Concepts) is a joint development by the U.S. Air Force and the University of Florida. It has been used as a simulation test bed for missiles, aircrafts, and spacecrafts. An example for the simulation of a three stage booster and for Generic Defense Missile simulation is given in [5] (Peter H. Zipfel, 2012).
- The JSBSim environment (see reference [6])
This is an open source aircraft simulation framework that is the basis of the Flight Gear Simulator.
- CMD C++ Model Developer
This is a framework created by the U.S. Army also with unrestricted distribution.

The use of these generic simulation platforms eliminates the need of implementing the core of the simulator (the numerical integrator and the 6 DoF propagation equations) at the price of adding complexity in the implementation of the algorithms with respect to using MATLAB®.

These simulation frameworks are in fact more focused in the development of commercial simulators, this is, in the Human-Machine Interface (HMI) than in developing a framework for testing algorithms. For this reason and taking into account that the main focus is on the development and testing of algorithms, these simulation platforms will not be used.

6.3.4 Comparison with Engineering simulators

There are also simulators provided by the missiles manufacturers or developed for the Military sector, like the MSIC++ Generic Simulation, created for the Missile and Space Intelligence Center (MSIC) for multi-purpose missile simulations.

However these simulators are not available to the public, being used in targeting tasks and system fine-tuning.

These simulators, having the real performance data of the system available are of course better than the simulator developed within this thesis, but they are not used in a disclosed research environment.

6.3.5 Results of the comparison

Without taking the Engineering simulators into consideration, the simulator developed within this thesis can be considered to be at the state of the art referred to algorithms implementation in the ballistic missile interception problem.

A special remark should be made regarding the configuration of the LGM-30G Minuteman III ICBM and the GBI missile, since this simulator includes data for these missiles which is the result of extensive search and analysis.

As a consequence this simulator constitutes an advanced simulation platform that can provide meaningful results in the analysis of interception problems, which was one of the purposes of the thesis.

Chapter 6 references

- [1] 341st Operations Group. Operation of the Minuteman III ICBM. <https://www.youtube.com/watch?v=PJ9tgSgx3PY>, March 2008. [web page accessed on 04/08/2014]. 158, 159
- [2] W.J.Harlin and D.A.Cicci. Ballistic missile trajectory prediction using a state transition matrix. *Applied Mathematics and Computation*, 188:1832–1847, June 2007. doi: 10.1016/j.amc.2006.11.048. 159
- [3] Brian R. McDavid. Launch Vehicle Performance Enhancement Using Aerodynamic Assist. Master’s thesis, Auburn University, Auburn, Alabama, August 2008. viii, 160
- [4] William B. Blake. *AFRL-VA-WP-TR-1998-3009. Missile DATCOM User’s Manual - 1997 Fortran 90 Revision*. Dover Publications, Inc., Wright-Patterson Air Force Base, Ohio, first edition, February 1998. 160
- [5] Peter H. Zipfel. CADAC: Multi-use Architecture for Constructive Aerospace Simulations. *Journal of Defense Modeling and Simulation: Applications, Methodology, Technology*, 9(2):129–145, April 2012. doi: 10.1177/1548512910395641. 161
- [6] Jon S. Berndt. *JSBSim: An open source, platform-independent, flight dynamics model in C++*. 2011. 161

Part III

Guidance algorithms

The guidance algorithms that allow aiming the ICBM to its target, or aiming the GBI missile to the incoming ICBM are analysed in this part of the thesis.

Conventional guidance techniques are explained and implemented in the simulator and optimal guidance strategies are developed, both for the exoatmospheric ascent phase and for the terminal phase.

This page intentionally left blank.

Chapter 7

Guidance strategies and aiming

This chapter indicates the guidance strategies used by the missiles in each phase.

It also analyses the basic aiming algorithms that provide the required velocity vector at cutoff to reach a certain target and that will be used in chapter 8 as a base for the conventional ascent guidance algorithms. These aiming algorithms are based in the solution of the Lambert problem.

7.1 Guidance strategies

The guidance strategies applied in the different phases of the missiles are briefly explained herein.

7.1.1 Atmospheric phase

7.1.1.1 Launch

In the case of the ICBM, once the launch starts a sequenced series of automatic operations begins (see [1]):

1. A final check of the system for combat readiness is made
2. The launcher closure door is removed
3. The upper umbilical which connects the missile with the silo is retracted from the missile
4. The first stage rocket motor is ignited.

After ignition, the missile starts moving and leaves the silo without performing any action for several seconds, in order to ensure a perfect vertical initial manoeuvre and leave the silo harmlessly.

In the case of the GBI we do not have a reference that indicates the actions related to the GBI launch. However we can assume that the actions are similar to the ones performed in any typical vertical launch: after ignition the GBI missile starts moving and leaves the silo without performing any action for several seconds, in order to ensure a perfect vertical initial manoeuvre and leave the silo harmlessly.

7.1.1.2 Gravity turn

The structure of large missiles is designed to provide axial strength and thus stand the thrust and drag forces on the missile. However it is not designed to stand important bending moments and lateral forces. For this reason these missiles are not capable of flying through the atmosphere at an angle of attack except at relatively low speeds. The aerodynamic loads generated when flying through the atmosphere at an angle of attack at several times the speed of sound would result in a catastrophic missile booster structural failure (see [2]).

In order to avoid this, a trajectory called *gravity turn* or *zero lift* (first discussed in [3] (Culler and Fried, 1957)) was designed. This trajectory is based on keeping the missile thrust vector always aligned with the vehicle's velocity vector starting from some non-zero, non-vertical initial velocity.

This trajectory offers two main advantages over a trajectory controlled solely through the vehicle's own thrust:

1. Firstly, the thrust doesn't need to be used to change the missile's direction so more of it can be used to accelerate the vehicle into orbit.
2. Secondly, and more importantly, during the initial ascent phase the vehicle can maintain low or even zero angle of attack. This minimizes transverse aerodynamic stress on the launch vehicle, allowing for a lighter launch vehicle ([4] (Sharaf and Alaqa, 2012)).

The gravity turn is performed as follows:

1. Initially the rocket ascends vertically and the flight path angle is constant $\gamma = 90^\circ$
2. A deviation from the vertical path is commanded by a pitch manoeuvre after lift-off. This is called *initial kick*.
3. Once the initial kick is commanded and the path angle is not vertical we maintain at all times the thrust vector parallel to the velocity vector.

In a very simplified way (equations in an inertial frame in a vertical plane) this would lead to the following system dynamics:

Figure 7.1: Simplified geometry for gravity turn equations

The related equations are:

$$\begin{aligned} E - D - M \cdot g \cdot \sin \gamma &= M \frac{dV}{dt} \\ M \cdot g \cdot \cos \gamma &= M \cdot V \cdot \frac{d\gamma}{dt} \Rightarrow \dot{\gamma} = -\frac{g}{V} \cdot \cos \gamma \end{aligned} \quad (7.1)$$

where:

Forces: $\vec{E}$ = thrust, $\vec{D}$ = drag force, $\vec{g}$ = gravity field acceleration

M = missile mass

γ = flight path angle

$\vec{V}$ = missile velocity

It can be observed in equation 7.1 that the gravity force is the only force that affects the missile turn, reducing the path angle. Performing a gravity turn quickly reduces the path angle while allowing the use of the whole thrust to increase the speed of the missile.

It is important to note that all this manoeuvre happens in a vertical plane defined by the initial heading requested for the missile (which is a result of the guidance module and decided according to the target position).

This is done in the real case by adjusting the controls so that the pitch is commanded within this heading. In the simulation we will achieve the same result by considering that the missile is placed before launch so that the angle $\phi = 0^\circ$ is aligned with the desired initial heading and then we can always command the controls within the gravity turn manoeuvre in order to keep $\phi = 0^\circ$. This allows not implementing some complex control transformations while keeping the same behaviour in the simulated missile as in the real one.

7.1.1.3 Yaw and roll control

Once the gravity turn manoeuvre is started, the yaw and roll controls are activated. These will be secondary controls since the main attitude change in the manoeuvre in this phase is the change in pitch.

Regarding the control in the horizontal plane, for the same reasons for which a very low angle of attack has to be kept while the missile is inside the atmosphere, the sideslip angle has to be kept very low at all times, while at the same time the heading is oriented to its desired value.

Regarding the roll angle, in the case of the ICBM it can be controlled and the objective is to keep it close to 0° .

In the case of the GBI, since the roll angle is not controlled, it is unavoidable that a certain roll spin appears. This means that the roll has to be taken into account when computing the required nozzle motions to reach the desired pitch and yaw angles. A certain deflection will be computed in the Oy_b and the Oz_b axes. These deflections will be combined having as a result a unique motion of the nozzle that allows controlling simultaneously the vertical and horizontal channels.

7.1.2 Outer space phase

7.1.2.1 Final flight path angles

Once the missile crosses the Kármán line that customarily represents the boundary between the Earth's atmosphere and the outer space and that is placed at an altitude of 100 km above the sea level, we can neglect the aerodynamic forces on the missile.

As a consequence once this level has been reached it is not necessary for the trajectory to keep a zero angle of attack or a null sideslip angle, and we can focus on achieving the required injection parameters for reaching the target.

Different guidance algorithms exist for this ascent phase outside the atmosphere (see chapter 8) and new ones can be created (see chapter 10). They all provide the desired attitude angles for the missile so that the injection parameters are reached:

- required pitch (θ_{ref})
- required yaw (ψ_{ref})

7.1.2.2 Boost flight termination

The boost flight can be terminated before the third stage burnout as mission requirements dictate (for example for a shorter range mission). As a consequence another control, also an output of the guidance module, is the time in which the boost flight is terminated. In the case of the ICBM this can happen either within stage 3 or within the post-boost phase. In the case of the GBI this can only happen in stage 3 since there is no post-boost phase.

7.1.2.3 Reentry preparation (ICBM)

In the case of the ICBM in the post-boost phase and before each reentry vehicle (RV) is detached from the deployment module, the PSRE has the capability (10 attitude-control engines according to [1] and [2]) of providing the adequate attitude to each reentry vehicle in order to avoid instabilities in the reentry when crossing again the Kármán line.

This is simulated by setting a new yaw and pitch adequate for the reentry, so that the axis of the reentry vehicle is aligned with the flight path angle when performing the reentry.

These manoeuvres do not change the flight path of the missile, since they are done once the boost phase of the missile has finished. This is, we are basically rotating the body axes of the missile while keeping the original flight path.

Figure 7.2: Preparation for reentry

It can be observed in figure 7.2 that a very large angle of attack appears on the missile at the injection point because of this reentry preparation. Since in this phase the missile is in the outer space and there are no aerodynamic forces having a large angle of attack is irrelevant, but it can be observed when plotting the values of the angle of attack of the missile in this phase.

7.1.2.4 EKV guidance (GBI)

Once the boost phase of the missile has finished, the exoatmospheric kill vehicle will perform 2 kind of manoeuvres:

1. It will change its attitude in order to align the EO/IR sensor with the position of the ICBM, so that it can be tracked. The EKV has a sensor with a high sensibility designed to be able to discriminate the ICBM warhead from debris and decoys and this sensor has to be aligned with the estimated position of the missile in order to perform this task.

Figure 7.3: Rendered image of the EKV
(graph from reference [5])

2. It will divert its trajectory using the divert thrusters in order to perform a final encounter with the ICBM and destroy it.

7.1.3 Reentry phase (ICBM)

The reentry phase is an uncontrolled phase but errors in the desired trajectory because of aerodynamic forces are minimized by providing the reentry vehicle with a high spin rate, activated before the reentry through a hot gas spin system located in the aft section.

In the simulation this spinning effect is simulated (see appendix G.1.2.5) and its consequences in the aerodynamic forces and moments are taken into account as indicated in section 4.3.2.2.

7.2 The Lambert problem

The Lambert problem, sometimes referred to as orbital boundary value problem, basically consists in finding the velocity vectors $\vec{v}_1(t_1)$ and $\vec{v}_2(t_2)$ knowing the position vectors $\vec{r}_1(t_1)$ and $\vec{r}_2(t_2)$ and the time of flight $(t_2 - t_1)$ between these 2 points.

The problem is explained in Appendix section F.2.2 where the existence of a unique orbit as a solution of the problem is demonstrated.

The solution of Lambert's problem directly provides the velocity vector required at an initial point to reach a target in a certain time of flight, and will be the base for the algorithms used for aiming the missiles indicated in section 7.3.

7.2.1 Solutions of Lambert's problem

As indicated in reference [6] there exists a plethora of literature discussing different approaches developed over the years to solve the Lambert problem. The problem has attracted the interest of many scientists and engineers for years, not only because of its practicality in astronomy and aerospace engineering, but also because of the challenge to find its solution.

In the absence of an analytical solution, all the approaches discussed in the literature are based on a numerical procedure where the value of the free parameter is searched iteratively.

The different Lambert solvers all differ in at least one of the following subjects (see reference [7]):

- The iteration variable
- The iteration algorithm
- The initial guess
- The reconstruction of the terminal velocity vectors

The following classification of methods can be made according to the iteration variable:

- Methods that iterate in the conventional space of orbit elements to solve some equivalent Lambert equation.

Among these methods the following can be highlighted:

- The classical method (by Gauss)

It was Gauss who first developed an iterative procedure to solve Lambert's problem and published it in 1809 in his *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium*. He selected the ratio of the area swept by the desired orbit during the transfer time to the area of the triangle determined by the two positions as an iteration variable, and used a successive substitution technique to find its value.

His algorithm is efficient and converges rapidly for small transfer angles but, as indicated in reference [8], it is slow when the transfer angle is large, fails to converge for a wide spectrum of hyperbolic cases and has a singularity for a 180° transfer case.

- The method by Nelson and Zarchan ([9] (Nelson and Zarchan, 1992))

In this method the iteration variable is the vertical flight path angle so the approach is very intuitive and easily related to the physical parameters of the trajectory. This method has been improved in reference [10] (Ahn and Lee, 2013) modifying the strategy for the update.

The main drawback of this method is its accuracy, which is much worse than the one provided by Gooding's method. Also, the method has a singularity for 0° and 360° .

- The eccentricity-vector based method ([11] (Avanzini, 2008)).

This method exploits properties of the eccentricity vector $\vec{e}$ to parametrize the problem and offers a behaviour similar to the one by the Battin in terms of accuracy, convergence speed and numerical efficiency, while offering a much simpler approach (a simple Newton-Raphson iterative scheme is used to solve the equation).

The main advantage of these methods is that they are easy to understand since they are based on physical parameters. However, these methods suffer from three main drawbacks (see reference [6]):

1. They are only valid for elliptical orbits
2. They iterate with unbounded parameters which can lead to numerical problems
3. They offer a tedious implementation for the multi-revolution case.

- Methods based on universal variables

These methods are valid for all conics and offer a numerically robust solution. They initially perform a transformation from orbital parameters to an auxiliary variable that is better behaved.

Among these methods the following can be highlighted:

- The method by Lancaster-Blanchard

Lancaster and Blanchard (reference [12]) provided in 1969 the first universal variables solution valid for the multi-revolution case. This method reduced the solution of Lambert's problem to performing iterations each one requiring the computation of only one inverse trigonometric or hyperbolic function. It uses either of two iteration methods, depending upon whether or not the transfer time is near that for minimum energy. As a drawback, its solution for the velocity vector is singular for transfer angles that are a multiple of 180° , including 0° .

- The method by Battin

This method makes use of hypergeometric functions. The historical success of Battin's method is a consequence of its computational efficiency and wellposedness of the resulting equation.

As a drawback if on the one side the derivation of the equation for this method is relatively simple, its solution is far from trivial, requiring relatively cumbersome transformations and the evaluation of a hypergeometric function in terms of its (truncated) continued fraction expansion. Moreover, care is needed when dealing with very short arcs to provide a satisfactory first guess for the algorithm (see reference [11] (Avanzini, 2008)).

- The method by Gooding

R. H. Gooding (reference [13]) improved the Lancaster method focusing on the initial estimates for the iteration process and the iteration procedure (Gooding algorithm employs Halley iterations).

The resulting procedure is extremely efficient achieving high precision in only 3 iterations for all geometries (this method always takes at least 3 iterations).

- The method by **Arora and Russell** ([6])
This method uses the cosine of the change in eccentric anomaly as iteration variable. The method has shown improvements in the execution time with respect to the Gooding's approach, especially for the multi-revolution case.
- The method by **Izzo** (reference [7])
This method, recently published, improves the Lancaster-Blanchard approach modifying the initial guess and the iteration scheme (a House-holder scheme is used).
This method was compared to the Gooding's procedure finding that the obtained accuracy was similar but the execution time was faster by a factor of 1.25 in the case of a single revolution since it converges to the solution in only 2 iterations on average compared to the 3 iterations needed for the Gooding case.
As a main advantage the method has a smaller computational complexity, being easy to implement.

The following figure from reference [6] briefly summarizes some of the major developments over the years related to solving the Lambert problem:

Figure 7.4: Timeline of major works related to solving the Lambert Problem (picture from [6])

The numerical convergence of the different methods and their overall computational cost depend on both the chosen parametrization and the numerical technique used for solving the resulting equation. Singularities exist, which prevent some of the algorithms from converging for particular cases or make convergence extremely slow.

As indicated in references [14] and [6], several studies comparing Lambert formulations suggest that Gooding's implementation is the most accurate, robust and fastest implementation available.

7.2.2 Solution of the Lambert problem used in the simulation

It is important to choose an appropriate Lambert algorithm since the solution of the Lambert problem acts as a building block for conventional guidance algorithms that will be calling this algorithm several times each computation cycle within the simulator. As a consequence, the faster and more accurate that the implemented algorithm is, the better.

Taking the considerations indicated in section 7.2.1 into account only Gooding's implementation (considered as being the state of the art in the existing implementations) and modern implementations that claim to improve Gooding's performance are taken into account. This is:

- Gooding's algorithm ([13])
- Arora's algorithm ([6])
- Izzo's algorithm ([7])

After some initial analyses Izzo's algorithm has been finally chosen because of its easy implementation.

The algorithm is detailed in reference [7] and in fact the code is available in C++ as part of the open source project PyKEP from the European Space Agency github repository <https://github.com/esa/pykep/>.

Within this thesis the implemented algorithm will be noted as $f_{Lambert}$. The inputs of this function will always be the 2 position vectors ($\vec{r}_1$ and $\vec{r}_2$) in the inertial reference frame, and the time of flight between them. The considered output will be the required velocity vector at t_1 , given with flight path angles and the modulus of the velocity vector, or given directly in a vector expression:

$$\begin{aligned} [\vec{v}_1] &= f_{Lambert}(\vec{r}_1, \vec{r}_2, tof) \\ \text{or} \\ [||\vec{v}_1||, \gamma_z, \gamma_y] &= f_{Lambert}(\vec{r}_1, \vec{r}_2, tof) \end{aligned} \tag{7.2}$$

7.2.3 Errors because of using a solution of the Lambert problem for aiming

All the solutions of Lambert's problem indicated in 7.2.1 consider a central Newtonian gravitational field, this is:

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^3} \cdot \vec{r} \quad (7.3)$$

However the Earth's gravitational field is more complex and includes zonal and tesseral terms (see equation E.21).

The differences between the ideal Lambert case and the real case for the trajectory of a point injected fulfilling perfectly Lambert's problem will be:

- Different reentry point
- Different time of flight

The considered reentry point in a central Newtonian gravitational field would be given from the injection point by:

$$\vec{r}_{reentry} = \vec{r}_{injection} + \vec{v}_{injection} \cdot t_f + \int_0^{t_f} \left(\int_0^t -\mu \frac{\vec{r}}{r^3} dt \right) dt \quad (7.4)$$

where t_f is the time of flight to the reentry point considered for solving Lambert's problem.

However, since the real gravitational field is different to $-\mu \frac{\vec{r}}{r^3}$ the reentry point (considered as a point with a certain constant height above ground, like for example the Kármán line) will be achieved in a different position and time of flight:

$$\int_0^{t_f} \left(\int_0^t -\mu \frac{\vec{r}}{r^3} dt \right) dt \neq \int_0^{t_f} \left(\int_0^t \vec{g} dt \right) dt \quad (7.5)$$

If along the trajectory the gravitational field is greater than the considered central one, the reentry point will be achieved earlier than expected.

If along the trajectory the gravitational field is smaller than the considered central one the reentry point will be achieved later than expected:

- Case 1:

Target position (in ECI frame) is not reached and time of flight is smaller than t_f

$$\left\| \int_0^{t_f} \left(\int_0^t \vec{g} dt \right) dt \right\| > \left\| \int_0^{t_f} \left(\int_0^t -\mu \frac{\vec{r}}{r^3} dt \right) dt \right\| \quad (7.6)$$

- Case 2:

Target position (in ECI frame) is overpassed and time of flight is larger than t_f

$$\left\| \int_0^{t_f} \left(\int_0^t \vec{g} dt \right) dt \right\| < \left\| \int_0^{t_f} \left(\int_0^t -\mu \frac{\vec{r}}{r^3} dt \right) dt \right\| \quad (7.7)$$

The effect can be observed in the following figure:

Figure 7.5: Difference between a theoretical Lambert trajectory and real trajectories

It can be observed that because all the considered orbits (case 1, case 2 and Lambert) start in the same point the final angles between the trajectory and the Kármán line are different. This means that case 2 leads to larger errors over the surface of the Earth than case 1. The error will be bigger for low trajectories. This is, the smaller the apogee, the bigger error in case 2.

The difference in the time of flight is also relevant since it adds another source of error. When computing the reentry point in a moving Earth environment the motion of the Earth during t_f is considered in order to transform the coordinates of the target from ECEF to ECI. If the desired reentry point (in terms of ECI coordinates) were achieved but not in the desired time, the Earth would move the target position and the following error would appear:

$$\Delta\lambda = \lambda_{impact} - \lambda_{target} \simeq \omega_{ei} * (t_{f_{real}} - t_{f_{Lambert}}) \quad (7.8)$$

where ω_{ei} is the angular rotation velocity of the Earth (see Appendix C)

This is, slow trajectories with respect to the Lambert trajectory make the target be to the West of the impact point and fast trajectories with respect to the Lambert trajectory make the target be to the East of the impact point.

Combining both effects we reach the following conclusions:

- Case A: direct orbit (to the East)
 - Case 1: trajectory shorter than the required one
Because the gravity is larger than expected we reach a reentry point to the West of the expected one. Since this reentry point is reached before it was expected (trajectory faster than expected) the impact point is also to the West of the target position.
This is, both effect combine increasing the error.
 - Case 2: trajectory larger than the required one
Because the gravity is smaller than expected we reach a reentry point to the East of the expected one. Since this reentry point is reached after it was expected (trajectory slower than expected) the impact point is also to the East of the target position.
This is, both effect combine increasing the error.
- Case B: retrograde orbit (to the West)
 - Case 1: trajectory shorter than the required one
Because the gravity is larger than expected we reach a reentry point to the East of the expected one. Since this reentry point is reached before it was expected (trajectory faster than expected) the impact point is to the West of the target position.
This is, the second effect reduces the error of the first one.
 - Case 2: trajectory larger than the required one
Because the gravity is smaller than expected we reach a reentry point to the West of the expected one. Since this reentry point is reached after it was expected (trajectory slower than expected) the impact point is to the East of the target position.
This is, the second effect reduces the error of the first one.

As a consequence the errors related to using the solution of Lambert's problem as a base for the guidance algorithms will always be worse in cases with a direct orbit.

It is difficult to obtain analytical expressions related to these errors but some figures can be obtained from the missiles simulator to get an idea of the order of magnitude of the error.

With this aim several simulations are executed with the ICBM launched from Shahrud, Iran and aiming at several target locations, and with the GBI used as an ICBM (for the reentry phase the aerodynamic data of the ICBM is used) launched from Fort Greely, Alaska, and aiming at several target locations.

The simulation for each missile against a certain target location is executed twice with only one difference:

1. The first execution is done using the gravitational acceleration normally used in the simulator (see section 4.2.1), which is provided by the WGS84 model.
2. The second execution is done using a central Newtonian gravitational field: $\vec{g} = -\mu \cdot \frac{\vec{r}}{r^3}$

The distance between the achieved impact points in both cases is computed. This distance, together with other relevant parameters such as apogee height of the trajectory, time of flight used as input for the Lambert algorithm, and resulting time of flight are given in table 7.1.

Table 7.1: Distance between impact points using the Lambert algorithm considering WGS84 gravity and central Newtonian gravity

Case	Type of orbit	Apogee height km	<i>TOF</i> used for Lambert algorithm s	<i>TOF</i> real (at impact) s	Distance to point obtained using using $\vec{g} = -\mu \frac{\vec{r}}{r^3}$ (km)
ICBM to N.Y. City	Retrograde	1,000.4	1,835.5	1,841.8	22.21
ICBM to L.A.	Retrograde	763.0	2,011.0	2,024.7	70.22
ICBM to Tokyo	Direct	1,736.0	2,036.0	2,037.8	21.29
ICBM to Atka Island	Direct	1,450.9	2,041.0	2,045.6	15.34
ICBM to Capetown	Retrograde	1,256.8	1,832.5	1,837.3	7.10
GBI as ICBM to La Coruña	Direct	824.7	1,510.5	1,534.9	70.08
GBI as ICBM to Reykjavik	Direct	1,003.0	1,355.0	1,367.6	26.62
GBI as ICBM to Kwajalein Atoll	Retrograde	756.3	1,378.5	1,397.2	48.38
GBI as ICBM to Rome	Direct	733.9	1,521.5	1,551.6	89.32
GBI as ICBM to Tokyo	Retrograde	865.5	1,314.0	1,328.2	35.76

These data are depicted hereafter separating the cases of direct and retrograde orbits:

Figure 7.6: Distance between impact points using the Lambert algorithm considering Newtonian gravity and WGS84 gravity (retrograde orbits)

Figure 7.7: Distance between impact points using the Lambert algorithm considering Newtonian gravity and WGS84 gravity (direct orbits)

It can be observed that a certain trend exists and the errors are higher with lower apogees. This was anticipated when explaining figure 7.5 and is also due to higher differences between a central Newtonian gravitational field and the real gravitational field in these cases. Also, and as anticipated, for similar apogees the errors are higher for direct orbits.

It has to be noted that these data were obtained using for the gravity the WGS84 model instead of using equation E.33 that uses the gravitational potential V and harmonic terms from an accurate model like the EGM2008 gravitational model for a 'zero tide' system (reference [15]). This means that there will be differences between the provided data in table 7.1 and the ones that would be achieved in a real case. Nevertheless the values provided in table 7.1 highlight the problem sufficiently and the order of magnitude of the errors can be observed.

The shown errors are bigger than the ones related to using a Lambert's problem solver with a worse accuracy and, as will be indicated in chapter 8, bigger than the errors related to using a worse ascent guidance algorithm.

As a consequence it would be ideal to have a Lambert's problem solver that includes, at least, the most relevant term among the harmonic coefficients (which is J_2 , as can be seen in table E.1). However, among the studied solvers only reference [14] claims to be able to take J_2 into consideration in its algorithm, but does not provide enough information to do so. This means that a Lambert's problem solver that does not include this effect is used (see section 7.2.2) and an iteration scheme for targeting will have to be used so that the ICBM reaches its target with the desired accuracy (see section 7.3.1.2).

This is not necessary in the case of the GBI missile since for this missile, once the ascent guidance phase finishes, a final terminal phase towards the ICBM exists in which the missile can compensate initial guidance errors (see chapter 9).

7.3 Aiming algorithms in the simulator

The aiming algorithms used in the simulator will be indicated herein, both for the ICBM and for the interceptor missile.

7.3.1 Initial aiming for the ICBM

In this case the approach usually consists of 2 phases:

- An aiming algorithm is used to estimate initial guidance parameters.
- An iterative algorithm is used making use of a simulator in order to slightly modify the previously computed initial guidance parameter so that the target is reached in the simulator with the desired accuracy: $\|\vec{r}_{target}^e - \vec{r}_{impact}^e\| < \epsilon$.

The algorithms used in the simulator will be detailed hereafter. It has to be noted that for the sake of simplicity the iterative algorithm described in section 7.3.1.2 has not been used when computing the aiming parameters for the ICBM in the interception problems.

7.3.1.1 Basic algorithm

The initial aiming in the simulator is performed using the Lambert algorithm detailed in section 7.2.2:

$$(V_{cutoff}, \gamma_z, \gamma_y) = f_{Lambert}(\vec{r}_1, \vec{r}_2, t_f) \quad (7.9)$$

where:

- V_{cutoff} is the required speed at $\vec{r}_1$ to reach $\vec{r}_2$ in a time t_f .
- γ_z is the required vertical flight path angle at $\vec{r}_1$ to reach $\vec{r}_2$ in a time t_f .
This angle is measured in the vertical plane with respect to the local vertical line (this is, the radius vector from the origin of the ECI reference frame to the $\vec{r}_1$ point).
- γ_y is the required horizontal flight path angle at $\vec{r}_1$ to reach $\vec{r}_2$ in a time t_f .
This is the angle between the vertical plane of the trajectory and the Oz vertical axis of the ECI reference frame (see Appendix A).

and using tables of the time of flight of the ICBM as a function of the achieved γ_z at injection obtained by launching simulations in a non-moving Earth environment:

$$TOF_{non\ moving\ Earth} = g_{tables}(\gamma_z) \quad (7.10)$$

Using these functions ($f_{Lambert}$ and g_{tables}) we implement an iteration loop that provides the desired initial aiming angles.

We start the loop considering an initial $\gamma_{z_{tables0}}$ angle of 30° .

From this point:

1. We compute from g_{tables} the time of flight related to the indicated $\gamma_{z_{tables}k}$:

$$TOF|_{tables_k} = g_{tables}(\gamma_{z_{tables}k})$$
2. We compute from g_{tables} the time of flight related to the angle $\gamma_{z_{tables}k} + \delta\gamma_{z_{tables}}$, where $\delta\gamma_{z_{tables}}$ is a small increment of the angle:

$$TOF|_{tables_{k+\delta}} = g_{tables}(\gamma_{z_{tables}k} + \delta\gamma_{z_{tables}})$$
3. Having these times of flight we obtain the derivative of the time of flight in the tables with respect to γ_z at this point:

$$\left. \frac{dTOF}{d\gamma_{z_{tables}}} \right|_{\gamma_{z_k}} = \frac{TOF|_{tables_{k+\delta}} - TOF|_{tables_k}}{\delta\gamma_{z_{tables}}} \quad (7.11)$$

4. From the geodetic coordinates of the launching site we compute the coordinates of the launching site in the ECEF reference frame ($\vec{r}_{\text{launching site}}^e$) using equation A.27.
5. We compute the launching point in the inertial (ECI) reference frame considering that the launch will be performed now:

$$\text{launch now (t)} \implies C_e^i(t) \implies \vec{r}_{\text{launching site}}^i = C_e^i(t) \cdot \vec{r}_{\text{launching site}}^e \quad (7.12)$$

6. We compute the target position at impact time in the inertial reference frame considering a time of flight of $TOF|_{tables_k}$:

$$TOF|_{tables_k} \implies C_e^i(t + TOF|_{tables_k}) \implies \vec{r}_{\text{target}}^i|_k = C_e^i(t + TOF|_{tables_k}) \cdot \vec{r}_{\text{target}}^e \quad (7.13)$$

7. With these inputs we obtain the required $\gamma_{z_{Lambert}}|_k$ from the Lambert algorithm:

$$(V_{\text{cutoffLambert}}|_k, \gamma_{z_{Lambert}}|_k, \gamma_{y_{Lambert}}|_k) = f_{\text{Lambert}}(\vec{r}_{\text{launching site}}^i, \vec{r}_{\text{target}}^i|_k, TOF|_{tables_k}) \quad (7.14)$$

8. We obtain the solution of Lambert's problem considering a time of flight of $TOF|_{tables_k+dt}$ where dt is a small increment of time:

$$\begin{aligned} (V_{\text{cutoffLambert}}|_{k+dt}, \gamma_{z_{Lambert}}|_{k+dt}, \gamma_{y_{Lambert}}|_{k+dt}) = \\ f_{\text{Lambert}}(\vec{r}_{\text{launching site}}^i, \vec{r}_{\text{target}}^i|_{k+dt}, TOF|_{tables_{k+dt}}) \end{aligned} \quad (7.15)$$

9. Having the solutions for both times of flight we obtain the derivative of $\gamma_{z_{Lambert}}$ with respect to the time of flight:

$$\left. \frac{d\gamma_{z_{Lambert}}}{dTOF} \right|_k = \frac{\gamma_{z_{Lambert}}|_{k+dt} - \gamma_{z_{Lambert}}|_k}{dt} \quad (7.16)$$

10. We now use a Newton method to obtain the following value for γ_z .

With this aim we consider the following function we want to be zero:

$$F = \gamma_{z\text{tables}} - \gamma_{z\text{Lambert}} \quad (7.17)$$

so according to the Newton method the next value of $\gamma_{z\text{tables}}$ would be given by:

$$\gamma_{z\text{tables}} = \gamma_{z\text{tables}} - \frac{F}{\frac{dF}{d\gamma_{z\text{tables}}}} \quad (7.18)$$

where according to equation 7.17:

$$\frac{dF}{d\gamma_{z\text{tables}}} = 1 - \frac{d\gamma_{z\text{Lambert}}}{d\gamma_{z\text{tables}}} = 1 - \frac{d\gamma_{z\text{Lambert}}}{dTOF} \cdot \frac{dTOF}{d\gamma_{z\text{tables}}} \quad (7.19)$$

where the term $\frac{d\gamma_{z\text{Lambert}}}{dTOF}$ has been computed in equation 7.16 and $\frac{dTOF}{d\gamma_{z\text{tables}}}$ has been computed in equation 7.11.

11. The loop is then repeated from step 1 until F is below a certain desired error.

Once the convergence is achieved we provide as solution for the initial aiming of the missile the values:

$$\begin{aligned} \psi_{\text{initial}} &= \gamma_y \\ \theta_{\text{initial}} &= \gamma_z \end{aligned} \quad (7.20)$$

where:

- ψ_{initial} is the initial yaw angle to be used at launch
- θ_{initial} is a reference pitch angle to be used during the ascent guidance phase
(the missiles are all launched from a vertical position so the initial pitch angle is always $\pi/2$).

At the same time the time of flight to be used for the ICBM (TOF) is obtained.

7.3.1.2 Final State Transition Matrix

When computing the initial aiming using a solver of Lambert's problem there are several sources of errors that avoid the ICBM from reaching its desired target:

- The errors related to the hypothesis of central Newtonian gravitational field to solve Lambert's problem. As indicated in section 7.2.3 these errors are basically:
 - Error in the achieved reentry point in the inertial reference frame.
As explained in section 7.2.3 this error is greater when the trajectory is larger than expected and the apogee is low.
 - Error in the time of flight that creates a certain error related to the rotation of the Earth in that time difference.
- Aerodynamic deflections in the reentry
When performing the injection the missile is rotated at the injection point so that when it reenters the atmosphere it is aligned with its velocity vector then. Any error in the estimation of the injection point or a wrong compensation of the rotations within the midcourse guidance (basically due to the gravity torque, see section 4.2.2) becomes an angle of attack that deflects the missile from the desired trajectory. This effect is minimized by adding a spinning motion to the missile, so that when reentering the lateral aerodynamic forces are counteracted. However even though the effects are minimized there will appear in any case initial lateral deflections that deviate the missile from the target.
- The effect of the aerodynamic drag in the time of flight
If we estimate the time of flight for Lambert's problem as the required from the injection point to the target position on ground considering that there is no atmosphere, the solution of Lambert's problem does not take into account the drag that the missile will suffer in its reentry. This drag will reduce the speed of the missile drastically and modify consequently the time of flight, so the initially computed value will not be valid and a certain error related to the rotation of the Earth in that time difference will appear.
- Errors due to winds and gusts in the atmospheric initial and final phases.
These errors can be estimated and counteracted using gusts models and wind estimations if meteorological data of the launching site and the target position is available.

Since a solution including all these factors would be impossible to obtain analytically, an algorithm based on supposing a linear variation of the impact position with respect to injection parameters is used.

The operation consists in creating a final state transition matrix or sensitivity matrix that relates the difference between the impact point and desired target positions with a variation of parameters at the injection with respect to the parameters provided by solving Lambert's problem.

This procedure is used by S.McFarland (reference [16]) who uses the variations of the initial velocity vector of the missile in the inertial reference frame and by [17] (W.J.Harlin and D.A.Cicci, 2007) who uses the variations of the initial missile quaternion in a simplified example.

Taking into account that what we can control is the state at the injection point, it makes more sense to create this sensibility matrix using the variation of the velocity vector at injection with respect to the values provided by solving Lambert's problem:

$$\vec{r}_{target}^e - \vec{r}_{impact}^e = \begin{bmatrix} \frac{\partial r_x^e}{\partial \Delta(v_x^i)_i} & \frac{\partial r_x^e}{\partial \Delta(v_y^i)_i} & \frac{\partial r_x^e}{\partial \Delta(v_z^i)_i} \\ \frac{\partial r_y^e}{\partial \Delta(v_x^i)_i} & \frac{\partial r_y^e}{\partial \Delta(v_y^i)_i} & \frac{\partial r_y^e}{\partial \Delta(v_z^i)_i} \\ \frac{\partial r_z^e}{\partial \Delta(v_x^i)_i} & \frac{\partial r_z^e}{\partial \Delta(v_y^i)_i} & \frac{\partial r_z^e}{\partial \Delta(v_z^i)_i} \end{bmatrix} \cdot \begin{Bmatrix} \Delta(v_x^i)_i \\ \Delta(v_y^i)_i \\ \Delta(v_z^i)_i \end{Bmatrix} = S \cdot \begin{Bmatrix} \Delta(v_x^i)_i \\ \Delta(v_y^i)_i \\ \Delta(v_z^i)_i \end{Bmatrix} \quad (7.21)$$

where the e superscript indicates coordinates in the ECEF reference frame, the i superscript indicates coordinates in the ECI reference frame, the i subscript indicates that the vector is measured from the ECI reference frame (see Appendix A) and S is the sensitivity matrix.

The procedure to obtain the variations to be applied in order to reach a certain desired accuracy is:

1. Initial values are given for the sensitivity factors $\frac{\delta \vec{r}^e}{\delta \Delta(v_x^i)_i}$, $\frac{\delta \vec{r}^e}{\delta \Delta(v_y^i)_i}$, $\frac{\delta \vec{r}^e}{\delta \Delta(v_z^i)_i}$.

These values can be obtained easily from new impact points obtained by solving the initial value problem three times with modified injection parameters:

$$\begin{aligned} \bullet (v_x^i)_{i \text{ Lambert}} + \delta(v_x^i)_i &\implies \frac{\partial \vec{r}^e}{\partial \Delta(v_x^i)_i} \simeq \frac{\vec{r}_{target}^e - \vec{r}_{impact}^e}{\delta(v_x^i)_i} \\ \bullet (v_y^i)_{i \text{ Lambert}} + \delta(v_y^i)_i &\implies \frac{\partial \vec{r}^e}{\partial \Delta(v_y^i)_i} \simeq \frac{\vec{r}_{target}^e - \vec{r}_{impact}^e}{\delta(v_y^i)_i} \\ \bullet (v_z^i)_{i \text{ Lambert}} + \delta(v_z^i)_i &\implies \frac{\partial \vec{r}^e}{\partial \Delta(v_z^i)_i} \simeq \frac{\vec{r}_{target}^e - \vec{r}_{impact}^e}{\delta(v_z^i)_i} \end{aligned}$$

where small random (non zero) values are used for $\delta(v_x^i)_i$, $\delta(v_y^i)_i$ and $\delta(v_z^i)_i$.

It has to be noted that solving the initial value problem is a very fast and straightforward procedure that does not involve launching the missile simulation, but simply solving the Kepler problem until the computed position is on the Earth's surface.

Starting the loop with $\Delta t = 0$:

- (a) We propagate the position of the missile using the solution of Kepler's problem (see Appendix section F.2.1):

$$\vec{r}^i(t + \Delta t) = f_{Kepler}(\vec{r}_0^i, \vec{v}_0^i, \Delta t) \quad (7.22)$$

where:

- t is the launching time
- $\vec{r}_0^i$ is the launching position
- $\vec{v}_0^i$ is computed from $\vec{v}_{Lambert} + (\delta(v_x^i)_i, \delta(v_y^i)_i, \delta(v_z^i)_i)$.

- (b) We check the height of the obtained vector: $\vec{r}^n = C_i^n \cdot \vec{r}^i(t + \Delta t)$

If this height is almost equal to the target height then Δt is the time of flight and $\vec{r}^i(t + \Delta t)$ is the impact point ($\vec{r}_{impact}^i$).

Otherwise, we increase Δt and go back to step a .

- (c) Once we have a solution we transform the impact point to the ECEF reference frame:

$$\vec{r}_{impact}^e = C_i^e(t + \Delta t) \cdot \vec{r}_{impact}^i \quad (7.23)$$

2. We now launch the missile simulation using null variations (a purely Lambert solution) and obtain the difference between the impact point and the target point in the simulator in the ECEF reference frame: $\vec{Y} = \vec{r}_{target}^e - \vec{r}_{impact}^e$
3. The vector $\vec{Y}$ will be considered as a measurement input for the following observation problem in which the state vector is composed of the elements of the sensitivity matrix:

state equations:

$$\begin{aligned}\vec{X} &= \{S11, S12, S13, S21, S22, S23, S31, S32, S33\}^T \\ \vec{X}_{k+1} &= I \cdot \vec{X}_k \\ \vec{X}_0 &= \left\{ \frac{\delta x^e}{\delta(v_x^i)_i}, \frac{\delta x^e}{\delta(v_y^i)_i}, \frac{\delta x^e}{\delta(v_z^i)_i}, \frac{\delta y^e}{\delta(v_x^i)_i}, \frac{\delta y^e}{\delta(v_y^i)_i}, \frac{\delta y^e}{\delta(v_z^i)_i}, \frac{\delta z^e}{\delta(v_x^i)_i}, \frac{\delta z^e}{\delta(v_y^i)_i}, \frac{\delta z^e}{\delta(v_z^i)_i} \right\}^T\end{aligned}\quad (7.24)$$

measurement equations:

$$\begin{aligned}\vec{Y}_{k+1} &= \vec{Y}_k + H \cdot \vec{X}_k \\ H &= \begin{bmatrix} [(\delta(v_x^i)_i)_{k+1} - (\delta(v_x^i)_i)_k], [(\delta(v_y^i)_i)_{k+1} - (\delta(v_y^i)_i)_k], [(\delta(v_z^i)_i)_{k+1} - (\delta(v_z^i)_i)_k], 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, [(\delta(v_x^i)_i)_{k+1} - (\delta(v_x^i)_i)_k], [(\delta(v_y^i)_i)_{k+1} - (\delta(v_y^i)_i)_k], [(\delta(v_z^i)_i)_{k+1} - (\delta(v_z^i)_i)_k], 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, [(\delta(v_x^i)_i)_{k+1} - (\delta(v_x^i)_i)_k], [(\delta(v_y^i)_i)_{k+1} - (\delta(v_y^i)_i)_k], [(\delta(v_z^i)_i)_{k+1} - (\delta(v_z^i)_i)_k] \end{bmatrix}\end{aligned}\quad (7.25)$$

This observation problem is completely set by estimating a covariance matrix for the state propagation (Q) and a covariance matrix for the error in the measurements (R).

The state propagation matrix should be large enough to include within the state errors the uncertainties of the model. This is, it should be able to allow all the changes in the variations of the parameters that the real effects indicated at the beginning of this section (7.3.1.2) could require. For the sake of simplicity a diagonal covariance matrix made from a 1% of the maximum values of the initial S matrix could be used.

R would be given by the maximum possible errors when computing $\vec{r}_{impact}^e$ in the simulator. Having these matrices we can obtain a better estimation of the $\vec{X}$ vector using a Kalman filter by:

$$\vec{X}_{k+1} = \vec{X}_k + P_k \cdot H_k^T \cdot (H_k \cdot P_k \cdot H_k^T + R)^{-1} \cdot \vec{Y}_k \quad (7.26)$$

4. With the new obtained $\vec{X}_{k+1}$ we have a new sensitivity matrix S and we can estimate the variations that would be required to reach the target since we want $\vec{Y}_{k+1}$ to be zero:

$$\begin{Bmatrix} (\delta(v_x^i)_i)_{k+1} - (\delta(v_x^i)_i)_k \\ (\delta(v_y^i)_i)_{k+1} - (\delta(v_y^i)_i)_k \\ (\delta(v_z^i)_i)_{k+1} - (\delta(v_z^i)_i)_k \end{Bmatrix} = S^{-1} \cdot \vec{Y}_k \quad (7.27)$$

5. The missile simulation would be executed using the variations from the Lambert solution computed in equation 7.27 obtaining a new impact point (and a new $\vec{Y}$).
6. With the previously computed $\vec{Y}$ the loop is repeated from step 3.
7. The procedure stops when the exit condition $\|\vec{r}_{target}^e - \vec{r}_{impact}^e\| < \epsilon$ is achieved

This method is very powerful since, on the one hand we are always improving the transition matrix in the last obtained point, so the linearization is always considered in the achieved solution, which avoids the possibility of divergence. On the other hand using observation techniques allows considering, if desired, errors in the simulation, and a good solution would still be achieved.

As an example the convergence steps for several cases are indicated in tables 7.2, 7.3 and 7.4 using as stop criteria $\|\bar{r}_{target}^e - \bar{r}_{impact}^e\| < 1 \text{ km}$:

Table 7.2: Case ICBM to NYC without adding errors in the simulation

step	$\Delta(v_x^i)_i$ m/s	$\Delta(v_y^i)_i$ m/s	$\Delta(v_z^i)_i$ m/s	$\ \bar{r}_{target}^e - \bar{r}_{impact}^e\ $ m
0	0.0	0.0	0.0	19,108
1	40.7	-0.3	19.3	8,765.9
2	47.8	-0.3	21.3	9,690.4
3	47.3	-0.4	22.9	7,758.0
4	49.4	-0.5	23.1	6,997.6
5	51.1	-0.6	23.3	846.4

Table 7.3: Case ICBM to Tokyo adding Gaussian errors (2% 3 σ) in the thrust force

step	$\Delta(v_x^i)_i$ m/s	$\Delta(v_y^i)_i$ m/s	$\Delta(v_z^i)_i$ m/s	$\ \bar{r}_{target}^e - \bar{r}_{impact}^e\ $ m
0	0.0	0.0	0.0	21,256.9
1	-13.6	0.5	13.3	2,194.8
2	-13.7	0.0	12.9	713.3

Table 7.4: Case ICBM to Cape Town adding Gaussian errors (2% 3 σ) in the thrust force

step	$\Delta(v_x^i)_i$ m/s	$\Delta(v_y^i)_i$ m/s	$\Delta(v_z^i)_i$ m/s	$\ \bar{r}_{target}^e - \bar{r}_{impact}^e\ $ m
0	0.0	0.0	0.0	39,113.9
1	30.1	0.9	56.1	40,596.3
2	14.6	0.7	34.2	26,868.1
3	9.5	0.2	24.0	15,499.2
4	10.2	-0.5	20.8	6,151.5
5	12.5	-1.3	20.9	736.8

7.3.2 Initial aiming for the GBI missile

In this case instead of aiming to a fixed position the missile is aimed to a Predicted Intercept Point (PIP) that has to be previously computed.

The solution will be obtained using the Lambert algorithm detailed in section 7.2.2:

$$(V_{cutoff}, \gamma_z, \gamma_y) = f_{Lambert}(\vec{r}_1, \vec{r}_2, t_f) \quad (7.28)$$

and using the Kepler algorithm detailed in Appendix section F.2.1:

$$\vec{r}^i(t + \Delta t) = f_{Kepler}(\vec{r}_0^i, \vec{v}_0^i, \Delta t) \quad (7.29)$$

and using tables of the cutoff speed of the GBI as a function of the achieved γ_z at injection obtained by launching simulations in a non-moving Earth environment:

$$V_{cutoff_{non\ moving\ Earth}} = h_{tables}(\gamma_z) \quad (7.30)$$

As an initial step we estimate the impact point of the ICBM by propagating its trajectory until the height is below 100 m. This is done propagating from the ICBM present position and velocity vector by solving Kepler's problem (see Appendix section F.2.1) iteratively until the obtained vector is on the Earth's surface. This impact position is then transformed to geodetic coordinates since the geodetic latitude and longitude of the target of the ICBM will be used within the main loop.

The loop starts considering a time-to-go (t_{go0}) for the GBI of 1000 seconds (which is the minimum considered interception time in the simulator). The procedure is detailed herein:

1. From the geodetic coordinates of the launching site we compute the coordinates of the launching site in the ECEF reference frame ($\vec{r}_{launching\ site}^e$) using equation A.27.
2. We compute the launching point in the inertial (ECI) reference frame considering that the launch will be performed now:

$$\text{launch now } (t) \implies C_e^i(t) \implies \vec{r}_{launching\ site}^i = C_e^i(t) \cdot \vec{r}_{launching\ site}^e \quad (7.31)$$

3. We compute the future ICBM position in $t + t_{go_k}$ in the inertial reference frame using orbital mechanics:

$$\vec{r}_{ICBM}^i(t + t_{go_k}) = f_{Kepler}(\vec{r}_{ICBM}^i(t), \vec{v}_{ICBM}^i(t), t_{go_k}) \quad (7.32)$$

4. We then obtain the ECEF coordinates of the ICBM in $t + t_{go_k}$

$$t + t_{go_k} \implies C_i^e(t + t_{go_k}) \implies \vec{r}_{ICBM}^e(t + t_{go_k}) = C_i^e \cdot \vec{r}_{ICBM}^i(t + t_{go_k}) \quad (7.33)$$

5. We compute the geodetic coordinates of the ICBM in t_{go_k} from the position in the ECEF frame applying equations A.32 and A.33.

6. From the geodetic coordinates of the ICBM in $t + t_{go_k}$ and the geodetic coordinates of its target we compute the distance along the Earth surface between the ICBM and its target in $t + t_{go_k}$: $distICBMtarget_k$.
7. We get the required speed and flight path angles for the GBI to reach the ICBM position in t_{go_k} from the Lambert algorithm:

$$(V_{cutoffLambert}|_k, \gamma_{zLambert}|_k, \gamma_{yLambert}|_k) = f_{Lambert}(\bar{r}_{launching\ site}^i, \bar{r}_{ICBM}|_k, t_{go_k}) \quad (7.34)$$

8. We check if this solution can be achieved by the GBI:
 - We check that the required vertical flight path angle is between 5° and 55° (We discard angles that lead to unusual trajectories for the GBI).
 - We check that the speed that can be achieved by the GBI with $\gamma_{zLambert}|_k$ is greater than the required plus a margin.

With this aim we first obtain from the tables of the missile the cutoff speed that can be achieved with $\gamma_{zLambert}|_k$:

$$V_{cutofftables}|_k = h_{tables}(\gamma_{zLambert}|_k) \quad (7.35)$$

and then we check that:

$$V_{cutofftables}|_k + V_{Earthrotation} > V_{cutoffLambert}|_k + margin \quad (7.36)$$

where $V_{Earthrotation}$ is a component that can be positive or negative depending on the required $\gamma_{yLambert}|_k$ and takes into account the rotation of the Earth since it can allow trajectories otherwise impossible when using direct orbits, and it could avoid some retrograde trajectories otherwise possible.

- If a certain trajectory is possible we store the obtained parameters $\gamma_{yLambert}|_k$, $\gamma_{zLambert}|_k$ and $distICBMtarget_k$ and the time for which these parameters were obtained (t_{go_k}).
9. We repeat the loop in step 3 increasing the time-to-go an amount dt . The loop finishes when t_{go} equals the time in which the ICBM reaches its target.

Among all the different possible trajectories we choose the one with a larger value for $distICBMtarget_k$. This is, we are maximizing the distance ICBM-target in the interception, since that will minimize the effect of a possible blast in the target location:

$$\begin{aligned} k_{optimum} \backslash distICBMtarget_k & \text{ is maximum} \\ \psi_{initial} &= \gamma_{yLambert}|_{k_{optimum}} \\ \theta_{initial} &= \gamma_{yLambert}|_{k_{optimum}} \end{aligned} \quad (7.37)$$

where:

- $\psi_{initial}$ is the initial yaw angle to be used at launch by the GBI
- $\theta_{initial}$ is a reference pitch angle to be used during the ascent guidance phase (the missiles are all launched from a vertical position so the initial pitch angle is always $\pi/2$).

At the same time the time of flight of the GBI missile ($t_{go_{k_{optimum}}}$) is obtained.

7.3.3 Aiming after launch

Once the missile has been launched the guidance algorithms (see chapter 8 and chapter 10) define the required guidance parameters so that it reaches its target.

These algorithms require as input in many cases aiming parameters along the trajectory. The cases for the ICBM and the GBI missile will be detailed hereafter:

7.3.3.1 ICBM

It is considered that the target for the ICBM does not change along its trajectory. As a consequence in this case the aiming parameters will be directly obtained each computation cycle from the Lambert algorithm taking the present position of the ICBM each time into account:

$$(V_{cutoff}, \gamma_z, \gamma_y)_{ICBM}(t) = f_{Lambert}(\vec{r}_{ICBM}^i(t), \vec{r}_{target}^i, TOF_{remaining}) \quad (7.38)$$

where $TOF_{remaining}$ is the remaining time of flight (the TOF computed at launch minus the time elapsed since the ICBM was launched).

The time of flight for the missile is kept as constant and equal to the value initially computed in section 7.3.1 since, being this parameter an input to other guidance algorithms, keeping it constant makes the solution more robust (modifying the required time of flight would imply changes in the required flight path angles).

7.3.3.2 GBI

Since the predicted intercept point (PIP) is computed when the GBI is launched which could happen when the ICBM is still thrusting, the PIP is recomputed each computation cycle by solving Kepler's problem from the present ICBM position and velocity vector, obtaining the future ICBM position in the remaining time to go.

$$\begin{aligned} [\vec{r}_{ICBM}^i, \vec{v}_{ICBM}^i](t + t_{go_{remaining}}) &= f_{Kepler}(\vec{r}_{ICBM}^i(t), \vec{v}_{ICBM}^i(t), t_{go_{remaining}}) \\ \implies \overrightarrow{PIP}(t) &= \vec{r}_{ICBM}^i(t + t_{go_{remaining}}) \end{aligned} \quad (7.39)$$

where $t_{go_{remaining}}$ is the remaining time of flight for the GBI (the time of flight computed at launch minus the time elapsed since the GBI was launched).

As in the case of the ICBM, the time of flight for the missile is kept as constant and equal to the value initially computed in section 7.3.2 since, being this parameter an input to other guidance algorithms, keeping it constant makes the solution more robust.

Once the PIP has been modified the Lambert algorithm is used to update the aiming parameters each computation cycle:

$$(V_{cutoff}, \gamma_z, \gamma_y)_{GBI}(t) = f_{Lambert}(\vec{r}_{GBI}^i(t), \overrightarrow{PIP}(t), t_{go_{remaining}}) \quad (7.40)$$

Chapter 7 references

- [1] ICBM Prime Team, Prime - 19378, TRW Systems. *Minuteman Weapon System History and Description*. ICBM System Program Office, Hill Air Force Base Utah, second edition, July 2001. 166, 169
- [2] David P. Blanks, Anthony M. Logue, Stephen J. Skotte, Douglas M. Bruce, Ralph A. Sandfry, and Michael L. Zywiec. A Two-Stage Intercontinental Ballistic Missile (ICBM) Design Optimization Study and Life Cycle Cost Analysis. Master's thesis, Faculty of the School of Engineering of the Air Force Institute of Technology Air University, Wright-Patterson Air Force Base, Ohio, December 1992. 166, 169
- [3] G.J. Culler and B.D. Fried. Universal Gravity Turn Trajectories. *Journal of Applied Physics*, 28(6):672–676, June 1957. doi: 10.1063/1.1722828. 166
- [4] M. A. Sharaf and L.A. Alaqa. Computational Algorithm for Gravity Turn Maneuver. *Global Journal of Science Frontier Research*, 12:59–64, 2012. 166
- [5] Jim Mallard. Web home of Jim Mallard. <http://www.jimmallard3.com>. [web page accessed on 02/01/2015]. 170
- [6] Nitin Arora and Ryan P. Russell. A Fast and Robust Multiple Revolution Lambert Algorithm Using a Cosine Transformation. http://russell.ae.utexas.edu/FinalPublications/ConferencePapers/2013Aug_HiltonHead_AAS-13-728_LambertAlgorithm.pdf, University of Texas at Austin, August 2013. 171, 172, 173, 174
- [7] Dario Izzo. Revisiting Lambert's Problem. <http://arxiv.org/pdf/1403.2705.pdf>, Cornell University Library, June 2014. 171, 173, 174
- [8] Richard H. Battin. *An Introduction to the Mathematics and Methods of Astrodynamics*. American Institute of Aeronautics and Astronautics Inc., Alexander Bell Drive, Reston, VA, revised edition edition, 1999. 171
- [9] Steven L. Nelson and Paul Zarchan. Alternative Approach to the Solution of Lambert's Problem. *Journal of Guidance, Control, and Dynamics*, 15(4):1003–1009, July-August 1992. doi: 10.2514/3.20935. 171
- [10] Jaemyung Ahn and Sang-Il Lee. Lambert Algorithm Using Analytic Gradients. *Journal of Guidance, Control, and Dynamics*, 36(6):1751–1761, November-December 2013. doi: 10.2514/1.62091. 171
- [11] Giulio Avanzini. A Simple Lambert Algorithm. *Journal of Guidance, Control and Dynamics*, 31(6):1832–1847, November-December 2008. doi: 10.2514/1.36426. 172
- [12] E.R. Lancaster and R.C. Blanchard. *Technical Note NASA TN D-5368. A Unified Form of Lambert's Theorem*. NASA, Goddard Space Flight Center, Greenbelt, Maryland, September 1969. 172

- [13] R H. Gooding. A Procedure for the Solution of Lambert’s Orbital Boundary-Value Problem. *Celestial Mechanics and Dynamical Astronomy*, 15(2):145–165, July-August 1990. [172](#), [174](#)
- [14] Gim J. Der. The Superior Lambert Algorithm. <http://www.amostech.com/TechnicalPapers/2011/Poster/DER.pdf>, 2011. [173](#), [180](#)
- [15] Nikolaus K. Pavlis, Simon A. Holmes , Steve C. Kenyon, and John K. Factor. An Earth Gravitational Model to Degree 2160: EGM2008. *General Assembly of the European Geosciences Union*, April 2008. [180](#)
- [16] Justin S.McFarland. Modelling the Ballistic Missile Problem with the State Transition Matrix: A Comparative Analysis. AIAA Region I-MA Student Conference, Paper 04-22666, April 2004. [184](#)
- [17] W.J.Harlin and D.A.Cicci. Ballistic missile trajectory prediction using a state transition matrix. *Applied Mathematics and Computation*, 188:1832–1847, June 2007. doi: 10.1016/j.amc.2006.11.048. [184](#)

Chapter 8

Conventional ascent guidance

This chapter analyses the more common ascent guidance algorithms, indicates the ones considered as relevant and describes the implementation performed in the missiles simulator.

These algorithms will be later used as a base for a comparison with the optimal guidance algorithms implemented in chapter [10](#).

8.1 Introduction

The purpose of the ascent guidance algorithms is to modify the trajectory during the ascent phase of the missile so that it reaches the target.

Ascent guidance algorithms can be classified in the following groups ([1] (Braun et al., 2013)):

- According to the information loop:
 - Open-loop (implicit)
Open-loop guidance systems do not take into account information regarding the actual flight dynamics of the vehicle when computing the commands for the system. Instead, the vehicle flies a scheduled set of commands.
 - Closed-loop (explicit)
Closed-loop guidance systems sense the vehicle's state and alter the guidance commands appropriately.

Open-loop guidance algorithms are simpler and easier to test and implement. However they generally result in reduced accuracy with respect to closed-loop guidance algorithms. Also, closed-loop guidance allows setting steering vectors and engine cutoff times according to a certain optimal law, like minimum fuel consumption or minimum injection time.

- According to the onboard computational sophistication:
 - Model-based guidance (predictive)
Model based guidance attempts to predict the future dynamics based on onboard models that are updated in flight.
 - Reference path guidance
In this case the missile is forced to follow an a priori defined reference path.

Generally, reference path algorithms are easier to implement onboard, but require many ground-based computations to create reference data. Model-based guidance solutions are harder to implement onboard, but can accommodate a larger range of initial conditions and inflight uncertainties.

- According to the source to detect the target:
 - Passive
Passive guidance systems use an external source to detect the target.
 - Active
Active guidance uses onboard navigational sensors to detect the target.

In the case of ICBMs, it is considered that they have active guidance when they include a complete navigation system so that they can accurately compute their own position.

Active guidance results in improved performance, but requires onboard sensing capabilities.

Ascent guidance algorithms have generally increased in complexity and capability as more powerful flight computers have become available.

Accuracy has improved with time, an advance generally attributed to the transition from passive and open-loop guidance systems to active, closed-loop guidance systems. This paradigm shift was enabled by the development of onboard inertial navigation systems, which in turn, were enabled by miniaturized sensors and computing hardware.

A basic evolution of ascent guidance algorithms can be given as:

1. Open loop

These schemes are used when there is not much room for optimization because of the constraints in the formulation. As an example this is usually the scheme followed for silo or tower clearance and during the atmospheric part of the ascent (see section 8.2)

2. Delta guidance

This is a closed loop scheme in which a reference path computed on ground is followed. Delta guidance is explained in section 8.3.1.

3. Path adaptive guidance

This is a closed loop scheme in which the most convenient among several reference paths computed on ground is followed. Path adaptive guidance is explained in section 8.3.2.

4. Lambert guidance

This is a closed loop scheme in which the *velocity to be gained*, this is, the difference between the present velocity and the required velocity to reach the target in an orbital motion from the present position, is computed continuously and then used as a base for the guidance scheme. Lambert guidance is explained in section 8.3.3.

5. Q guidance

This is a closed loop algorithm initially conceived for systems with a limited onboard computation capability in which all the complexity stays within the terms of a matrix that can be computed on ground. It allows a guidance theoretically more accurate than the Lambert guidance. Q guidance is explained in section 8.3.4.

6. Iterative Guidance Mode (IGM)

This is a closed loop algorithm in which an iteration (prediction-correction) is done each cycle in order to obtain the attitude of the missile that would allow reaching the target. This method is based on the concept of optimal linear tangent steering (see section 8.3.5). IGM is explained in section 8.3.5.1.

7. Powered Explicit Guidance (PEG)

This can be considered as a vector representation of the IGM (and as such is also based on the optimal linear tangent steering). An iteration (prediction-correction) is also performed each cycle in order to obtain the required attitude for the missile. PEG is explained in section 8.3.5.2.

8. Optimal guidance

Optimal guidance is explained in chapter 10 and will not be considered within this chapter.

Even though an evolution exists, ascent guidance algorithms are still a subject of intense study and future developments are to be expected to increase accuracy, flexibility and robustness.

Moreover, present systems make use of heritage algorithms and schemes that have a dependence on mission-specific constants. For instance, the trajectory and load design process for a given Space Shuttle mission started 303 days prior to launch. Later revisions to this process reduced the required time to 198 days for standard flights to the ISS. This significantly reduced mission flexibility throughout the life of the Space Shuttle program ([1] (Braun et al., 2013)).

The main guidance algorithms summarized before will be detailed hereafter.

8.2 Atmospheric ascent guidance

8.2.1 Conventional approaches

As explained in section 7.1.1.2, the structure of large missiles is not designed to stand important bending moments and lateral forces so a null angle of attack is usually forced when flying through the atmosphere, obtaining a trajectory called *gravity turn* or *zero lift*.

Having as constraint a null angle of attack, this manoeuvre can only be optimized for variable thrust missiles (see [2] (Large, 1979), which is not the case, or considering a certain maximum allowable bending moment or lateral force (see [3]), which could be risky.

Moreover, as indicated in [4] (Hanson et al., 1995), closed-loop atmospheric schemes that involve linear tangent steering did not yield any improvement with respect to open-loop designs as a function of altitude or velocity, and the evaluation of a closed-loop optimization scheme for flying through the atmosphere shows no advantages over open-loop optimization.

As a consequence, the traditional approach is to operate in an open-loop mode during the (early) high dynamic pressure portion of flight and then, based on a pre-determined time or event, to switch to a closed-loop in vacuo guidance scheme which operates on the premise that aerodynamic forces can be neglected. The open-loop mode typically makes use of pre-loaded tables of optimal steering angles versus time or speed.

For example, as indicated in [1] (Braun et al., 2013), the guidance for the Space Shuttle was open-loop during the first stage (flight with the solid rocket boosters), using pregenerated profiles for thrust and attitude based on constraints in order to achieve predetermined values.

8.2.2 Atmospheric ascent guidance in the simulator

In the case of the simulator, instead of imposing an open-loop trajectory, we will make use of the control strategy explained in section 4.5 imposing simultaneously a *zero lift* trajectory, and a certain reference pitch angle provided by:

$$\begin{aligned} \text{if } t_{\text{since lift off}} \leq 20s &\Rightarrow \theta_{ref} = 90.0^\circ + \frac{\theta_{ref \text{ guidance}} - 90.0^\circ}{20} \cdot t_{\text{since lift off}} \\ \text{if } t_{\text{since lift off}} > 20s &\Rightarrow \theta_{ref} = \theta + \frac{\Delta\theta}{\Delta t} \cdot dt \end{aligned} \quad (8.1)$$

where:

- $\Delta\theta = \theta_{ref \text{ guidance}} - \theta$
- $\theta_{ref \text{ guidance}}$ = the desired value for the vertical flight path angle given by the Lambert problem solver (see section 7.3.3)
- θ = present value of the pitch angle
- $\Delta t = 30$ seconds
- dt = time step

During the first 20 seconds we request a fast change in the pitch angle (*initial kick*).

After these 20 seconds and within the atmosphere we set a reference pitch angle ($\theta_{ref \text{ guidance}}$) for which we request a very slow convergence (in 30 seconds). This reference pitch angle is intended to be maintained by the control algorithm while keeping a null angle of attack in this phase (a *zero lift* trajectory) is prioritized, as indicated in section 4.5.

A similar approach will be used in the horizontal channel, providing the following reference yaw angle:

$$\psi_{ref} = \psi + \frac{\Delta\psi}{\Delta t} \cdot dt \quad (8.2)$$

where:

- $\Delta\psi = \psi_{ref \text{ guidance}} - \psi$
- $\psi_{ref \text{ guidance}}$ = the desired value for the horizontal flight path angle given by the Lambert problem solver (see section 7.3.3)
- ψ = present value of the yaw angle
- $\Delta t = 30$ seconds
- dt = time step

In the horizontal channel within the atmosphere we set a reference yaw angle ($\psi_{ref \text{ guidance}}$) for which we request a very slow convergence (in 30 seconds). This yaw angle will be achieved while maintaining at all times an almost null sideslip angle, which is the horizontal control that is prioritized in this phase.

These reference angles will be used until the air density is very low. The transition criteria has been set so that this occurs approximately when crossing the Kármán line that customarily represents the boundary between the Earth's atmosphere and the outer space, and that is placed at an altitude of 100 km above the sea level.

It is possible to start the vacuum guidance strategy before. In fact the comparison of cases performed by [4] (Hanson et al., 1995) indicates that an early release of vacuum closed-loop guidance yields some improvements. However in order to do this we would need to know the maximum allowable bending moment or lateral force in the missile in order to keep it always within a safe margin. Since this information was not available we decided to start the exoatmospheric ascent guidance once we could safely neglect any aerodynamic force.

8.3 Exoatmospheric ascent guidance

8.3.1 Delta guidance

8.3.1.1 Formulation

The Delta guidance algorithm starts with the development of reference trajectories computed on ground. During the flight, deviations from this reference trajectory caused by external disturbances as well as inaccurate performance, tolerances, misalignments, etc, are sensed and the vehicle is returned to the reference.

One possible formulation is given in reference [5]. The reference (required) velocity vector is represented by expansions of the type:

$$\begin{aligned} V_{Rx} &= V_{Rxn} + A_x \cdot (x - x_n) + B_x \cdot (y - y_n) + C_x \cdot (z - z_n) + D_x \cdot (t - t_n) \\ V_{Ry} &= V_{Ryn} + A_y \cdot (x - x_n) + B_y \cdot (y - y_n) + C_y \cdot (z - z_n) + D_y \cdot (t - t_n) \\ V_{Rz} &= V_{Rzn} + A_z \cdot (x - x_n) + B_z \cdot (y - y_n) + C_z \cdot (z - z_n) + D_z \cdot (t - t_n) \end{aligned} \quad (8.3)$$

where $(V_{Rxn}, V_{Ryn}, V_{Rzn})$ is the position of the reference trajectory at the cutoff point.

The coefficients $(A_x, A_y, A_z \dots)$ are usually obtained by a technique known as *targeting*, this is, using a simulation, or using least squares (*curve fitting*) (see reference [6]).

Taking the present missile position (which provides the required velocity using equation 8.3) and velocity $(\vec{v})$ into account the *velocity to be gained* is computed:

$$\vec{v}_g = \vec{v}_R - \vec{v} \quad (8.4)$$

The direction in which the thrust vector has to be directed (the unit thrust vector) is then obtained as:

$$\vec{\xi} = \frac{K}{a_T} \vec{v}_g + \frac{L}{a_T} \int_{t_0}^t \vec{v}_g dt \quad (8.5)$$

where a_T is the thrust acceleration.

The integral term in the steering equation is used to compensate thrust misalignments.

A similar approach but with a different formulation was developed by Richard H. Battin and J. Halcombe Laning Jr. in the 1950s (see [7] (Battin, 1982)) and consisted in considering a functional of the missile position vector, the missile velocity vector and the target position vector such that fulfilling this condition in free fall the missile reaches the target:

$$F(\vec{r}, \vec{v}, \vec{r}_T) = 0 \quad (8.6)$$

where the subscript T refers to the target.

The reference trajectory computed on ground fulfills the functional at cut-off $(\vec{r}_0$ and $\vec{v}_0)$.

If F is expanded in a Taylor series about the reference cut-off values we get:

$$F(\vec{r}_0 + \Delta r, \vec{v}_0 + \Delta v, \vec{r}_T) = F(\vec{r}_0, \vec{v}_0, \vec{r}_T) + \left. \frac{\partial F}{\partial \vec{r}} \right|_0 \cdot \Delta \vec{r} + \left. \frac{\partial F}{\partial \vec{v}} \right|_0 \cdot \Delta \vec{v} + H.O.T. \quad (8.7)$$

Since the reference trajectory fulfills the functional at cut-off we need for cutoff to fulfill:

$$0 = \left. \frac{\partial F}{\partial \vec{r}} \right|_0 \cdot \Delta \vec{r} + \left. \frac{\partial F}{\partial \vec{v}} \right|_0 \cdot \Delta \vec{v} \quad (8.8)$$

which is the condition requested to the missile autopilot (each point of the reference trajectory is treated as a potential cut-off point).

The functional used by Battin and Laning, derived from orbital mechanics, was:

$$F(\vec{r}, \vec{v}, \vec{r}_T) = (\vec{v} \times \vec{r}) \cdot [\vec{v} \times (\vec{r}_T - \vec{r}) + \mu \vec{r}_T \cdot \left(\frac{\vec{r}_T}{r_T} - \frac{\vec{r}}{r} \right)] \quad (8.9)$$

where μ is the Earth's gravitational constant.

Delta guidance is simple in concept and was used for several space missions like, for instance, the Juno launch vehicles which carried the Explorer satellites ([8]), and for the Atlas missile ([9]).

However, it requires considerable reference data created on ground and it is only valid as long as the deviations from the reference trajectory are kept sufficiently small. Unexpected disturbances would lead to a mission failure once the reference trajectory cannot be recovered.

Also, it requires a full navigation system that provides the missile position (in order to obtain the required velocity in equation 8.3). This requires computing the Earth's gravity along the trajectory. Polynomial expressions for the gravity were used with this aim.

8.3.1.2 Delta guidance in the missiles simulator

Delta guidance became obsolete once the on board computational capability increased, and it will not be considered in the missiles simulator.

8.3.2 Path-Adaptive guidance

8.3.2.1 Formulation

If large deviations have occurred between the present trajectory and the reference one, it may be very uneconomical or even impossible to return to the reference trajectory. In such a case it would be required to compute a new trajectory from the actual conditions to the prescribed final state.

As a consequence, a large family of trajectories was precalculated and evaluated before the flight and stored in the on-board guidance computer ([8]).

This is, the delta guidance approach is basically used but many reference trajectories are computed instead of only 1. An algorithm is implemented for deciding when to switch from one reference trajectory to another.

This procedure reduces the risk of mission failures due to unexpected disturbances, but at the cost of increasing the required computations on ground.

8.3.2.2 Path-Adaptive guidance in the missiles simulator

In the same way as delta guidance, path-adaptive guidance became obsolete once the on board computational capability increased, and will not be considered in the missiles simulator.

8.3.3 Lambert guidance

8.3.3.1 Formulation

In this case it is considered that the computational capability of the onboard guidance computer is enough to solve at a high rate the Lambert problem with a certain optimized solver (see section 7.2.1).

This solver directly provides a required velocity $\vec{v}_R$ that, in contrast to delta guidance, takes into account the actual trajectory with its disturbances, and not a reference one.

The *velocity to be gained* is then obtained:

$$\vec{v}_g = \vec{v}_R - \vec{v} \quad (8.10)$$

where $\vec{v}$ is the present velocity vector of the missile.

The guidance strategy to be applied with this type of guidance consists in keeping the unit thrust vector of the missile parallel to this velocity to be gained vector at all times:

$$\vec{\xi} = \frac{\vec{v}_g}{\|\vec{v}_g\|} \quad (8.11)$$

A similar approach can be used providing as guidance variable, instead of the desired unit thrust vector $\vec{\xi}$, the angular velocity vector to be used, defined as:

$$\vec{\omega} = \kappa(\vec{a}_T \times \vec{v}_g) \quad (8.12)$$

When this approach is used it is said that a *cross product steering* is being used.

The results are the same with both expressions and the use of one or another formulation depends mainly on the control scheme used within the missile.

It has to be highlighted that with this guidance algorithm on the orbital injection point the required velocity is just defined by the velocity of the local orbit, with no constraint on the desired height. As a consequence, this approach will cause an injection with height error.

This means that this algorithm is more suitable for guiding missiles (where the height at injection is not relevant) than for placing spacecrafts in a certain orbit.

8.3.3.2 Lambert guidance in the missiles simulator

In the simulator developed within this thesis the control system is decoupled from the guidance system having as inputs the required missile attitude (see section 4.5), so we will use equation 8.11 to implement the Lambert guidance, directly providing the pitch and yaw angles associated to the unit thrust vector $\vec{\xi}$ to the control system so that the missile achieves the required attitude.

8.3.4 Q guidance

8.3.4.1 Formulation

This method was developed by Richard H. Battin and J. Halcombe Laning Jr. in 1955 (see reference [7] (Battin, 1982)).

A good explanation of the procedure is given in reference [6] and will be briefly summarized herein.

Let us suppose that at time t the missile is located at a point M and that a *correlated missile* is simultaneously located at the same position.

The correlated missile is assumed to move at all times with the required velocity vector to reach the desired target in the desired time of flight, this is, with the correlated velocity ($\vec{v}_c = \vec{v}_R$) and it is assumed to be accelerated by the force of gravity only.

The equations of motion of these 2 missiles in Δt would be:

$$\Delta \vec{v}_m = \vec{a}_T \cdot \Delta t + \vec{g} \cdot \Delta t \quad (8.13)$$

$$\Delta \vec{v}_c = \vec{g} \cdot \Delta t \quad (8.14)$$

where the subscript m denotes the missile and the subscript c denotes the correlated missile.

The relative position vector between the 2 missiles in Δt would be:

$$\Delta \vec{R}_{m-c} = \vec{r}_c - \vec{r}_m = (\vec{v}_c - \vec{v}_m) \cdot \Delta t = \vec{v}_g \cdot \Delta t \quad (8.15)$$

This means that if we want to place the correlated missile in the position of the missile in Δt , an additional acceleration would be required:

$$\Delta \vec{v}_c^* = \vec{g} \cdot \Delta t + \frac{\partial \vec{v}_c}{\partial \vec{r}} \cdot \Delta \vec{R}_{c-m} = \vec{g} \cdot \Delta t - \frac{\partial \vec{v}_c}{\partial \vec{r}} \cdot \vec{v}_g \cdot \Delta t \quad (8.16)$$

where $*$ indicates a correlated missile always placed in the position of the missile.

The Q matrix is defined as:

$$Q = \left. \frac{\partial \vec{v}_c}{\partial \vec{r}} \right|_{tof, \vec{R}_T} \quad (8.17)$$

(the time of flight tof and the target position is kept constant in this process)

so the equation of motion for a correlated missile always placed in the missile position would be (from equation 8.16):

$$\Delta \vec{v}_c^* = \vec{g} \cdot \Delta t - Q \cdot \vec{v}_g \cdot \Delta t \quad (8.18)$$

Subtracting equation 8.13 from equation 8.18 we get:

$$\Delta \vec{v}_c^* - \Delta \vec{v}_m = \vec{g} \cdot \Delta t - Q \cdot \vec{v}_g \cdot \Delta t - \vec{a}_T \cdot \Delta t - \vec{g} \cdot \Delta t \quad (8.19)$$

which leads to:

$$\dot{\vec{v}}_g = -\vec{a}_T - Q \cdot \vec{v}_g \quad (8.20)$$

The beauty of equation 8.20 lies in its simplicity, and in the fact that the necessity to compute Earth's gravity, an implied feature of Delta guidance, has vanished. This is, using Q guidance the missile does not need to know its position vector in order to determine $\vec{v}_g$.

All the difficulties of this guidance method lie in the computation of the matrix Q . For example, Battin states in [7] (Battin, 1982) that in their final report the equations for the terms of the Q matrix took 14 pages.

Once the Q matrix is obtained, the guidance strategy to be applied with this type of guidance consists in keeping the unit thrust vector of the missile with a direction such that $\vec{v}_g \times \frac{d\vec{v}_g}{dt} = \vec{0}$. (This approach provides the optimal steering within this guidance algorithm as shown in reference [10]).

This is obtained with the following expression (see [11] (Bhat and Shrivastava, 1987)):

$$\vec{\xi} = \frac{\alpha \cdot \vec{v}_g + \vec{v}_g \times \vec{y}}{\|\vec{v}_g\|^2} \quad (8.21)$$

where:

$$\begin{aligned} \vec{y} &= \vec{v}_g \times Q \cdot \vec{v}_g \\ \alpha &= a_T^2 \|\vec{v}_g\|^2 - \|\vec{y}\|^2 \end{aligned} \quad (8.22)$$

A similar approach can be used providing as guidance variable instead of the desired unit thrust vector $\vec{\xi}$, the angular velocity vector to be used which in this case would be:

$$\vec{\omega} = \kappa(\vec{v}_g \times \frac{d\vec{v}_g}{dt}) \quad (8.23)$$

When this approach is used it is said that a *cross product steering* is being used.

8.3.4.2 Behaviour and problems of the Q guidance algorithm

The Q guidance algorithm was initially conceived to improve the results of the Delta guidance method while keeping a simplified algorithm on board. Most of the computations (especially the terms of the Q matrix) were initially done on ground, which reduced the computational requirements on board. Also, and since it did not require a complete navigation system on board, it was much easier to mechanize.

For this reason it was used in the 1950s on the Thor IRBM and on the Polaris SLBM.

The behaviour shown by the Q guidance algorithm was good. This is because it was found to be an optimal guidance scheme when a flat Earth hypothesis is used (see reference [7] (Battin, 1982)), so in fact is a pseudo-optimal guidance technique.

However, as with Lambert guidance, on the orbital injection point the required velocity is just defined by the velocity of the local orbit, with no constrain on the desired height. As a consequence, this approach will cause injection into an orbit just near the desired one, but with height error ([12] (Mohammadi et al., 2010)).

This is also highlighted in reference [13], where it is indicated that with this algorithm it is impossible to obtain all orbital parameters for the final orbit.

8.3.4.3 Q guidance in the missiles simulator

As the onboard computational capability improved, the advantages of having a simplified algorithm on board were less appealing. At the same time it was increasingly important for the space vehicles to contain instantaneous position and velocity information, which made the advantage of the Q mechanization less attractive.

However, recent researches on this guidance method were found in the literature and, as indicated in [1] (Braun et al., 2013), Q Guidance remains a topic of study and continues to form the basis of new guidance schemes. Also, as indicated before, this guidance techniques is theoretically better than Lambert guidance.

Taking this into account, this guidance method will be implemented in the simulator.

The analytic expressions indicated in reference [14] will be used to obtain the terms of the Q matrix.

In this source a reference frame, which we will denote as u , given by the following versors:

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{r}}{\|\vec{r}\|} \\ \vec{u}_2 &= \frac{(\vec{r} \times \vec{r}_T) \times \vec{r}}{\|(\vec{r} \times \vec{r}_T) \times \vec{r}\|} \\ \vec{u}_2 &= \vec{u}_1 \times \vec{u}_2\end{aligned}\tag{8.24}$$

is used, where $\vec{r}$ is the present position vector and $\vec{r}_T$ is the target position vector, both given in the inertial reference frame.

Using this reference frame we define:

$$\begin{aligned}V_{CR} &= \vec{V}_c \cdot \vec{u}_1 \\ V_{C\theta} &= \vec{V}_c \cdot \vec{u}_2\end{aligned}\tag{8.25}$$

where $\vec{V}_c$ is the required velocity (also called correlated velocity) to reach the target $\vec{r}_T$ from the present position $\vec{r}$ in a certain given time of flight tof . This velocity vector is given by a Lambert's problem solver as indicated in chapter 7.

Reference [14] indicates that the Q^u matrix in this reference frame is:

$$Q^u = \begin{bmatrix} \frac{\partial V_{CR}}{\partial r} & -\frac{1}{r} \left(\frac{\partial V_{CR}}{\partial \theta_R} + V_{C\theta} \right) & 0 \\ \frac{\partial V_{C\theta}}{\partial r} & -\frac{1}{r} \left(\frac{\partial V_{C\theta}}{\partial \theta_R} - V_{CR} \right) & 0 \\ 0 & 0 & \frac{1}{r} (V_{CR} - V_{C\theta} \cdot \cot \theta_R) \end{bmatrix}\tag{8.26}$$

where θ_R is the angle between $\vec{r}$ and $\vec{r}_T$, and $r = \|\vec{r}\|$.

The expressions to compute these terms are ([14]):

Initial data:

$$\begin{aligned}
 \delta &= \frac{r}{\|\vec{r}_T\|} \\
 \frac{\partial f}{\partial \theta_R} &= \frac{\mu}{r \cdot V_{C\theta}} - V_{C\theta} - V_{CR} \cdot \cot \theta_R \\
 \frac{\partial f}{\partial r} &= \frac{1}{r} \cdot (V_{C\theta} \cdot (\cos \theta_R - 2\delta) \cdot \csc \theta_R - V_{CR}) \\
 \frac{\partial f}{\partial V_{C\theta}} &= 2 \csc \theta_R \cdot (\cos \theta_R - \delta) - \frac{V_{CR}}{V_{C\theta}} \\
 V_{CR}^* &= \frac{\left(\delta \cdot \cos \theta_R - 1 \right) \cdot V_{C\theta} + \frac{\mu}{r \cdot V_{C\theta}} \cdot (1 - \cos \theta_R)}{\sin \theta_R}
 \end{aligned} \tag{8.27}$$

where μ is the Earth's gravitational parameter.

Derivatives of $V_{C\theta}$:

$$\begin{aligned}
 \frac{\partial V_{C\theta}}{\partial \theta_R} &= \frac{\frac{1}{2} \cdot V_{C\theta} \cdot V_{CR} - \left[r + V_{CR} \cdot \left(\frac{\|\vec{r}_T\|}{V_{CR}^*} + \frac{3}{2} \cdot tof \right) \right] \cdot \frac{\partial V_{C\theta}}{\partial tof}}{V_{C\theta} \cdot \left(1 + \delta \cdot \frac{V_{CR}}{V_{CR}^*} \right)} \\
 \frac{\partial V_{C\theta}}{\partial r} &= -\frac{1}{r} \left(\frac{\partial f}{\partial \theta_R} + \frac{\partial f}{\partial V_{C\theta}} \cdot \frac{\partial V_{C\theta}}{\partial \theta_R} + V_{C\theta} \right)
 \end{aligned} \tag{8.28}$$

Derivatives of V_{CR} :

$$\begin{aligned}
 \frac{\partial V_{CR}}{\partial \theta_R} &= \frac{\partial f}{\partial \theta_R} + \frac{\partial f}{\partial V_{C\theta}} \cdot \frac{\partial V_{C\theta}}{\partial \theta_R} \\
 \frac{\partial V_{CR}}{\partial r} &= \frac{\partial f}{\partial r} + \frac{\partial f}{\partial V_{C\theta}} \cdot \frac{\partial V_{C\theta}}{\partial r}
 \end{aligned} \tag{8.29}$$

The only term missing in these equations is $\frac{\partial V_{C\theta}}{\partial tof}$, which we will compute by numerical differentiation of the velocity $\vec{V}_c$ given by the Lambert's solver.

Once the Q^u matrix is obtained from equation 8.26 we can obtain Q in the inertial reference frame using the following conversion:

$$Q = C_u^i \cdot Q^u \cdot C_i^u \tag{8.30}$$

where the change of base matrix C_u^i can be easily obtained from the versors of the base (equation 8.24) as indicated in equation B.5.

8.3.5 Linear Tangent Guidance (LTG)

When optimal control (see chapter 10) is applied to a simplified ascent guidance problem with the following assumptions:

- Planar motion (all the motion is considered within a vertical plane)
- Constant gravity (flat Earth)
- Constant specific impulse

and a functional that only depends on the final time and final state of the missile is used ($J = f(\vec{X}(t_f), t_f)$) it is found (see references [15] and [16]) that the optimal guidance strategy is to apply a bilinear tangent law for the thrust direction ξ :

$$\tan \xi = \frac{a + bt}{c + dt} \quad (8.31)$$

where the angle is computed in the inertial reference frame.

This concept led to the idea of applying a bilinear tangent law for the thrust direction of the missile also in normal conditions (considering that it will be a pseudo-optimal guidance law in these conditions).

The thrust direction with this law is obtained imposing that the integration of the accelerations along the trajectory leads to the required velocity vector and position at cut-off.

The first approach in this sense was done purely in terms of angular magnitudes, obtaining the Iterative Guidance Mode (IGM) (see section 8.3.5.1). The same approach was later applied directly in a vector form, obtaining the Power Explicit Guidance (PEG) (see section 8.3.5.2). These algorithms will be detailed hereafter.

8.3.5.1 Iterative Guidance Mode (IGM)

8.3.5.1.1 Formulation

The equations for the Iterative Guidance Mode (IGM) are detailed in reference [17] and are analysed in reference [18] (Song et al., 2015).

The scheme starts considering a linear variation for the pitch and yaw attitude commands relative to the guidance reference system G :

$$\begin{aligned} \theta_G &= \theta_{Gv} - (\theta_{Gp} - \dot{\theta}_G t) \\ \psi_G &= \psi_{Gv} - (\psi_{Gp} - \dot{\psi}_G t) \end{aligned} \quad (8.32)$$

where as a simplification the present time is set to 0.

The guidance reference system G is defined with origin at the center of the Earth, the Ox_G axis toward the predicted orbit injection point, the Oy_G axis in the normal direction of the orbit plane, and Oz_G axis completing a right-handed coordinate system.

The guidance parameters to be obtained are θ_{Gv} , θ_{Gp} , $\dot{\theta}_G$, ψ_{Gv} , ψ_{Gp} , $\dot{\psi}_G$, the time to go (t_f) and the angle between the current position and the predicted target injection point (ξ_g) that defines the G reference frame.

An analytic integration of the following point-mass equations of motion applicable to a vehicle in vacuum with the assumption of global thrust constant in magnitude ($a_T = \text{constant}$) is performed:

$$\begin{aligned}\ddot{x}^G &= a_T \sin \theta_G \cos \psi_G + g_x^G \\ \ddot{y}^G &= a_T \sin \psi_G + g_y^G \\ \ddot{z}^G &= a_T \cos \theta_G \cos \psi_G + g_z^G\end{aligned}\tag{8.33}$$

where g_x^G , g_y^G and g_z^G are the coordinates of the gravity acceleration in the G reference frame.

The time to go (t_f) can be estimated using approximations for the integrals of the gravity terms as:

$$\begin{aligned}\Delta \vec{v}^G &= \vec{v}_f^G - \vec{v}_0^G - \vec{v}_{grav}^G \\ \implies \|\Delta \vec{v}^G\| &= \int_0^{t_f} a_T dt \quad \text{provides } t_f\end{aligned}\tag{8.34}$$

being $\vec{v}_f^G$ the desired injection velocity vector.

Having t_f , and assuming constant thrust direction (θ_G and ψ_G constant) an integration of equation 8.33 leads to:

$$\begin{aligned}\dot{x}_f^G &= \dot{x}_0^G + \sin \theta_{Gv} \cos \psi_{Gv} \int_0^{t_f} a_T dt + v_{grav_x}^G \\ \dot{y}_f^G &= \dot{y}_0^G + \sin \psi_{Gv} \int_0^{t_f} a_T dt + v_{grav_y}^G \\ \dot{z}_f^G &= \dot{z}_0^G + \cos \theta_{Gv} \cos \psi_{Gv} \int_0^{t_f} a_T dt + v_{grav_z}^G\end{aligned}\tag{8.35}$$

θ_{Gv} and ψ_{Gv} are obtained so that they satisfy the terminal velocity constraints:

$$\theta_{Gv} = \text{atan} \left(\frac{\Delta \vec{v}^G |x}{\Delta \vec{v}^G |z} \right) \quad \psi_{Gv} = \text{atan} \left(\frac{\Delta \vec{v}^G |y}{\sqrt{(\Delta \vec{v}^G |x)^2 + (\Delta \vec{v}^G |z)^2}} \right)\tag{8.36}$$

The remaining attitude guidance parameters (θ_{Gp} , $\dot{\theta}_G$, ψ_{Gp} , $\dot{\psi}_G$) are derived to satisfy the terminal position constraints in the Ox_G and Oy_G axes. With this aim the terms $(\theta_{Gp} - \dot{\theta}_G t)$ and $(\psi_{Gp} - \dot{\psi}_G t)$ are assumed to be small enough so that:

$$\begin{aligned}\sin(\theta_{Gp} - \dot{\theta}_G t) &\approx (\theta_{Gp} - \dot{\theta}_G t) \\ \cos(\theta_{Gp} - \dot{\theta}_G t) &\approx 1 - \frac{(\theta_{Gp} - \dot{\theta}_G t)^2}{2!} \\ \sin(\psi_{Gp} - \dot{\psi}_G t) &\approx (\psi_{Gp} - \dot{\psi}_G t) \\ \cos(\psi_{Gp} - \dot{\psi}_G t) &\approx 1 - \frac{(\psi_{Gp} - \dot{\psi}_G t)^2}{2!}\end{aligned}\tag{8.37}$$

With these approximations the original IGM formulation ([17]) considers:

$$\begin{aligned}
\sin\theta_G &= \sin(\theta_{Gv} - (\theta_{Gp} - \dot{\theta}_G t)) \approx \sin\theta_{Gv} - (\theta_{Gp} - \dot{\theta}_G t)\cos\theta_{Gv} \\
\cos\psi_G &= \cos(\psi_{Gv} - (\psi_{Gp} - \dot{\psi}_G t)) \approx \cos\psi_{Gv} + (\psi_{Gp} - \dot{\psi}_G t)\sin\psi_{Gv} \\
\sin\theta_G \cos\psi_G &\approx \sin\theta_{Gv} \cos\psi_{Gv} - (\theta_{Gp} - \dot{\theta}_G t)\cos\theta_{Gv} \cos\psi_{Gv} + \\
&(\psi_{Gp} - \dot{\psi}_G t)\sin\theta_{Gv} \sin\psi_{Gv} - (\theta_{Gp} - \dot{\theta}_G t)(\psi_{Gp} - \dot{\psi}_G t)\cos\theta_{Gv} \sin\psi_{Gv}
\end{aligned} \tag{8.38}$$

which in fact is a wrongful approximation as highlighted in reference [18] (Song et al., 2015) since, in order to keep in equation 8.38 second order terms, higher order terms from equation 8.37 should have been kept. The correct approximation to be used for $\sin\theta_G \cos\psi_G$ is given in reference [18] (Song et al., 2015) .

From these approximations we can integrate equations 8.33 obtaining:

$$\begin{aligned}
\dot{x}_f^G &= \dot{x}_0^G + f_1(L, J, P, \theta_{Gv}, \theta_{Gp}, \dot{\theta}_G, \psi_{Gv}, \psi_{Gp}, \dot{\psi}_G) + v_{grav_x}^G \\
x_f^G &= x_0^G + \dot{x}_0^G \cdot t_f + f_2(S, Q, U, \theta_{Gv}, \theta_{Gp}, \dot{\theta}_G, \psi_{Gv}, \psi_{Gp}, \dot{\psi}_G) + r_{grav_x}^G \\
\dot{y}_f^G &= \dot{y}_0^G + f_3(L, J, P, \theta_{Gv}, \theta_{Gp}, \dot{\theta}_G, \psi_{Gv}, \psi_{Gp}, \dot{\psi}_G) + v_{grav_y}^G \\
y_f^G &= y_0^G + \dot{y}_0^G \cdot t_f + f_4(S, Q, U, \theta_{Gv}, \theta_{Gp}, \dot{\theta}_G, \psi_{Gv}, \psi_{Gp}, \dot{\psi}_G) + r_{grav_y}^G
\end{aligned} \tag{8.39}$$

which allows solving the unknowns θ_{Gp} , $\dot{\theta}_G$, ψ_{Gp} and $\dot{\psi}_G$.

Expressions for f_1 , f_2 , f_3 and f_4 are given in reference [18] (Song et al., 2015) (the term P is denoted as H there and the term U is denoted as P).

L , J , P , S , Q and U are integrals involving the missile thrust for which expressions are provided in section 8.3.5.2.1.

The remaining guidance parameter (ξ_g) is obtained from the position in the Oz_G axis:

$$z_f^G = z_0^G + \dot{z}_0^G \cdot t_f + f_6(S, Q, U, \theta_{Gv}, \theta_{Gp}, \dot{\theta}_G, \psi_{Gv}, \psi_{Gp}, \dot{\psi}_G) + r_{grav_z}^G \tag{8.40}$$

where the expression for f_6 is given in reference [18] (Song et al., 2015) (the term P is denoted as H there and the term U is denoted as P).

The original formulation of IGM avoided this latter equation considering a mission dependent equation for ξ_g :

$$\xi_g = \frac{1}{r_f}(V \cdot t_f + S + K_g \cdot t_f^2) \tag{8.41}$$

where r_f is the magnitude of the position vector at injection and K_g was calculated before each flight with computer simulations of the flight trajectories.

A predictor-corrector scheme is used to solve the equations:

- The prediction step calculates an approximation of the desired angles
- The correction step refines the initial approximation

The original formulation of IGM used t_f as iteration variable

8.3.5.1.2 Behaviour and problems of the IGM guidance algorithm

With human payloads in mind, the Saturn vehicles were designed for maximum reliability. This necessitated the use of a closed-loop guidance scheme which could handle off-nominal scenarios automatically during flight. To provide this capability, the Iterative Guidance Mode, or IGM, was developed ([1] (Braun et al., 2013)).

IGM was started once the open-loop first stage time-tilt steering program was complete. It was also used for in-space manoeuvres on the Saturn V with minor modifications. The flexible and iterative nature of IGM allowed it to easily compensate for in-flight engine-out scenarios through modification of guidance constants.

Even though the method worked relatively well, it has several problems:

- The original formulation of PEG method includes some simplifications in the trigonometrical terms according to the hypothesis of low values for $(\theta_{Gp} - \dot{\theta}_G t)$ and $(\psi_{Gp} - \dot{\psi}_G t)$. This is valid for small times to go, but becomes less accurate for long ranges. Reference [18] (Song et al., 2015) indicates a solution for this problem consisting in including higher order terms in the formulation of IGM, achieving this way an accuracy similar to that of PEG.
- The formulation for IGM is completely based in trigonometrical expressions. This makes the implementation of this algorithm difficult and prone to error.

8.3.5.1.3 Iterative Guidance Mode (IGM) in the missiles simulator

The IGM guidance algorithm, being based in trigonometrical expressions, is difficult to implement. PEG applies the same approach than IGM (guidance based in linear tangent steering) in a vector formulation (see section 8.3.5.2), which is easier to implement.

At the same time, according to the results provided in section 8.3.6, IGM is less accurate than PEG (as anticipated in section 8.3.5.1.2) and only including higher order terms in the formulation of IGM makes it reach accuracies comparable to the normal PEG algorithm.

Taking all these considerations into account, we will not include this guidance algorithm in the missiles simulator. We will consider that LTG algorithms are properly represented by the PEG algorithm that will be detailed hereafter.

8.3.5.2 Power Explicit Guidance (PEG)

8.3.5.2.1 Formulation

The application of the bilinear tangent law for the thrust direction can be expressed with the following equation by R.L. McHenry et al. as indicated in [19] (Song et al., 2011):

$$\vec{u}_F = \frac{\vec{\lambda}_F}{\|\vec{\lambda}_F\|} = \frac{\vec{\lambda}_v + \dot{\vec{\lambda}} \cdot (t - t_\lambda)}{\sqrt{1 + \dot{\lambda}^2 (t - t_\lambda)^2}} \quad (8.42)$$

where:

- $\vec{u}_F$ is the unit thrust vector (previously denoted as $\vec{\xi}$)
- $\vec{\lambda}_F$ is a vector in the direction of the thrust
- $\vec{\lambda}_v$ is a unit vector in the direction of the velocity to be gained ($\vec{v}_g$)
- $\dot{\vec{\lambda}}$ is a vector normal to $\vec{\lambda}_v$ representing the rate of change of $\vec{\lambda}_F$
- t is the present time
- t_λ denotes the time when $\vec{\lambda}_F$ is parallel to $\vec{v}_g$ and it is an unknown of the problem.

The following figure explains the terms in this equation:

Figure 8.1: Geometry of $\vec{\lambda}$ vectors in the PEG algorithm

It is easy to check that equation 8.42 leads to a bilinear tangent law in inertial coordinates. For example if we consider only the z and x coordinates of the equation we have:

$$\begin{aligned} \vec{u}_F|_z &= \frac{\vec{\lambda}_v|_z + \dot{\vec{\lambda}}|_z \cdot (t - t_\lambda)}{\sqrt{1 + \dot{\lambda}^2 (t - t_\lambda)^2}} \\ \vec{u}_F|x &= \frac{\vec{\lambda}_v|x + \dot{\vec{\lambda}}|x \cdot (t - t_\lambda)}{\sqrt{1 + \dot{\lambda}^2 (t - t_\lambda)^2}} \end{aligned} \quad (8.43)$$

so the angle between them would be given by:

$$\tan(\xi) = \frac{\vec{\lambda}_v|_z + \dot{\vec{\lambda}}|_z \cdot (t - t_\lambda)}{\vec{\lambda}_v|x + \dot{\vec{\lambda}}|x \cdot (t - t_\lambda)} \quad (8.44)$$

Expression 8.42 can be approximated using only linear terms as:

$$\vec{u}_F \simeq \vec{\lambda}_v + \dot{\vec{\lambda}}(t - t_\lambda) \quad (8.45)$$

in the classical approach for PEG.

When quadratic terms are kept the method is called Power Explicit Guidance with Higher Order Terms (PEG H.O.T.):

$$\vec{u}_F \simeq \vec{\lambda}_v \left\{ 1 - \frac{1}{2} \dot{\lambda}^2 (t - t_\lambda)^2 \right\} + \dot{\vec{\lambda}}(t - t_\lambda) \quad (8.46)$$

The PEG algorithm consists in using expression 8.45 to integrate the thrust force (considered as constant in magnitude) from the present position in order to obtain the velocity vector and the position at the end of the remaining thrust time, imposing then that these vectors have the desired values:

$$\begin{aligned} \vec{E} &= E \cdot \vec{u}_F = E \cdot \vec{\lambda}_v + E \cdot \dot{\vec{\lambda}}(t - t_\lambda) \implies \\ \vec{v}(t) &= \vec{v}_0 + \int_0^t \frac{E}{m} dt \vec{\lambda}_v + \left[\int_0^t \frac{E}{m} t dt - t_\lambda \cdot \int_0^t \frac{E}{m} dt \right] \cdot \dot{\vec{\lambda}} + \int_0^t \vec{g} dt \end{aligned} \quad (8.47)$$

$$\begin{aligned} \vec{r}(t) &= \vec{r}_0 + \vec{v}_0 \cdot t + \int_0^t \left\{ \int_0^t \frac{E}{m} dt \right\} dt \cdot \vec{\lambda}_v + \\ &\left[\int_0^t \left\{ \int_0^t \frac{E}{m} t dt \right\} dt - t_\lambda \cdot \int_0^t \left\{ \int_0^t \frac{E}{m} dt \right\} dt \right] \cdot \dot{\vec{\lambda}} + \int_0^t \left\{ \int_0^t \vec{g} dt \right\} dt \end{aligned}$$

where the present time is considered as $t = 0$ as a simplification.

Defining the following constants:

$$\begin{aligned} v_{ex} &= \frac{E}{\dot{m}} \\ \tau &= \frac{m_0}{\dot{m}} \end{aligned} \quad (8.48)$$

the integral terms in equation 8.47 can be estimated as:

$$\begin{aligned} L(t) &= \int_0^t \frac{E}{m} dt = -v_{ex} \cdot \ln \left(1 - \frac{t}{\tau} \right) \\ J(t) &= \int_0^t \frac{E}{m} t dt = -v_{ex} \cdot \left[t + \tau \cdot \ln \left(1 - \frac{t}{\tau} \right) \right] \\ S(t) &= \int_0^t \left\{ \int_0^t \frac{E}{m} dt \right\} dt = -v_{ex} \cdot \left[(t - \tau) \cdot \ln \left(1 - \frac{t}{\tau} \right) - t \right] = L(t) \cdot (t - \tau) + v_{ex} \cdot t \\ Q(t) &= \int_0^t \left\{ \int_0^t \frac{E}{m} t dt \right\} dt = -v_{ex} \cdot \frac{t^2}{2} - v_{ex} \cdot \tau \left[(t - \tau) \cdot \ln \left(1 - \frac{t}{\tau} \right) - t \right] = S(t) \cdot \tau - v_{ex} \cdot \frac{t^2}{2} \end{aligned} \quad (8.49)$$

In the case of several stages these expressions have to be computed integrating for the remaining times of every stage. This is:

$$L(t) = \sum_i^N L_i(t) \quad J(t) = \sum_i^N J_i(t) \quad S(t) = \sum_i^N S_i(t) \quad Q(t) = \sum_i^N Q_i(t) \quad (8.50)$$

where i is the active stage and N is the number of stages for the ascent phase.

If instead of the classical PEG method higher order terms are taken into account (PEG H.O.T.) the approach is similar, but more integral terms appear:

$$\begin{aligned} P(t) &= \int_0^t \frac{E}{m} t^2 dt = v_{ex} \cdot \left\{ \tau \cdot t + \frac{t^2}{2} + \tau^2 \cdot \ln \left(1 - \frac{t}{\tau} \right) \right\} \\ U(t) &= \int_0^t \left\{ \int_0^t \frac{E}{m} t^2 dt \right\} dt = -v_{ex} \cdot \left[\frac{t^3}{6} + \tau \cdot \frac{t^2}{2} + \tau^2 \cdot \left\{ (t - \tau) \cdot \ln \left(1 - \frac{t}{\tau} \right) - t \right\} \right] \end{aligned} \quad (8.51)$$

Using the previous integral expressions in equation 8.47 the position and velocity vectors of the missile in t would be:

$$\vec{v}(t) = \vec{v}_0 + L(t) \vec{\lambda}_v + [J(t) - t_\lambda \cdot L(t)] \cdot \dot{\vec{\lambda}} + \int_0^t \vec{g} dt \quad (8.52)$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 \cdot t + S(t) \cdot \vec{\lambda}_v + [Q(t) - t_\lambda \cdot S(t)] \cdot \dot{\vec{\lambda}} + \int_0^t \left\{ \int_0^t \vec{g} dt \right\} dt \quad (8.53)$$

The part of the integration that is due to the gravity term can be denoted as:

$$\int_0^t \vec{g} dt = \vec{v}_{grav}(t) \quad (8.54)$$

$$\int_0^t \left\{ \int_0^t \vec{g} dt \right\} dt = \vec{r}_{grav}(t) \quad (8.55)$$

The part of the integration that is due to the thrust can be denoted as:

$$\vec{v}_{thrust}(t) = L(t) \vec{\lambda}_v + [J(t) - t_\lambda \cdot L(t)] \cdot \dot{\vec{\lambda}} \quad (8.56)$$

$$\vec{r}_{thrust}(t) = S(t) \cdot \vec{\lambda}_v + [Q(t) - t_\lambda \cdot S(t)] \cdot \dot{\vec{\lambda}} \quad (8.57)$$

so equations 8.52 and 8.53 can be written as:

$$\vec{v}(t) = \vec{v}_0 + \vec{v}_{thrust}(t) + \vec{v}_{grav}(t) \quad (8.58)$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 \cdot t + \vec{r}_{thrust}(t) + \vec{r}_{grav}(t) \quad (8.59)$$

The gravity integrals are computed using approximations.

- For example the following approximation is suggested by R.L. McHenry et al. as indicated in [20]:

The missile trajectory without considering gravity can be approximated by a polynomial given by:

$$\begin{aligned}\vec{r}_p(t) &= \vec{\nu}_0 + \vec{\nu}_1 t + \vec{\nu}_2 t^2 + \vec{\nu}_3 t^3 \\ \vec{v}_p(t) &= \vec{\nu}_1 + 2\vec{\nu}_2 t + 3\vec{\nu}_3 t^2\end{aligned}\tag{8.60}$$

where the constants are obtained imposing that:

$$\begin{aligned}\vec{r}_p(t=0) &= \vec{r}_0 \\ \vec{v}_p(t=0) &= \vec{v}_0 \\ \vec{r}_p(t=t_f) &= \vec{r}_0 + \vec{v}_0 t_f + \vec{r}_{thrust}(t_f) \\ \vec{v}_p(t=t_f) &= \vec{v}_0 + \vec{v}_{thrust}(t_f)\end{aligned}\tag{8.61}$$

where $\vec{r}_{thrust}(t_f)$ and $\vec{v}_{thrust}(t_f)$ are the contribution of gravity to the final position and velocity vector.

With these constraints we obtain:

$$\begin{aligned}\vec{\nu}_0 &= \vec{r}_0 \\ \vec{\nu}_1 &= \vec{v}_0 \\ \vec{\nu}_2 &= \frac{3\vec{r}_{thrust}(t_f) - \vec{v}_{thrust}(t_f) \cdot t_f}{t_f^2} \\ \vec{\nu}_3 &= \frac{-2\vec{r}_{thrust}(t_f) + \vec{v}_{thrust}(t_f) \cdot t_f}{t_f^3}\end{aligned}\tag{8.62}$$

At the same time a coasting trajectory is approximated by:

$$\begin{aligned}\vec{r}_c(t) &= \vec{r}_{c1} + \vec{v}_{c1} t \\ \vec{v}_c(t) &= \vec{v}_{c1}\end{aligned}\tag{8.63}$$

and we obtain these new constants $\vec{r}_{c1}$ and $\vec{v}_{c1}$ imposing that the coasting trajectory is close to the thrust one with:

$$\begin{aligned}\int_0^{t_f} [\vec{r}_c(t) - \vec{r}_p(t)] dt &= 0 \\ \int_0^{t_f} [\vec{r}_c(t) - \vec{r}_p(t)] (t_f - t) dt &= 0\end{aligned}\tag{8.64}$$

which leads to:

$$\begin{aligned}\vec{r}_{c1} &= \vec{r}_0 - \frac{1}{10} \vec{r}_{thrust}(t_f) - \frac{1}{30} \vec{v}_{thrust}(t_f) \cdot t_f \\ \vec{v}_{c1} &= \vec{v}_0 + \frac{6}{5} \frac{\vec{r}_{thrust}(t_f)}{t_f} - \frac{1}{10} \vec{v}_{thrust}(t_f)\end{aligned}\tag{8.65}$$

Having $\vec{r}_{c1}$ and $\vec{v}_{c1}$ we propagate the coasting trajectory using orbital motion during t_f obtaining $\vec{r}_{c2}$ and $\vec{v}_{c2}$, and then we can estimate the effect of gravity as:

$$\int_0^{t_f} \vec{g} dt = \vec{v}_{grav}(t_f) \simeq \vec{v}_{c2} - \vec{v}_{c1} \quad (8.66)$$

$$\int_0^{t_f} \left\{ \int_0^t \vec{g} dt \right\} dt = \vec{r}_{grav}(t_f) \simeq \vec{r}_{c2} - \vec{r}_{c1} - \vec{v}_{c1} \cdot t_f \quad (8.67)$$

- A different approximation, developed by M. Delporte and F. Sauvient, is suggested in [18] (Song et al., 2015):

The gravity vector is approximated by a polynomial expression:

$$\vec{g}(t) = \vec{a}_0 + \vec{a}_1 t + \vec{a}_2 t^2 + \vec{a}_3 t^3 \quad (8.68)$$

where the constants are obtained by forcing:

$$\begin{aligned} \vec{g}(t=0) &= \vec{g}_0 \\ \dot{\vec{g}}(t=0) &= \dot{\vec{g}}_0 \\ \vec{g}(t=t_f) &= \vec{g}_f \\ \dot{\vec{g}}(t=t_f) &= \dot{\vec{g}}_f \end{aligned} \quad (8.69)$$

This approximations leads to the following values of the gravity integrals:

$$\int_0^{t_f} \vec{g} dt = \frac{1}{2} [\vec{g}_0 t_f + \vec{g}_f t_f] + \frac{1}{12} [\dot{\vec{g}}_0 t_f^2 + \dot{\vec{g}}_f t_f^2] \quad (8.70)$$

$$\int_0^{t_f} \left\{ \int_0^t \vec{g} dt \right\} dt = \frac{1}{20} [7\vec{g}_0 t_f^2 + 3\vec{g}_f t_f^2] + \frac{1}{60} [3\dot{\vec{g}}_0 t_f^3 - 2\dot{\vec{g}}_f t_f^3] \quad (8.71)$$

Having all these integral expressions the thrust direction that provides the desired injection values $\vec{r}_{inj}$ and $\vec{v}_{inj}$ is computed using a predictor-corrector scheme starting with $\vec{v}_{grav}(t_f) = \vec{0}$ and $\vec{r}_{grav}(t_f) = \vec{0}$:

1. Prediction step

- (a) $\vec{v}_{thrust}(t_f)$ is obtained from equation 8.58 taking the injection velocity $\vec{v}_{inj}$ into account:

$$\vec{v}_{thrust}(t_f) = \vec{v}_{inj} - \vec{v}_0 - \vec{v}_{grav}(t_f) \quad (8.72)$$

- (b) From equation 8.56 and forcing a value of t_λ such that $J(t_f) - t_\lambda \cdot L(t_f) = 0$ we get:

$$\vec{v}_{thrust}(t_f) = L(t_f)\vec{\lambda}_v \implies L(t_f) = \|\vec{v}_{thrust}(t_f)\| \quad (8.73)$$

This equation allows obtaining t_f from the equation for $L(t)$ in 8.50.

At the same time the value for $\vec{\lambda}_v$ is obtained:

$$\vec{v}_{thrust}(t_f) = L(t_f)\vec{\lambda}_v \implies \vec{\lambda}_v = \frac{\vec{v}_{thrust}(t_f)}{\|\vec{v}_{thrust}(t_f)\|} \quad (8.74)$$

- (c) Having t_f we obtain $L(t_f)$, $J(t_f)$, $S(t_f)$ and $Q(t_f)$ by integration up to t_f from equation 8.50.

- (d) We now obtain t_λ using the previously used hypothesis:

$$t_\lambda = \frac{J(t_f)}{L(t_f)} \quad (8.75)$$

- (e) We obtain $\vec{r}_{thrust}(t_f)$ from equation 8.59 taking the injection position $\vec{r}_{inj}$ into account:

$$\vec{r}_{thrust}(t_f) = \vec{r}_{inj} - \vec{r}_0 - \vec{v}_0 \cdot t_f - \vec{r}_{grav}(t_f) \quad (8.76)$$

- (f) We compute $\dot{\vec{\lambda}}$ from equation 8.57

$$\dot{\vec{\lambda}} = \frac{\vec{r}_{thrust}(t_f) - S(t_f) \cdot \vec{\lambda}_v}{Q(t_f) - t_\lambda \cdot S(t_f)} \quad (8.77)$$

- (g) We compute $\vec{v}_{grav}(t_f)$ and $\vec{r}_{grav}(t_f)$ using approximations like the ones explained before (using for example the expressions by R.L. McHenry or the expressions by M. Delporte)

2. Correction step

- (a) We now recompute the value for $\vec{v}_{thrust}(t_f)$ as:

$$\vec{v}_{thrust}(t_f) = \vec{v}_{inj} - \vec{v}_0 - \vec{v}_{grav}(t_f) \quad (8.78)$$

- (b) The prediction step is then repeated using this vector in step 1b

- (c) The prediction-correction loop is stopped when the obtained value for $\vec{v}_{thrust}(t_f)$ is almost equal to the previously computed one.

8.3.5.2.2 Behaviour and problems of the PEG guidance algorithm

The Space Launch System (SLS) in which the Space Shuttle was installed had an open-loop boost guidance followed by PEG ascent guidance ([1] (Braun et al., 2013)).

The PEG algorithm was also used by the Space Shuttle for orbit manoeuvring guidance ([1] (Braun et al., 2013)), including deorbit and all abort scenarios (return to launch site (RTLS), transoceanic abort landing (TAL), abort once around (AOA), and abort to orbit (ATO)).

Even though the method provides reliability and robustness covering a large flight envelope, it has several problems, as indicated in reference [18] (Song et al., 2015):

- The PEG algorithm has a general stability problem

When the vehicle is close to the injection point t_f tends to 0 and the obtained value for the integral terms tend to 0 as well, which leads to unstable solutions for $\vec{\lambda}_v$ (equation 8.74) and $\dot{\vec{\lambda}}$ (equation 8.77).

Different possible solutions for this problem are provided in reference [18] (Song et al., 2015). However, the usual procedure is to freeze the value for $\vec{\lambda}_v$ when t_f is low, which leads to errors with respect to the desired injection values.

- There could be convergence problems in the algorithm, especially when the initial guess is not very good.

Solutions for this problem are also analysed in reference [18] (Song et al., 2015) consisting in ways to improve the initial guess from the solutions obtained in the previous time step.

It has to be noted that the PEG algorithm uses as initial hypothesis the approximation $\vec{u}_F \simeq \vec{\lambda}_v + \dot{\vec{\lambda}}(t - t_\lambda)$, which is similar to the supposition for low values of $(\theta_{Gp} - \theta_{Gt})$ and $(\psi_{Gp} - \psi_{Gt})$ in the IGM algorithm. However it has been checked that in real cases for the Space Shuttle $\|\vec{\lambda}_v + \dot{\vec{\lambda}}(t - t_\lambda)\| \leq 1.1$ (see [20]) so the supposition in the case of the PEG is more applicable to the real case than the one used in the IGM and does not involve an accuracy problem.

8.3.5.2.3 Power Explicit Guidance (PEG) in the missiles simulator

The formulation for PEG given in section 8.3.5.2.1 is valid for achieving a certain injection point with the desired injection velocity. Achieving a certain injection position can be interesting in the case of a satellite, but it is irrelevant in the case of a missile where what is paramount is achieving the required velocity vector according to a Lambert's problem solver at the injection point.

With this aim the PEG algorithm will be modified in the missiles simulator so that the focus is achieving the required velocity vector, and the instability problem of the classical PEG formulation near the injection point (where as explained in section 8.3.5.2.2 $\vec{\lambda}_v$ has to be kept as constant) is avoided.

The procedure is described hereafter.

However it has to be noted that, being the achieved injection position irrelevant in this case, the main advantage of this guidance algorithm is lost.

A predictor-corrector scheme is used starting with an estimation for t_f (for example the time in which stage 3 finishes), $\vec{v}_{thrust}(t_f) = \vec{0}$, $\vec{v}_{grav}(t_f) = \vec{0}$ and $\vec{r}_{grav}(t_f) = \vec{0}$:

1. Prediction step

(a) We estimate the injection point as:

$$\vec{r}_{inj} = \vec{r}_0 + \vec{v}_0 \cdot t_f + \vec{r}_{grav}(t_f) + \vec{r}_{thrust}(t_f) \quad (8.79)$$

(b) The required velocity vector ($\vec{v}_{inj}$) to reach the target position from $\vec{r}_{inj}$ is computed using a Lambert's problem solver (see chapter 7). The time of flight initially computed for reaching the target is used in this solver in order to increase the robustness of the guidance algorithm, as was explained in section 7.3.3.

(c) $\vec{v}_{thrust}(t_f)$ is obtained from equation 8.58:

$$\vec{v}_{thrust}(t_f) = \vec{v}_{inj} - \vec{v}_0 - \vec{v}_{grav}(t_f) \quad (8.80)$$

(d) From equation 8.56 and forcing a value of t_λ such that $J(t_f) - t_\lambda \cdot L(t_f) = 0$ we get:

$$\vec{v}_{thrust}(t_f) = L(t_f)\vec{\lambda}_v \implies L(t_f) = \|\vec{v}_{thrust}(t_f)\| \quad (8.81)$$

This equation allows obtaining t_f from the equation for $L(t)$ in 8.50.

At the same time the value for $\vec{\lambda}_v$ is obtained:

$$\vec{v}_{thrust}(t_f) = L(t_f)\vec{\lambda}_v \implies \vec{\lambda}_v = \frac{\vec{v}_{thrust}(t_f)}{\|\vec{v}_{thrust}(t_f)\|} \quad (8.82)$$

(e) Having t_f we obtain $L(t_f)$, $J(t_f)$, $S(t_f)$ and $Q(t_f)$ by integration up to t_f from equation 8.50.

(f) We now obtain t_λ using the previously used hypothesis:

$$t_\lambda = \frac{J(t_f)}{L(t_f)} \quad (8.83)$$

(g) We create versors in the injection point:

$$\vec{u}_\perp = \frac{\vec{r}_{inj} \times \vec{v}_{thrust}(t_f)}{\|\vec{r}_{inj} \times \vec{v}_{thrust}(t_f)\|} \quad \vec{u}_{\perp\lambda} = \frac{\vec{\lambda}_v \times \vec{u}_\perp}{\|\vec{\lambda}_v \times \vec{u}_\perp\|} \quad (8.84)$$

(h) We directly create $\dot{\vec{\lambda}}$ as a vector that fulfills the conditions:

$$\begin{aligned} &\bullet \dot{\vec{\lambda}} \perp \vec{\lambda}_v \\ &\bullet \|\vec{\lambda}_v + \dot{\vec{\lambda}}(t_f - t_\lambda)\| \leq 1.0001 \\ &\implies \dot{\vec{\lambda}} = A \cdot \vec{u}_{\perp\lambda} \quad \text{with} \quad A \leq \frac{\sqrt{1.0001^2 - 1}}{(t_f - t_\lambda)} \end{aligned} \quad (8.85)$$

- (i) We obtain $\vec{r}_{thrust}(t_f)$ from equation 8.59:

$$\vec{r}_{thrust}(t_f) = \vec{r}_{inj} - \vec{r}_0 - \vec{v}_0 \cdot t_f - \vec{r}_{grav}(t_f) \quad (8.86)$$

- (j) We compute $\vec{v}_{grav}(t_f)$ and $\vec{r}_{grav}(t_f)$ using approximations like the ones explained before (using for example the expressions by R.L. McHenry or the expressions by M. Delporte).

2. Correction step

- (a) We use these values (t_f , $\vec{r}_{thrust}(t_f)$ and $\vec{r}_{grav}(t_f)$) in order to compute again the estimated injection point, going back to the step 1a of the prediction loop.
- (b) The correction step is then repeated.
- (c) The prediction-correction loop is stopped when the obtained value for $\vec{v}_{thrust}$ is almost equal to the previously computed one.

The cutoff condition within this scheme is when $\vec{v} \simeq \vec{v}_{inj}$.

An analysis was done in the simulator in order to assess whether including higher order terms in the loop (this is, using PEG H.O.T. instead of PEG) could increase the accuracy. This analysis concluded that with the implementation provided before, using higher order terms is irrelevant, since the error related to using $\vec{\lambda}_v + \dot{\vec{\lambda}}(t - t_\lambda)$ as $\vec{u}_F$ is directly set with this scheme, so including quadratic terms will not reduce the error of this hypothesis. As a consequence, higher order terms have not been considered in the simulator for this guidance algorithm.

For the gravity expressions both the formulation by R.L. McHenry and by M. Delporte have been included and briefly compared in the missiles simulator, concluding that the formulation by M. Delporte provides better results. As a consequence this will be the formulation used in the simulator for the gravity terms.

8.3.6 Comparison of exoatmospheric ascent guidance algorithms

It is hard to find open literatures comparing the performance of conventional ascent guidance algorithms.

[18](Song et al., 2015) compares the behaviour of different implementations of IGM and PEG, including and not higher order terms, modifying the initial guess for PEG, and using a linear differential corrector as well as the classical prediction-correction method for PEG. Within this comparison, a Monte-Carlo execution of 500 cases with different implementations of IGM and PEG showing the errors achieved in the injection point for a case with $V_{thrust}(t_f) \leq 750m/s$ was done. In these executions the main uncertainties within the flight were modelled with a normal distribution in which the 3σ value is assumed to be 2 %. Due to its relevance, a figure from reference [18](Song et al., 2015) with the results of these executions is repeated herein and shown in figure 8.2.

Figure 8.2: Injection errors with different IGM and PEG implementations (picture from reference [18])

From a theoretical point of view and being PEG a kind of vector form of IGM the performance of these 2 algorithms should be nearly identical. However, [18] shows that the performance of IGM can be more deteriorated than that of PEG, especially in terms of slower convergence in the time to go prediction. Also it was found that IGM can produce erroneous engine shutdown time with an early termination of the guidance algorithm, resulting in orbit injection errors (as can be seen in figure 8.2). Including the higher order terms to IGM allows reaching similar performance than that of PEG.

In the case of ballistic missiles, the errors in the position at the injection point are irrelevant and the relevant variable is the error with respect to the theoretical injection velocity vector given by a Lambert's problem solver for the actual injection position (for that reason the PEG algorithm was modified in the simulator as indicated in section 8.3.5.2.3).

Having that in mind, a comparison of the guidance algorithms implemented in the simulator (Lambert, Q and PEG) has been performed focusing on the errors in the velocity vector.

30 executions have been launched with the simulator for each considered case, computing the achieved error with respect to the theoretical velocity at the injection point (given by a Lambert solver).

A gaussian error of 2 % 3σ was introduced in the total thrust in order to align the comparison with that of [18] (Song et al., 2015).

The considered cases have been:

- ICBM launched from Shahrud, Iran and aiming at:
 - New York city, U.S.
 - Los Angeles, U.S.
 - Tokyo, Japan
 - Atka Island, U.S.
 - Capetown, South Africa
- GBI used as an ICBM (for the reentry phase the aerodynamic data of the ICBM is used) launched from Fort Greely, Alaska, and aiming at:
 - La Coruña, Spain
 - Reykjavik, Iceland
 - Kwajalein Atoll, Republic of the Marshall Islands
 - Rome, Italy
 - Tokyo, Japan

The objective is to cover all types of different possible trajectories (direct, retrograde, shorter range, larger range, short apogee, large apogee...) in case this affects the results provided by the algorithms.

The obtained results (average error in speed with respect to the value provided by a Lambert solver at the injection point, and standard deviation for the 30 executions) with each guidance algorithm within each considered case are given in table 8.1.

Table 8.1: Speed error at the injection point for Lambert guidance, Q guidance and Power Explicit Guidance (PEG)

Case	Type of orbit	Lambert guidance		Q guidance		Power Explicit Guidance	
		average $\ \Delta\vec{v}\ $ m/s	σ m/s	average $\ \Delta\vec{v}\ $ m/s	σ m/s	average $\ \Delta\vec{v}\ $ m/s	σ m/s
ICBM to N.Y.City	Retrograde	0.636	0.371	0.710	0.360	0.089	0.011
ICBM to L.A.	Retrograde	0.015	0.005	0.013	0.004	0.112	0.134
ICBM to Tokyo	Direct	0.543	0.261	0.455	0.262	1.090	0.803
ICBM to Atka Island	Direct	0.599	0.355	0.513	0.343	1.167	1.027
ICBM to Capetown	Retrograde	0.580	0.299	0.591	0.323	1.437	1.027
GBI to La Coruña	Direct	0.263	0.106	0.493	0.057	0.243	0.203
GBI to Reykjavik	Direct	0.232	0.073	0.489	0.027	0.160	0.148
GBI to Kwajalein Atoll	Retrograde	0.309	0.107	0.437	0.057	0.311	0.275
GBI to Rome	Direct	0.288	0.129	0.559	0.045	0.325	0.262
GBI to Tokyo	Retrograde	0.258	0.096	0.367	0.516	0.197	0.185

Table 8.2: Speed error at the injection point for Lambert guidance, Q guidance and Power Explicit Guidance (PEG)
Average values for all ICBM and GBI cases

Case	Lambert guidance		Q guidance		Power Explicit Guidance	
	average $\ \Delta\vec{v}\ $ m/s	σ m/s	average $\ \Delta\vec{v}\ $ m/s	σ m/s	average $\ \Delta\vec{v}\ $ m/s	σ m/s
ICBM cases	0.474	0.371	0.456	0.374	0.783	0.936
GBI cases	0.270	0.107	0.470	0.081	0.247	0.229

It can be seen in table 8.1 that the results obtained with all the methods are very good, with very low maximum errors (less than 1.5 m/s in average for all cases).

From the theoretical point of view, Q guidance should be better than Lambert guidance, since it was found to be an optimal guidance scheme when a flat Earth hypothesis is used (see reference [7] (Battin, 1982)). However, the results obtained with Lambert guidance and Q guidance are very similar and Q guidance is only slightly better for ICBM launches, being Lambert guidance clearly better for GBI launches in the simulations (see table 8.2).

The results obtained for the implementation of PEG used in the simulator are much worse for ICBM launches. Being the method based on an integration of the thrust vector with time, it behaves much worse when a post-boost phase is required, since performing a correct estimation of the integrals in this case is more difficult (the post-boost phase involves a low thrust vector and a large integration time). In the case of GBI launches, since there is no post-boost phase, the integration method proves to be more reliable and it is the most accurate method among the 3 implemented ones (see table 8.2).

8.3.7 Algorithm to be used in the simulator for conventional ascent guidance

As indicated before, the following conventional ascent guidance algorithms have been implemented in the simulator and can be used to launch executions:

- Lambert guidance
- Q guidance
- Powered Explicit Guidance (PEG)

Taking the results of the comparisons into account, Q guidance will be used as the conventional ascent guidance within this thesis for the ICBM, and PEG will be used as the conventional ascent guidance for the GBI, when comparisons between different guidance algorithms are made (chapter 11).

Chapter 8 references

- [1] Robert D. Braun, Zachary R. Putnam, and Bradley A. Steinfeldt. Advances in Inertial Guidance Technology for Aerospace Systems. *AIAA Guidance, Navigation and Control (GNC) Conference*, August 2013. doi: 10.2514/6.2013-5123. [194](#), [195](#), [196](#), [203](#), [208](#), [215](#)
- [2] Edward Large. Optimal Climb Paths for a Ballistic Rocket within the Atmosphere. *Journal of Spacecraft and Rockets*, 16(4):307–326, July 1979. doi: 10.2514/3.57658. [196](#)
- [3] Greg A. Dukeman. *Closed-Loop Nominal and Abort Atmospheric Ascent Guidance for Rocket-Powered Launch Vehicles*. PhD thesis, Georgia Institute of Technology, May 2005. [196](#)
- [4] John M. Hanson, M. Wade Shrader, and Craig A. Cruzen. NASA-TM-112493. Ascent Guidance Comparisons. *Journal of Astronautical Sciences*, 43(3):307–326, July-September 1995. doi: 10.2514/6.1994-3568. [196](#), [197](#)
- [5] D. W. Whitcombe. *Air Force Report SAMSO-TR-72-299. Present and Advanced Guidance Techniques*. Engineering Science Operations. The Aerospace Corporation, Huntsville, Alabama, November 1972. [198](#)
- [6] George M. Siouris. *Missile Guidance and Control Systems*. Springer-Verlag New York, Inc., New York, 2004. [198](#), [201](#)
- [7] Richard H. Battin. Space guidance evolution - a personal narrative. *Journal of Guidance, Control, and Dynamics*, 5(2):97–110, March-April 1982. doi: 10.2514/3.19761. [198](#), [201](#), [202](#), [221](#)
- [8] Helmut J. Horn. *NASA Technical Note D-2967. Application of an Iterative Guidance Mode to a Lunar Landing*. George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, November 1965. [199](#)
- [9] Richard H. Battin. *An Introduction to the Mathematics and Methods of Astrodynamics*. American Institute of Aeronautics and Astronautics Inc., Alexander Bell Drive, Reston, VA, revised edition edition, 1999. [199](#)
- [10] G.E. Townsend, A.S. Abbot, and R.R. Palmer. *NASA Contract Report CR-1007. Guidance, Flight Mechanics and Trajectory Optimization. Volume VIII - Boost Guidance Equations*. National Aeronautics and Space Administration, Washington D.C., April 1968. [202](#)
- [11] M. Seetharama Bhat and Shashi K. Shrivastava. An Optimal Q-Guidance Scheme for Satellite Launch Vehicles. *Journal of Guidance, Control, and Dynamics*, 10(1):53–60, January-February 1987. doi: 10.2514/3.20180. [202](#)
- [12] Aidin Mohammadi, Jafar Roshanian, Mohsen Bahrani, and Behrouz Ebrahimi. On Application of Q-guidance Method for Satellite Launch Systems. *IEEE Conference Publications. 3rd International Symposium on Systems and Control in Aeronautics and Astronautics*, 48: 1314–1319, June 2010. doi: 10.1109/ISSCAA.2010.5632295. [202](#)

- [13] C. Circi. Hybrid methods and Q-guidance for rocket performance optimization. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, pages 353–359, May 2004. 202
- [14] W. L. Davis. *An Efficient Algorithm for Computing the Q-Guidance Matrix*. Naval Surface Weapons Center, Dahlgren, Virginia, January 1979. 203, 204
- [15] Frank L. Lewis, Draguna L. Vrabie, and Vassilis L. Syrmos. *Optimal Control*. John Wiley & Sons Inc., Hoboken, New Jersey, 2012. 205
- [16] Arthur E. Bryson and Yu-Chi Ho. *Applied Optimal Control: optimization, estimation, and control*. Taylor & Francis Group, New York, 1975. 205
- [17] I.E.Smith. *Technical Memorandum X-53414. General Formulation of the Iterative Guidance Mode*. George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, March 1966. 205, 207
- [18] Eun-Jung Song, Sang bum Cho, and Woong-Rae Roh. A comparison of iterative explicit guidance algorithms for space launch vehicles. *Advances in Space Research*, 55:463–476, January 2015. doi: 10.1016/j.asr.2014.09.025. 205, 207, 208, 213, 215, 218, 219
- [19] Eun-Jung Song, Sang bum Cho, Woong-Rae Roh, and Miok Joh. Stability analysis of explicit guidance laws for space launch vehicles with varying thrust integrals. *Advances in Space Research*, 48:133–145, July 2011. doi: 10.1016/j.asr.2011.02.014. 209
- [20] Philip N. Springmann. Lunar Descent Using Sequential Engine Shutdown. Master’s thesis, Massachusetts Institute of Technology, February 2006. 212, 215
- [21] Keum-Seong Kim, Jung-Woo Park, Min-Jea Tahk, and Han-Lim Choi. A PEG-Based Ascending Guidance Algorithm for Ramjet-Powered Vehicles. http://www.icas.org/ICAS_ARCHIVE/ICAS2012/PAPERS/611.PDF, September 2012.
- [22] Frank M. Perkins. *Air Force Report SSD-TR-66-211. Derivation of Linear-Tangent Steering Laws*. Aerospace Corporation, El Segundo. California, November 1966.
- [23] Peter H. Zipfel. Orbital Insertion Control of a Three-stage Solid Rocket Booster Modeled in Six Degrees-of-freedom. *Journal of Modeling, Simulation, Identification, and Control*, 2(1): 31–44, September 2014. doi: 10.7726/jmsic.2014.1003.
- [24] Pedro Sanz Aránguez. Resúmenes de Optimización de Trayectorias y Leyes de Guiado printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, 2003.

This page intentionally left blank.

Chapter 9

Conventional terminal guidance

This chapter analyses the more common terminal guidance algorithms, adapts them to the simulator, and indicates the one to be used in the final phase of the interception problem by the EKV when a conventional terminal guidance is requested, once the EKV has been deployed by the GBI towards the PIP or directly towards the incoming ICBM.

9.1 The EKV during the terminal guidance

If the ascent guidance phase worked properly, once the EKV has been released by the GBI, its velocity vector is aiming towards the PIP or towards the incoming ICBM, which could be more than 7000 km apart.

The ICBM is in these conditions approaching with a speed of about 7 km/s and the EKV has a speed of about 5 km/s. This means that the interception would take place in about 700 seconds so, on the one hand, there is still time to properly manoeuvre the EKV. On the other hand and taking the closing speed into account, any guidance error would lead to a huge miss distance.

Figure 9.1: The EKV in space

As indicated in Appendix G there are 4 divert thrusters and 2 attitude control systems in the EKV (see figure 4.7). This provides the capability to control the EKV attitude (using the attitude control systems) and modify the direction of the trajectory in the most suitable way for the interception (using the divert thrusters).

9.2 EKV attitude control

The EKV is basically a big infrared sensor traveling through space. Even though the EKV keeps getting information from the Command and Control, Battle Management and Communications Center about the state of the incoming ICBM (see chapter 2.3.2), it is paramount that its own infrared sensor detects the threat, since that way the detection errors in the estimation of the ICBM state, which would be relatively constant using only information from ground radars and satellite sensors, can decrease with time and become very small in the final interception moments, when any detection error could lead to a failed interception.

As a consequence we will use the attitude control systems to rotate the EKV so that its Ox_b axis is always aiming towards the ICBM estimated position.

This procedure will be done by providing to the EKV control system the desired attitude angles (roll, pitch and yaw) so that it can achieve them.

The procedure to obtain the desired angles is indicated hereafter.

The first step is computing the Line of Sight vector (LOS). This is a unit vector that starting from the EKV always points to the estimated position of the ICBM.

From the estimated position of the ICBM and the position of the EKV in the inertial reference frame (ECI, see Appendix A) we get:

$$\overrightarrow{LOS}^i = \frac{\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i}{\|\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i\|} \quad (9.1)$$

where the superscript i indicates the inertial reference frame.

We then transform this $\overrightarrow{LOS}^i$ vector to the ECEF frame:

$$\overrightarrow{LOS}^e = C_i^e \cdot \overrightarrow{LOS}^i \quad (9.2)$$

where the superscript e indicates the ECEF reference frame and the transformation matrix C_i^e only depends on the present time (see Appendix section A.3).

From the geodetic coordinates of the EKV we obtain the transformation matrix from the ECEF frame to the navigation frame (equation A.35), which allows obtaining the LOS vector in the navigation frame:

$$\overrightarrow{LOS}^n = C_e^n \cdot \overrightarrow{LOS}^e \quad (9.3)$$

where the superscript n indicates the Navigation reference frame.

Having the LOS vector in the navigation frame we can easily obtain the required yaw and pitch angles by:

$$\begin{aligned} \psi_{reference} &= \text{atan} \frac{\overrightarrow{LOS}^n|_{East}}{\overrightarrow{LOS}^n|_{North}} \\ \theta_{reference} &= \text{atan} \frac{-\overrightarrow{LOS}^n|_{Down}}{\overrightarrow{LOS}^n|_{Hor}} \\ \overrightarrow{LOS}^n|_{Hor} &= \sqrt{(\overrightarrow{LOS}^n|_{East})^2 + (\overrightarrow{LOS}^n|_{North})^2} \end{aligned} \quad (9.4)$$

which will be the reference angles provided to the EKV control system.

A null roll reference angle will also be provided:

$$\phi_{reference} = 0.0 \quad (9.5)$$

This means that we demand the EKV control system to stop spinning. The GBI did not have any roll control system so residual spins could have appear through the ascent phase. While this spinning did not affect much the GBI performance, it could deteriorate the performance of the EKV that would have to counteract otherwise the rotation motion when commanding lateral forces.

9.3 EKV divert control

The most relevant guidance techniques that can be used by the EKV in the terminal guidance phase will be classified herein (this classification is based on reference [1]):

- Command guidance

In this case the missile is not creating the guidance commands but using ground-generated guidance commands to reach the target.

There are several types of command guidance:

- Beam riding

In this case the target is illuminated from a ground base with an electromagnetic beam (infrared, laser or radio signal) and the missile tries to keep as close to the center of the beam as possible.

- Command to line of sight (also called *3 points guidance*)

In this case the missile is commanded to remain as close as possible to the line of sight between the tracking station and the target.

- Active guidance or homing guidance

In this case the missile is using its own sensors to locate the target and generate the guidance commands.

There are several types of homing guidance:

- Line of sight guidance (also called *pure pursuit* and *2 points guidance*)

In this case the nose of the missile always directs itself towards the present position of the target.

- Deviated pursuit

In this case the nose of the missile is always pointing a certain angle ahead (*lead pursuit*) or behind (*lag pursuit*) the target.

- Proportional Navigation (also known as *collision homing*)

In this case the lead angle is changed proportionally to the angular rate of the line of sight to the target. This guidance law is detailed in section 9.6

- Optimal guidance

In this case the guidance law is obtained by minimizing a functional. These types of guidance laws will be analysed in chapter 10

In this case the EKV will not use a command guidance. On the one hand the Beam Riding guidance cannot be applied since it would be difficult to point a beam to the incoming ICBM which is too far away and moving very fast, and the geometry ground-target-EKV would avoid a successful use of this beam. In the same way a Command to line of sight guidance would require a single tracking station, which is not the case.

This means that the EKV will use homing guidance techniques. These techniques will be based in all the information it gets through an uplink from the target from the Command and Control, Battle Management and Communications Center (see chapter 2.3.2) and its own infrared sensor. In fact in the first part of the interception, when the ICBM is too far away, it will have to trust the information about the target from the ground tracking stations. In any case, based on this information, it will have to use homing guidance techniques.

Among these homing guidance techniques line of sight guidance will not be considered. As indicated in reference [2]) command to line of sight guidance results in high lateral accelerations in fast-dynamics situations, so its use is limited in practice to relatively close-range applications against a slow target like anti-tank missile systems.

In the same way pure pursuit guidance and deviated pursuit guidance will not be considered. As indicated in reference [1], pursuit guidance is considered impractical as a homing guidance law against moving targets because the manoeuvres required of the missile become increasingly hard during the last, critical, stages of the flight. The most favorable application of the pursuit course guidance law is against slow-moving targets.

As a consequence, the focus will be in different implementations of proportional navigation guidance, that will be analysed hereafter, and optimal guidance techniques (that will be analysed in chapter 10).

9.3.1 Proportional Navigation (PN)

Proportional Navigation is the most widespread guidance technique used for terminal guidance when pursuing targets. It was probably invented by C.L. Yuan at the RCA Laboratories in the U.S. in 1943 (see reference [2]) and since it was first used in the 50s in the Lark missile, it has been used in virtually all the world's tactical guided missiles due to its simplicity, effectiveness and ease of implementation (see reference [3]).

The guidance equation in this case is very simple and consists in providing acceleration commands perpendicular to the instantaneous missile-target line of sight which are proportional to the line of sight variation with time and to the closing velocity:

$$\vec{n}_c|_{PN} = N' \cdot V_c \cdot \dot{\overrightarrow{LOS}} \quad (9.6)$$

where:

- $\vec{n}_c|_{PN}$ is the acceleration command using this guidance law to reach the target (perpendicular to the $\overrightarrow{LOS}$ vector).
- N' is a dimensionless gain for the guidance scheme (known as effective navigation ratio). Values of N' are usually between 3 and 5.
- V_c is the missile-target closing speed:

$$V_c = \|(\vec{v}_{target} - \vec{v}_{Missile}) \cdot \overrightarrow{LOS}\| \quad (9.7)$$

- $\overrightarrow{LOS}$ is the Line of Sight vector as computed in equation 9.1

Making the command proportional to $\dot{\overrightarrow{LOS}}$ in 9.6 leads quickly to a constant direction for the missile in case the target is not manoeuvring nor accelerating. This is, it leads to a *parallel navigation* technique (see reference [2]).

The derivative of the $\overrightarrow{LOS^i}$ vector can be easily obtained:

$$\begin{aligned}\overrightarrow{LOS^i} &= \frac{d}{dt}(\overrightarrow{LOS^i}) = \frac{d}{dt} \left(\frac{\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i}{\|\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i\|} \right) = \\ &= \frac{\vec{v}_{ICBM}^i - \vec{v}_{EKV}^i}{\|\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i\|} - (\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i) \cdot \frac{(\vec{v}_{ICBM}^i - \vec{v}_{EKV}^i) \cdot (\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i)}{\|\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i\|^3}\end{aligned}\quad (9.8)$$

In our interception problem this guidance law would be applied to the divert thrusters taking the EKV attitude into account:

$$\begin{aligned}\vec{n}_c^i &= N' \cdot V_c \cdot \overrightarrow{LOS^i} \\ \vec{n}_c^b &= C_i^b \cdot \vec{n}_c^i \\ \vec{U}^b &= M_{EKV} \cdot \vec{n}_c^b\end{aligned}\quad (9.9)$$

where the superscript b indicates the body reference frame and the matrix C_i^b can be easily obtained using equation 3.17 from the EKV state vector since it contains the rotation quaternion from the inertial to the body reference frame (q_{bi}).

M_{EKV} is the mass of the EKV and it is required to transform the acceleration commands to force commands in the thrusters. This magnitude can be estimated by the EKV taking into account the forces requested to the thrusters throughout the trajectory, or in a more simplified way. The resulting force, in the body reference frame ($\vec{U}^b$) is directly the control command to be obtained from the divert thrusters.

9.3.2 Augmented Proportional Navigation and gravity compensation

The Proportional navigation scheme previously indicated does not take into account any knowledge of the dynamics of the system. It seems sensible that if the dynamics of the target is known and taken into account a better guidance law could be obtained.

Reference [3] provides a way to take this into account with the following guidance law (Augmented Proportional Navigation) obtained from the minimization of the zero error miss (ZEM) vector (see section 9.3.4):

$$\vec{n}_c|_{APN} = N' \cdot V_c \cdot \overrightarrow{LOS} + N' \cdot \frac{\vec{n}_T}{2} \quad (9.10)$$

where:

- $\vec{n}_c|_{APN}$ is the acceleration command using this guidance law to reach the target (perpendicular to the $\overrightarrow{LOS}$ vector).
- N' is the effective navigation ratio.
- V_c is the missile-target closing speed (equation 9.7)
- $\vec{n}_T$ is the part of the acceleration of the target perpendicular to the line of sight.

In our case the target is in a free-falling trajectory with only the gravity acting on it so:

$$\vec{n}_T = \vec{g}_T - \overrightarrow{LOS} \cdot (\vec{g}_T \cdot \overrightarrow{LOS}) \quad (9.11)$$

where $\vec{g}_T$ can be estimated from the position of the target in a simplified way ($\vec{g}_T \simeq -\mu \frac{\vec{r}}{r^3}$) or using a more complex expression like equation 4.1 as explained in section 4.2.1.

If we are taking into account the dynamics of the target system, it also seems sensible to take into account the own dynamics of the missile, which is also within a gravitational field. This leads to the Proportional Navigation law with gravity compensation:

$$\vec{n}_c|_{PNgrav} = N' \cdot V_c \cdot \overrightarrow{LOS} + N' \cdot \frac{\vec{n}_T - \vec{n}_M}{2} \quad (9.12)$$

where:

- $\vec{n}_c|_{PNgrav}$ is the acceleration command using this guidance law to reach the target (perpendicular to the $\overrightarrow{LOS}$ vector).
- N' is the effective navigation ratio.
- V_c is the missile-target closing speed (equation 9.7)
- $\vec{n}_T$ is the part of the acceleration of the target perpendicular to the line of sight (equation 9.11).
- $\vec{n}_M$ is the part of the non commanded acceleration of the missile perpendicular to the line of sight.

As with the target the non commanded acceleration of the missile comes from the gravity acceleration:

$$\vec{n}_M = \vec{g}_M - \overrightarrow{LOS} \cdot (\vec{g}_M \cdot \overrightarrow{LOS}) \quad (9.13)$$

where $\vec{g}_M$ can be obtained from the missile position using equation 4.1 as explained in section 4.2.1.

In our interception problem this guidance law would be applied to the divert thrusters taking the EKV attitude into account:

$$\begin{aligned} \vec{n}_c^i &= N' \cdot V_c \cdot \overrightarrow{LOS}^i + N' \cdot \frac{\vec{n}_T^i - \vec{n}_M^i}{2} \\ \vec{n}_c^b &= C_i^b \cdot \vec{n}_c^i \\ \vec{U}^b &= M_{EKV} \cdot \vec{n}_c^b \end{aligned} \quad (9.14)$$

where the superscript b indicates the body reference frame and:

- $\overrightarrow{LOS}^i$ is given by equation 9.8
- $\vec{n}_T$ is given by equation 9.11
- $\vec{n}_M$ is given by equation 9.13
- C_i^b can be easily obtained using equation 3.17 from the EKV state vector since it contains the rotation quaternion from the inertial to the body reference frame (q_{bi}).
- M_{EKV} is the mass of the EKV.

This magnitude can be estimated by the EKV taking into account the forces requested to the thrusters throughout the trajectory, or in a more simplified way.

- $\vec{U}^b$ is directly the control command to be obtained from the divert thrusters.

9.3.3 Performance of Proportional Guidance

In this section we generically talk about "Proportional Guidance" referring to conventional "Proportional Guidance" (section 9.3.1), to "Augmented Proportional Guidance" (section 9.3.2) and to "Proportional Guidance with gravity compensation" (also section 9.3.2).

The proportional navigation law is very simple and elegant and being so simple it may seem easy to improve it with more sophisticated guidance techniques.

However that is not the case and it continues being used extensively.

On the one hand its simplicity allows a very easy implementation with a very low computational demand since it only requires the variation of the line of sight (which is easy to obtain from the missile sensors directly in the body frame, avoiding the conversions we did in equation 9.9) and the closing speed (which can be estimated).

On the other hand the algorithm is very robust, avoiding singularities near the interception, present in other guidance schemes.

Finally, its accuracy is very good.

The accuracy comes from the fact that the proportional guidance is a pseudo-optimal guidance law, as shown first by Arthur E. Bryson and Yu-Chi Ho (see reference [4]): if we apply an optimal problem for an interception missile-target in which the following hypotheses are made:

1. Planar motion
2. The coordinates away from the line of sight are small (near-collision course assumptions)
3. There are no delays in the missile commands
4. No saturation is considered for the missile commands

and we then minimize the functional:

$$J = \int_0^{t_F} \|\vec{n}_c\|^2 dt \quad (9.15)$$

we get the law related to the Augmented Proportional Navigation with $N' = 3$ (see pages 171-173 of [3] for the demonstration). A different effective navigation ratio would be obtained if a different function of the acceleration instead of $\|\vec{n}_c\|^2$ were to be minimized.

This explains why this algorithm is so widely used and why it is so difficult to improve.

The only known limitations of the proportional navigation algorithms (that allow room for improvement) are:

- It shows problems when it is applied to a system with significant delays in the controls (as studied in reference [3])
- It shows sensitivity to noise in the target position
- It fails against certain manoeuvres carried out by the target (as extensively studied in reference [3])

9.3.4 Predictive Guidance

A way to obtain the Proportional navigation guidance law is to consider the estimated Zero Error Miss, this is, the projected vector missile-target in the final interception point and minimizing its value.

This guidance law would be given by the following equation:

$$\vec{n}_c|_{PNPRED} = N' \cdot \frac{\overrightarrow{ZEM}_{\perp LOS}}{t_{go}^2} \quad (9.16)$$

where:

- $\vec{n}_c|_{PNPRED}$ is the acceleration command using this guidance law to reach the target.
- N' is the effective navigation ratio.
- t_{go} is the remaining time for the interception
- $\overrightarrow{ZEM}$ is the vector missile-target in t_{go} obtained using models for the target and the missile motion, and using a numerical integration

From this equation, when the following expression is used for t_{go} :

$$t_{go} = \frac{\|\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i\|}{\|(\vec{v}_{target} - \vec{v}_{Missile}) \cdot \overrightarrow{LOS}\|} = \frac{\|\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i\|^2}{\|(\vec{v}_{target}^i - \vec{v}_{Missile}^i) \cdot (\vec{r}_{ICBM}^i - \vec{r}_{EKV}^i)\|} \quad (9.17)$$

and no acceleration is considered for the target and the missile, the proportional navigation law as indicated in equation 9.6 is obtained.

When an acceleration is considered for the target, the augmented proportional navigation law given in equation 9.10 is obtained.

When an acceleration is considered for the missile and for the target, the proportional navigation law with gravity compensation given in equation 9.12 is obtained.

If better approximations are used to obtain t_{go} or to predict the future position of the ICBM and the EKV, using expression 9.16 could improve the performance of proportional navigation guidance laws. The accuracy depends on the models used.

As in the previous cases, this guidance law would be applied to the divert thrusters taking the EKV attitude into account:

$$\begin{aligned}\vec{n}_c^i &= N' \cdot \frac{\overrightarrow{ZEM}_{\perp LOS}^i}{t_{go}^2} \\ \vec{n}_c^b &= C_i^b \cdot \vec{n}_c^i \\ \vec{U}^b &= M_{EKV} \cdot \vec{n}_c^b\end{aligned}\tag{9.18}$$

where the superscript b indicates the body reference frame and:

- $\overrightarrow{ZEM}^i = \vec{r}_T(t_{go}) - \vec{r}_M(t_{go})$ is to be computed from a model of the motion of the missile and the target
- $\overrightarrow{LOS}^i$ is given by equation 9.8
- $\overrightarrow{ZEM}_{\perp LOS}^i = \overrightarrow{ZEM}^i - \overrightarrow{LOS}^i \cdot (\overrightarrow{ZEM}^i \cdot \overrightarrow{LOS}^i)$
- t_{go} is computed using equation 9.17
- C_i^b can be easily obtained using equation 3.17 from the EKV state vector since it contains the rotation quaternion from the inertial to the body reference frame (q_{bi}).
- M_{EKV} is the mass of the EKV.
This magnitude can be estimated by the EKV taking into account the forces requested to the thrusters throughout the trajectory, or in a more simplified way.
- $\vec{U}^b$ is directly the control command to be obtained from the divert thrusters.

9.3.5 Comparison of terminal guidance algorithms

Comparisons of the behaviour of the proportional navigation scheme versus augmented proportional navigation and proportional navigation with gravity compensation are done in reference [3] showing that, as expected, the gravity compensation reduces the required commands, especially at the end of the interception, being this a more suitable algorithm in interception scenarios where the time to interception is large, as it is the case.

Regarding predictive guidance, this algorithm only improves the results of other proportional navigation schemes when better models for the dynamics of the target or the missile are considered. For example reference [3] indicates its advantages versus proportional navigation with gravity compensation when a boosting target is considered, since in this case a better estimation of the $\overrightarrow{ZEM}$ vector can be done versus what would be obtained supposing only gravitational forces. With the same approach [5](Hablani, 2006) uses predictive guidance in a strategic interception case taking into account system delays.

In our case and for this terminal phase a model considering the gravitational accelerations of both the missile and the target seems enough so predictive guidance will not be used.

9.3.6 Algorithm to be used in the simulator for conventional terminal guidance

Taking the previous analyses into account, proportional navigation guidance with gravity compensation will be the conventional terminal guidance algorithm to be used in the simulator when comparisons between different guidance algorithms are made (chapter 11).

Chapter 9 references

- [1] George M. Siouris. *Missile Guidance and Control Systems*. Springer-Verlag New York, Inc., New York, 2004. [228](#), [229](#)
- [2] N.A.Shneydor. *Missile Guidance and Pursuit. Kynematics, Dynamics and Control*. Woodhead Publishing Limited, Cambridge, United Kingdom, first edition, 1998. [229](#)
- [3] Paul Zarchan. *Tactical and Strategic Missile Guidance*. American Institute of Aeronautics and Astronautics Inc., Alexander Bell Drive, Reston, Virginia, sixth edition, February 2012. [229](#), [230](#), [232](#), [234](#)
- [4] Arthur E. Bryson and Yu-Chi Ho. *Applied Optimal Control: optimization, estimation, and control*. Taylor & Francis Group, New York, 1975. [232](#)
- [5] Hari B. Hablani. Endgame Guidance and Relative Navigation of Strategic Interceptors with Delays. *Journal of Guidance, Control, And Dynamics*, 29(1):82–94, January-February 2006. doi: 10.2514/1.12748. [234](#)
- [6] Paul Zarchan. Ballistic Missile Defense. Guidance and Control Issues. <http://scienceandglobalsecurity.org/archive/sgs08zarchan.pdf>, 1998.
- [7] Rafael Yanushevsky. *Modern Missile Guidance*. CRC Press. Taylor & Francis Group, Boca Raton, Florida, first edition, 2008.

This page intentionally left blank.

Chapter 10

Optimal guidance

This chapter develops optimal guidance algorithms for the ascent phase since the end of the atmospheric ascent, and for the terminal phase.

These algorithms will be implemented in the simulator in order to be compared with the conventional guidance algorithms.

10.1 Optimal control theory

10.1.1 Historical note

The calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals. This is, obtaining extremal functions that make these functional attain a maximum or minimum value.

It could be considered that the calculus of variations started with the brachistochrone curve problem raised by Johann Bernoulli (1667-1748) in 1696 (finding the shape of the curve $y = f(x)$ connecting two points A and B in a uniform gravity field so that a particle starting in A in a frictionless track over the curve reaches point B in the minimum time). These problems were analysed by Leonhard Euler (1707-1783) who gave name to the science with his book *Elementa Calculi Variationum*. Euler's work was continued by Joseph-Louis Lagrange (1736-1813), who contributed extensively to the theory. Other renowned contributors from the 18th and 19th century were Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1755-1855), Siméon Denis Poisson (1781-1840), Augustin-Louis Cauchy (1789-1857), Mikhail Ostrogradsky (1801-1862), Carl Gustav Jacob Jacobi (1804-1851), William Rowan Hamilton (1805-1865), and especially Karl Theodor Wilhelm Weierstrass (1815-1897). In the 20th century the works by David Hilbert (1862-1943), Jacques Hadamard (1865-1963) and Henri Lebesgue (1875-1941) can be highlighted.

The optimal control theory can be considered as an extension of the calculus of variations, in which the main purpose is to obtain control policies.

The methods applied in optimal control theory are largely due to the work of Lev Pontryagin (1908-1988) and Richard E. Bellman (1920-1984).

10.1.2 Basic equations

10.1.2.1 Problem formulation

The optimal control theory starts with a functional that includes the control vector for a system and that we want to minimize:

$$J = \phi(\vec{X}_f, t_f) + \int_{t_0}^{t_f} \mathcal{L}(\vec{X}, \vec{U}, t) dt \quad (10.1)$$

subject to:

$$\begin{aligned} \dot{\vec{X}}(t) &= \vec{f}(\vec{X}, \vec{U}, t) && \text{system equation (} n \text{ components for } \vec{X} \text{ and } m \text{ components for } \vec{U}) \\ \vec{X}(t_0) &= \vec{X}_0 && \text{initial conditions} \\ \vec{\Psi}(\vec{X}_f, t_f) &= \vec{0} && \text{terminal constraints (} q \text{ equations with } q \leq n) \end{aligned}$$

where the term $\phi(\vec{X}_f, t_f)$ is a terminal cost.

When the functional is expressed as in equation 10.1 the optimal problem is called a *Bolza problem*. When it only includes the terminal cost the optimal problem is called a *Mayer problem*. When it does not include a terminal cost the optimal problem is called a *Lagrange problem*.

When t_f is considered as fixed the problem is called a *brachistochrone problem*. When the t_f is not fixed the problem is called a *launch problem*.

10.1.2.2 Euler-Lagrange theorem

The Euler-Lagrange Theorem (see reference [1] and [2] for the demonstration) states that if $\Phi, \mathcal{L}, f, \vec{\Psi} \in \mathcal{C}^1$, and the control of the system ($\vec{U}$) is unbounded, there exists a time-varying multiplier vector $\vec{\lambda}^T(t)$ (n elements) (called *costate vector*) and a constant multiplier vector $\vec{v}^T$ (q elements) such that with the Hamiltonian function:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \mathcal{L}(\vec{X}, \vec{U}, t) + \vec{\lambda}^T \cdot \vec{f}(\vec{X}, \vec{U}, t) \quad (10.2)$$

and with the terminal function:

$$\Phi(\vec{X}_f, t_f) = \phi(\vec{X}_f, t_f) + \vec{v}^T \cdot \vec{\Psi}(\vec{X}_f, t_f) \quad (10.3)$$

the following necessary conditions must hold:

$$\begin{aligned} \dot{\vec{\lambda}}^T &= - \frac{\partial H}{\partial \vec{X}} \\ \vec{\lambda}^T(t_f) &= \frac{\partial \Phi}{\partial \vec{X}} \Big|_f \\ \frac{\partial H}{\partial \vec{U}} &= \vec{0} \end{aligned} \quad (10.4)$$

and the following *transversality condition* (called *algebraic form of the transversality condition*):

$$\Omega(\vec{X}_f, \vec{U}_f, t_f) = \mathcal{L}_f + \frac{d\Phi}{dt} \Big|_f = 0 \quad (10.5)$$

which only applies if t_f is unspecified (in *launch problems*).

With this formulation we have up to $2n + m + q + 1$ unknowns and enough equations and boundary conditions to solve them:

- n unknowns for the components of $\vec{X}$
with n system equations: $\dot{\vec{X}}(t) = \vec{f}(\vec{X}, \vec{U}, t)$
and with n initial conditions: $\vec{X}_0$
- n unknowns for the components of $\vec{\lambda}$
with n costate equations: $\dot{\vec{\lambda}}^T = - \frac{\partial H}{\partial \vec{X}}$
and with n boundary conditions: $\vec{\lambda}^T(t_f) = \frac{\partial \Phi}{\partial \vec{X}} \Big|_f$
- m unknowns for the components of $\vec{U}$
with m algebraic equations: $\frac{\partial H}{\partial \vec{U}} = \vec{0}$
- q unknowns for the components of $\vec{v}$
with q terminal conditions: $\vec{\Psi}(\vec{X}_f, t_f) = \vec{0}$
- 1 unknowns for the final time t_f (if it is unknown) with 1 boundary equations:
 $\Omega(\vec{X}_f, \vec{U}_f, t_f) = \mathcal{L}_f + \frac{d\Phi}{dt} \Big|_f = 0$

Note that in this formulation t_f is considered as a constraint independent of $\vec{\Psi} = \vec{0}$.

The formulation indicated in equations 10.4 includes the constant multiplier vector $\vec{v}$. For doing this, this formulation is called the *adjointed method*. A different approach (*un-adjointed method*) does not include them, but the formulation is less systematic and will not be used herein.

10.1.2.3 About the transversality condition

The transversality condition 10.5 can be modified applying the differentiation of Φ :

$$\frac{d\Phi}{dt} = \frac{d\phi}{dt} + \vec{\nu}^T \cdot \frac{d\vec{\Psi}}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial \vec{X}} \cdot \dot{\vec{X}} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial \vec{X}} \cdot \dot{\vec{X}} \quad (10.6)$$

and taking into account that from equation 10.2 $\mathcal{L} = H - \vec{\lambda}^T \cdot \vec{f}$, which leads to the *differential form of the transversality condition*:

$$\left(\frac{\partial\phi}{\partial \vec{X}} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} - \vec{\lambda}^T \right) \Big|_f \cdot d\vec{X}_f + \left(\frac{\partial\phi}{\partial t} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} + H \right) \Big|_f \cdot dt_f = 0 \quad (10.7)$$

The following considerations can be made about the transversality condition 10.7:

- For brachistochrone problems (t_f known) $dt_f = 0$ and the terminal condition becomes:

$$\left(\frac{\partial\phi}{\partial \vec{X}} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} - \vec{\lambda}^T \right) \Big|_f = \vec{0} \quad (10.8)$$

- For launch problems:

- If the final state is known ($d\vec{X}_f = \vec{0}$) the transversality condition becomes:

$$\left(\frac{\partial\phi}{\partial t} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} + H \right) \Big|_f = 0 \quad (10.9)$$

- If the final state and the final time are independent then both equations have to be fulfilled simultaneously:

$$\begin{aligned} \left(\frac{\partial\phi}{\partial \vec{X}} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} - \vec{\lambda}^T \right) \Big|_f \cdot d\vec{X}_f &= 0 \\ \left(\frac{\partial\phi}{\partial t} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} + H \right) \Big|_f &= 0 \end{aligned} \quad (10.10)$$

- If the final state and the final time are not independent equation 10.7 can be re-written as:

$$\left(\frac{\partial\phi}{\partial \vec{X}} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} - \vec{\lambda}^T \right) \Big|_f \cdot \dot{\vec{X}}_f + \left(\frac{\partial\phi}{\partial t} + \vec{\nu}^T \cdot \frac{\partial\vec{\Psi}}{\partial t} + H \right) \Big|_f = 0 \quad (10.11)$$

In this case, if H does not explicitly contain t then using the considerations for the Hamiltonian indicated in section 10.1.2.4 can simplify the formulation.

10.1.2.4 Pontryagin's Minimum Principle

An interesting remark can be made about H from equations 10.4: the derivative of H with time would be given by:

$$\frac{dH}{dt} = \frac{\partial H}{\partial \vec{X}} \cdot \dot{\vec{X}} + \frac{\partial H}{\partial \vec{U}} \cdot \dot{\vec{U}} + \frac{\partial H}{\partial \vec{\lambda}} \cdot \dot{\vec{\lambda}} + \frac{\partial H}{\partial t} = -\dot{\vec{\lambda}} \cdot \dot{\vec{X}} + \vec{f} \cdot \dot{\vec{\lambda}} + \frac{\partial H}{\partial t} = -\dot{\vec{\lambda}} \cdot \dot{\vec{X}} + \dot{\vec{X}} \cdot \dot{\vec{\lambda}} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad (10.12)$$

This means that if H does not explicitly contain t in its formulation, H is constant:

$$\text{if } \frac{\partial H}{\partial t} = 0 \implies H = \text{constant} \quad (10.13)$$

This condition is expanded by Pontryagin's Minimum Principle (see reference [3] for the demonstration) that states that in the case of bounded controls equations 10.4 still hold but the stationary condition ($\frac{\partial H}{\partial \vec{U}} = \vec{0}$) does not apply and has to be substituted by the following condition: for all "measurable" control functions $\vec{U}(t)$, the optimal control $\vec{U}^*$ in an optimal trajectory $\vec{X}^*$ is such that:

$$H(\vec{X}^*, \vec{U}^*, \vec{\lambda}, t) \leq H(\vec{X}^*, \vec{U}, \vec{\lambda}, t) \quad (10.14)$$

Equation 10.14 will be used to obtain the applicable control vector $\vec{U}$ when there are constraints in the controls.

10.1.3 Linear systems

10.1.3.1 Linear quadratic regulator (LQR)

In this environment we will consider that a regulator is a control scheme designed to make the system state vector reach zero values.

When the system we are trying to control is linear and the functional we use in the control scheme is quadratic the resulting control scheme is called a linear quadratic regulator (LQR).

10.1.3.1.1 Case with fixed final time

If we consider a problem given by:

- A linear system state:

$$\begin{aligned}\dot{\vec{X}} &= A \cdot \vec{X} + B \cdot \vec{U} \\ \vec{X}(t_0) &= \vec{X}_0\end{aligned}\tag{10.15}$$

- A quadratic functional given by the following expression:

$$J = \frac{1}{2} \vec{X}_f^T \cdot S_f \cdot \vec{X}_f + \int_{t_0}^{t_f} \left(\frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right) dt\tag{10.16}$$

(where R is a symmetric positive definite matrix and S_f , Q are symmetric positive semidefinite matrices).

- No terminal conditions ($\vec{\Psi}(\vec{X}_f, t_f)$),
- A fixed final time ($dt_f = 0$)

then the application of the Euler-Lagrange theorem for this case leads to:

Hamiltonian:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^T \cdot (A \cdot \vec{X} + B \cdot \vec{U})\tag{10.17}$$

Terminal function:

$$\Phi(\vec{X}_f, t_f) = \frac{1}{2} \vec{X}_f^T \cdot S_f \cdot \vec{X}_f\tag{10.18}$$

and the following necessary conditions must hold:

$$\begin{aligned}\dot{\vec{\lambda}}^T &= -\frac{\partial H}{\partial \vec{X}} = -\vec{X}^T \cdot Q - \vec{\lambda}^T \cdot A \implies \dot{\vec{\lambda}} = -Q \cdot \vec{X} - A^T \cdot \vec{\lambda} \\ \vec{\lambda}^T(t_f) &= \frac{\partial \Phi}{\partial \vec{X}} \Big|_f \implies \vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f \\ \frac{\partial H}{\partial \vec{U}} &= \vec{0} \implies \vec{U}^T R + \vec{\lambda}^T \cdot B = \vec{0} \implies \vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda}\end{aligned}\tag{10.19}$$

In this case no transversality condition applies.

This can be written in a matrix form as:

$$\begin{Bmatrix} \dot{\vec{X}} \\ \dot{\vec{\lambda}} \end{Bmatrix} = \begin{bmatrix} A & -B \cdot R^{-1} \cdot B^T \\ -Q & -A^T \end{bmatrix} \cdot \begin{Bmatrix} \vec{X} \\ \vec{\lambda} \end{Bmatrix} = H \cdot \begin{Bmatrix} \vec{X} \\ \vec{\lambda} \end{Bmatrix} \quad (10.20)$$

with boundary conditions:

- $\vec{X}(t_0) = \vec{X}_0$
- $\vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f$

where the matrix H is called the *Hamiltonian matrix*.

Taking the final value of $\vec{\lambda}$ into account ($\vec{\lambda}(t_f) = S_f$) we can consider that $\vec{\lambda}(t)$ will have the following linear form:

$$\vec{\lambda}(t) = S(t) \cdot \vec{X} \quad (10.21)$$

Using expression 10.21 in the equations 10.20 leads to the Riccati matrix differential equation:

$$-\dot{S} = A^T \cdot S + S \cdot A - S \cdot B \cdot R^{-1} \cdot B^T \cdot S + Q \quad (10.22)$$

with boundary condition:

- $S(t_f) = S_f$

Ways to solve this equation will be explained in section 10.1.3.5

Once $S(t)$ is obtained, $\vec{\lambda}(t)$ is known at each point using equation 10.21, and the control vector is obtained from equation 10.19:

$$\vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda} = -R^{-1} \cdot B^T \cdot S(t) \cdot \vec{X} \quad (10.23)$$

It can be observed that the solution for the LQR case is very systematic and simple to pose. For this reason the linear quadratic regulator is extensively used in the design of linear controllers.

10.1.3.1.2 Case with free final time

When the final time is free we usually want it to be minimum. With this aim it is customary to include in the functional a term proportional to the final time ($J_t = \int_{t_0}^{t_f} \rho dt$)

If we considered the following case:

- A linear system state:

$$\begin{aligned}\dot{\vec{X}} &= A \cdot \vec{X} + B \cdot \vec{U} \\ \vec{X}(t_0) &= \vec{X}_0\end{aligned}\tag{10.24}$$

- A quadratic functional given by the following expression:

$$J = \frac{1}{2} \vec{X}_f^T \cdot S_f \cdot \vec{X}_f + \int_{t_0}^{t_f} \left(\rho + \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right) dt \tag{10.25}$$

(where R is a symmetric positive definite matrix and S_f , Q are symmetric positive semidefinite matrices).

- No terminal conditions ($\vec{\Psi}(\vec{X}_f, t_f)$),
- Unknown final time t_f ($dt_f \neq 0$)

then the application of the Euler-Lagrange theorem leads to:

Hamiltonian:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \rho + \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^T \cdot (A \cdot \vec{X} + B \cdot \vec{U}) \tag{10.26}$$

Terminal function:

$$\Phi(\vec{X}_f, t_f) = \frac{1}{2} \vec{X}_f^T \cdot S_f \cdot \vec{X}_f \tag{10.27}$$

and the following necessary conditions must hold:

$$\begin{aligned}\dot{\vec{\lambda}}^T &= -\frac{\partial H}{\partial \vec{X}} = -\vec{X}^T \cdot Q - \vec{\lambda}^T \cdot A \implies \dot{\vec{\lambda}} = -Q \cdot \vec{X} - A^T \cdot \vec{\lambda} \\ \vec{\lambda}^T(t_f) &= \frac{\partial \Phi}{\partial \vec{X}} \Big|_f \implies \vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f \\ \frac{\partial H}{\partial \vec{U}} &= \vec{0} \implies \vec{U}^T \vec{R} + \vec{\lambda}^T \cdot B = \vec{0} \implies \vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda}\end{aligned}\tag{10.28}$$

Since the final time is free we have to apply the transversality condition:

$$\left(\vec{X}^T \cdot S_f - \vec{\lambda}^T \right) \Big|_f \cdot d\vec{X}_f + (H) \Big|_f \cdot dt_f = 0 \tag{10.29}$$

In order to fulfill equation 10.29 we can try to make both terms zero simultaneously, this is:

$$\begin{aligned}\vec{\lambda} \Big|_f &= S_f \cdot \vec{X} \Big|_f \\ H \Big|_f &= 0\end{aligned}\tag{10.30}$$

Equation 10.30 leads to the following considerations:

- Since $\vec{\lambda}\big|_f = S_f \cdot \vec{X}\big|_f$ as it was in the case for the LQR with a fixed final time, we can apply the same procedure as in that case: consider that $\vec{\lambda}(t) = S(t) \cdot \vec{X}$ and solve S using the Riccati matrix differential equations.
The only difference is that in this case we don't know t_f . This is, if we are obtaining S by backwards integration, we don't know when to stop the integration.
- $H = 0$ would then be used to obtain t_f , this is, it would tell when to stop the backwards integration for S .

A way to obtain t_f from H is explained in [4]. In this reference it is shown how the Hamiltonian function (equation 10.26) can be expressed using S as:

$$H = \rho - \vec{X}_0^T \cdot \dot{S} \cdot \vec{X}_0 \quad (10.31)$$

Using this expression we could solve the Riccati matrix differential equation backwards until the following condition holds:

$$\rho = \vec{X}_0^T \cdot \dot{S} \cdot \vec{X}_0 \quad (10.32)$$

When this condition is fulfilled we could get the final time from the integration step ($t_f = N \cdot dt$).

10.1.3.2 Linear quadratic tracking (LQT)

In this environment we will consider that a tracker is as a control scheme designed to make the system state vector reach a reference value.

When the system we are trying to control is linear and the functional we use in the control scheme is quadratic the resulting control scheme is called a linear quadratic tracking (LQT).

10.1.3.2.1 Case with fixed final time

If we consider a problem given by:

- A linear system state:

$$\begin{aligned}\dot{\vec{X}} &= A \cdot \vec{X} + B \cdot \vec{U} \\ \vec{X}(t_0) &= \vec{X}_0\end{aligned}\tag{10.33}$$

- A quadratic functional given by the following expression:

$$J = \frac{1}{2}(\vec{X} - \vec{r})_f^T \cdot S_f \cdot (\vec{X} - \vec{r})_f + \int_{t_0}^{t_f} \frac{1}{2}(\vec{X} - \vec{r})^T \cdot Q \cdot (\vec{X} - \vec{r}) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} dt \tag{10.34}$$

where R is a symmetric positive definite matrix and S_f , Q are symmetric positive semidefinite matrices.

- No terminal conditions ($\vec{\Psi}(\vec{X}_f, t_f)$)
- A fixed final time ($dt_f = 0$)

then the application of the Euler-Lagrange theorem for this case leads to:

Hamiltonian:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \frac{1}{2}(\vec{X} - \vec{r})^T \cdot Q \cdot (\vec{X} - \vec{r}) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^T \cdot (A \cdot \vec{X} + B \cdot \vec{U}) \tag{10.35}$$

Terminal function:

$$\Phi(\vec{X}_f, t_f) = \frac{1}{2}(\vec{X} - \vec{r})_f^T \cdot S_f \cdot (\vec{X} - \vec{r})_f \tag{10.36}$$

and the following necessary conditions must hold:

$$\begin{aligned}\dot{\vec{\lambda}}^T &= -\frac{\partial H}{\partial \vec{X}} = -(\vec{X} - \vec{r})^T \cdot Q - \vec{\lambda}^T \cdot A \implies \dot{\vec{\lambda}} = -Q \cdot (\vec{X} - \vec{r}) - A^T \cdot \vec{\lambda} \\ \vec{\lambda}^T(t_f) &= \frac{\partial \Phi}{\partial \vec{X}} \Big|_f \implies \vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f - S_f \cdot \vec{r} \\ \frac{\partial H}{\partial \vec{U}} &= \vec{0} \implies \vec{U}^T \vec{R} + \vec{\lambda}^T \cdot B = \vec{0} \implies \vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda}\end{aligned}\tag{10.37}$$

In this case no transversality condition applies.

This can be written in a matrix form as:

$$\begin{Bmatrix} \dot{\vec{X}} \\ \dot{\vec{\lambda}} \end{Bmatrix} = \begin{bmatrix} A & -B \cdot R^{-1} \cdot B^T \\ -Q & -A^T \end{bmatrix} \cdot \begin{Bmatrix} \vec{X} \\ \vec{\lambda} \end{Bmatrix} + \begin{Bmatrix} \vec{0} \\ Q \cdot \vec{r} \end{Bmatrix} \quad (10.38)$$

with boundary conditions:

- $\vec{X}(t_0) = \vec{X}_0$
- $\vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f - S_f \cdot \vec{r}$

Taking the final value of $\vec{\lambda}$ into account ($\vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f - S_f \cdot \vec{r}$) we can consider that $\vec{\lambda}(t)$ will have the following linear form:

$$\vec{\lambda}(t) = S(t) \cdot \vec{X} - \vec{V} \quad (10.39)$$

Using expression 10.39 in the equations 10.38 leads to the following matrix differential equations:

$$\begin{aligned} -\dot{S} &= A^T \cdot S + S \cdot A - S \cdot B \cdot R^{-1} \cdot B^T \cdot S + Q \\ -\dot{\vec{V}} &= (A^T - S \cdot B \cdot R^{-1} \cdot B^T) \cdot \vec{V} + Q \cdot \vec{r} \end{aligned} \quad (10.40)$$

with boundary conditions:

- $S(t_f) = S_f$
- $V(t_f) = S_f \cdot \vec{r}$

This is, we have again a Riccati matrix differential equation for S , and a first order matrix differential equation for $\vec{V}$, that can be solved by backwards integration as will be explained in section 10.1.3.5.

Once $S(t)$ and $\vec{V}(t)$ are obtained, $\vec{\lambda}(t)$ is known at each point using equation 10.39, and the control vector is obtained from equation 10.37:

$$\vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda} = -R^{-1} \cdot B^T \cdot (S(t) \cdot \vec{X} - \vec{V}(t)) \quad (10.41)$$

10.1.3.2.2 Case with free final time

When the final time is free we usually want it to be minimum. With this aim it is customary to include in the functional a term proportional to the final time ($J_t = \int_{t_0}^{t_f} \rho dt$)

If we consider a problem given by:

- A linear system state:

$$\begin{aligned}\dot{\vec{X}} &= A \cdot \vec{X} + B \cdot \vec{U} \\ \vec{X}(t_0) &= \vec{X}_0\end{aligned}\tag{10.42}$$

- A quadratic functional given by the following expression:

$$J = \frac{1}{2}(\vec{X} - \vec{r})_f^T \cdot S_f \cdot (\vec{X} - \vec{r})_f + \int_{t_0}^{t_f} \rho + \frac{1}{2}(\vec{X} - \vec{r})^T \cdot Q \cdot (\vec{X} - \vec{r}) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} dt \tag{10.43}$$

where R is a symmetric positive definite matrix and S_f, Q are symmetric positive semidefinite matrices.

- No terminal conditions ($\vec{\Psi}(\vec{X}_f, t_f)$)
- A fixed final time ($dt_f = 0$)

then the application of the Euler-Lagrange theorem for this case leads to:

Hamiltonian:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \rho + \frac{1}{2}(\vec{X} - \vec{r})^T \cdot Q \cdot (\vec{X} - \vec{r}) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^T \cdot (A \cdot \vec{X} + B \cdot \vec{U}) \tag{10.44}$$

Terminal function:

$$\Phi(\vec{X}_f, t_f) = \frac{1}{2}(\vec{X} - \vec{r})_f^T \cdot S_f \cdot (\vec{X} - \vec{r})_f \tag{10.45}$$

and the following necessary conditions must hold:

$$\begin{aligned}\dot{\vec{\lambda}}^T &= -\frac{\partial H}{\partial \vec{X}} = -(\vec{X} - \vec{r})^T \cdot Q - \vec{\lambda}^T \cdot A \implies \dot{\vec{\lambda}} = -Q \cdot (\vec{X} - \vec{r}) - A^T \cdot \vec{\lambda} \\ \vec{\lambda}^T(t_f) &= \frac{\partial \Phi}{\partial \vec{X}} \Big|_f \implies \vec{\lambda}(t_f) = S_f \cdot \vec{X} \Big|_f - S_f \cdot \vec{r} \\ \frac{\partial H}{\partial \vec{U}} &= \vec{0} \implies \vec{U}^T R + \vec{\lambda}^T \cdot B = \vec{0} \implies \vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda}\end{aligned}\tag{10.46}$$

Since the final time is free we have to apply the transversality condition:

$$\left((\vec{X} - \vec{r})^T \cdot S_f - \vec{\lambda}^T \right) \Big|_f \cdot d\vec{X}_f + (H) \Big|_f \cdot dt_f = 0 \tag{10.47}$$

In order to fulfill equation 10.47 we can try to make both terms zero simultaneously, this is:

$$\begin{aligned}\vec{\lambda} \Big|_f &= S_f \cdot \vec{X} \Big|_f - S_f \cdot \vec{r} \\ H \Big|_f &= 0\end{aligned}\tag{10.48}$$

Equation 10.48 leads to the following considerations:

- Since $\vec{\lambda}\Big|_f = S_f \cdot \vec{X}\Big|_f - S_f \cdot \vec{r}$ as it was in the case for the LQT with a fixed final time, we can apply the same procedure as in that case: consider that $\vec{\lambda}(t) = S(t) \cdot \vec{X} - \vec{V}$ and solve S and $\vec{V}$ by backwards integration as explained in section 10.1.3.5.
The only difference is that in this case we don't know t_f . This is, we don't know when to stop the integration.
- $H = 0$ would then be used to obtain t_f , this is, it would tell when to stop the backwards integration for S and $\vec{V}$.

Basically the procedure consists in obtaining each step S and $\vec{V}$ by backwards integration from their final values and with this information computing $\vec{\lambda}$ (equation 10.48) and $\vec{U}$ (equation 10.46) which allow computing the value of the Hamiltonian in equation 10.44.

The integration would continue until the value obtained for the Hamiltonian is null. When this condition is fulfilled we would get the final time from the integration step ($t_f = N \cdot dt$).

10.1.3.3 Fixed final state (LQ)

In this case we consider a linear system and a quadratic functional.

However since the final state is fixed the functional only contains the integral part. Also it is customary in this case to set $Q = 0$ since the final state is to be obtained anyway so what happens along the trajectory is not so relevant.

If we consider a problem given by:

- A linear system state:

$$\begin{aligned}\dot{\vec{X}} &= A \cdot \vec{X} + B \cdot \vec{U} \\ \vec{X}(t_0) &= \vec{X}_0\end{aligned}\tag{10.49}$$

- A quadratic functional given by the following expression:

$$J = \int_{t_0}^{t_f} \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} dt\tag{10.50}$$

where R is a symmetric positive definite matrix.

- No terminal conditions ($\vec{\Psi}(\vec{X}_f, t_f)$)
- A fixed final time ($dt_f = 0$)

then the application of the Euler-Lagrange theorem for this case leads to:

Hamiltonian:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^T \cdot (A \cdot \vec{X} + B \cdot \vec{U})\tag{10.51}$$

Terminal condition:

$$\vec{X}_f = \vec{r}_f\tag{10.52}$$

and the following necessary conditions must hold:

$$\begin{aligned}\dot{\vec{\lambda}}^T &= -\frac{\partial H}{\partial \vec{X}} = -\vec{\lambda}^T \cdot A \implies \dot{\vec{\lambda}} = -A^T \cdot \vec{\lambda} \\ \frac{\partial H}{\partial \vec{U}} &= \vec{0} \implies \vec{U}^T \vec{R} + \vec{\lambda}^T \cdot B = \vec{0} \implies \vec{U} = -R^{-1} \cdot B^T \cdot \vec{\lambda}\end{aligned}\tag{10.53}$$

We can see that in this case $\vec{\lambda}$ and $\vec{X}$ are decoupled.

$\vec{\lambda}$ can be directly obtained from equation 10.53 as:

$$\vec{\lambda}(t) = e^{A^T \cdot (t_f - t)} \cdot \vec{\lambda}(t_f)\tag{10.54}$$

where $\vec{\lambda}(t_f)$ is unknown until the terminal condition is applied.

Using this value for $\vec{\lambda}(t)$ in the state equation leads to:

$$\dot{\vec{X}} = A \cdot \vec{X} - B \cdot R^{-1} \cdot B^T \cdot \vec{\lambda}(t) = A \cdot \vec{X} - B \cdot R^{-1} \cdot B^T \cdot e^{A^T \cdot (t_f - t)} \cdot \vec{\lambda}(t_f) \quad (10.55)$$

with the following solution:

$$\vec{X}(t) = e^{A \cdot (t - t_0)} \cdot \vec{X}(t_0) + \int_{t_0}^{t_f} e^{A \cdot (t - \tau)} B R^{-1} B^T e^{A^T \cdot (t_f - \tau)} \vec{\lambda}(t_f) d\tau \quad (10.56)$$

that can be expressed in terms of the *continuous reachability Gramian* as:

$$\vec{X}(t) = e^{A \cdot (t - t_0)} \cdot \vec{X}(t_0) - G(t_0, t) \vec{\lambda}(t) \quad (10.57)$$

where:

$$G(t_0, t) = \int_{t_0}^t e^{A \cdot (t - \tau)} B R^{-1} B^T e^{A^T \cdot (t - \tau)} d\tau \quad (10.58)$$

The reachability Gramian function is usually computed by solving the Lyapunov equation (see reference [4]):

$$\dot{P} = AP + PA^T + BR^{-1}B^T \quad (10.59)$$

since the solution of this equation is:

$$G(t_0, t) = P(t) \quad \text{if } P(t_0) = 0 \quad (10.60)$$

and performing the integration of equation 10.58 is very difficult.

Obtaining $\vec{X}$ in t_f , which is known to be equal to $\vec{r}_f$, we can obtain $\vec{\lambda}(t_f)$:

$$\vec{\lambda}(t_f) = -G^{-1}(t_0, t_f) \left[\vec{r}(t_f) - e^{A \cdot (t_f - t_0)} \cdot \vec{X}_0 \right] \quad (10.61)$$

Note that this is an open loop control since $\vec{U}(t)$ does not depend on the current state $\vec{X}(t)$.

10.1.3.4 Constraints in the controls in quadratic regulators

When using a quadratic regulator the functional has the following general form:

$$J = \frac{1}{2}(\vec{X} - \vec{r})_f^T \cdot S_f \cdot (\vec{X} - \vec{r})_f + \int_{t_0}^{t_f} \left(\rho + \frac{1}{2}(\vec{X} - \vec{r})^T \cdot Q \cdot (\vec{X} - \vec{r}) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} \right) dt \quad (10.62)$$

so when using a quadratic regulator the following generic expression can be provided for the Hamiltonian function of a linear system:

$$H(\vec{X}, \vec{U}, \vec{\lambda}, t) = \rho + \frac{1}{2}(\vec{X} - \vec{r})^T \cdot Q \cdot (\vec{X} - \vec{r}) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^T \cdot (A \cdot \vec{X} + B \cdot \vec{U}) \quad (10.63)$$

where $\vec{r}$ is $\vec{0}$ in the LQR case, and ρ is 0 in the case for fixed final time.

According to the Pontryagin's minimum principle (equation 10.14) the optimal control function will fulfill:

$$\begin{aligned} H(\vec{X}^*, \vec{U}^*, \vec{\lambda}^*, t) &\leq H(\vec{X}^*, \vec{U}, \vec{\lambda}^*, t) \implies \\ \frac{1}{2}\vec{U}^{*T} \cdot R \cdot \vec{U}^* + \vec{\lambda}^{*T} \cdot B \cdot \vec{U}^* &\leq \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} + \vec{\lambda}^{*T} \cdot B \cdot \vec{U} \end{aligned} \quad (10.64)$$

over all admissible $\vec{U}$ for optimal values of the state ($\vec{X}^*$) and costate ($\vec{\lambda}^*$).

Reference [4] indicates how to solve equation 10.64 obtaining the result:

$$\vec{U}_i^* = -sat \left(\left[R^{-1} \cdot B^T \cdot \vec{\lambda}(t) \right]_i \right) \quad i = 1 \dots m \quad (10.65)$$

where the *sat* function is a function that saturates the value of each i coordinate of $\vec{U}$ to its maximum allowed value $Umax_i$:

$$sat(U_i) = \begin{cases} -Umax_i & \text{if } U_i < -Umax_i \\ U_i & \text{if } |U_i| \leq Umax_i \\ Umax_i & \text{if } U_i > Umax_i \end{cases} \quad (10.66)$$

10.1.3.5 Integration of the equations for the LQR and LQT

10.1.3.5.1 Riccati equations with eigenvectors

When the system matrix A is non singular it is possible to obtain an analytic solution for the Riccati equations based on the eigenvalues and eigenvectors of the Hamiltonian matrix.

The procedure is explained in reference [4] (pages 158-160) and consists of the following steps:

1. We obtain the eigenvalues of the Hamiltonian matrix H
2. We rearrange the array of eigenvalues so that all the eigenvalues with negative real part are placed before (in the first n positions)
3. We form a diagonal matrix with the first n eigenvalues (the ones with negative real part):

$$M = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad (10.67)$$

4. We now obtain the eigenvectors associated to each eigenvalue in the order set before
5. The eigenvectors are placed in a matrix that can be divided in 4 parts:

$$W = \text{Matrix of eigenvectors} = \begin{bmatrix} W11 & W12 \\ W21 & W22 \end{bmatrix} \quad (10.68)$$

6. The solution of the Riccati equations in each time is given by:

$$\begin{aligned} V_f &= -(W22 - S_f \cdot W12)^{-1} \cdot (W21 - S_f \cdot W11) \\ V(t) &= e^{M \cdot (t_f - t)} \cdot V_f \cdot e^{M \cdot (t_f - t)} \\ S(t) &= (W21 + W22 \cdot V(t)) \cdot (W11 + W12 \cdot V(t))^{-1} \end{aligned} \quad (10.69)$$

10.1.3.5.2 Riccati equations by backwards integration

Another approach (the only that can be applied with A singular) is to perform a backwards integration from t_f obtaining each time step the new value of S from the posterior one according to the Riccati matrix differential equation. The procedure, in the discrete domain, is detailed in reference [4] and consists of the following steps:

1. We start with the matrix in the final state:

$$S(N) = S_f \quad (10.70)$$

2. We propagate the result N times ($N = t_f/dt$) with the algorithm:

$$\begin{aligned} K(i) &= (B_{discrete}^T \cdot S(i+1) \cdot B_{discrete} + R)^{-1} \cdot (B_{discrete}^T \cdot S(i+1) \cdot A_{discrete}) \\ S(i) &= A_{discrete}^T \cdot S(i+1) \cdot (A_{discrete} - B_{discrete} \cdot K(i)) + Q \end{aligned} \quad (10.71)$$

10.1.3.5.3 Backwards integration for the equations of the LQT

In the case of the LQT problem (section 10.1.3.2) together with the Riccati equations we have to obtain each step the vector $\vec{V}(t)$. The procedure, in the discrete domain, is detailed in reference [4] and consists of the following steps:

1. We start with the matrices in the final state:

$$\begin{aligned} S(N) &= S_f \\ V(N) &= S_f \cdot \vec{X}_T \end{aligned} \tag{10.72}$$

2. We propagate the result N times ($N = t_f/dt$) with the algorithm:

$$\begin{aligned} K(i) &= (B_{discrete}^T \cdot S(i+1) \cdot B_{discrete} + R)^{-1} \cdot (B_{discrete}^T \cdot S(i+1) \cdot A_{discrete}) \\ KV(i) &= (B_{discrete}^T \cdot S(i+1) \cdot B_{discrete} + R)^{-1} \cdot B_{discrete}^T \\ S(i) &= A_{discrete}^T \cdot S(i+1) \cdot (A_{discrete} - B_{discrete} \cdot K(i)) + Q \\ V(i) &= (A_{discrete} - B_{discrete} \cdot K(i))^T \cdot V(i+1) + Q \cdot \vec{X}_T \end{aligned} \tag{10.73}$$

10.1.4 Linearization

The linear quadratic methods previously analysed are very powerful and can be applied in a systematic way.

In order to extend the applicability of these techniques to non linear systems a linearization can be made, obtaining the associated formulation:

$$\dot{\overrightarrow{\Delta X}}(t) = A \cdot \overrightarrow{\Delta X}(t) + B \cdot \overrightarrow{\Delta U}(t) \quad (10.74)$$

with:

- $A = \left. \frac{\partial f}{\partial \vec{X}} \right|_{\vec{X}_0, \vec{u}_0, t}$
- $B = \left. \frac{\partial f}{\partial \vec{u}} \right|_{\vec{X}_0, \vec{u}_0, t}$
- $\overrightarrow{\Delta X} = \vec{X} - \vec{X}_0$
- $\overrightarrow{\Delta U} = \vec{U} - \vec{U}_0$

where $(\vec{X}_0, \vec{U}_0)$ is the linearization point.

In the case of the missile, the system equations (equations 3.33) will be linearized each simulation step around the present state, as indicated in section 3.3.6. As a difference, a value of $\vec{U}_0 = \vec{0}$ is chosen in this case.

After this linearization, the missile problem will be treated as a linear problem and optimal methods for linear systems will be applied. The approach is similar to the linearization approach for the Kalman Filter that leads to the Extended Kalman Filter in observation problems, a widely spread technique.

The obtained results will be in terms of position with respect to the linearization point $(\overrightarrow{\Delta X})$ and required control with respect to the one used for the linearization $(\overrightarrow{\Delta U})$, which is directly applicable as final control vector in this case since $\vec{U}_0 = \vec{0}$ was chosen.

Since the real system is not linear, the solution obtained applying a linear quadratic functional to the linearized system will be different to the solution that would be obtained for the non linear system with the same quadratic functional.

The following considerations can be made:

- The approximation depends on the non-linearity of the system. If the considered system is almost linear, the obtained solution will be very close to the optimal one.
- Even though the solution obtained with this linearization is not optimal, it provides optimal solutions in the vicinity of each linearization point (since it provides an optimal solution for each linear problem we are analysing). As the system state evolves and it gets closer to the final state, the solution collapses to an optimal one (valid for the linear system around the final state)

This means that this method provides a solution made of quasi-optimal solutions and collapses to a final optimal solution for the real state, so a good behaviour of the method is to be expected. The more linear the system behaviour is, the better that the solution provided by this method will be.

10.2 Optimal guidance algorithms for the interception problem

10.2.1 General considerations

The optimal guidance strategies implemented for the interceptor missile take into account the general equations and methods explained in section 10.1, adapting the methods to this particular case.

The considerations applied and their justification is given hereafter:

- We will linearize the system and apply techniques applicable only in theory to linear systems. As indicated in section 10.1.4 the solution obtained with this approach will be better if the non linearities of the system are reduced. In our case, even though the missile system equations (equations 3.33) are obviously non-linear, the non-linearities concentrate in the attitude and the angular velocity vector terms.

The position equations show a linear behaviour between the position and the velocity, and the velocity vector is linear with the external forces.

This means that for non rotating missiles where the attitude quaternion would remain constant and considering the external forces as the control vector, the equations would be linear. It has to be noted that since quaternions were used instead of angles for representing the attitude, the behaviour of the attitude equations is more linear than the one that would have been obtained using trigonometrical expressions.

Taking these considerations into account instead of using the real control vector of the GBI missile as indicated in section 4.5.2.1 we will consider fictional controls equal to the external forces of the missile in the three axes. These fictional controls will be transformed to the real control vector afterwards, in order to obtain for the missile an external force equal to the requested one.

With this control vector and since the missile is basically following a quasi linear path with a small angular velocity vector, the linearization of the system will provide good results in terms of position and velocity, which are the most interesting part of the solution in an interception case.

The only exception is the axial rotation of the missile, that appears in the ascent phase of the GBI since there is no control in roll. This creates a non linearity that affects mostly the quaternion and the components v and w of the velocity vector (in the body reference frame). The linearization will add in this case a certain barrel roll manoeuvre to the optimal solution, since each linearization point the provided optimal solution is linearized about a rotating state.

This factor does not appear in the case of the EKV, where the roll motion is stopped (see chapter 9), so in this case the system behaviour is almost entirely linear and the optimality of the solution will be higher.

- We will consider free final state ($d\vec{X}_f \neq 0$).

The justification for this choice comes from the fact that considering a fixed final state leads to open loop control strategies (see section 10.1.3.3) in which the present state of the system is not taken into account. Since the open loop control would be repeated every time step the provided solution would tend to the target position, but of course a closed loop strategy is more powerful. For example, the behaviour of the fixed final state scheme is only good as long as the system (A, B) is *reachable* (see reference [4]). However, the free final state schemes always provide the best achievable solution to make the functional J small, even when the system is not reachable.

At the same time it can be highlighted that we can get as close to the fixed final state as desired by increasing the S_f final state matrix gain.

- We will try to optimize the final time in the ascent phase.

When in the ascent phase the rocket motor has enough power to optimize the interception time instead of using a value obtained considering Lambert and tables from the missile (see section 7.3.2).

However we will not apply the method explained in sections 10.1.3.1.2 and 10.1.3.2.2 since these methods are in fact tricky in the sense that there is no criterion to set ρ in equations 10.25 or equation 10.43, so the condition could in fact be impossible to reach or, on the other hand, it could be too easy to achieve and not fully benefit from the available thrust.

Instead, we will integrate backwards the Riccati matrix differential equations from S_f obtaining each step the value of $\vec{\lambda}(t)$ and from that value the control vector $\vec{U}$. Once in an integration step a value for $\vec{U}$ equal to the maximum thrust of the missile is obtained, we have reached the first attainable optimal trajectory, so this is the first solution that could actually lead the missile to the target in the indicated time. The time in which this is achieved will be the chosen value for t_f :

$$\text{when } \|\vec{U}_k\| = F_{Thrust} \implies t_f = \frac{k}{dt} \quad (10.75)$$

It has to be indicated that, even though this method is much better than adding a parameter to be set in the functional, it has been found that in this case any kind of final time optimization in this problem is difficult to apply.

This is because in this case the requested control provided by optimal guidance algorithm shows a quick saturation with the final time:

Figure 10.1: $\|\vec{U}\|$ from optimal guidance as a function of the final time during ascent guidance

As a consequence any optimization of the final time is difficult to apply. We will perform a time optimization whenever the variation of $\|\vec{U}\|$ with respect to the final time is not negligible, and use a constant final time as in the terminal case otherwise.

- We will not optimize the final time in the terminal phase.

When the ascent phase has finished and the EKV has been deployed, this is, in the terminal phase, we will consider that the final time is fixed and available from the simple equation:

$$t_{go} = \frac{\|\vec{X}_{ICBM} - \vec{X}_{EKV}\|}{V_c} \quad (10.76)$$

where V_c is the closing speed (the relative speed in the line of sight direction).

The reason for this is that the divert thrusters of the EKV, which provide the available control in the terminal phase, are not designed to speed up the interception, but to deflect the missile effectively to achieve the ICBM when it is possible.

The equation applied for t_{go} is the same one that is used when obtaining the conventional terminal guidance laws from the ZEM vector (see section 9.3.4). This final time is recomputed in any computation cycle so that the computed value collapses to the real one as the missiles get closer.

- We will consider saturation in the controls according to Pontryagin's Minimum Principle. Since we will be using quadratic regulators, equation 10.66 will be used for the controls.

10.2.2 Overall description of the optimal guidance algorithms that have been implemented

Taking the considerations of section 10.2.1 into account 4 optimal guidance algorithms have been implemented:

1. Optimal interception guidance for the terminal phase

This case is only applicable for the terminal interception phase.

The implemented algorithm is an LQR algorithm in which $\vec{X}_{ICBM} - \vec{X}_{EKV}$ is a part of the state vector that we want to regulate and a linear formulation of the interception equations is made.

As indicated before, fixed final time is considered in this case.

This case has been implemented in order to be able to compare exclusively the conventional terminal guidance (proportional navigation guidance with gravity compensation) detailed in section 9.3.2) with optimal terminal guidance.

2. Global optimal interception guidance

This case is applicable since the missile leaves the atmosphere during the ascent phase.

This case is an extension of the previous one. An LQR algorithm is used in which $\vec{X}_{ICBM} - \vec{X}_{EKV}$ is a part of the state vector that we want to regulate and a linear formulation of the interception equations is made.

As indicated before, during the ascent phase, we consider free final time but once the terminal phase starts, the final time will be considered as fixed in the algorithm.

3. Global optimal tracking guidance

This case is applicable since the missile leaves the atmosphere during the ascent phase.

The implemented algorithm is an LQT in which $\vec{X}_{ICBM}$ is the position we want to track. In this case the complete state vector of the missile X_{GBI} is used and a linearization of the system through the A, B matrices (see section 3.3.6).

As indicated before, during the ascent phase, we consider free final time but once the terminal phase starts, the final time will be considered as fixed in the algorithm.

4. Global optimal guidance using an augmented state vector

This case is applicable since the missile leaves the atmosphere during the ascent phase.

The implemented algorithm is an LQR in which in which $\vec{X}_{ICBM} - \vec{X}_{EKV}$ is a part of the state vector that we want to regulate but at the same time $\vec{X}_{EKV}$ is also part of the state vector. A linearization of the system through the A, B matrices is used (see section 3.3.6).

As indicated before, during the ascent phase we consider free final time but once the terminal phase starts, the final time will be considered as fixed in the algorithm.

These guidance algorithms will be described in detail hereafter.

10.2.3 Optimal terminal guidance

The first algorithm to be developed for the EKV-ICBM interception problem is an optimal terminal guidance whose behaviour will be later compared to the one obtained using the proportional navigation guidance with gravity compensation algorithm explained in section 9.3.2.

The considered state vector in this problem is:

$$\vec{X} = \begin{pmatrix} X_T^i - X_M^i \\ Y_T^i - Y_M^i \\ Z_T^i - Z_M^i \\ Vx_T^i - Vx_M^i \\ Vy_T^i - Vy_M^i \\ Vz_T^i - Vz_M^i \end{pmatrix} \quad (10.77)$$

where the superscript i indicates the inertial reference frame (see Appendix A), the subscript T indicates the target (the ICBM), the subscript M indicates the missile (the EKV) and:

- $\{X_T^i - X_M^i, Y_T^i - Y_M^i, Z_T^i - Z_M^i\}^T$ is the relative EKV-ICBM position vector
- $\{Vx_T^i - Vx_M^i, Vy_T^i - Vy_M^i, Vz_T^i - Vz_M^i\}^T$ is the relative EKV-ICBM velocity vector

In this case we will consider as control vector for the system:

$$\vec{U}^i = \begin{pmatrix} Ux^i \\ Uy^i \\ Uz^i \end{pmatrix} = \begin{pmatrix} Ax_T^i - Ax_M^i \\ Ay_T^i - Ay_M^i \\ Az_T^i - Az_M^i \end{pmatrix} = \vec{A}_T^i - \vec{A}_M^i \quad (10.78)$$

where:

- $\{Ax_T^i, Ay_T^i, Az_T^i\}^T$ is the acceleration of the ICBM (the gravitational acceleration):

$$\{Ax_T^i, Ay_T^i, Az_T^i\}^T = \{gx_T^i, gy_T^i, gz_T^i\}^T = \vec{g}_T^i \quad (10.79)$$

which can be estimated from the position of the target in a simplified way ($\vec{g}_T^i \simeq -\mu \frac{\vec{r}_T^i}{r_T^{i3}}$) or using a more complex expression like equation 4.1 as explained in section 4.2.1.

- $\{Ax_M^i, Ay_M^i, Az_M^i\}^T$ is the acceleration of the EKV (gravitational acceleration plus thrust).

Using the state vector given in 10.77 and the control vector given in equation 10.78, the interception problem would be represented by the following state equation:

$$\dot{\vec{X}} = A \cdot \vec{X} + B \cdot \vec{U} \quad (10.80)$$

with:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (10.81)$$

The following functional is considered for the optimization:

$$J = \int_0^{t_f} \left\{ \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right\} dt + \frac{1}{2} \vec{X}^T \cdot S_f \cdot \vec{X} \Big|_{t_f} \quad (10.82)$$

where:

- $\int_0^{t_f} \left\{ \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} \right\} dt$ is a term of the functional related to the difference of states between the missile and the target during the interception.
- $\int_0^{t_f} \left\{ \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right\} dt$ is a term of the functional related to the requested controls during the interception.
- $\frac{1}{2} \vec{X}^T \cdot S_f \cdot \vec{X} \Big|_{t_f}$ is a term of the functional related to the final difference of states between the missile and the target.
- t_f is the interception time

As indicated in section 10.2.1 we will consider a fixed interception time t_f (given by equation 10.76) and free final state.

Since we are only interested in minimizing the distance missile-target during the interception trajectory and especially in the final point (the relative velocity cannot be zero in this case since the ICBM target is approaching the EKV) we will use the following weight matrices:

$$Q = \begin{bmatrix} Qd & 0 & 0 & 0 & 0 & 0 \\ 0 & Qd & 0 & 0 & 0 & 0 \\ 0 & 0 & Qd & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad S_f = \begin{bmatrix} Sd & 0 & 0 & 0 & 0 & 0 \\ 0 & Sd & 0 & 0 & 0 & 0 \\ 0 & 0 & Sd & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10.83)$$

After some tests the following values have been selected:

- $Sd = 1000.0 \frac{1}{m^2}$
- $Qd = 10.0 \frac{1}{m^2}$

Since we want to minimize all the controls without any preference for a certain direction we will use a symmetric weight matrix for the controls:

$$R = \begin{bmatrix} Rd & 0 & 0 \\ 0 & Rd & 0 \\ 0 & 0 & Rd \end{bmatrix} \quad (10.84)$$

After some tests the following value has been selected: $Rd = 1.0 \frac{1}{m/s^2}$.

With this formulation this problem is an LQR with fixed final time and free final state whose formulation was given in section 10.1.3.1.1.

Since in this case the A state matrix is regular, this problem will be solved from the Riccati matrix differential equation for the auxiliary matrix S as indicated in section 10.1.3.5.1, obtaining as solution $\vec{U}^i = \vec{A}_T^i - \vec{A}_M^i$, which allows obtaining $\vec{A}_M^i$.

This acceleration vector cannot be commanded directly by the EKV, since the EKV has a gravitational acceleration that should be taken into account to have the global acceleration $\vec{A}_M^i$:

$$\vec{A}_{thrusters_0}^i = \vec{A}_M^i - \vec{g}_M^i \quad (10.85)$$

We will modify this requested acceleration vector since in this terminal phase we don't want the thrusters to request forces in the direction of the line of sight (the thrusters have not been designed to accelerate the interception) so the final commands to be requested to the thrusters will be:

$$\begin{aligned} \overrightarrow{LOS}^i &= \frac{\vec{X}_{ICBM}^i - \vec{X}_{EKV}^i}{\|\vec{X}_{ICBM}^i - \vec{X}_{EKV}^i\|} \\ \vec{A}_{thrusters}^i(t) &= \vec{A}_{thrusters_0}^i(t) - \overrightarrow{LOS}^i \cdot \frac{\vec{A}_{thrusters_0}^i(t) \cdot \overrightarrow{LOS}^i}{\|\overrightarrow{LOS}^i\|^2} \end{aligned} \quad (10.86)$$

On the other hand the force to be obtained from the divert thrusters has to be requested in the body reference frame, but it is easy to obtain from $\vec{A}_{thrusters}^i$ the real control forces to be requested from the thrusters in the body reference frame with the following expressions:

$$\vec{U}_{thrusters}^b = M_{EKV} \cdot C_i^b \cdot \vec{A}_{thrusters}^i \quad (10.87)$$

where the superscript b indicates the body reference frame and the matrix C_i^b can be easily obtained using equation 3.17 from the EKV state vector since it contains the rotation quaternion from the inertial to the body reference frame (q_{bi}).

M_{EKV} is the mass of the EKV and it is required to transform the acceleration commands to force commands in the thrusters. This magnitude can be estimated by the EKV taking into account the forces requested to the thrusters throughout the trajectory, or in a more simplified way.

The resulting force, in the body reference frame ($\vec{U}_{thrusters}^b$) is directly the control command to be obtained from the divert thrusters.

10.2.4 Global optimal interception guidance

This algorithm is basically an extension of the previous case (section 10.2.3) applicable also to the ascent phase. This is, we will consider the state vector of an interception problem and a control vector with the accelerations in 3 axes:

$$\vec{X} = \begin{Bmatrix} X_T^i - X_M^i \\ Y_T^i - Y_M^i \\ Z_T^i - Z_M^i \\ Vx_T^i - Vx_M^i \\ Vy_T^i - Vy_M^i \\ Vz_T^i - Vz_M^i \end{Bmatrix} \quad \vec{U}^i = \begin{Bmatrix} Ux^i \\ Uy^i \\ Uz^i \end{Bmatrix} = \begin{Bmatrix} Ax_T^i - Ax_M^i \\ Ay_T^i - Ay_M^i \\ Az_T^i - Az_M^i \end{Bmatrix} = \vec{A}_T^i - \vec{A}_M^i \quad (10.88)$$

where the superscript i indicates the inertial reference frame (see Appendix A), the subscript T indicates the target (the ICBM) and the subscript M indicates the interceptor missile.

Using the state vector and the control vector given in 10.88, the interception problem would be represented by the following state equation:

$$\dot{\vec{X}} = A \cdot \vec{X} + B \cdot \vec{U} \quad (10.89)$$

where the A and B matrices are the same as in equation 10.80.

Different approaches to obtain the optimal guidance law will be given for the ascent phase and for the terminal phase.

10.2.4.1 In the ascent phase

The following functional is considered for the optimization:

$$J = \int_0^{t_f} \left\{ \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right\} dt + \frac{1}{2} \vec{X}^T \cdot S_f \cdot \vec{X} \Big|_{t_f} \quad (10.90)$$

Since we are only interested in minimizing the distance missile-target during the interception trajectory and especially in the final point (the relative velocity cannot be zero in this case since the ICBM target is approaching the EKV) the weight matrices Q and S_f given in equation 10.83 will also be used, applying also the same constants Qd , Sd as in section 10.2.3

Since we want to minimize all the controls without any preference for a certain direction a symmetric weight matrix for the controls (R) as in equation 10.84 will be used. We will also apply the same constant Rd as in section 10.2.3.

With this formulation this problem is an LQR with free final time and free final state whose formulation was given in section 10.1.3.1.2.

Since in this case the A state matrix is regular, this problem will be solved from the Riccati matrix differential equation for the auxiliary matrix S as indicated in section 10.1.3.5.1, obtaining as solution $\vec{U}^i = \vec{A}_T^i - \vec{A}_M^i$, which allows obtaining $\vec{A}_M^i$.

The acceleration to be requested from the motors would then be obtained as:

$$\vec{A}_{thrust}^i = \vec{A}_M^i - \vec{g}_M^i \quad (10.91)$$

When possible (this is, outside saturation), the final time t_f is obtained as indicated in section 10.2.1 from considering that the maximum possible thrust has to be applied:

$$\text{when } M \cdot \|\vec{A}_{thrust}^i(t_{final})\| = F_{thrust} \implies t_f = t_{final} \quad (10.92)$$

When this optimization cannot be applied, we will use for the final time equation 10.76.

$\vec{A}_{thrust}^i$ cannot be directly applied to the missile, since in the ascent phase, as indicated in section 4.5.2.1, the real controls are the angles that the exhaust nozzle can be pivoted from null. We could try to set pivoting angles that approach the resulting forces to the ones indicated by $\vec{A}_{thrust}^i$, but that would make the missile unstable, since when deflecting the nozzle the main result is not the lateral force that is obtained, but rotating the missile, which will then align its main force (the thrust is always in the Ox_b direction) in a different axis.

The solution for this is to consider that the required acceleration $\vec{A}_{thrust}^i$ provides an attitude to place the missile.

The procedure would be as follows:

1. We transform the $\vec{A}_{thrust}^i(t)$ vector to the ECEF frame:

$$\vec{A}_{thrust}^e(t) = C_i^e \cdot \vec{A}_{thrust}^i(t) \quad (10.93)$$

where the superscript e indicates the ECEF reference frame and the transformation matrix C_i^e only depends on the present time (see Appendix section A.3).

2. From the geodetic coordinates of the GBI we obtain the transformation matrix from the ECEF frame to the navigation frame (equation A.35), which allows obtaining the $\vec{U}$ vector in the navigation frame:

$$\vec{A}_{thrust}^n(t) = C_e^n \cdot \vec{A}_{thrust}^e(t) \quad (10.94)$$

where the superscript n indicates the Navigation reference frame.

3. Having the $\vec{A}_{thrust}(t)$ vector in the navigation frame we can easily obtain the required yaw and pitch angles so that the Ox_b axis is aligned with it:

$$\begin{aligned}\psi_{reference}(t) &= atan \frac{\vec{A}_{thrust}^n(t)|_{East}}{\vec{A}_{thrust}^n(t)|_{North}} \\ \theta_{reference} &= atan \frac{-\vec{A}_{thrust}^n(t)|_{Down}}{\vec{A}_{thrust}^n(t)|_{Hor}} \\ \vec{A}^n(t)|_{Hor} &= \sqrt{(\vec{A}_{thrust}^n(t)|_{East})^2 + (\vec{A}_{thrust}^n(t)|_{North})^2}\end{aligned}\tag{10.95}$$

which will be the reference angles provided to the GBI control system.

This approach leads the missile to a situation in which the Ox_b axis of the missile is placed in such a way that the acceleration provided by the motor plus the gravity is in the direction of the acceleration requested by the optimal guidance algorithm.

10.2.4.2 In the terminal phase

In this case we consider fixed final time (given by equation 10.76) and the formulation is the same as in section 10.2.3.

10.2.5 Optimal tracking guidance

In this case we consider the GBI state vector as indicated in section 3.2.2 and the following control vector:

$$\vec{U}^b = \begin{Bmatrix} Fx_b \\ Fy_b \\ Fz_b \end{Bmatrix} \quad (10.96)$$

where Fx_b, Fy_b, Fz_b are the forces obtained from the engines in the body reference frame. The system would be represented by:

$$\dot{\vec{X}}(t) = A \cdot \vec{X} + B \cdot \vec{U} \quad (10.97)$$

where A is the GBI state matrix made from a linearization of the system equations 3.33 (see section 3.3.6) and the B matrix has to be computed for the considered control vector (equation 10.96).

Different approaches to obtain the optimal guidance will be given for the ascent phase and for the terminal phase.

10.2.5.1 In the ascent phase

The following functional is considered for the optimization:

$$J = \int_0^{t_f} \left\{ \frac{1}{2}(\vec{X} - \vec{X}_T)^T \cdot Q \cdot (\vec{X} - \vec{X}_T) + \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} \right\} dt + \frac{1}{2}(\vec{X} - \vec{X}_T)^T \cdot S_f \cdot (\vec{X} - \vec{X}_T) \Big|_{t_f} \quad (10.98)$$

where:

- $\vec{X}_T$ is the state vector of the target missile (the ICBM).
- t_f is the interception time
- $\int_0^{t_f} \left\{ \frac{1}{2}(\vec{X} - \vec{X}_T)^T \cdot Q \cdot (\vec{X} - \vec{X}_T) \right\} dt$ is a term of the functional related to the difference of states between the missile and the target during the interception.
- $\int_0^{t_f} \left\{ \frac{1}{2}\vec{U}^T \cdot R \cdot \vec{U} \right\} dt$ is a term of the functional related to the requested controls during the interception.
- $\frac{1}{2}(\vec{X} - \vec{X}_T)^T \cdot S_f \cdot (\vec{X} - \vec{X}_T) \Big|_{t_f}$ is a term of the functional related to the final difference of states between the missile and the target.

Since we are only interested in minimizing the distance missile-target during the interception trajectory and especially in the final point (the relative velocity cannot be zero in this case since the ICBM target is approaching the EKV and the attitude and rotation rates are irrelevant for us) we will use the following weight matrices:

$$Q = \begin{bmatrix} Qd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Qd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Qd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10.99)$$

$$S_f = \begin{bmatrix} Sd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Sd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Sd & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10.100)$$

Since we want to minimize all the controls without any preference for a certain direction a symmetric weight matrix for the controls (R) as in equation 10.84 will be used.

We will use in this case the same values for Qd , Sd and R as in the optimal terminal guidance algorithm (section 10.2.3).

With this formulation this problem is an LQT with free final time and free final state whose formulation was given in section 10.1.3.2. This problem will be solved from the Riccati matrix differential equation for the auxiliary matrices S and V by backwards integration as indicated in section 10.1.3.5.3. Note that since this method is applied in the discrete domain the system matrices have to be converted first to this domain by:

$$\begin{aligned} A_{discrete} &= A + I \cdot dt \\ B_{discrete} &= B \cdot dt \end{aligned} \quad (10.101)$$

When possible (this is, outside saturation), the final time t_f will be obtained as indicated in section 10.2.1 from considering that the maximum possible thrust has to be applied:

$$\text{when } \|\vec{U}(N - k)\| = F_{thrust} \implies t_f = k \cdot dt \quad (10.102)$$

When this optimization cannot be applied, we will use for the final time equation 10.76.

The obtained solution is $\vec{U}^b = \begin{Bmatrix} Fx_b \\ Fy_b \\ Fz_b \end{Bmatrix} = \vec{F}^b$, that already takes into account the possible saturation in the controls from equation 10.66.

This control vector cannot be directly applied to the missile, since in the ascent phase, as indicated in section 4.5.2.1, the real controls are the angles that the exhaust nozzle can be pivoted from null. We could try to set pivoting angles that approach the resulting forces to the ones indicated by $\vec{U}^b$, but that would make the missile unstable, since when deflecting the nozzle the main result is not the lateral force that is obtained, but rotating the missile, which will then align its main force (the thrust is always in the Ox_b direction) in a different axis.

The solution will be again, as we did in section 10.2.4.1, to consider that the required force $\vec{F}^b$ provides an attitude to place the missile.

The procedure will be as follows:

1. We transform the $\vec{U}^b(t)$ vector to the inertial reference frame:

$$\vec{U}^i(t) = C_b^i \vec{U}^b(t) \quad (10.103)$$

where the superscript i indicates the ECI reference frame and the transformation matrix C_b^i can be easily obtained using equation 3.17 from the GBI state vector since it contains the rotation quaternion from the inertial to the body reference frame (q_{bi}).

2. We transform the $\vec{U}^i(t)$ vector to the ECEF frame:

$$\vec{U}^e(t) = C_i^e \vec{U}^i(t) \quad (10.104)$$

where the superscript e indicates the ECEF reference frame and the transformation matrix C_i^e only depends on the present time (see Appendix section A.3).

3. From the geodetic coordinates of the GBI we obtain the transformation matrix from the ECEF frame to the navigation frame (equation A.35), which allows obtaining the $\vec{U}$ vector in the navigation frame:

$$\vec{U}^n(t) = C_e^n \cdot \vec{U}^e(t) \quad (10.105)$$

where the superscript n indicates the Navigation reference frame.

4. Having the $\vec{U}(t)$ vector in the navigation frame we can easily obtain the required yaw and pitch angles so that the Ox_b axis of the missile is aligned with it:

$$\begin{aligned} \psi_{reference}(t) &= \text{atan} \frac{\vec{U}^n(t)|_{East}}{\vec{U}^n(t)|_{North}} \\ \theta_{reference} &= \text{atan} \frac{-\vec{U}^n(t)|_{Down}}{\vec{U}^n(t)|_{Hor}} \\ \vec{U}^n(t)|_{Hor} &= \sqrt{(\vec{U}^n(t)|_{East})^2 + (\vec{U}^n(t)|_{North})^2} \end{aligned} \quad (10.106)$$

which will be the reference angles provided to the GBI control system.

This approach leads the missile to a situation in which the Ox_b axis of the missile is placed in such a way that the thrust provided by the motor is in the direction of the thrust force requested by the optimal guidance algorithm.

10.2.5.2 In the terminal phase

The procedure is similar with the following differences:

- The final time is considered as fixed (given by equation 10.76). As a consequence, the backwards integration to solve from the Riccati matrix differential equation the auxiliary matrices S and V (using the procedure indicated in section 10.1.3.5.3) stops exactly when the integration time is t_f .
- Since in the terminal phase the thrusters allow controlling in the 3 axis of the EKV, $\vec{F}^b$ is directly applied in this case as control vector.

10.2.6 Global optimal guidance using an augmented state vector

The procedure used in section 10.2.5 allowed the possibility of using the real state vector of the GBI missile and its state transition matrix A . However, this procedure considered in all cases the target as a static position. It seems sensible to assume that including information about the target dynamics, as considered in section 10.2.4, while using the real state vector of the GBI missile and its state transition matrix will improve the optimization strategy.

In order to do so it is necessary to make use of an augmented state vector defined as:

$$\vec{X}_{aug} = \begin{pmatrix} X_T^i - X_M^i \\ Y_T^i - Y_M^i \\ Z_T^i - Z_M^i \\ Vx_T^i - Vx_M^i \\ Vy_T^i - Vy_M^i \\ Vz_T^i - Vz_M^i \\ Ax_T^i \\ Ay_T^i \\ Az_T^i \\ \vec{X} \end{pmatrix} \quad (10.107)$$

where the subscript T indicates the target (the ICBM), the subscript M indicates the GBI missile, the superscript i indicates the inertial reference frame and:

- $\{X_T^i - X_M^i, Y_T^i - Y_M^i, Z_T^i - Z_M^i\}^T$ is the relative GBI-ICBM position vector
- $\{Vx_T^i - Vx_M^i, Vy_T^i - Vy_M^i, Vz_T^i - Vz_M^i\}^T$ is the relative GBI-ICBM velocity vector
- $\{Ax_T^i, Ay_T^i, Az_T^i\}^T$ is the acceleration of the ICBM (the gravitational acceleration):

$$\{Ax_T^i, Ay_T^i, Az_T^i\}^T = \{gx_T^i, gy_T^i, gz_T^i\}^T = \vec{g}_T^i \quad (10.108)$$

which can be estimated from the position of the target in a simplified way ($\vec{g}_T^i \simeq -\mu \frac{\vec{r}_T^i}{r_T^{i3}}$) or using a more complex expression like equation 4.1 as explained in section 4.2.1.

- $\vec{X}$ is the GBI state vector as provided in section 3.2.2.

The following control vector is considered:

$$\vec{U}^b = \begin{pmatrix} Fx_b \\ Fy_b \\ Fz_b \end{pmatrix} \quad (10.109)$$

where Fx_b, Fy_b, Fz_b are the forces obtained from the engines in the body reference frame.

The system would be represented by:

$$\dot{\vec{X}}(t) = A_{aug} \cdot \vec{X} + B_{aug} \cdot \vec{U} \quad (10.110)$$

where A_{aug} can be obtained from the GBI system state matrix A as:

$$A_{aug} = \left[\begin{array}{ccc|cccccc} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & & & & & \\ \hline 0 & 0 & 0 & & & & & \end{array} \right] \quad (10.111)$$

A

and B_{aug} is given by:

$$B_{aug} = \left[\begin{array}{c} -\frac{\partial \vec{r}_M^b}{\partial \vec{U}} \\ \hline -C_b^i \cdot \frac{\partial \vec{v}_M^b}{\partial \vec{U}} \\ \hline \begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \\ \hline B \end{array} \right] \quad (10.112)$$

where B is the control matrix of the GBI missile system computed for the considered control vector (equation 10.109) as indicated in section 3.3.6 and $\frac{\partial \vec{r}_M^b}{\partial \vec{U}}$ and $\frac{\partial \vec{v}_M^b}{\partial \vec{U}}$ are the first 6 rows of that matrix.

Different approaches to obtain the optimal guidance will be given for the ascent phase and for the terminal phase.

10.2.6.1 In the ascent phase

The following functional is considered for the optimization:

$$J = \int_0^{t_f} \left\{ \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} + \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right\} dt + \frac{1}{2} \vec{X}^T \cdot S_f \cdot \vec{X} \Big|_{t_f} \quad (10.113)$$

where:

- t_f is the interception time
- $\int_0^{t_f} \left\{ \frac{1}{2} \vec{X}^T \cdot Q \cdot \vec{X} \right\} dt$ is a term of the functional related to the difference of states between the missile and the target during the interception.
- $\int_0^{t_f} \left\{ \frac{1}{2} \vec{U}^T \cdot R \cdot \vec{U} \right\} dt$ is a term of the functional related to the requested controls during the interception.
- $\frac{1}{2} \vec{X}^T \cdot S_f \cdot \vec{X} \Big|_{t_f}$ is a term of the functional related to the final difference of states between the missile and the target.

Since we are only interested in minimizing the distance missile-target during the interception trajectory and especially in the final point (the relative velocity cannot be zero in this case since the ICBM target is approaching the EKV) we will use the following weight matrices:

$$Q = \begin{bmatrix} Qd & 0 & 0 & 0 & \cdots & 0 \\ 0 & Qd & 0 & 0 & \cdots & 0 \\ 0 & 0 & Qd & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad S_f = \begin{bmatrix} Sd & 0 & 0 & 0 & \cdots & 0 \\ 0 & Sd & 0 & 0 & \cdots & 0 \\ 0 & 0 & Sd & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (10.114)$$

Since we want to minimize all the controls without any preference for a certain direction a symmetric weight matrix for the controls (R) as in equation 10.84 will be used.

We will use in this case the same values for Qd , Sd and R as in the optimal terminal guidance algorithm (section 10.2.3).

With this formulation this problem is an LQR with free final time and free final state whose formulation was given in section 10.1.3.1.1. This problem will be solved from the Riccati matrix differential equation for the auxiliary matrix S by backwards integration as indicated in section 10.1.3.5.2. Note that since this method is applied in the discrete domain the system matrices have to be converted first to this domain by:

$$\begin{aligned} A_{discrete} &= A + I \cdot dt \\ B_{discrete} &= B \cdot dt \end{aligned} \quad (10.115)$$

When possible (this is, outside saturation), the final time t_f will be obtained as indicated in section 10.2.1 from considering that the maximum possible thrust has to be applied:

$$\text{when } \|\vec{U}(N - k)\| = F_{thrust} \implies t_f = k \cdot dt \quad (10.116)$$

When this optimization cannot be applied, we will use for the final time equation 10.76.

The obtained solution is $\vec{U}^b = \begin{Bmatrix} Fx_b \\ Fy_b \\ Fz_b \end{Bmatrix} = \vec{F}^b$, that already takes into account the possible saturation in the controls from equation 10.66.

This control vector cannot be directly applied to the missile, since in the ascent phase, as indicated in section 4.5.2.1, the real controls are the angles that the exhaust nozzle can be pivoted from null. We could try to set pivoting angles that approach the resulting forces to the ones indicated by $\vec{U}^b$, but that would make the missile unstable, since when deflecting the nozzle the main result is not the lateral force that is obtained, but rotating the missile, which will then align its main force (the thrust is always in the Ox_b direction) in a different axis.

The solution will be again, as we did in section 10.2.4.1, to consider that the required force $\vec{F}^b$ provides an attitude to place the missile.

The procedure is the same as in section 10.2.5:

1. We transform the $\vec{U}^b(t)$ vector to the inertial reference frame:

$$\vec{U}^i(t) = C_b^i \vec{U}^b(t) \quad (10.117)$$

where the superscript i indicates the ECI reference frame and the transformation matrix C_b^i can be easily obtained using equation 3.17 from the GBI state vector since it contains the rotation quaternion from the inertial to the body reference frame (q_{bi}).

2. We transform the $\vec{U}^i(t)$ vector to the ECEF frame:

$$\vec{U}^e(t) = C_i^e \vec{U}^i(t) \quad (10.118)$$

where the superscript e indicates the ECEF reference frame and the transformation matrix C_i^e only depends on the present time (see Appendix section A.3).

3. From the geodetic coordinates of the GBI we obtain the transformation matrix from the ECEF frame to the navigation frame (equation A.35), which allows obtaining the $\vec{U}$ vector in the navigation frame:

$$\vec{U}^n(t) = C_e^n \cdot \vec{U}^e(t) \quad (10.119)$$

where the superscript n indicates the Navigation reference frame.

4. Having the $\vec{U}(t)$ vector in the navigation frame we can easily obtain the required yaw and pitch angles so that the Ox_b axis of the missile is aligned with it:

$$\begin{aligned}\psi_{reference}(t) &= \text{atan} \frac{\vec{U}^n(t)|_{East}}{\vec{U}^n(t)|_{North}} \\ \theta_{reference} &= \text{atan} \frac{-\vec{U}^n(t)|_{Down}}{\vec{U}^n(t)|_{Hor}} \\ \vec{U}^n(t)|_{Hor} &= \sqrt{(\vec{U}^n(t)|_{East})^2 + (\vec{U}^n(t)|_{North})^2}\end{aligned}\tag{10.120}$$

which will be the reference angles provided to the GBI control system.

This approach leads the missile to a situation in which the Ox_b axis of the missile is placed in such a way that the thrust provided by the motor is in the direction of the thrust force requested by the optimal guidance algorithm.

10.2.6.2 In the terminal phase

The procedure is similar with the following differences:

- The final time is considered as fixed (given by equation 10.76). As a consequence, the backwards integration to solve from the Riccati matrix differential equation the auxiliary matrix S (using the procedure indicated in section 10.1.3.5.2) stops exactly when the integration time is t_f .
- Since in the terminal phase the thrusters allow controlling in the 3 axis of the EKV, $\vec{F}^b$ is directly applied in this case as control vector.

Chapter 10 references

- [1] James M. Longuski, José J. Guzmán, and John E. Prussing. *Optimal Control with Aerospace Applications*. Springer, New York, 2014. [239](#)
- [2] Arthur E. Bryson and Yu-Chi Ho. *Applied Optimal Control: optimization, estimation, and control*. Taylor & Francis Group, New York, 1975. [239](#)
- [3] Hans P. Geering. *Optimal Control with Engineering Applications*. Springer-Verlag, Berlin, 2007. [241](#)
- [4] Frank L. Lewis, Draguna L. Vrabie, and Vassilis L. Syrmos. *Optimal Control*. John Wiley & Sons Inc., Hoboken, New Jersey, 2012. [245](#), [251](#), [252](#), [253](#), [254](#), [257](#)
- [5] Donald E. Kirk. *Optimal Control Theory. An Introduction*. Dover Publications Inc., Mineola, New York, 2004.
- [6] Joseph Z. Ben-Asher and Isaac Yaesh. *Advances in Missile Guidance Theory*. American Institute of Aeronautics and Astronautics Inc., Alexander Bell Drive, Reston. Virginia, 1998.
- [7] J.A. McMahan Jr. Linear Systems and Optimal Control Condensed Notes. <http://www4.ncsu.edu/~jamcmaha/control/control.pdf>, November 2010.
- [8] Pedro Sanz Aránguez. Resúmenes de Optimización de Trayectorias y Leyes de Guiado printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, 2003.
- [9] Randal J. Barnes. Matrix Differentiation. <http://www.atmos.washington.edu/~dennis/MatrixCalculus.pdf>, 2006.

This page intentionally left blank.

Chapter 11

Comparison of guidance algorithms

Several interception cases will be executed in this chapter using different guidance algorithms for the interceptor missile: conventional ascent guidance plus conventional terminal guidance, conventional ascent guidance plus optimal terminal guidance, global optimal interception guidance, optimal tracking guidance and global optimal guidance using an augmented state vector.

The results of these executions will then be compared in order to identify the behaviour of the different guidance algorithms.

11.1 Introduction

11.1.1 Parameters to be compared

The following parameters will be used for comparing the guidance algorithms:

- Accuracy

This will be compared in terms of the minimum distance achieved at the interception between the ICBM and the EKV.

It has to be noted that, even though this parameter is commonly used for the comparison of guidance algorithms (see for example [1] (Jang et al., 2008) and [2]), it is very difficult to indicate the algorithm that has a better accuracy when they both behave properly just by comparing this distance, since in this case the results are very similar and the results could be the consequence of numerical errors.

In order to minimize the numerical errors we will reduce the time step in the simulation according to the following sequence:

- Relative distance > 100 km $\implies$ normal Δt (usually 0.1 s unless there are stability issues). With this Δt and taking the relative velocity vector into account the relative distance decreases about 1 km in one Δt .
- Relative distance < 100 km $\implies \Delta t = 0.01$ s. With this Δt the relative distance decreases about 100 m in one Δt .
- Relative distance < 30 km $\implies \Delta t = 0.0001$ s. With this Δt the relative distance decreases about 1 m in one Δt .

- Consumed fuel

This parameter is directly provided by the simulator. It has to be noted that the consumption in the 4 divert thrusters and 2 attitude control systems in the EKV per obtained force is unknown, so an estimation of 0.466 kg/s per 1400 N is used, based in the consumption of the Rocketdyne RS-14 motor used in the post-boost phase of the ICBM, which is known (see Appendix section G.2.1.4).

Even though the exact consumption per force is unknown for the EKV, since the same value for $\dot{M}_{\text{EKV}}$ is applied to all the cases, the analysis of the obtained consumed fuel provides a valid ranking for the results.

- Horizontal distance to the ICBM target at the interception point or time since liftoff to the interception point

The time since liftoff to the interception point directly indicates the algorithm that provides a faster interception. In order to highlight the importance of this parameter it can also be provided in terms of the horizontal distance to the target city of the ICBM at the interception point. The fastest the interception, the further away from the target city that the missile will be destroyed, which is a very interesting result since that reduces possible contamination issues, and provides more time for secondary measures in case of a failed interception, like launching another GBI missile.

11.1.2 Cases to be analysed

11.1.2.1 Geographical cases

The American National Missile Defense (NMD) system (see chapter 2) is designed for the defence of the U.S. against incoming ICBMs. This means that we can only analyse cases in which the target is in the U.S. Target cities that are too far away from the U.S. soil cannot be reached by a GBI launched from Fort Greely, Alaska, or Vandenberg Air Force Base, California, where these missiles are based. In fact the NMD is designed for the defence of American continental targets and places like Hawaii, and other American possessions in the Pacific, cannot be defended nowadays by this system.

Taking this constraint into account, only interception scenarios with the target city in the continental U.S. will be considered. Since the simulations require a very long time we will limit the cases to be studied to launches from Fort Greely to 2 target cities one on each coast: New York City and Los Angeles.

11.1.2.2 Guidance algorithms to be compared

As indicated in chapter 8, Powered Explicit Guidance (PEG) will be used as the conventional ascent guidance within this chapter for the GBI.

As indicated in chapter 9, proportional navigation guidance with gravity compensation will be used in the simulator when a conventional terminal guidance is requested.

These conventional guidance algorithms will be compared with optimal algorithms, starting with conventional ascent guidance plus the optimal terminal guidance detailed in section 10.2.3.

When using a conventional ascent guidance, it can be considered as unnecessary to request force from the divert thrusters when the target is too far away, since the ascent guidance in this case has been designed so that the EKV would ideally reach the target without applying any force (a maximum distance of about 10 km at the closest point of approach would appear in the real case because of different errors). This is, only small adjustments are required to correct the nominal trajectory and these adjustments could be applied only at the end of the trajectory.

As a consequence, we will also consider the cases of conventional ascent (PEG) plus proportional navigation with gravity compensation activated when the distance between the missiles is smaller than 1000 km, and conventional ascent (PEG) plus optimal terminal guidance activated when the distance between the missiles is smaller than 1000 km.

Global optimal guidance algorithms will also be considered: global optimal interception guidance (section 10.2.4), optimal tracking guidance (section 10.2.5) and global optimal guidance using an augmented state vector (section 10.2.6).

The guidance algorithms to be compared are indicated in the following table:

Table 11.1: Guidance algorithms to be analysed

Case	Ascent guidance algorithm	Terminal guidance algorithm
1	Powered Explicit Guidance (PEG)	PN with gravity compensation
2	Powered Explicit Guidance (PEG)	PN with gravity compensation active last 1000 km
3	Powered Explicit Guidance (PEG)	Optimal terminal guidance (section 10.2.3)
4	Powered Explicit Guidance (PEG)	Optimal terminal guidance (section 10.2.3) active last 1000 km
5	Global optimal interception guidance (section 10.2.4)	
6	Optimal tracking guidance (section 10.2.5)	
7	Global optimal guidance using an augmented state vector (section 10.2.6)	

11.1.2.3 Number of executions

In order to have representative results from the statistical point of view the number of executed simulations should be as high as possible.

However we have noticed that the variance among executions is not very high. Taking this into account and being the executions as time consuming as they are, we will limit the executions to a minimum of 10 for each case.

11.2 Results obtained with each guidance algorithm

11.2.1 Conventional ascent guidance with conventional terminal guidance

In this case we will launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses PEG guidance for the ascent phase and proportional navigation with gravity compensation for the terminal phase.

These simulations will be launched considering gaussian errors ($2\% \text{ } 3\sigma$) in the thrust force.

Gaussian errors will also be considered in the gravity force according to the indications in section 4.6 ($\sigma = 29 \text{ mGals}$).

The obtained results are provided in the following table:

Table 11.2: Results with PEG ascent guidance and proportional navigation with gravity compensation for terminal guidance

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	PEG + PN with grav. comp.	0.48 $\sigma = 0.073$	8.58 $\sigma = 0.017$	3398.38 $\sigma = 2.128$	1284.66 $\sigma = 0.337$
ICBM to L.A.	PEG + PN with grav. comp.	0.38 $\sigma = 0.093$	8.41 $\sigma = 0.004$	5430.52 $\sigma = 0.144$	1190.35 $\sigma = 0.002$

The required force in the divert thrusters in this case (which is directly related to the fuel consumption) for a case against an ICBM heading to NYC can be seen in the following figure:

Figure 11.1: Divert thrusters force in conventional terminal guidance after conventional ascent against an ICBM heading to N.Y.C.

A close up of the final interception moments is provided:

Figure 11.2: Divert thrusters force in conventional terminal guidance after conventional ascent against an ICBM heading to N.Y.C. (closeup of final moments)

It can be seen that the required force decreases smoothly along the trajectory and increases only when the time step is reduced (which would not happen in the real case) and when the interception happens.

11.2.2 Conventional ascent guidance with conventional terminal guidance active only the last 1000 km

In this case we will launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses PEG guidance for the ascent phase and proportional navigation with gravity compensation for the terminal phase, only active when the distance between the missiles is smaller than 1000 km.

These simulations will be launched considering gaussian errors ($2\% \ 3\sigma$) in the thrust force and gaussian errors ($\sigma = 29$ mGals) in the gravity force, as justified in section 4.6.

The obtained results are provided in the following table:

Table 11.3: Results with PEG ascent guidance and proportional navigation with gravity compensation for terminal guidance active only the last 1000 km

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	PEG + PN with grav. comp. active only last 1000 km	0.48 $\sigma = 0.067$	3.62 $\sigma = 0.026$	3402.26 $\sigma = 2.085$	1283.94 $\sigma = 0.335$
ICBM to L.A.	PEG + PN with grav. comp. active only last 1000 km	0.37 $\sigma = 0.090$	1.73 $\sigma = 0.013$	5433.06 $\sigma = 0.127$	1189.91 $\sigma = 0.002$

The required force in the divert thrusters in this case (which is directly related to the fuel consumption) for a case against an ICBM heading to NYC can be seen in the following figure:

Figure 11.3: Divert thrusters force in conventional terminal guidance active the last 1000 km after conventional ascent against an ICBM heading to N.Y.C.

It can be seen that in this case the required force decreases initially very quickly and increases only when the time step is reduced (which would not happen in the real case) and just when the interception happens:

Figure 11.4: Divert thrusters force in conventional terminal guidance active the last 1000 km after conventional ascent against an ICBM heading to N.Y.C. (closeup of final moments)

11.2.3 Conventional ascent guidance with optimal terminal guidance

In this case we will launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses PEG guidance for the ascent phase and optimal terminal guidance for the terminal phase (as detailed in section 10.2.3).

These simulations will be launched considering gaussian errors ($2\% \ 3\sigma$) in the thrust force and gaussian errors ($\sigma = 29$ mGals) in the gravity force, as justified in section 4.6.

The obtained results are provided in the following table:

Table 11.4: Results with PEG ascent guidance and optimal terminal guidance

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	PEG + optimal terminal	0.47 $\sigma = 0.057$	16.46 $\sigma = 0.013$	3361.56 $\sigma = 1.214$	1291.21 $\sigma = 0.215$
ICBM to L.A.	PEG + optimal terminal	0.38 $\sigma = 0.096$	15.86 $\sigma = 0.041$	5417.06 $\sigma = 2.254$	1192.57 $\sigma = 0.377$

The required force in the divert thrusters in this case (which is directly related to the fuel consumption) for a case against an ICBM heading to NYC can be seen in the following figure:

Figure 11.5: Divert thrusters force with optimal terminal guidance after conventional ascent against an ICBM heading to N.Y.C.

It can be seen that in this case there is an important force requirement just after the ascent phase finishes:

Figure 11.6: Divert thrusters force with optimal terminal guidance after conventional ascent against an ICBM heading to N.Y.C. (closeup of initial moments)

The final part is similar to previous cases, with changes each time the time step is changed (which would not happen in the real case) and just when the interception happens:

Figure 11.7: Divert thrusters force with optimal terminal guidance after conventional ascent against an ICBM heading to N.Y.C. (closeup of final moments)

11.2.4 Conventional ascent guidance with optimal terminal guidance active only the last 1000 km

In this case we will launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses PEG guidance for the ascent phase and optimal terminal guidance for the terminal phase (as detailed in section 10.2.3), only active when the distance between the missiles is smaller than 1000 km.

These simulations will be launched considering gaussian errors ($2\% \ 3\sigma$) in the thrust force and gaussian errors ($\sigma = 29$ mGals) in the gravity force, as justified in section 4.6.

The obtained results are provided in the following table:

Table 11.5: Results with PEG ascent guidance and optimal terminal guidance active only the last 1000 km

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	PEG + optimal terminal active only last 1000 km	0.45 $\sigma = 0.051$	2.82 $\sigma = 0.246$	3401.45 $\sigma = 1.449$	1284.01 $\sigma = 0.264$
ICBM to L.A.	PEG + optimal terminal active only last 1000 km	0.34 $\sigma = 0.032$	1.66 $\sigma = 0.010$	5433.04 $\sigma = 0.095$	1189.91 $\sigma = 0.002$

The required force in the divert thrusters in this case (which is directly related to the fuel consumption) for a case against an ICBM heading to NYC can be seen in the following figure:

Figure 11.8: Divert thrusters force with optimal terminal guidance active the last 1000 km after conventional ascent against an ICBM heading to N.Y.C.

It can be seen that in this case the force requirements when the terminal guidance is activated last only several seconds, reducing drastically the fuel consumption with respect to the algorithm in section 11.2.3:

Figure 11.9: Divert thrusters force with optimal terminal guidance active the last 1000 km after conventional ascent against an ICBM heading to N.Y.C. (closeup of initial moments)

The final part is similar to previous cases, with changes each time the time step is changed (which would not happen in the real case) and just when the interception happens:

Figure 11.10: Divert thrusters force with optimal terminal guidance active the last 1000 km after conventional ascent against an ICBM heading to N.Y.C. (closeup of final moments)

11.2.5 Global optimal interception

In this case we will launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses global optimal interception guidance for both the ascent phase and the terminal phase (as detailed in section 10.2.4).

These simulations will be launched considering gaussian errors ($2\% \ 3\sigma$) in the thrust force and gaussian errors ($\sigma = 29$ mGals) in the gravity force, as justified in section 4.6.

The obtained results are provided in the following table:

Table 11.6: Results with global optimal interception guidance

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	global interception guidance	0.54 $\sigma = 0.091$	51.39 $\sigma = 0.014$	4156.93 $\sigma = 1.030$	1145.54 $\sigma = 0.163$
ICBM to L.A.	global interception guidance	0.47 $\sigma = 0.124$	46.17 $\sigma = 0.018$	6703.08 $\sigma = 1.012$	978.25 $\sigma = 0.171$

The required force in the divert thrusters in this case (which is directly related to the fuel consumption) for a case against an ICBM heading to NYC can be seen in the following figure:

Figure 11.11: Divert thrusters force with global interception guidance against an ICBM heading to N.Y.C.

It can be seen that in this case there is an important force requirement just after the ascent phase finishes. This cannot be avoided like when a conventional ascent guidance algorithm was used, since the ascent guidance algorithm in this case does not lead the missile to a direct interception course like a Lambert-based guidance:

Figure 11.12: Divert thrusters force with global interception guidance against an ICBM heading to N.Y.C. (closeup of initial moments)

The final part is similar to previous cases, with changes each time the time step is changed (which would not happen in the real case) and just when the interception happens:

Figure 11.13: Divert thrusters force with global interception guidance against an ICBM heading to N.Y.C. (closeup of final moments)

11.2.6 Optimal tracking guidance

In this case we tried to launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses optimal tracking guidance for both the ascent phase and the terminal phase (as detailed in section 10.2.5).

However this algorithm fails in the interception. The reason for this is that this algorithm tries to reach a static target placed in its present position and for which no dynamics are considered. With the limitations of the algorithm considered in section 10.2.5, in which the curvature of the Earth and the variation of the gravitational field is not considered, the provided solution leads to a reentry in the atmosphere and fails in the interception.

This can be observed in figures 11.14 and 11.15:

Figure 11.14: Failed interception trajectory using optimal tracking guidance (1)
against an ICBM heading to N.Y.C.
(picture obtained with Google EarthTM mapping service)

Figure 11.15: Failed interception trajectory using optimal tracking guidance (2)
against an ICBM heading to N.Y.C.
(picture obtained with Google EarthTM mapping service)

Being this algorithm unable to provide an interception trajectory, it has only been kept herein because it shows some limitations that will be highlighted in chapter 13 when analysing the behaviour of the optimal guidance algorithms.

11.2.7 Global optimal guidance using an augmented state vector

In this case we will launch simulations in which the ICBM uses Q guidance for the ascent phase and the interceptor missile uses global optimal guidance using an augmented state vector for both the ascent phase and the terminal phase (as detailed in section 10.2.6).

These simulations will be launched considering gaussian errors ($2\% \ 3\sigma$) in the thrust force and gaussian errors ($\sigma = 29$ mGals) in the gravity force, as justified in section 4.6.

The obtained results are provided in the following table:

Table 11.7: Results with global optimal guidance using an augmented state vector

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	global optimal guidance with aug. state vector	2.36 $\sigma = 0.018$	51.31 $\sigma = 0.018$	4191.42 $\sigma = 1.066$	1139.13 $\sigma = 0.179$
ICBM to L.A.	global optimal guidance with aug. state vector	2.94 $\sigma = 0.018$	46.13 $\sigma = 0.019$	6728.76 $\sigma = 0.579$	973.95 $\sigma = 0.094$

The required force in the divert thrusters in this case (which is directly related to the fuel consumption) for a case against an ICBM heading to NYC can be seen in the following figure:

Figure 11.16: Divert thrusters force with global optimal guidance using an augmented state vector against an ICBM heading to N.Y.C.

It can be seen that in this case there is an important force requirement just after the ascent phase finishes. This phase cannot be avoided like when a conventional ascent guidance algorithm was used, since the ascent guidance algorithm in this case does not lead the missile to a direct interception course like a Lambert-based guidance:

Figure 11.17: Divert thrusters force with global optimal guidance using an augmented state vector against an ICBM heading to N.Y.C. (closeup of initial moments)

The final part is a bit different to previous cases. Apart from changes each time the time step is changed (which would not happen in the real case) and when the interception happens, this guidance algorithm is requesting non null values in the last interception moments:

Figure 11.18: Divert thrusters force with global optimal guidance using an augmented state vector against an ICBM heading to N.Y.C. (closeup of final moments)

11.3 Comparison of results

The parameters previously obtained with different guidance algorithms will be compared now. For this reason they will all be grouped in a single table:

Table 11.8: Results with different guidance algorithms

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C.	PEG + PN with grav. comp.	0.48 $\sigma = 0.073$	8.58 $\sigma = 0.017$	3398.38 $\sigma = 2.128$	1284.66 $\sigma = 0.337$
ICBM to N.Y.C.	PEG + PN with grav. comp. active only last 1000 km	0.48 $\sigma = 0.067$	3.62 $\sigma = 0.026$	3402.26 $\sigma = 2.085$	1283.94 $\sigma = 0.335$
ICBM to N.Y.C.	PEG + optimal terminal	0.47 $\sigma = 0.057$	16.46 $\sigma = 0.013$	3361.56 $\sigma = 1.214$	1291.21 $\sigma = 0.215$
ICBM to N.Y.C.	PEG + optimal terminal active only last 1000 km	0.45 $\sigma = 0.051$	2.82 $\sigma = 0.246$	3401.45 $\sigma = 1.449$	1284.01 $\sigma = 0.264$
ICBM to N.Y.C.	global interception guidance	0.54 $\sigma = 0.091$	51.39 $\sigma = 0.014$	4156.93 $\sigma = 1.030$	1145.54 $\sigma = 0.163$
ICBM to N.Y.C.	global optimal guidance with aug. state vector	2.36 $\sigma = 0.018$	51.31 $\sigma = 0.018$	4191.42 $\sigma = 1.066$	1139.13 $\sigma = 0.179$
ICBM to L.A.	PEG + PN with grav. comp.	0.38 $\sigma = 0.093$	8.41 $\sigma = 0.004$	5430.52 $\sigma = 0.144$	1190.35 $\sigma = 0.002$
ICBM to L.A.	PEG + PN with grav. comp. only last 1000 km	0.37 $\sigma = 0.090$	1.73 $\sigma = 0.013$	5433.06 $\sigma = 0.127$	1189.91 $\sigma = 0.002$
ICBM to L.A.	PEG + optimal terminal	0.38 $\sigma = 0.096$	15.86 $\sigma = 0.041$	5417.06 $\sigma = 2.254$	1192.57 $\sigma = 0.377$
ICBM to L.A.	PEG + optimal terminal active only last 1000 km	0.34 $\sigma = 0.032$	1.66 $\sigma = 0.010$	5433.04 $\sigma = 0.095$	1189.91 $\sigma = 0.002$
ICBM to L.A.	global interception guidance	0.47 $\sigma = 0.124$	46.17 $\sigma = 0.018$	6703.08 $\sigma = 1.012$	978.25 $\sigma = 0.171$
ICBM to L.A.	global optimal guidance with aug. state vector	2.94 $\sigma = 0.018$	46.13 $\sigma = 0.019$	6728.76 $\sigma = 0.579$	973.95 $\sigma = 0.094$

It can be observed that in terms of accuracy all the guidance algorithms provide similar results, being conventional ascent guidance plus optimal terminal guidance active only the last 1000 km slightly better. Since the difference is very small no great advantage from using one algorithm or the other can be deduced from this criterion.

The only difference in terms in accuracy is related to global optimal guidance with augmented state vector, for which the miss distance is clearly higher. This could be solved changing the parameters in this case (Q , R , S) or even transitioning from global optimal guidance with augmented state vector to optimal terminal guidance in the last moments of the interception.

In terms of fuel consumption there are great differences between the algorithms in which the ascent guidance leads to an interception course (the ones with conventional ascent guidance) and the algorithms for which an optimal algorithm is applied in the ascent phase, since in this latter case a much bigger fuel consumption has to be applied to change the trajectory from the injection point to the interception.

Among the algorithms with a conventional ascent guidance, the best results in terms of consumption are obtained again using optimal terminal guidance active only the last 1000 km. Applying this algorithm since the injection point, however, led to almost double fuel consumption than the proportional navigation algorithm. This is because the optimal terminal guidance does not consider the variation of the gravitational field, so applying it too far away from the target leads to unnecessary commands when the ascent guidance is placing the EKV within 10 km of the course of the ICBM.

In the case of optimal algorithms applied also in the ascent phase, the state at injection does not lead to a direct interception course so we cannot apply the divert thrusters only near the interception and much more fuel consumption is required.

The change in the trajectory when the terminal guidance starts can be observed in figures 11.19 and 11.20

Figure 11.19: Interception trajectory using optimal interception guidance against an ICBM heading to L.A.
(picture obtained with Google EarthTM mapping service)

Figure 11.20: Interception trajectory using global optimal guidance with augmented state vector against an ICBM heading to N.Y.C.
(picture obtained with Google EarthTM mapping service)

It has to be indicated that the fuel consumption in these cases is so high that in fact the present configuration for the GBI and the EKV could not be applied with these algorithms, since a mass of 64 kg is considered for the EKV including fuel mass, so not much more than 15 kg for the fuel would be possible.

This means that in order to apply the implemented global optimal guidance algorithms during the ascent phase: global optimal interception guidance (section 10.2.4) and global optimal guidance using an augmented state vector (section 10.2.6)), a different configuration for the EKV should be considered and much more fuel (probably around 200 kg) would be required.

Finally, regarding the time to interception, it is easy to check in table 11.8 that all the methods that use conventional ascent guidance provide similar results, being optimal terminal guidance active the last 1000 km slightly better.

In this case huge improvements appear when using optimal ascent guidance. An improvement of more than 2 minutes is achieved in the case of the ICBM heading to NYC and in the case of the ICBM heading to LA an improvement of 3.5 minutes is achieved.

This huge improvement can be observed in figures 11.21 and 11.22:

Figure 11.21: Interception trajectories with the ICBM heading to NYC (1)
(picture obtained with Google EarthTM mapping service)

Figure 11.22: Interception trajectory with the ICBM heading to LA (1)
(picture obtained with Google EarthTM mapping service)

A different view of the interception trajectories is provided in figures figures 11.23 and 11.24. It can be seen that there are very relevant differences between the optimal trajectories and the conventional ones, especially in terms of the height of apogee, which is much lower in the optimal cases.

Figure 11.23: Interception trajectories with the ICBM heading to NYC (2)
(picture obtained with Google EarthTM mapping service)

Figure 11.24: Interception trajectory with the ICBM heading to LA (2)
(picture obtained with Google EarthTM mapping service)

Chapter 11 references

- [1] Sang-Keun Jang, Robert G. Hutchins, and Phillip E. Pace. Boost Phase ICBM Engagement Using Space-Based Interceptors: A Comparison of Guidance Laws. *IEEE International Conference on System of Systems Engineering. SoSE '08*, pages 1–5, June 2008. doi: 10.1109/SYSOSE.2008.4724209. 278
- [2] Sang-Keun Jang, Phillip E. Pace, Robert G. Hutchins, and James B. Michael. Technical Report NPS-CS-08-009. A Comparative Analysis of Guidance Laws for Boost-Phase Ballistic Missile Intercept Using Exo-Atmospheric Kill Vehicles. <http://edocs.nps.edu/npspubs/scholarly/TR/2008/NPS-CS-08-009.pdf>, April 2008. 278

Part IV

Results and conclusions

The results of the comparison of different guidance algorithms will be assessed and, taking into account the observed behaviour of the guidance algorithms, the feasibility of the Ballistic Missile Defense system in terms of guidance will be indicated.

After this analysis, the main achievements of this thesis will be summarized and conclusions will be given.

This page intentionally left blank.

Chapter 12

Analysis of guidance algorithms

This chapter analyses the observed behaviour of the different guidance algorithms for the interception of ICBMs, indicating the problems found and the feasibility of the Ballistic Missile Defense system in terms of guidance.

12.1 Behaviour of the guidance algorithms without noise nor delays in the target estimation

The results provided in chapter 11 indicate that conventional guidance algorithms are able to direct the EKV to an interception course against the ICBM that would result in a successful hit, even when errors are considered in the control forces and the gravity forces.

This can be achieved with a very small fuel consumption (if the proportional navigation scheme is activated only when both missiles are less than 1000 km apart, less than 4 kg would be required in the most demanding case - ICBM heading to NYC - with the hypotheses considered in the simulation about fuel mass consumption).

The behaviour is even better when conventional ascent guidance is followed by optimal terminal guidance: the achieved miss distance is slightly smaller and the fuel consumption is much smaller (there is an average improvement in the fuel consumption of about a 28 % in the case of the ICBM heading to NYC, and an average improvement of about 4 % in the case of the ICBM heading to LA).

From the accuracy point of view the global interception guidance (section 10.2.4) also provides an adequate accuracy, but the global optimal guidance using an augmented state vector (section 10.2.6) leads to a higher miss distance. Some fine-tuning of the algorithms parameters could be required to improve the accuracy of this algorithm, or even a transition to an optimal terminal guidance in the last moments of the interception.

It has to be noted, however, that the optimal algorithms applied during the ascent phase lead to a fuel consumption that is not compatible with the present configuration of the interceptor missile. These algorithms also lead to a much faster interception (an improvement of more than 2 minutes is achieved in the case of the ICBM heading to NYC and in the case of the ICBM heading to LA an improvement of 3.5 minutes is achieved), so this change in the configuration to allow that the EKV is released at the injection point with a much higher fuel capacity could be a very convenient choice.

12.2 Behaviour of the guidance algorithms when noise is considered in the target state estimation

If small errors are contemplated, provided that the trajectory of the incoming ICBM could be accurately computed, the behaviour of the considered guidance algorithms is similar to the case without errors in the target state estimation except in the final terminal phase.

The behaviour in this final terminal phase greatly depends on the considered noise.

As an example, considering a gaussian distribution for the target state estimation with the following standard deviation values:

$$\begin{aligned} \text{if distance} > 1000 \text{ km} &\Rightarrow \begin{cases} \sigma_{pos} = 100m. \\ \sigma_{vel} = 1m/s \end{cases} \\ \text{if } 10 \text{ km} < \text{distance} < 1000 \text{ km} &\Rightarrow \begin{cases} \sigma_{pos} = 0.0001 \cdot \text{distance} \text{ m.} \\ \sigma_{vel} = 0.000001 \cdot \text{distance} \text{ m/s} \end{cases} \\ \text{if distance} < 10 \text{ km} &\Rightarrow \begin{cases} \sigma_{pos} = 1m. \\ \sigma_{vel} = 0.1m/s \end{cases} \end{aligned} \quad (12.1)$$

the following results are obtained for 10 executions with conventional ascent guidance plus conventional terminal guidance (proportional navigation with gravity compensation) and for conventional ascent guidance plus optimal terminal guidance (as indicated in section 10.2.3), both active only when the distance between the missiles is smaller than 1000 km, against an ICBM missile applying Q guidance and heading to NYC:

Table 12.1: Results with errors in the target state estimation

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C. with no errors in target estimation	PEG + PN with grav. comp. active only last 1000 km	0.48 $\sigma = 0.067$	3.62 $\sigma = 0.026$	3402.26 $\sigma = 2.085$	1283.94 $\sigma = 0.335$
ICBM to N.Y.C. with errors in target estimation	PEG + PN with grav. comp. active only last 1000 km	0.46 $\sigma = 0.079$	5.20 $\sigma = 0.051$	3401.14 $\sigma = 1.625$	1284.27 $\sigma = 0.279$
ICBM to N.Y.C. with no errors in target estimation	PEG + optimal terminal active only last 1000 km	0.45 $\sigma = 0.051$	2.82 $\sigma = 0.246$	3401.45 $\sigma = 1.449$	1284.01 $\sigma = 0.264$
ICBM to N.Y.C. with errors in target estimation	PEG + optimal terminal active only last 1000 km	0.46 $\sigma = 0.069$	10.87 $\sigma = 0.103$	3401.61 $\sigma = 0.112$	1284.16 $\sigma = 0.009$

There is a relevant increase (43.6%) in the fuel consumption of the divert thrusters in the case of proportional navigation because of the noise in the target estimation.

In the case of optimal terminal guidance the increase in the requested force from the divert thrusters is much higher (the fuel consumption basically multiplies $\times 3$ with respect to the case without noise). The requested force in this case is shown in the following figure:

Figure 12.1: Divert thrusters force with optimal terminal guidance after conventional ascent against an ICBM heading to N.Y.C. considering unfiltered noise in the target state estimation

This means that the optimal terminal guidance is more sensitive to noise than the proportional navigation algorithm. Reference [1] (pages 204-211) performs different comparisons of the effect of noise in optimal terminal guidance and proportional navigation, concluding that optimal terminal guidance yields smaller miss distance when small flight times are considered and when a large guidance system time constant is considered. This cannot be observed in our comparison since we are supposing in the simulation a small guidance system time constant (equal to the time step) and since the flight times are not small. In any case reference [1] is adding noise to the target position estimation and what worsens more the optimal terminal guidance algorithm is the noise in the velocity vector.

It can be concluded that in order to successfully apply the optimal terminal guidance algorithm a smoothing algorithm has to be implemented to reduce the fuel consumption as much as possible.

In any case it can be observed that the obtained miss distances with unfiltered noise are similar to the ones obtained without considering errors in the target estimation.

This means that if the sensors (ground radars, tracking satellites and the IR seeker on board the EKV) provide accurate results, there will be no problem in obtaining collision courses and destroying the incoming ICBM using the considered algorithms.

This is, however, a preliminary assessment. In order to properly assess this, we would have to know the exact accuracy of the sensors within the National Missile Defense (NMD) system.

12.3 Behaviour of the guidance algorithms when delays are considered in the target state estimation

Any delay in the target state estimation or in the system response worsens the system behaviour in the terminal phase either increasing the final miss distance or increasing the required fuel consumption.

As an example, the effect of several system delays in the terminal guidance for 10 executions with conventional ascent guidance plus conventional terminal guidance (proportional navigation with gravity compensation) and for conventional ascent guidance plus optimal terminal guidance (as indicated in section 10.2.3), both active only when the distance between the missiles is smaller than 1000 km, against an ICBM missile applying Q guidance and heading to NYC, is shown in the following table:

Table 12.2: Results with delays in the target state estimation

Case	Guidance algorithms (GBI-EKV)	Miss distance (m)	Fuel consumed (kg)	Distance to target (km)	Time from ICBM liftoff at intercept (s)
ICBM to N.Y.C. with a system delay of 0.0 s	PEG + PN with grav. comp. active only last 1000 km	0.48 $\sigma = 0.067$	3.62 $\sigma = 0.026$	3402.26 $\sigma = 2.085$	1283.94 $\sigma = 0.335$
ICBM to N.Y.C. with a system delay of 0.01 s	PEG + PN with grav. comp. active only last 1000 km	38.69 $\sigma = 0.004$	3.99 $\sigma = 0.045$	3400.48 $\sigma = 2.188$	1284.37 $\sigma = 0.380$
ICBM to N.Y.C. with a system delay of 0.1 s	PEG + PN with grav. comp. active only last 1000 km	384.44 $\sigma = 0.031$	4.50 $\sigma = 0.053$	3401.53 $\sigma = 0.094$	1284.18 $\sigma = 0.016$
ICBM to N.Y.C. with a system delay of 0.0 s	PEG + optimal terminal active only last 1000 km	0.45 $\sigma = 0.051$	2.82 $\sigma = 0.246$	3401.45 $\sigma = 1.449$	1284.01 $\sigma = 0.264$
ICBM to N.Y.C. with a system delay of 0.01 s	PEG + optimal terminal active only last 1000 km	38.59 $\sigma = 0.003$	3.30 $\sigma = 0.036$	3403.97 $\sigma = 1.678$	1283.75 $\sigma = 0.287$
ICBM to N.Y.C. with a system delay of 0.1 s	PEG + optimal terminal active only last 1000 km	383.72 $\sigma = 0.017$	3.97 $\sigma = 0.033$	3401.60 $\sigma = 0.091$	1284.17 $\sigma = 0.011$

It can be seen that any delay becomes a very relevant miss distance while the fuel consumption increases linearly and the time to interception remains basically unchanged. The increase in miss distance is due to the high closing speed of the missiles.

In order to keep the accuracy in the interception it is necessary to compute the system delay very accurately and propagate the state of the target to the future taking this delay into account (using for example an Extended Kalman Filter).

The system delay can also be alleviated in terms of required control forces if the system delay is known when an optimal guidance algorithm is used.

For example, reference [1] modifies the optimal terminal guidance by including an additional variable "achieved acceleration" related to the commanded one with a first order law:

$$n_L = \frac{n_c}{1 + sT} \quad (12.2)$$

where n_L is the achieved acceleration, n_c is the commanded acceleration, T is the system delay and s is equivalent to $\frac{d}{dt}$ in the Laplace domain.

12.4 Feasibility of the Ballistic Missile Defense system in terms of guidance

It can be deduced from the results provided in chapter 11 and from the previous analyses when errors are considered in the target state estimation and when system delays are considered, that the Ballistic Missile Defense system is feasible from the guidance point of view if a correct management of the system delays is made and a smoothing algorithm is used to reduce the noise in the target velocity estimation, and collision trajectories can always be obtained against an incoming ICBM.

There are open issues that can only be assessed having classified information from the system. For example the exact behaviour against target state estimation errors requires information from the system sensors that is nowadays unavailable. However, problems in this sense could always be solved by increasing the number of sensors or their accuracy, until the achieved accuracy is good enough to ensure collision trajectories.

In the same way the exact effect of system delays could only be assessed when the real system delay and the management of this delay within the control system is known. In any case any problem in this sense will tend to diminish as time goes by, as better computation capabilities are included on the onboard computers of the EKV.

If these open issues are properly handled, the problems of the Ballistic Missile Defense system regarding an accuracy interception are not in the guidance algorithms but in problems that would jeopardize their correct application:

- Reaction time

There is a limited reaction time in order to be able to destroy an incoming ICBM. Taking into account that launching a GBI missile would probably require political and military approval, it is likely that the launch of an ICBM heading to the U.S. without any previous warning or political tension, would result in a delay in the reaction time that would make the system useless.

This problem is difficult to solve since it has nothing to do with a technical problem itself, but depends on political and military decisions.

The application of optimal ascent guidance algorithms as indicated in chapter 11 can provide some extra minutes for this decision since it allows a faster interception than conventional guidance algorithms.

- MIRVs (Multiple Independently targetable Reentry vehicles)

As indicated in chapter 1 each ICBM usually carries several reentry vehicles that are independently deployed along the main trajectory, reentering at different points. In order to successfully intercept all the incoming MIRVs it would be required to deploy several GBIs for each detected ICBM launch. The exact number of GBIs to be launched in each case would depend on intelligence information about the most likely number of MIRVs, and again on political and military decisions, since many factors (desired probability of kill, PK, strategic considerations regarding expected future attacks, etc.) are to be taken into account.

The effects of MIRVs deployment has not been considered in this thesis, nor the possible targetting problems that may arise if several EKVs are pursuing the same ICBM before the MIRVs are deployed.

- Decoy discrimination

This is the biggest technical problem faced by the system. As indicated in chapter 1, the ICBM usually deploys decoys together with the real warhead. Being outside the atmosphere these decoys (simple balloons), will move close to the real warhead and are not easy to discriminate, since in mid-course guidance the warhead is cold. Some debris coming from previous stages of the missile could also accompany the real warheads acting as unintentional decoys.

In fact there are studies indicating that the EKV will never have the capability to successfully discriminate a warhead from a decoy ([2]).

This is one of the main areas of research within the Ballistic Missile Defense system and characterizations of the infrared signatures of different objects are being studied by the Missile Defense Agency (MDA) in order to solve this problem.

- Target manoeuvre

It would be possible to develop ICBMs with a midcourse trajectory modification (see for example [3]). This is for instance the case of the RT-2UTTKh Topol M ICBM which according to Russia is capable of making evasive manoeuvres.

The effect of the evasive manoeuvres on the final miss distance will depend on the manoeuvring capability of the ICBM. This, however, is a problem that can be probably tackled by the EKV since it is unlikely that the ICBM will have a high manoeuvring capability since that would lead to an uncertain target position.

Chapter 12 references

- [1] Paul Zarchan. *Tactical and Strategic Missile Guidance*. American Institute of Aeronautics and Astronautics Inc., Alexander Bell Drive, Reston, Virginia, sixth edition, February 2012. 306, 308
- [2] Theodore Postol. Explanation of Why the Sensor in the Exoatmospheric Kill Vehicle (EKV) Cannot Reliably Discriminate Decoys from Warheads. https://www.fas.org/spp/starwars/program/news00/postol_atta.pdf, May 2000. 310
- [3] Mathew P. Gillis III. Optimal Mid-Course Modifications of Ballistic Missile Trajectories. Master's thesis, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, December 1975. 310

This page intentionally left blank.

Chapter 13

Achievements and conclusions

This chapter indicates the main achievements of this thesis and provides the main conclusions that have been reached.

13.1 Achievements

Several achievements obtained with this thesis can be highlighted:

- There are countless studies in the literature about Lambert’s problem solvers. However, the errors in the targeting process because of using a Lambert solver are not highlighted in the literature.
In this thesis we performed an analysis of the source of errors when using this aiming algorithm (section 7.2.3), and error values for different cases were provided (table 7.1). These results indicate that for conventional ascent guidance, the errors due to using a Lambert solver are much higher than the errors due to the guidance algorithm (see table 8.1). This means that in the case of targeting, in terms of accuracy, it is more relevant to improve the initial aiming solution (for example using a Lambert algorithm that includes the J_2 terms), than improving the used guidance algorithm. This finding was not indicated in the undisclosed literature so it constitutes an original finding.
- A novel method for fine-tuning the initial targeting, not included in the undisclosed literature, was developed in section 7.3.1.2.
This method allows improving the final state transition matrix at the same time that a new achieved target is computed by simulation. The method is based on the application of observation techniques (i.e. Kalman filtering), to calibration problems.
Examples of this method for an ICBM heading to NYC, Tokyo and Cape Town respectively are provided in tables 7.2, 7.3 and 7.4. It can be observed that the method is robust and provides a good convergence.
- The analysis of conventional ascent algorithms indicated that one of the most commonly used algorithms, Power Explicit Guidance (PEG), could not be directly used for aiming intercontinental ballistic missiles, since it is conceived for reaching a certain position vector known beforehand.
We have adapted the implementation of this algorithm so that it can be used for missiles (see section 8.3.5.2.3).
The obtained results (see section 8.3.6) are comparable to those of Lambert guidance and Q guidance, even better in the case of the GBI missile so the implementation has been successful and again constitutes a novelty within the undisclosed literature.
- A comparison of several conventional ascent guidance algorithms has been made (section 8.3.6).
This is relevant since there are few comparisons of this type in the undisclosed literature (apart from reference [1] (Song et al., 2015)), and the available comparisons focus in the problem of spacecraft injection instead of the missile aiming problem. As a consequence, the comparison provided in table 8.1 constitutes a novelty.

- We have studied optimal guidance techniques and implemented optimal guidance algorithms, both for the terminal guidance phase and especially for the ascent phase.

This implementation required the use of innovative procedures so that these algorithms could lead to a successful interception.

For example, the idea of converting the desired forces provided by the optimal guidance algorithms to desired attitude is very innovative and very powerful in terms of increasing the linearity of the problem and avoiding instability problems.

- Comparisons of conventional guidance with optimal guidance algorithms have been performed in chapter 11.

Again, there are few of these comparisons in the undisclosed literature, even in cases where optimal algorithms have been implemented like in reference [2], so they can be considered as original analyses.

- The implemented optimal guidance algorithms, with the only exception of LQT, lead to successful interceptions.

In the case of optimal terminal guidance, a slight improvement in accuracy and a significant improvement in fuel consumption is achieved with respect to conventional terminal guidance when no noise is considered in the target estimation.

In the case of optimal ascent guidance, the algorithms lead to a huge increase in the fuel consumption so they couldn't be applied with the present configuration of the GBI. However, they also provide significant improvements in terms of time of interception (an improvement of more than 2 minutes is achieved in the case of the ICBM heading to NYC and in the case of the ICBM heading to LA an improvement of 3.5 minutes is achieved). This improvement is very relevant so a change in the present configuration of the interceptor missile to allow that the EKV is released at the injection point with a much higher fuel capacity could be a convenient choice.

- We have assessed the feasibility of the Ballistic Missile Defense system in terms of guidance (section 12.4).

The conclusion is that the system is feasible and that the main problems are not in the guidance algorithms but in problems that would jeopardize their correct application, especially related to the reaction time, the accuracy of the sensors, the correct management of the system delays and, especially, to the problem of the discrimination of warheads from decoys.

13.2 Conclusions

This thesis had a very ambitious approach and many different subjects were analysed.

On the one hand it was decided to develop a highly representative simulator for the involved missiles instead of analysing guidance algorithms with a purely mathematical approach. The justification for this is given in section 6.3.1: it allows obtaining representative results whereas the findings obtained using simplified simulators are only useful to show the general concept and cannot be directly applied to real cases in many occasions. As a drawback, this approximation, requiring the development of a realistic simulator, implied a huge effort.

On the other hand, an analysis of the existing conventional guidance algorithms had to be undertaken to know their formulation, develop an implementation suitable for the interception scenario, and find the better ones in order to compare them with optimal guidance algorithms. This also implied an important additional effort.

As a consequence, the development, implementation and analysis of optimal guidance algorithms was only the last step of a long way.

Despite this ambitious approach, the main objectives of the thesis have been successfully achieved:

- As indicated in section 6.3.5, a state of the art simulator has been developed in terms of algorithms implementation in the ballistic missile interception problem.

A special mention should be made regarding the configuration of the LGM-30G Minuteman III ICBM and the GBI missile, since this simulator includes data for these missiles which is the result of extensive search and analysis. It can be concluded that an advanced simulation platform that can provide meaningful results in the analysis of interception problems was developed.

- Existing conventional guidance algorithms have been analysed, implemented and compared (see chapters 8 and 9 in part III), in order to find the better ones and obtain the performance they provide in an interception scenario.

These analyses led to several relevant achievements, as indicated in section 13.1, like for example the relevance of the errors because of using a Lambert's problem solver versus the errors due to the chosen guidance algorithm (for conventional guidance), the adaptation of the PEG algorithm for the intercontinental missiles problem, or the comparison of exoatmospheric ascent guidance algorithms performed in section 8.3.6.

- Optimal guidance algorithms for the interception problem were developed and implemented (see chapter 10 in part III).

This development led to relevant implementations like the use of the forces obtained from the optimal algorithms not as forces to be directly commanded, but as an input to compute reference guidance angles. This approach allowed keeping a high linearity when implementing the optimal problems, and allowed avoiding at the same time instability problems.

- The behaviour of conventional guidance algorithms and optimal guidance algorithms has been compared (see chapter 11 in part III) and analysed (see chapter 12 in part IV).

In terms of accuracy and required fuel consumption it has been concluded that when no noise is considered in the target state estimation, the combination of conventional ascent guidance (PEG) plus optimal terminal guidance is the most convenient algorithm to be used (compared with a pure conventional guidance algorithm the achieved miss distance is slightly better and the fuel consumption is much smaller with an average improvement in the fuel consumption of about a 28 % in the case of the ICBM heading to NYC, and an average improvement of about 4 % in the case of the ICBM heading to LA).

In terms of interception time, using optimal guidance algorithms during the ascent phase is the best choice (an improvement of more than 2 minutes is achieved in the case of the ICBM heading to NYC and in the case of the ICBM heading to LA an improvement of 3.5 minutes is achieved). However these algorithms require much more fuel for the divert thrusters of what is available in the EKV, so a change in the present configuration of the interceptor missile to allow that the EKV is released at the injection point with a much higher fuel capacity would be required.

- The feasibility of the Ballistic Missile Defense system in terms of guidance has been assessed (see chapter 12 in part IV).

It has been concluded that the system is feasible and that the main problems are not in the guidance algorithms but in problems that would jeopardize their correct application, especially related to the reaction time, the accuracy of the sensors, the correct management of the system delays and, especially, to the problem of the discrimination of warheads from decoys.

As a secondary objective the topic of Ballistic Missile Defense has been analysed and briefly depicted to the reader.

This system is the focus of political controversy but nevertheless it is very interesting from the technical point of view since it tackles with problems of incredible complexity, trying to destroy an incoming missile launched thousands of kilometers away and traveling at almost orbital speed with another missile that has to hit it directly.

13.3 Possible future research

Since the National Missile Defense (NMD) is a very complex system of systems, there are many aspects that could be included in future studies. Among them:

- Optimal algorithms applied during the ascent phase lead to a fuel consumption that is not compatible with the present configuration of the interceptor missile. These algorithms also lead to a much faster interception (an improvement of more than 2 minutes is achieved in the case of the ICBM heading to NYC and in the case of the ICBM heading to LA an improvement of 3.5 minutes is achieved), so this change in the configuration to allow that the EKV is released at the injection point with a much higher fuel capacity could be a very convenient choice.

It would be interesting to perform a reconfiguration of the GBI missile and the EKV to allow this increase in the fuel capacity.

- As indicated in chapter 2, Boost-Phase Interception based on surface-based interceptors is nowadays considered as unpractical (see pages 4-13 of [3]).

An alternative based on airborne interceptors was first considered in reference [4] (Wilkening, 2004) and later analysed in detail by Paul Zarchan in [5] (Zarchan, 2011).

It can be argued that this approach is a breach to the Outer Space Treaty (1967) that bars states party to the treaty from placing nuclear weapons or any other weapons of mass destruction in the orbit of the Earth, installing them on the Moon or any other celestial body, or to otherwise station them in outer space (see chapter 2), but being the EKV's purely kinetic vehicles without any kind of explosive inside, it can also be argued that having them in orbit is not a breach of any kind.

This approach would allow trying to destroy the ICBMs in their boost phase, when they are easy to detect and when decoys are difficult to employ as countermeasures.

A deep analysis of this Ballistic Missile configuration constitutes a whole new area of research.

- It would be interesting to implement for this interception problem optimal guidance algorithms based on non linear methods like:
 - Sliding mode
 - Non linear programming (NLP)
 - State Dependent Riccati Equation (SDRE)
 - Control Lyapunov Functions
- It would also be interesting to implement guidance algorithms for this interception problem based on genetic algorithms.

These interesting topics of future research imply each of them an extensive work that could be the topic of several PhD theses. These researches could benefit from the use of the interception simulator developed for this thesis.

Chapter 13 references

- [1] Eun-Jung Song, Sang bum Cho, and Woong-Rae Roh. A comparison of iterative explicit guidance algorithms for space launch vehicles. *Advances in Space Research*, 55:463–476, January 2015. doi: 10.1016/j.asr.2014.09.025. [314](#)
- [2] Greg A. Dukeman. *Closed-Loop Nominal and Abort Atmospheric Ascent Guidance for Rocket-Powered Launch Vehicles*. PhD thesis, Georgia Institute of Technology, May 2005. [315](#)
- [3] Committee on an Assessment of Concepts and Systems for U.S. Boost-Phase Missile Defense in Comparison to Other Alternatives. Division on Engineering and Physical Sciences. *Making Sense of Ballistic Missile Defense*. The National Academies Press, Washington DC, first edition, 2012. [318](#)
- [4] Dean A. Wilkening. Airborne Boost-Phase Ballistic Missile Defense. *Science and Global Security*, 12(2):1–67, June 2004. doi: 10.1080/08929880490464649. [318](#)
- [5] Paul Zarchan. Kill Vehicle Guidance and Control Sizing for Boost-Phase Intercept. *Journal of Guidance, Control, and Dynamics*, 34(2):513–521, March-April 2011. doi: 10.2514/1.50927. [318](#)

This page intentionally left blank.

Appendices

This page intentionally left blank.

Appendix A

Frames of reference

This appendix details all the frames of reference to be used in the simulation of the ICBM and its interceptor.

Also, the most important formulae for changing from one frame to another are detailed.

The expressions indicated in this appendix are used in chapters [3](#), [4](#), [5](#), [7](#), [9](#) and [10](#).

A.1 ECI (Earth-Centered Inertial)

This is an inertial Cartesian coordinate system, with origin in the centre of mass of the Earth, and fixed orientation with respect to fixed stars. The Oz axis is parallel to the rotation axis of the Earth, while the Ox axis is directed towards the vernal point.

ECI coordinate frames are not truly inertial since the Earth itself is accelerating as it travels in its orbit about the Sun but in many cases it may be assumed that the ECI frame is inertial without adverse effect.

For the purposes of this thesis, the influence of the acceleration of the centre of mass of the Earth on the dynamics of the considered missiles will be neglected and as a consequence these axes will be considered as inertial.

Figure A.1: ECI frame

Since the rotation axis of the Earth wobbles due to gravitational perturbations (precession and nutation), the equatorial plane of the Earth is constantly changing over time. Thus, the vernal equinox and the equatorial plane of the Earth vary according to date and are specified for a particular epoch. Since there are also short-term periodic oscillations in the rotation axis of the Earth (polar motion) it is usual to provide a mean position within a certain amount of time, instead of the true position, which would only apply for an instant.

One commonly used ECI frame is defined with the Earth's Mean Equator and Equinox at 12:00 Terrestrial Time on 1 January 2000 (Epoch J2000). It can be referred to as EME2000 and it is referred as Conventional Inertial frame (CIS). This is in fact the "inertial" reference frame considered in document [1] from 1987.

The IAU decided to refine the definition of the inertial reference frame in 1997 and 2000 so the inertial reference frame now considered by the international community is different.

However, we will still use the ECI frame defined in [1] in this thesis since the transformation equations from ECI to ECEF, detailed in [1] are easier than the ones to be used using the IAU inertial reference system, as detailed in [2], and achieving a greater accuracy in this transformation is not the objective of this thesis.

A.2 ECEF (Earth-Centered Earth-Fixed)

This coordinate system is similar to ECI, but in this case the axes are fixed to the Earth, and rotate with it.

The WGS84 coordinate system will be used. This frame is set with the following criteria:

- The origin is located in the center of mass of the Earth including oceans and atmosphere.
- The OZ -axis is in the direction of the IERS reference pole (IRP).
- The OX -axis is in the intersection of the IERS reference meridian (IRM) and the plane passing through the origin and normal to the OZ axis.
- The OY -axis completes a right handed Earth-Centered Earth-Fixed (ECEF, also called CTS: Conventional Terrestrial System) orthogonal coordinate system.

Figure A.2: ECEF frame

A.3 Conversion between ECI coordinates and ECEF coordinates

This conversion is quite complex, since it takes into account different motions of the Earth. It can be argued that taking into account the times of motion of the missiles this conversion is irrelevant, but it is included so that for any launching date for the missiles, the transformation matrices are correct.

The required transformations indicated herein are derived from [1]. There are 4 required steps:

1. Conversion between CIS coordinates and Mean Earth-Centered Inertial of Date coordinates
This transformation changes the axes from the Mean Inertial frame of J2000.0 ($X_1Y_1Z_1$) to the Mean Earth-Centered Inertial of the required Date ($X_2Y_2Z_2$). This is, the precession motion of the Earth from J2000.0 to the applicable date is taken into account.

2. Conversion between Mean Earth-Centered Inertial of Date coordinates ($X_2Y_2Z_2$) and Mean True Earth-Centered Inertial of Date coordinates ($X_3Y_3Z_3$).
The second required transformation takes into account the nutation motion of the Earth in order to transform from mean to true coordinates of date.

3. Conversion between Mean True Earth-Centered Inertial of Date coordinates ($X_3Y_3Z_3$) and True Earth-Centered Earth-Fixed coordinates ($X_4Y_4Z_4$).
This transformation takes into account the rotational motion of the Earth around its axis in order to change to Earth-Centered frames.

4. Conversion between True Earth-Centered Earth-Fixed coordinates ($X_4Y_4Z_4$) and Mean Earth-Centered Earth-Fixed coordinates (Conventional Terrestrial, CTS).
This transformation takes into account the movement of the rotation axis with respect to the Earth's crust. This is, the Polar motion.

A.3.1 Conversion between CIS coordinates and Mean Earth-Centered Inertial of Date coordinates

This transformation takes into account the precession motion of the Earth from the 1 January 2000 at 12 h (J2000) to the instant when the Mean Earth-Centered Inertial frame is required.

This transformation can be achieved by means of 3 Givens rotations ([1]):

1. A positive rotation about the CIS Z axis (Z_1) through the angle $(90^\circ - \zeta)$ resulting in the axes $X'_1 Y'_1 Z'_1$
2. A positive rotation about the X'_1 axis through the angle θ resulting in the axes $X''_1 Y''_1 Z''_1$
3. A negative rotation about the Z''_1 axis through the angle $(90^\circ + z)$ resulting in the axes $X_2 Y_2 Z_2$

The result of these rotations is provided in the following transformation matrix:

$$D = R_Z^*[-(90^\circ + z)] \cdot R_X^*[\theta] \cdot R_Z^*[90^\circ - \zeta] =$$

$$\begin{bmatrix} \cos z \cos \theta \cos \zeta - \sin z \sin \zeta & -\cos z \cos \theta \sin \zeta - \sin z \cos \zeta & -\cos z \sin \theta \\ \sin z \cos \theta \cos \zeta + \cos z \sin \zeta & -\sin z \cos \theta \sin \zeta + \cos z \cos \zeta & -\sin z \sin \theta \\ \sin \theta \cos \zeta & -\sin \theta \sin \zeta & \cos \theta \end{bmatrix} \quad (\text{A.1})$$

The parameters ζ , θ and z can be obtained as a function of time with the following equations according to [1]:

$$\begin{aligned} \zeta &= \frac{\pi}{648000} \cdot (2306.2181 \cdot T + 0.30188 \cdot T^2 + 0.017998 \cdot T^3) \quad \text{rad} \\ z &= \frac{\pi}{648000} \cdot (2306.2181 \cdot T + 1.09468 \cdot T^2 + 0.018203 \cdot T^3) \quad \text{rad} \\ \theta &= \frac{\pi}{648000} \cdot (2004.3109 \cdot T - 0.42665 \cdot T^2 - 0.041833 \cdot T^3) \quad \text{rad} \end{aligned} \quad (\text{A.2})$$

where T is the Julian time in centuries from:

$$T = \frac{JED - 2451545.0}{36525} \quad (\text{A.3})$$

A.3.2 Conversion between Mean Earth-Centered Inertial of Date coordinates and Mean True Earth-Centered Inertial of Date coordinates

This transformation takes into account the nutation motion of the Earth in order to transform from mean to true coordinates of date.

This transformation can be achieved by means of 3 Givens rotations ([1]):

1. A positive rotation about the X_2 axis through the angle $\bar{\epsilon}$ (mean obliquity of ecliptic) resulting in the axes $X'_2 Y'_2 Z'_2$
2. A negative rotation about the Z'_2 axis through the angle $\Delta\psi$ (nutation in longitude) resulting in the axes $X''_2 Y''_2 Z''_2$
3. A negative rotation about the Z''_2 axis through the angle ϵ (true obliquity of ecliptic) resulting in the axes $X_3 Y_3 Z_3$

The result of these rotations is provided in the following transformation matrix:

$$C = R_X^*[-\epsilon] \cdot R_Z^*[-\Delta\psi] \cdot R_X^*[\bar{\epsilon}] =$$

$$\begin{bmatrix} \cos \Delta\psi & -\sin \Delta\psi \cos \bar{\epsilon} & -\sin \Delta\psi \sin \bar{\epsilon} \\ \cos \epsilon \sin \Delta\psi & \cos \epsilon \cos \Delta\psi \cos \bar{\epsilon} + \sin \epsilon \sin \bar{\epsilon} & \cos \epsilon \cos \Delta\psi \sin \bar{\epsilon} - \sin \epsilon \cos \bar{\epsilon} \\ \sin \epsilon \sin \Delta\psi & \sin \epsilon \cos \Delta\psi \cos \bar{\epsilon} - \cos \epsilon \sin \bar{\epsilon} & \sin \epsilon \cos \Delta\psi \sin \bar{\epsilon} + \cos \epsilon \cos \bar{\epsilon} \end{bmatrix} \quad (\text{A.4})$$

The parameters $\bar{\epsilon}$, $\Delta\psi$ and ϵ can be obtained as a function of time with the following equations according to [1] from the 1980 IAU theory of nutation ([3] (Seidelmann, 1982)):

$$\epsilon = \bar{\epsilon} + \Delta\epsilon = \text{True Obliquity of ecliptic}$$

$$\bar{\epsilon} = \frac{\pi}{648000} \cdot (\epsilon_0 - 46.8150 \cdot T - 0.00059 \cdot T^2 + 0.001813 \cdot T^3) \quad \text{rad}$$

$$\text{with } \epsilon_0 = 23^\circ 26' 21.448'' = 84381.448 \quad \text{arc sec}$$

$$(\bar{\epsilon} = \text{mean obliquity of ecliptic}) \quad (\text{A.5})$$

$$\Delta\epsilon = \sum_{i=1}^{106} \Delta\epsilon_i = \text{Nutation in Obliquity}$$

$$\text{with } \Delta\epsilon_i = (C_i + D_i \cdot T) \cdot \cos(a_{1i} \cdot l + a_{2i} \cdot l' + a_{3i} \cdot F + a_{4i} \cdot D + a_{5i} \cdot \Omega)$$

$$\Delta\psi = \sum_{i=1}^{106} \Delta\psi_i = \text{Nutation in Longitude} \quad (\text{A.6})$$

$$\text{with } \Delta\psi_i = (A_i + B_i \cdot T) \cdot \sin(a_{1i} \cdot l + a_{2i} \cdot l' + a_{3i} \cdot F + a_{4i} \cdot D + a_{5i} \cdot \Omega)$$

where T is the Julian time in centuries (equation A.3).

The coefficients a_{1i} , a_{2i} , a_{3i} , a_{4i} and a_{5i} can be found in reference [1] for up to $i = 106$.

In the same way the coefficients A_i , B_i , C_i , D_i can be found in reference [1] in 0.0001 seconds of arc (they have to be multiplied by $0.0001 \cdot \frac{\pi}{648000}$ in order to convert $\Delta\psi$ and $\Delta\epsilon$ to radians).

The parameters ℓ , ℓ' , F , D and Ω are angles which are functions of the position of the Sun and the Moon and can be approximated using the following expressions ([1]):

$$\begin{aligned}
 \ell &= \frac{\pi}{648000} \cdot (485866.733 + (1325^r + 715922.633) \cdot T + 31.310 \cdot T^2 + 0.064 \cdot T^3) \quad \text{rad} \\
 &= \text{mean anomaly of the Moon} \\
 \ell' &= \frac{\pi}{648000} \cdot (1287099.804 + (99^r + 1292581.244) \cdot T - 0.577 \cdot T^2 - 0.012 \cdot T^3) \quad \text{rad} \\
 &= \text{mean anomaly of the Sun} \\
 F &= \frac{\pi}{648000} \cdot (335778.877 + (1342^r + 295263.137) \cdot T - 13.257 \cdot T^2 + 0.011 \cdot T^3) \quad \text{rad} \\
 &= \text{mean longitude of the Moon} \\
 D &= \frac{\pi}{648000} \cdot (1072261.307 + (1236^r + 1105601.328) \cdot T - 6.891 \cdot T^2 + 0.019 \cdot T^3) \quad \text{rad} \\
 &= \text{mean elongation of the Moon from the Sun} \\
 \Omega &= \frac{\pi}{648000} \cdot (450160.280 - (5^r + 482890.539) \cdot T + 7.455 \cdot T^2 + 0.008 \cdot T^3) \quad \text{rad} \\
 &= \text{longitude of the ascending node of the lunar mean orbit on the ecliptic} \\
 &\quad \text{measured from the mean equinox of date}
 \end{aligned} \tag{A.7}$$

where T is the Julian time in centuries (equation A.3).

In these expressions the superscript "r" means that the affected number is provided in radians and should be converted to seconds of arc multiplying by 1296000.

A.3.3 Conversion between Mean True Earth-Centered Inertial of Date coordinates and True Earth-Centered Earth-Fixed coordinates

This transformation takes into account the rotational motion of the Earth around its axis in order to change to Earth-Centered frames, going from the $X_3Y_3Z_3$ axes to the $X_4Y_4Z_4$ axes.

The transformation matrix consists of a positive rotation of an angle Λ about the Z_3 axis:

$$B = R_Z^*[\Lambda] = \begin{bmatrix} \cos \Lambda & \sin \Lambda & 0 \\ -\sin \Lambda & \cos \Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{A.8})$$

Λ is the longitude of the Greenwich Meridian from the true vernal equinox of date (where the axis OX_3 points). It can be represented as the addition of the Greenwich Mean Sidereal Time (GMST) at the beginning of the day (H_0), plus the angle of the apparent minus mean sidereal time (ΔH , equation of the equinoxes), plus the rotation rate in a precessing reference frame (ω^*) multiplied by the UTC time within the day (t) minus a correction factor (Δt) that corrects for the irregular rotation of the Earth (difference between the coordinated time and the mean solar time UT1).

As a summary Λ can be obtained from the following equation ([1]):

$$\Lambda = H_0 + \Delta H + \omega^* \cdot (t - \Delta t) \quad \text{seconds} \quad (\text{A.9})$$

with:

$$\begin{aligned} \Delta H &= \arctan(\cos \epsilon \tan \Delta \psi) \\ (\text{equation of the equinoxes} &= \Upsilon_{\text{true equinox of the date}} - \Upsilon_{\text{mean equinox of the date}}) \\ \epsilon &= \text{True obliquity, see equation A.5} \\ \Delta \psi &= \text{Nutation in longitude, see equation A.6} \\ H_0 &= 24110.54841 + 8640184.812866 \cdot T_u + 0.093104 \cdot T_u^2 - 6.2 \cdot 10^{-6} \cdot T_u^3 \quad \text{seconds} \\ t &= \text{time within day (UTC) in seconds} \\ \Delta t &= UTC - UT1 \quad \text{in seconds} \end{aligned} \quad (\text{A.10})$$

The rotation rate in a precessing reference frame (ω^*) is the sum of the Earth's inertial rotation rate and the rate of precession in the right ascension of the mean equinox (the Earth rotates to the East while the mean equinox precesses to the west):

$$\omega^* = \omega' + m = 7.2921158553 \cdot 10^{-5} + 4.3 \cdot 10^{-15} \cdot T_u \quad \text{rad/s} \quad (\text{A.11})$$

with:

$$\begin{aligned} \omega' &= \text{Earth's inertial rotation rate} = 7.2921151467 \cdot 10^{-5} \quad \text{rad/s} \\ m &= \text{rate of precession in right ascension} = 7.086 \cdot 10^{-12} + 4.3 \cdot 10^{-15} \cdot T_u \quad \text{rad/s} \end{aligned} \quad (\text{A.12})$$

In these equations:

$$T_u = \frac{d_u}{36525} \quad (\text{A.13})$$

with:

$$d_u = JED - 2451545 \quad \text{at } 0^h \text{ UT1} \quad (\text{A.14})$$

d_u must end in an 0.5 because the Julian day starts at noon.

Values for $\Delta t = UTC - UT1$ are given by the United States Naval Observatory (USNO) in [4].

For recent or future moments with respect to the present day when launching the simulations, data from the Bulletin A from [4] will be used within this thesis. This bulletin contains $UT1 - UTC$ data and their errors at daily intervals and predictions for 1 year into the future.

For past moments with respect to the present day when launching the simulations the data from Bulletin B from [4] will be used. This bulletin reports the "final" determinations for $UT1 - UTC$ at five-day intervals. Smoothed one-day values are also provided. IERS Bulletin B is updated monthly.

These data (bulletin A and bulletin B) is combined in the file "finals.all", also available in [4] that contains all the $UT1 - UTC$ data since 02 January 1973 with 1 year of predictions.

Taking into account equation A.9 it is possible to obtain the derivative of matrix B from equation A.8:

$$\begin{aligned} \Lambda &= H_0 + \Delta H + \omega^* \cdot (t - \Delta t) \quad \text{seconds} \\ \implies \dot{\Lambda} &= \omega^* \end{aligned} \quad (\text{A.15})$$

$$\begin{aligned} B &= R_Z^*[\Lambda] = \begin{bmatrix} \cos \Lambda & \sin \Lambda & 0 \\ -\sin \Lambda & \cos \Lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \implies \dot{B} &= \begin{bmatrix} -\sin \Lambda \cdot \omega^* & \cos \Lambda \cdot \omega^* & 0 \\ -\cos \Lambda \cdot \omega^* & -\sin \Lambda \cdot \omega^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (\text{A.16})$$

Since apart from the Earth's rotation all the other motions involved in the transformations required in section A.3 (precession, nutation and polar motion) are very slow (precession and nutation) or hard to predict (polar motion), $\dot{B}$ will be the only derivative considered for the transformation matrices and the rest (A , C , D) will be considered as constant with time once computed for the initial time of the missile motion.

A.3.4 Conversion between True Earth-Centered Earth-Fixed coordinates and Mean Earth-Centered Earth-Fixed coordinates (CTS)

This transformation takes into account the movement of the rotation axis (CEP) with respect to the Earth's crust. This is, the Polar motion. This transformation goes from True Earth-Centered Earth-Fixed coordinates ($X_4Y_4Z_4$) to CTS coordinates ($X_5Y_5Z_5$):

- The X_5 axis is in the mean astronomic equator, and it is positive toward the Zero Meridian.
- The Z_5 axis is perpendicular to the CTS equator and positive north toward the Conventional Terrestrial Pole (CTP) as defined by the BIH (Bureau International de l'Heure or International Time Bureau), on the basis of the coordinates adopted for the BIH stations.
- The Y_5 axis is in the plane of the CTS equator completing a right-handed orthogonal coordinate system.

Two rotations are required:

1. A negative rotation about the X_4 axis through the angle y_p
 y_p is the angular displacement of the CEP from the mean terrestrial pole measured normal to the Zero Meridian (positive West).
2. A negative rotation about the Y_4 axis through the angle x_p
 x_p is the angular displacement of the CEP from the mean terrestrial pole measured along the Zero Meridian (positive South).

No intermediate $X'_4Y'_4Z'_4$ is required since x_p and y_p are small angles. Also, approximations can be used for the sine and cosine of the angles.

The result of these rotations is provided in the following transformation matrix:

$$A = R_Y^*[-x_p] \cdot R_X^*[-y_p] = \begin{bmatrix} \cos x_p & -\sin x_p \cdot \sin y_p & \sin x_p \cdot \cos y_p \\ 0 & \cos y_p & -\sin y_p \\ -\sin x_p & \cos x_p \cdot \sin y_p & \cos x_p \cdot \cos y_p \end{bmatrix} \quad (\text{A.17})$$

Values for x_p and y_p are given by the United States Naval Observatory (USNO) in [4].

For recent or future moments with respect to the present day when launching the simulations, data from the Bulletin A from [4] will be used within this thesis. This bulletin contains Earth orientation parameters x_p , y_p and their errors at daily intervals and predictions for 1 year into the future.

For past moments with respect to the present day when launching the simulations the data from Bulletin B from [4] will be used. This bulletin reports the "final" determinations for polar motion at five-day intervals. Smoothed one-day values are also provided. IERS Bulletin B is updated monthly.

These data is combined in the file "finals.all", also available in [4] that contains all the x_p , y_p data since 02 January 1973 with 1 year of predictions.

A.3.5 Summary of transformations ECI-ECEF

Once all the transformation matrices computed in sections A.3.1, A.3.2, A.3.3 and A.3.4 have been generated for a certain time, the global transformation from ECI axes to ECEF (CTS) axes can be performed in the following way:

$$\begin{aligned}\vec{r}_{ECEF} &= [A \cdot B \cdot C \cdot D] \cdot \vec{r}_{ECI} \\ \vec{r}_{ECI} &= [A \cdot B \cdot C \cdot D]^T \cdot \vec{r}_{ECEF}\end{aligned}\tag{A.18}$$

Since $\dot{B}$ is the only derivative considered for the transformation matrices, the derivatives would be:

$$\begin{aligned}\dot{\vec{r}}_{ECEF} &= [A \cdot B \cdot C \cdot D] \cdot \dot{\vec{r}}_{ECI} + [A \cdot \dot{B} \cdot C \cdot D] \cdot \vec{r}_{ECI} \\ \dot{\vec{r}}_{ECI} &= [A \cdot B \cdot C \cdot D]^T \cdot \dot{\vec{r}}_{ECEF} + [A \cdot \dot{B} \cdot C \cdot D]^T \cdot \vec{r}_{ECEF}\end{aligned}\tag{A.19}$$

A.4 Geodetic coordinates

The position of the missiles will be given in the ECEF frame of reference, using Cartesian coordinates (that is, using directly the x, y, z coordinates) and also using geodetic coordinates: latitude, longitude and height with respect to the WGS84 ellipsoid of reference.

The WGS84 ECEF coordinate system origin is also the origin of the ellipsoid and the OZ axis serves as the rotational axis of the ellipsoid.

The following notation will be used:

- φ : geodetic latitude
- λ : longitude
- h : height

The definition of these parameters is shown in the following figure:

Figure A.3: Geodetic coordinates

The ellipsoid is completely determined by its semi-major axis a and its semi-minor axis b .

The WGS84 model provides a value for these magnitudes and for several derived magnitudes (reference [5]):

$$a = 6378137.0$$

$$b = 6356752.3142$$

$$e^2 = 1 - (b^2/a^2) = 6.69437999014 \cdot 10^{-3}$$

$$1/f = \frac{a}{a-b} = 298.257223563$$

A.5 Conversion between geodetic coordinates and ECEF Cartesian coordinates

A.5.1 From geodetic coordinates to ECEF coordinates

In order to transform from the geodetic latitude to the ECEF coordinates a special care must be taken since the ellipsoid is a pole-flattened (oblate) spheroid.

A vertical plane within this ellipsoid will produce an ellipse:

Figure A.4: Intersection of the ellipsoid with a vertical plane

Because of the flattened shape of the ellipsoid the perpendicular in a surface point to the local horizon does not reach the center.

As a consequence we can distinguish among:

- φ : geodetic latitude (the one that is commonly used)
- φ' : geocentric latitude
- β : reduced latitude

Figure A.5: Latitude angles

The angle β is the eccentric anomaly of the ellipse so the following expressions apply:

$$\begin{aligned} X &= a \cdot \cos \beta \\ Z &= b \cdot \sin \beta = a \cdot \sqrt{1 - e^2} \cdot \sin \beta \end{aligned} \tag{A.20}$$

Taking into account the definition of the geodetic latitude (measured from the perpendicular line to the local horizon) there will be a relationship between this angle and the step of the curve (the angle of the tangent of the ellipse with the Ox axis), this is, with the derivative of the ellipse in that point:

Figure A.6: Interpretation of the geodetic latitude in terms of derivatives

From figure A.6 the following equation can be deduced:

$$\tan \varphi = -\frac{dX}{dZ} = -\frac{-a \cdot \sin \beta}{a \cdot \sqrt{1-e^2} \cdot \cos \beta} \Rightarrow \tan \beta = \sqrt{1-e^2} \cdot \tan \varphi \quad (\text{A.21})$$

From equation A.21 and since $\sin^2 \beta + \cos^2 \beta = 1$ we can obtain the following relationships:

$$\begin{aligned} \cos \beta &= \frac{1}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} \cdot \cos \varphi \\ \sin \beta &= \frac{\sqrt{1-e^2}}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} \cdot \sin \varphi \end{aligned} \quad (\text{A.22})$$

Using these expressions in equation A.20 we have:

$$\begin{aligned} X &= a \cdot \cos \beta = \frac{a}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} \cdot \cos \varphi \\ Z &= a \cdot \sqrt{1-e^2} \sin \beta = \frac{a \cdot (1-e^2)}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} \cdot \sin \varphi \end{aligned} \quad (\text{A.23})$$

If the considered point is at a height h with respect to the Earth surface the following expressions have to be added (the height is measured in the perpendicular to the local horizon):

$$\begin{aligned} \Delta X &= h \cdot \cos \varphi \\ \Delta Z &= h \cdot \sin \varphi \end{aligned} \quad (\text{A.24})$$

As a consequence it can be concluded that in a vertical plane of the ellipsoid the following formulae apply:

$$\begin{aligned} X &= \left(\frac{a}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} + h \right) \cdot \cos \varphi \\ Z &= \left(\frac{a \cdot (1-e^2)}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} + h \right) \cdot \sin \varphi \end{aligned} \quad (\text{A.25})$$

X can be considered as the radius of a local circumference in a horizontal plane so we only need the longitude to provide the right position of the object within this plane.

Defining:

$$N(\varphi) := \frac{a}{\sqrt{1-e^2 \cdot \sin^2 \varphi}} \quad (\text{A.26})$$

the conversion is given by:

$$\begin{aligned} x &= (N(\varphi) + h) \cdot \cos \varphi \cdot \cos \lambda \\ y &= (N(\varphi) + h) \cdot \cos \varphi \cdot \sin \lambda \\ z &= (N(\varphi) \cdot (1-e^2) + h) \cdot \sin \varphi \end{aligned} \quad (\text{A.27})$$

A.5.2 From ECEF coordinates to geodetic coordinates

The transformation from ECEF Cartesian coordinates to geodetic coordinates has been typically done by numerical methods.

The longitude can be directly obtained from:

$$\left. \begin{aligned} x &= (N(\varphi) + h) \cdot \cos \varphi \cdot \cos \lambda \\ y &= (N(\varphi) + h) \cdot \cos \varphi \cdot \sin \lambda \end{aligned} \right\} \implies \tan \lambda = \frac{y}{x} \implies \lambda = \arctan \left(\frac{y}{x} \right) \quad (\text{A.28})$$

The latitude can be obtained from the following expression:

$$\left. \begin{aligned} x &= (N(\varphi) + h) \cdot \cos \varphi \cdot \cos \lambda \\ y &= (N(\varphi) + h) \cdot \cos \varphi \cdot \sin \lambda \\ z &= (N(\varphi) \cdot (1-e^2) + h) \cdot \sin \varphi \end{aligned} \right\} \implies \tan \varphi = \frac{N(\varphi) + h}{N(\varphi) \cdot (1-e^2) + h} \cdot \frac{z}{\sqrt{x^2 + y^2}} \quad (\text{A.29})$$

Equation A.29 can be solved in an iterative way starting with the solution for $h = 0$:

$$\tan \varphi_0 = \frac{1}{(1-e^2)} \cdot \frac{z}{\sqrt{x^2 + y^2}} \quad (\text{A.30})$$

Finally the altitude can be obtained from:

$$\begin{aligned} \left. \begin{aligned} x &= (N(\varphi) + h) \cdot \cos \varphi \cdot \cos \lambda \\ y &= (N(\varphi) + h) \cdot \cos \varphi \cdot \sin \lambda \end{aligned} \right\} \implies \\ \sqrt{x^2 + y^2} &= (N(\varphi) + h) \cdot \cos \varphi \implies h = \frac{\sqrt{x^2 + y^2}}{\cos \varphi} - N(\varphi) \end{aligned} \quad (\text{A.31})$$

However it is possible to perform this transformation directly using algebraic methods that convert the latitude equation into a quadratic equation.

We will use for this purpose the equations indicated in reference [6] (Vermeille, 2000) and reference [7] (Vermeille, 2004) which are valid for any point which is further away from the Earth Center than 43 km.

The first step is to obtain the following parameters in a sequential way:

$$\begin{aligned}
 p &= \frac{x^2 + y^2}{a^2} \\
 q &= \frac{1 - e^2}{a^2} \cdot z^2 \\
 r &= \frac{p + q - e^4}{6} \\
 s &= \frac{e^4 \cdot p \cdot q}{4 \cdot r^3} \\
 t &= \sqrt[3]{1 + s + \sqrt{s(2 + s)}} \\
 u &= r \cdot \left(1 + t + \frac{1}{t}\right) \\
 v &= \sqrt{u^2 + e^4 \cdot q} \\
 w &= \frac{e^2 \cdot (u + v - q)}{2 \cdot v} \\
 k &= \sqrt{u + v + w^2} - w
 \end{aligned} \tag{A.32}$$

Once these parameters have been computed the geodetic coordinates can be obtained (Vermeille, 2004) using:

$$\begin{aligned}
 D = \frac{k \cdot (x^2 + y^2)}{k + e^2} &\implies \begin{cases} \varphi = 2 \cdot \arctan\left(\frac{z}{D + \sqrt{D^2 + z^2}}\right) \\ h = \frac{k + e^2 - 1}{k} \cdot \sqrt{D^2 + z^2} \end{cases} \\
 \begin{cases} \text{if } y \geq 0 \implies \lambda = \frac{\pi}{2} - 2 \cdot \arctan\left(\frac{x}{\sqrt{x^2 + y^2} + y}\right) \\ \text{if } y \leq 0 \implies \lambda = -\frac{\pi}{2} + 2 \cdot \arctan\left(\frac{x}{\sqrt{x^2 + y^2} - y}\right) \end{cases} & \tag{A.33}
 \end{aligned}$$

A.6 The Navigation frame

The Navigation frame is a frame based on the local horizon in the considered point. The Ox axis is tangent to the surface of the ellipsoid, pointing to the geographic North, the Oy axis is tangent to the surface of the ellipsoid pointing to the East direction and the Oz axis is perpendicular to the surface of the ellipsoid, and directed towards the centre of the Earth.

Consequently, the navigation frame is also called the NED frame, according to the orientation of the axes (North, East, Down).

Figure A.7: The Navigation frame

A.7 Conversion between ECEF coordinates and Navigation coordinates

The versors of the navigation axes can be easily obtained with respect to the ECEF versors.

The meridian ellipse and parallel circumference are obtained from the intersection of the ellipsoid with a vertical (for meridian) and horizontal (for parallel) plane:

Figure A.8: Meridian and parallel planes in the ellipsoid

The versors of the navigation axes are tangent to the meridian curve ($\vec{i}_n$), perpendicular to the meridian curve ($\vec{k}_n$) and tangent to the parallel curve ($\vec{j}_n$) at the point, so it is very easy to relate these versors to the ECEF versors if we draw the versors in the meridian and parallel planes.

The following relationships can be found within the parallel circumference:

Figure A.9: Versors in a parallel plane

The following relationships can be found within the meridian ellipse:

Figure A.10: Versors in a meridian plane

Taking figures A.9 and A.10 into account the following expressions are obtained:

$$\begin{aligned}
 \vec{i}_{North} &= -\sin \varphi \cdot \cos \lambda \cdot \vec{i}_e - \sin \varphi \cdot \sin \lambda \cdot \vec{j}_e + \cos \varphi \cdot \vec{k}_e \\
 \vec{j}_{East} &= -\sin \lambda \cdot \vec{i}_e + \cos \lambda \cdot \vec{j}_e \\
 \vec{k}_{Down} &= -\cos \varphi \cdot \cos \lambda \cdot \vec{i}_e - \cos \varphi \cdot \sin \lambda \cdot \vec{j}_e - \sin \varphi \cdot \vec{k}_e
 \end{aligned} \tag{A.34}$$

Equations B.5 and A.34 allow providing the transformation matrix between the ECEF frame and the Navigation frame:

$$C_e^n = \begin{bmatrix} -\cos \lambda \cdot \sin \varphi & -\sin \lambda \cdot \sin \varphi & \cos \varphi \\ -\sin \lambda & \cos \lambda & 0 \\ -\cos \lambda \cdot \cos \varphi & -\sin \lambda \cdot \cos \varphi & -\sin \varphi \end{bmatrix} \tag{A.35}$$

A.8 The Body frame

This reference frame is rigidly attached to the missile.

We will place this frame with the Ox axis pointing forward in the symmetry axis and the Oz axis pointing downwards on a normal attitude of flight. The Oy axis points to the right of the missile so as to make a right-handed frame. The Oy axis and the Oz axis have much more physical sense in the case of aircrafts. In the case of missiles with a rotational symmetry it is less important where to place these axes.

The origin of this frame will be located in the beginning of the missile at launch (the point with lowest height). It has to be noted that since the considered missiles have several stages this origin will be outside the missile in several phases of flight. This origin allows providing an X coordinate to the center of gravity of the missile, that varies during the flight.

Figure A.11: Body axes

The position of the origin of the body frame will be given in the navigation frame by geodetic coordinates.

The attitude of the body frame with respect to the navigation frame will be given using quaternions. That avoids the singularity with Euler angles at pitch angles of 90° .

A.9 Conversion between Navigation coordinates and Body coordinates

This conversion is detailed in chapter B.3 where the following expression (equation B.25) is provided as a function of Euler angles:

$$C_n^b = \begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \cos \theta \\ \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \cos \theta \end{bmatrix} \quad (\text{A.36})$$

At the same time the rotation from the navigation frame to the body-fixed frame is provided as a function of rotation quaternions in chapter B.4.2 (equation B.48) so that the following change of basis matrix can be obtained:

$$C_n^b = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_0q_3 + q_1q_2) & 2(-q_0q_2 + q_1q_3) \\ 2(-q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(q_0q_1 + q_2q_3) \\ 2(q_0q_2 + q_1q_3) & 2(-q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \quad (\text{A.37})$$

Appendix A references

- [1] Defense Mapping Agency, Department of Defense. *Technical Report: DMA TR 8350.2-A: Appendix: Transformation of ECI (CIS, Epoch J2000.0) Coordinates to WGS 84 (CTS, ECEF) Coordinates*. National Geospatial-Intelligence Agency, December 1987. URL <http://earth-info.nga.mil/GandG/publications/tr8350.2/tr8350.2-a/Appendix.pdf>. 325, 326, 327, 328, 329, 330
- [2] George H. Kaplan. *Circular No 179, The IAU Resolutions on Astronomical Reference Systems, Time Scales and Earth Rotation Models, Explanation and Implementation*. United States Naval Observatory, Washington DC, first edition, October 2005. 325
- [3] P. K. Seidelmann. 1980 IAU theory of nutation - The final report of the IAU Working Group on Nutation. *Celestial Mechanics*, 27:79–106, May 1982. doi: 10.1007/BF01228952. 328
- [4] United States Naval Observatory (USNO. IERS Rapid Service/Prediction Center for Earth Orientation Parameters. <http://maia.usno.navy.mil/>, July 2013. [web page accessed on 02/08/2013]. 331, 332
- [5] National Imagery and Mapping Agency, Department of Defense. *Technical Report: NIMA TR8350.2: World Geodetic System 1984. Its Definition and Relationship with Local Geodetic Systems*. National Geospatial-Intelligence Agency, amendment 1, third edition, January 2000. URL <http://earth-info.nga.mil/GandG/publications/tr8350.2/wgs84fin.pdf>. 334
- [6] Hugues Vermeille. Direct transformation from geocentric coordinates to geodetic coordinates. *Journal of Geodesy*, 76:451–454, 2000. doi: 10.1007/s00190-002-0273-6. 338
- [7] Hugues Vermeille. Computing geodetic coordinates from geocentric coordinates. *Journal of Geodesy*, 78:94–95, 2004. doi: 10.1007/s00190-004-0375-4. 338
- [8] Archie E. Roy. *Orbital motion*. Adam Hilger Ltd., Bristol, Great Britain, second edition, 1982.
- [9] Tomás Elices. *Introducción a la Dinámica Espacial*. Instituto Nacional de Técnica Aeroespacial, 1991.
- [10] Jay H. Lieske. Precession matrix based on IAU (1976) system of astronomical constants. *Astronomy and Astrophysics*, 73:282–284, June 1979.

Appendix B

Change in the reference frame

This appendix details the operations for the change in the reference frame using change of basis matrices, rotation matrices, Euler angles and quaternions.

The relationship between the different operators is given.

These formulae were scattered in many references, in many cases with mistakes or without explaining how they had been obtained, so it has been considered as necessary to include this appendix in order to provide all the formulae related to the change in the reference frame in a systematic way that allows checking their correctness.

The expressions indicated in this appendix are used in [chapter 3](#) and [chapter 8](#).

B.1 Change of basis matrix

The matrix that allows changing from a reference frame a to a reference frame b is written as C_a^b :

$$\vec{v}^b = C_a^b \cdot \vec{v}^a \quad (\text{B.1})$$

C_a^b can be built from the following equation:

$$v_x^b \vec{i}_b + v_y^b \vec{j}_b + v_z^b \vec{k}_b = v_x^a \vec{i}_a + v_y^a \vec{j}_a + v_z^a \vec{k}_a \quad (\text{B.2})$$

The basis versors of frame a can be written using the basis versors of b :

$$\begin{aligned} \vec{i}_a &= i_{ax}^b \vec{i}_b + i_{ay}^b \vec{j}_b + i_{az}^b \vec{k}_b \\ \vec{j}_a &= j_{ax}^b \vec{i}_b + j_{ay}^b \vec{j}_b + j_{az}^b \vec{k}_b \\ \vec{k}_a &= k_{ax}^b \vec{i}_b + k_{ay}^b \vec{j}_b + k_{az}^b \vec{k}_b \end{aligned} \quad (\text{B.3})$$

Substituting and grouping we get:

$$\vec{v}^b = v_x^b \vec{i}_b + v_y^b \vec{j}_b + v_z^b \vec{k}_b = \begin{Bmatrix} (v_x^a i_{ax}^b + v_y^a j_{ax}^b + v_z^a k_{ax}^b) \cdot \vec{i}_b \\ (v_x^a i_{ay}^b + v_y^a j_{ay}^b + v_z^a k_{ay}^b) \cdot \vec{j}_b \\ (v_x^a i_{az}^b + v_y^a j_{az}^b + v_z^a k_{az}^b) \cdot \vec{k}_b \end{Bmatrix} \quad (\text{B.4})$$

This means that the change of basis matrix can be expressed as a function of the components of the versors of the old reference frame in the new one:

$$\vec{v}^b = \begin{bmatrix} i_{ax}^b & j_{ax}^b & k_{ax}^b \\ i_{ay}^b & j_{ay}^b & k_{ay}^b \\ i_{az}^b & j_{az}^b & k_{az}^b \end{bmatrix} \cdot \vec{v}^a \implies C_a^b = \begin{bmatrix} i_{ax}^b & j_{ax}^b & k_{ax}^b \\ i_{ay}^b & j_{ay}^b & k_{ay}^b \\ i_{az}^b & j_{az}^b & k_{az}^b \end{bmatrix} \quad (\text{B.5})$$

Obviously, the change of basis can be done in a sequential way. This is, a final change of basis matrix can be obtained by multiplying intermediate change of basis matrices:

$$C_c^a = C_b^a \cdot C_c^b \quad (\text{B.6})$$

B.2 Rotation matrix

B.2.1 Rotation

From the mathematical point of view a rotation within a Euclidean space is a transformation in a plane or in space that describes the motion of a rigid body around a fixed point. It is a global isometric transformation of the space (together with translation and reflection). That means that the relative distances are kept after the transformation. Unlike translation, it keeps a fixed point unchanged. Unlike reflection, it keeps orientation of the space.

The rotation of a vector $\vec{q}$ about a unit vector $\vec{p}$ by an angle θ to give the vector $\vec{q}'$ is given by Rodrigues' rotation formula:

$$\vec{q}' = (1 - \cos \theta)(\vec{q} \cdot \vec{p})\vec{p} + \cos \theta \vec{q} + (\vec{p} \wedge \vec{q}) \sin \theta \quad (\text{B.7})$$

Figure B.1: Rotation of a vector about an axis

B.2.2 Definition of rotation matrix

Taking into account the definition of the skew matrices it can be easily checked that the following relationship holds:

$$\vec{p} \wedge \vec{q} = \tilde{p} \cdot \vec{q} \quad (\text{B.8})$$

It can also be deduced that:

$$\begin{aligned} (\vec{q} \cdot \vec{p})\vec{p} &= (q_1 p_1 + q_2 p_2 + q_3 p_3) \cdot \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \\ &= \begin{Bmatrix} q_1 p_1 p_1 + q_2 p_2 p_1 + q_3 p_3 p_1 \\ q_1 p_1 p_2 + q_2 p_2 p_2 + q_3 p_3 p_2 \\ q_1 p_1 p_3 + q_2 p_2 p_3 + q_3 p_3 p_3 \end{Bmatrix} = \begin{bmatrix} p_1 p_1 & p_2 p_1 & p_3 p_1 \\ p_1 p_2 & p_2 p_2 & p_3 p_2 \\ p_1 p_3 & p_2 p_3 & p_3 p_3 \end{bmatrix} \cdot \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = [\vec{p} \cdot \vec{p}^T] \cdot \vec{q} \end{aligned} \quad (\text{B.9})$$

As a consequence, we can rewrite Rodrigues' formula in a matrix form:

$$\vec{q}' = [(1 - \cos \theta)\vec{p} \cdot \vec{p}^T + I \cos \theta + \sin \theta \tilde{p}] \cdot \vec{q} \quad (\text{B.10})$$

Equation B.10 means that any rotation of a vector $\vec{q}$ about a unit vector $\vec{p}$ by an angle θ can be expressed as the multiplication of the original vector by a rotation matrix R :

$$\vec{q}' = R \cdot \vec{q} \quad (\text{B.11})$$

with:

$$R = [(1 - \cos \theta)\vec{p} \cdot \vec{p}^T + I \cos \theta + \sin \theta \tilde{p}] \quad (\text{B.12})$$

B.2.3 Rotation matrix and change of basis matrix

Consider a rotation given by the matrix R that rotates a reference frame r into a rotated frame b . The coordinates of the rotation of the versors of r will generate 3 versors in the rotated frame that can be used as a basis for that frame. These rotated versors will satisfy:

$$\begin{aligned}\vec{i}_b &= R \cdot \vec{i}_r = R \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_{11} \\ R_{21} \\ R_{31} \end{pmatrix} \\ \vec{j}_b &= R \cdot \vec{j}_r = R \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R_{12} \\ R_{22} \\ R_{32} \end{pmatrix} \\ \vec{k}_b &= R \cdot \vec{k}_r = R \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} R_{13} \\ R_{23} \\ R_{33} \end{pmatrix}\end{aligned}\tag{B.13}$$

so we have that the rotation matrix is actually:

$$R = [\vec{i}_b^r \mid \vec{j}_b^r \mid \vec{k}_b^r]\tag{B.14}$$

This means, according to equation B.5, that the rotation matrix that rotates a reference frame r into a rotated frame b is the matrix that changes coordinates from the rotated frame b to the reference frame r :

$$R(\text{rotates from } r \text{ to } b) = C_b^r(\text{changes from } b \text{ to } r)\tag{B.15}$$

B.2.4 Givens rotation matrices

A Givens rotation matrix $R_\Gamma(\theta)$ is a rotation matrix related to a simple rotation of an angle θ about one of the main axes ($\Gamma = X, Y$ or Z) of the initial frame.

The 3 possible Givens rotation matrices in dimension 3 can be easily obtained applying equation B.14:

$$R_X(\theta) = C_{X'Y'X'}^{XYZ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}\tag{B.16}$$

$$R_Y(\theta) = C_{X'Y'X'}^{XYZ} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}\tag{B.17}$$

$$R_Z(\theta) = C_{X'Y'X'}^{XYZ} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\tag{B.18}$$

It has to be noted that certain references (as [1] in appendix A) define the Givens rotations matrices inverting the sign of the sines in equations B.16, B.17 and B.18 (rotating in the opposite sense as R). We will denote as R^* the Givens rotations matrices defined this way.

With this definition equation B.15 does not apply and the following expression has to be used:

$$R^*(\text{rotation from r to b}) = C_r^b(\text{changes from r to b}) \quad (\text{B.19})$$

B.2.5 Composition of Givens rotation matrices

The composition of Givens rotation matrices in order to obtain a global rotation matrix has to be done in a sequential way in the same order as the rotations applied.

This is, if we initially have a reference frame 0 that is rotated by $R_X(\theta)$ into frame 1 and then frame 1 is rotated by $R_Y(\lambda)$ into frame 2 the rotation matrix representing the global rotation would be:

$$R = R_X(\theta) \cdot R_Y(\lambda) \quad (\text{B.20})$$

This can be easily demonstrated. We know from B.15 that $R_X(\theta) = C_1^0$ and $R_Y(\lambda) = C_2^1$. Since it also has to be that $R = C_2^0$ and taking equation B.6 into account then:

$$R = C_2^0 = C_1^0 \cdot C_2^1 = R_X(\theta) \cdot R_Y(\lambda) \quad (\text{B.21})$$

When the other definition of Givens matrices is used the composition of rotations have to be applied starting with the last one. This is, if we initially have a reference frame 0 that is rotated by $R_X^*(\theta)$ into frame 1 and then frame 1 is rotated by $R_Y^*(\lambda)$ into frame 2 the rotation matrix representing the global rotation would be:

$$R^* = R_Y^*(\lambda) \cdot R_X^*(\theta) \quad (\text{B.22})$$

This can be demonstrated in a similar way as before. We know from B.19 that $R_X^*(\theta) = C_0^1$ and $R_Y^*(\lambda) = C_1^2$. Since it also has to be that $R^* = C_0^2$ and taking equation B.6 into account then:

$$R^* = C_0^2 = C_1^2 \cdot C_0^1 = R_Y^*(\lambda) \cdot R_X^*(\theta) \quad (\text{B.23})$$

B.3 Euler angles

Since rotation matrices must satisfy $R \cdot R^T = I$ any 3 dimensional rotation matrix is constrained with 6 equations and only has 3 degrees of freedom.

These degrees of freedom can be expressed as a sequence of rotation matrices using 3 sequential Givens rotations.

This means that the attitude of a rigid body, given by a general rotation from a frame of reference to a frame fixed to the body, can always be achieved by composing three elemental rotations. Since the composition of rotations is not commutative the order matters.

In this document this will be used for rotating from the navigation frame (see chapter A.6) to the body-fixed frame, and the order to be used will be:

1. $\psi := \text{Yaw}$: rotation about the Z_0 axis from the frame of reference 0 (navigation frame) to the intermediate frame 1 ($R_Z(\psi)$)
2. $\theta := \text{Pitch}$: rotation about the Y_1 axis from the intermediate frame 1 to the intermediate frame 2 ($R_Y(\theta)$)
3. $\phi := \text{Roll}$: rotation about the X_2 axis from the intermediate frame 2 to the body fixed frame 3 ($R_X(\phi)$)

Figure B.2: Euler angles definition

The resulting rotation matrix would be, according to section B.2.5:

$$R = R_Z(\psi) \cdot R_Y(\theta) \cdot R_X(\phi) =$$

$$\begin{bmatrix} \cos \theta \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \cos \theta \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi \\ -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix} \quad (\text{B.24})$$

This rotation matrix is, according to equation B.15, the change of basis matrix C_b^n so the matrix C_n^b would be:

$$C_n^b = \begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \cos \theta \\ \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \cos \theta \end{bmatrix} \quad (\text{B.25})$$

It has to be noted that this representation of the attitude has singularities when the pitch equals 90° (several combinations of roll and yaw lead to the same attitude).

B.4 Rotation quaternions

B.4.1 Basic formula

Rodrigues' formula can be written using quaternions. In order to do this we should first recall that we can think of a vector as a pure quaternion:

$$\begin{aligned} p &= 0 + \vec{p} \\ q &= 0 + \vec{q} \end{aligned} \tag{B.26}$$

When multiplying pure quaternions the following formula holds:

$$pq = -\vec{p} \cdot \vec{q} + \vec{p} \wedge \vec{q} \implies \begin{aligned} \vec{p} \cdot \vec{q} &= \frac{1}{2}(p^*q + q^*p) \\ \vec{p} \wedge \vec{q} &= \frac{1}{2}(pq - q^*p^*) \end{aligned} \tag{B.27}$$

Also, for pure quaternions the following formulae hold:

$$\begin{aligned} \vec{q}^* &:= -\vec{q} \\ \vec{q}^{-1} &:= \frac{-\vec{q}}{|\vec{q}|^2} \end{aligned} \tag{B.28}$$

Using B.27 and equations B.28 inside Rodrigues' formula (B.7) we get:

$$\begin{aligned} \vec{q}' &= (1 - \cos \theta)(\vec{q} \cdot \vec{p})\vec{p} + \cos \theta \vec{q} + (\vec{p} \wedge \vec{q}) \sin \theta = \\ &= (1 - \cos \theta) \cdot \frac{1}{2}(-pq - qp) \cdot p + \cos \theta \cdot q + \frac{1}{2}(pq - qp) \sin \theta = \\ &= \frac{1 + \cos \theta}{2}q + \frac{1}{2}(pq - qp) \sin \theta - \frac{1 - \cos \theta}{2}pqp \end{aligned} \tag{B.29}$$

which is a bilinear function on p .

This bilinear function can be factorized in the following way:

$$q' = (a + bp)q(c + dp) \tag{B.30}$$

where:

$$\begin{aligned} ac &= \frac{1 + \cos \theta}{2} & bc &= \frac{\sin \theta}{2} \\ ad &= -\frac{\sin \theta}{2} & bd &= -\frac{1 - \cos \theta}{2} \end{aligned} \tag{B.31}$$

This system of equations has multiple solutions since once a solution (a, b, c, d) is obtained we could get infinite solutions $(a' = \lambda a, b' = \lambda b, c' = \frac{1}{\lambda}c, d' = \frac{1}{\lambda}d)$.

In order to limit the number of solutions we will impose the additional condition:

$$|c + d\vec{p}| = 1 \tag{B.32}$$

Since we already have $|p| = 1$ this is equivalent to imposing that there exists some α such that c and d are:

$$c = \cos \alpha \quad d = \sin \alpha \tag{B.33}$$

Introducing these equalities in B.31 and solving we get:

$$a = c = \cos \frac{\theta}{2} \quad b = -d = \sin \frac{\theta}{2} \quad (\text{B.34})$$

And finally in B.33:

$$0 + \vec{q}' = \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \vec{p} \right) (0 + \vec{q}) \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \vec{p} \right) \quad (\text{B.35})$$

As a summary, given a unit vector $\vec{p}$ and an angle θ we can construct the following rotation quaternion:

$$r = \left[\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \vec{p} \right] \quad (\text{B.36})$$

and the rotation of a pure quaternion $\vec{q}$ will be given by:

$$q' = rqr^* \quad (\text{B.37})$$

If we compose a rotation given by r_1 followed by a rotation given by r_2 the result is:

$$\left. \begin{array}{l} q'_1 = r_1 \cdot q \cdot r_1^* \\ q'_2 = r_2 \cdot q'_1 \cdot r_2^* \end{array} \right\} \implies q'_2 = r_2 \cdot r_1 \cdot q \cdot r_1^* \cdot r_2^* \quad (\text{B.38})$$

so the composition of a rotation given by r_1 followed by a rotation r_2 is given by the quaternion $r_2 r_1$.

It has to be noticed that when defining the rotation quaternion, since the vector components are associated to a reference frame, the quaternion components are associated to the same reference frame, so when multiplying quaternions that represent rotations they must be put in a common reference frame:

$$q'^1 = r_2^1 \cdot r_1^1 \cdot q^1 \cdot r_1^{*1} \cdot r_2^{*1} \quad (\text{B.39})$$

Equation B.39 can be arranged in such a way that each rotation can be expressed in different axes.

Since there is a relation between change of basis matrix and rotation matrix (see equation B.15) we know that:

$$\vec{a}^1 = C_2^1 \cdot \vec{a}^2 = R_{2 \rightarrow 1}^T \cdot \vec{a}^2 = R_{1 \rightarrow 2} \cdot \vec{a}^2 = r_1^1 \cdot \vec{a}^2 \cdot r_1^{*1} \quad (\text{B.40})$$

That means that if we consider $\vec{r}_2^1$ as a vector we can use the same formula and apply:

$$\vec{r}_2^1 = r_1^1 \cdot \vec{r}_2^2 \cdot r_1^{*1} \quad (\text{B.41})$$

so the quaternion that represents a composed rotation can be written as:

$$r_1 = r_2^1 \cdot r_1^1 = r_1^1 \cdot r_2^2 \cdot r_1^{*1} \cdot r_1^1 = r_1^1 \cdot r_2^2 \quad (\text{B.42})$$

B.4.2 Quaternions and the rotation matrix

The multiplication of two quaternions is given by:

$$\begin{aligned} r = p \cdot q = & (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2) \cdot i \\ & + (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1) \cdot j + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0) \cdot k \end{aligned} \quad (\text{B.43})$$

This can be represented using matrices:

$$r = p \cdot q = P_L q = Q_R p \quad (\text{B.44})$$

where the L and R indices denote whether the element is at the left or at the right of the multiplication. P_L and Q_R are:

$$P_L = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \quad Q_R = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \quad (\text{B.45})$$

Using equation B.44 we can write the rotation of a pure quaternion p using a rotation quaternion q as:

$$p' = qpq^* = Q_L Q_R^* p \quad (\text{B.46})$$

Performing the matrix multiplication and using the fact that $|q| = 1$ we get:

$$\begin{bmatrix} p'_0 \\ p'_1 \\ p'_2 \\ p'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(q_2^2 + q_3^2) & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 0 & 2(q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(-q_0q_1 + q_2q_3) \\ 0 & 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \quad (\text{B.47})$$

The vector part of this equation provides the following rotation matrix:

$$R = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \quad (\text{B.48})$$

Taking equation B.48 into account it is very easy to obtain the rotation quaternion related to a certain rotation matrix.

If we denote the rotation matrix R as:

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \quad (\text{B.49})$$

it is easy to check that the following expressions hold:

$$\begin{aligned} 4q_0^2 &= 1 + (R_{11} + R_{22} + R_{33}) \\ 4q_1^2 &= 1 + (R_{11} - R_{22} - R_{33}) \\ 4q_2^2 &= 1 + (-R_{11} + R_{22} - R_{33}) \\ 4q_3^2 &= 1 + (-R_{11} - R_{22} + R_{33}) \end{aligned} \quad (\text{B.50})$$

The sign for the quaternion components can be obtained considering as a convention that the sign of q_0 will be positive. With that convention and taken into account that:

$$\begin{aligned} 4q_0q_1 &= R_{32} - R_{23} \\ 4q_0q_2 &= R_{13} - R_{31} \\ 4q_0q_3 &= R_{21} - R_{12} \end{aligned} \quad (\text{B.51})$$

we can conclude that:

$$\begin{aligned} q_0 &= \frac{1}{2} \sqrt{1 + (R_{11} + R_{22} + R_{33})} \\ q_1 &= \text{sign}(R_{32} - R_{23}) \cdot \frac{1}{2} \sqrt{1 + (R_{11} - R_{22} - R_{33})} \\ q_2 &= \text{sign}(R_{13} - R_{31}) \cdot \frac{1}{2} \sqrt{1 + (-R_{11} + R_{22} - R_{33})} \\ q_3 &= \text{sign}(R_{21} - R_{12}) \cdot \frac{1}{2} \sqrt{1 + (-R_{11} - R_{22} + R_{33})} \end{aligned} \quad (\text{B.52})$$

B.4.3 Quaternions and the angular velocity vector

As indicated in B.36 a rotation about the vector $\vec{q}$ of an angle θ can be represented by a rotation quaternion with the following expression:

$$q = \left[\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \vec{q} \right] \quad (\text{B.53})$$

We can consider that θ and $\vec{q}$ are a function of time. In this case the rotation of an arbitrary vector p would be:

$$\begin{aligned} \text{In } t &\Rightarrow p'(t) = q(t) \cdot p \cdot q^*(t) \\ \text{In } t + dt &\Rightarrow p'(t + dt) = q(t + dt) \cdot p \cdot q^*(t + dt) \end{aligned} \quad (\text{B.54})$$

The rotation from $p'(t)$ to $p'(t + dt)$ can be achieved using the rotation quaternion dq :

$$p'(t + dt) = dq(t) \cdot p'(t) \cdot dq^*(t) \quad (\text{B.55})$$

so we can express:

$$p'(t + dt) = dq(t) \cdot q(t) \cdot p \cdot q^*(t) \cdot dq^*(t) \quad (\text{B.56})$$

If we equal both expressions for $p'(t + dt)$ (equations B.54 and B.56):

$$q(t + dt) \cdot p \cdot q^*(t + dt) = dq(t) \cdot q(t) \cdot p \cdot q^*(t) \cdot dq^*(t) \Rightarrow q(t + dt) = dq(t) \cdot q(t) \quad (\text{B.57})$$

where dq would be a rotation quaternion related to a rotation about the vector $\vec{\omega}/\|\vec{\omega}\|$ of an angle $d\theta$:

$$dq = \left[\cos \frac{d\theta}{2}, \sin \frac{d\theta}{2} \cdot \frac{\vec{\omega}(t)}{\|\vec{\omega}(t)\|} \right] \simeq \left[1, \frac{d\theta}{2} \cdot \frac{\vec{\omega}(t)}{\|\vec{\omega}(t)\|} \right] \quad (\text{B.58})$$

The angular velocity of the rotation is given by the expression:

$$\|\vec{\omega}(t)\| = \frac{d\theta}{dt} \quad (\text{B.59})$$

As a consequence we can express:

$$dq \simeq \left[1, \frac{d\theta}{2} \cdot \frac{\vec{\omega}(t)}{\|\vec{\omega}(t)\|} \right] = \left[1, \frac{\vec{\omega}(t)}{2} dt \right] \quad (\text{B.60})$$

Using this in equation B.57 we get:

$$\begin{aligned} q(t + dt) &= dq \cdot q(t) = \left[1, \frac{\vec{\omega}(t)}{2} dt \right] \cdot q(t) = q(t) + \left[0, \frac{\vec{\omega}(t)}{2} dt \right] \cdot q(t) \Rightarrow \\ \frac{q(t + dt) - q(t)}{dt} &= \frac{dq(t)}{dt} = \left[0, \frac{\vec{\omega}(t)}{2} \right] \cdot q(t) \end{aligned} \quad (\text{B.61})$$

This product can be represented in a matricial form according to B.44:

$$\left[0, \frac{\vec{\omega}(t)}{2}\right] \cdot q(t) = \frac{1}{2} \cdot [0, \vec{\omega}(t)] \cdot q(t) = \frac{1}{2} \Omega_L \cdot q(t) \quad (\text{B.62})$$

so:

$$\frac{dq(t)}{dt} = \frac{1}{2} \Omega_L \cdot q(t) \quad (\text{B.63})$$

where the components of $\vec{\omega}$ are given in the non-rotating frame of reference in order to make it compatible with the expression for q .

Equation B.63 can be expressed, taking equation B.46 into account, as:

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & -\omega_3 & \omega_2 \\ \omega_2 & \omega_3 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{bmatrix} \cdot \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (\text{B.64})$$

We can obtain an equation similar to B.63 expressing the coordinates of $\vec{\omega}$ in the rotating frame of reference (we will notate it $\vec{\omega}'$ in this case).

According to equation B.41:

$$\left[0, \frac{\vec{\omega}}{2}\right] = q \cdot \left[0, \frac{\vec{\omega}'}{2}\right] \cdot q^* \quad (\text{B.65})$$

so:

$$\frac{dq}{dt}(t) = \left[0, \frac{\vec{\omega}}{2}\right] \cdot q = q \cdot \left[0, \frac{\vec{\omega}'}{2}\right] \cdot q^* \cdot q = q \cdot \left[0, \frac{\vec{\omega}'}{2}\right] = \frac{1}{2} \Omega'_R \cdot q(t) \quad (\text{B.66})$$

As a consequence we have that:

$$\frac{dq}{dt}(t) = \frac{1}{2} \Omega'_R \cdot q(t) \quad (\text{B.67})$$

so we can express:

$$\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega'_1 & -\omega'_2 & -\omega'_3 \\ \omega'_1 & 0 & \omega'_3 & -\omega'_2 \\ \omega'_2 & -\omega'_3 & 0 & \omega'_1 \\ \omega'_3 & \omega'_2 & -\omega'_1 & 0 \end{bmatrix} \cdot \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (\text{B.68})$$

or:

$$\begin{aligned} \dot{q}_0 &= \frac{1}{2} (-\omega'_1 \cdot q_1 - \omega'_2 \cdot q_2 - \omega'_3 \cdot q_3) \\ \dot{q}_1 &= \frac{1}{2} (\omega'_1 \cdot q_0 + \omega'_3 \cdot q_2 - \omega'_2 \cdot q_3) \\ \dot{q}_2 &= \frac{1}{2} (\omega'_2 \cdot q_0 - \omega'_3 \cdot q_1 + \omega'_1 \cdot q_3) \\ \dot{q}_3 &= \frac{1}{2} (\omega'_3 \cdot q_0 + \omega'_2 \cdot q_1 - \omega'_1 \cdot q_2) \end{aligned} \quad (\text{B.69})$$

The inverse transformation that provides the angular velocity vector from the quaternion derivatives can be obtained easily from equations B.63 and B.67:

$$\begin{aligned}\frac{dq}{dt} &= \left[0, \frac{\vec{\omega}}{2}\right] \cdot q(t) \implies [0, \vec{\omega}] = 2 \frac{dq}{dt} \cdot q^* \\ \frac{dq}{dt} &= q(t) \cdot \left[0, \frac{\vec{\omega}'}{2}\right] \implies [0, \vec{\omega}'] = 2q^* \cdot \frac{dq}{dt}\end{aligned}\tag{B.70}$$

In matrix form:

$$\begin{aligned}[0, \vec{\omega}] &= 2 \frac{dq}{dt} \cdot q^* \implies \vec{\omega} = 2Q_R^* \cdot \frac{dq}{dt} \\ [0, \vec{\omega}'] &= 2q^* \cdot \frac{dq}{dt} \implies \vec{\omega}' = 2Q_L^* \cdot \frac{dq}{dt}\end{aligned}\tag{B.71}$$

This is:

$$\vec{\omega} = 2Q_R^* \cdot \frac{dq}{dt} = 2 \begin{bmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \cdot \begin{Bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}\tag{B.72}$$

$$\vec{\omega}' = 2Q_L^* \cdot \frac{dq}{dt} = 2 \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \cdot \begin{Bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix}\tag{B.73}$$

B.4.4 Quaternions and Euler angles

The rotation matrix from the reference frame (navigation frame) to the body-fixed frame can be represented using Euler angles by equation B.24.

At the same time this rotation matrix can be represented using quaternions according to equation B.48.

As a consequence we can equal both expressions in order to relate the quaternion representation of the attitude of the body-fixed frame with the Euler representation of the attitude:

$$\begin{aligned}
 R = C_b^n &= \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(-q_0q_3 + q_1q_2) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & 1 - 2(q_1^2 + q_3^2) & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} = \\
 &= \begin{bmatrix} \cos \theta \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \cos \theta \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi \\ -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix}
 \end{aligned} \tag{B.74}$$

This allows obtaining the following expressions for the Euler angles from the quaternions:

If $\theta \simeq \frac{\pi}{2}$:

$$\begin{aligned}
 \sin \theta &= -2(-q_0q_2 + q_1q_3) \\
 \theta &= \text{sign}(\sin \theta) \cdot \frac{\pi}{2} \\
 \psi &= \psi_{old} \\
 \text{if } \sin \theta \geq 0.0 &\implies \phi = \text{atan2}(2(-q_0q_3 + q_1q_2), 1 - 2(q_1^2 + q_3^2)) \\
 \text{else if } \sin \theta < 0.0 &\implies \phi = \text{atan2}(-2(-q_0q_3 + q_1q_2), 1 - 2(q_1^2 + q_3^2)) - \psi
 \end{aligned} \tag{B.75}$$

If $\theta \neq \frac{\pi}{2}$:

$$\begin{aligned}
 \psi_1 &= \text{atan2}(2(q_1q_2 + q_0q_3), 1 - 2(q_2^2 + q_3^2)) \\
 \psi_2 &= \text{atan2}(-2(q_1q_2 + q_0q_3), -(1 - 2(q_2^2 + q_3^2))) \\
 \implies \psi &\text{ is chosen from } \psi_1 \text{ and } \psi_2 \text{ as the closest to } \psi_{old} \\
 \text{if } \|\cos \psi\| > \|\sin \psi\| &\implies \cos \theta = \left(\frac{1 - 2(q_2^2 + q_3^2)}{\cos \psi} \right) \\
 \text{else if } \|\sin \psi\| \geq \|\cos \psi\| &\implies \cos \theta = \left(\frac{2(q_0q_3 + q_1q_2)}{\sin \psi} \right) \\
 \text{sign} &= \text{sign}(\cos \theta) \\
 \phi &= \text{atan2}(\text{sign} \cdot (2(q_0q_1 + q_2q_3)), \text{sign} \cdot (1 - 2(q_1^2 + q_2^2))) \\
 \theta &= \text{atan2}(2(q_0q_2 - q_1q_3), \cos \theta)
 \end{aligned} \tag{B.76}$$

being ψ_{old} a previous value for ψ from a previous step, close to the present value.

In order to obtain the quaternions from the Euler angles we can recall that the Euler angles are given by 3 rotations in 3 different axes (see figure B.2) that can be represented by a rotation quaternion:

1. ψ around $\vec{k} \Rightarrow r_1 = \left[\cos \frac{\psi}{2}, \sin \frac{\psi}{2} \vec{k} \right]$
2. θ around $\vec{j}_1 \Rightarrow r_2 = \left[\cos \frac{\theta}{2}, \sin \frac{\theta}{2} \vec{j}_1 \right]$
3. ϕ around $\vec{i}_2 \Rightarrow r_3 = \left[\cos \frac{\phi}{2}, \sin \frac{\phi}{2} \vec{i}_2 \right]$

so the rotation quaternion will be given by $q = r_3 \cdot r_2 \cdot r_1$.

We can make use of equation B.42 in order to simplify the multiplication:

$$q^1 = r_3^1 \cdot r_2^1 \cdot r_1^1 = r_1^1 \cdot r_2^2 \cdot r_3^3 \quad (\text{B.77})$$

which leads to:

$$\begin{aligned} q_0^1 &= \cos \frac{\psi}{2} \cos \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\psi}{2} \sin \frac{\theta}{2} \sin \frac{\phi}{2} \\ q_1^1 &= \cos \frac{\psi}{2} \cos \frac{\theta}{2} \sin \frac{\phi}{2} - \sin \frac{\psi}{2} \sin \frac{\theta}{2} \cos \frac{\phi}{2} \\ q_2^1 &= \cos \frac{\psi}{2} \sin \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\psi}{2} \cos \frac{\theta}{2} \sin \frac{\phi}{2} \\ q_3^1 &= \sin \frac{\psi}{2} \cos \frac{\theta}{2} \cos \frac{\phi}{2} - \cos \frac{\psi}{2} \sin \frac{\theta}{2} \sin \frac{\phi}{2} \end{aligned} \quad (\text{B.78})$$

The chosen quaternion will be the closest from $[q_0^1, q_1^1, q_2^1, q_3^1]$ and $-[q_0^1, q_1^1, q_2^1, q_3^1]$ to the previous quaternion vector.

Appendix B references

- [1] Defense Mapping Agency, Department of Defense. *Technical Report: DMA TR 8350.2-A: Appendix: Transformation of ECI (CIS, Epoch J2000.0) Coordinates to WGS 84 (CTS, ECEF) Coordinates*. National Geospatial-Intelligence Agency, December 1987. URL <http://earth-info.nga.mil/GandG/publications/tr8350.2/tr8350.2-a/Appendix.pdf>. 349
- [2] M. Prieto-Alberca. *Apuntes de Mecánica Racional, Cinemática y Estática* printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, November 1991.
- [3] Miguel Ángel Gómez Tierno, Manuel Pérez Cortés, and César Puentes Márquez. *Mecánica del Vuelo*. Instituto Universitario de Microgravedad Ignacio da Riva (IDR/UPM), Madrid, first edition, 2009.
- [4] James Diebel. *Representing Attitude: Euler Angles, Unit Quaternions, and Rotation Vectors*. Stanford University, Standford, California 94301-9010, October 2006.

This page intentionally left blank.

Appendix C

Angular velocity vectors

In this appendix the angular velocity vectors that relate the different reference frames used within this document will be deduced.

The expressions indicated in this appendix are used in [chapter 3](#).

C.1 Angular velocity of the ECEF frame w.r.t. the ECI frame

According to section A.3 the transformation between the ECI reference frame and the ECEF reference frame requires 4 steps:

1. Conversion between CIS coordinates and Mean Earth-Centered Inertial of Date coordinates
This transformation changes the axes from the Mean Inertial frame of J2000.0 ($X_1Y_1Z_1$) to the Mean Earth-Centered Inertial of the required Date ($X_2Y_2Z_2$) taking into account the precession motion of the Earth.
2. Conversion between Mean Earth-Centered Inertial of Date coordinates ($X_2Y_2Z_2$) and Mean True Earth-Centered Inertial of Date coordinates ($X_3Y_3Z_3$).
This transformation takes into account the nutation motion of the Earth in order to transform from mean to true coordinates of date.
3. Conversion between Mean True Earth-Centered Inertial of Date coordinates ($X_3Y_3Z_3$) and True Earth-Centered Earth-Fixed coordinates ($X_4Y_4Z_4$).
This transformation takes into account the rotational motion of the Earth around its axis in order to change to Earth-Centered frames.
4. Conversion between True Earth-Centered Earth-Fixed coordinates ($X_4Y_4Z_4$) and Mean Earth-Centered Earth-Fixed coordinates (Conventional Terrestrial, CTS).
This transformation takes into account the movement of the rotation axis with respect to the Earth's crust. This is, the Polar motion.

As a consequence the angular velocity of the ECEF frame with respect to the ECI frame could be decomposed as:

$$\vec{\omega}_{ei} = \vec{\omega}_{e4} + \vec{\omega}_{43} + \vec{\omega}_{32} + \vec{\omega}_{2i} \quad (\text{C.1})$$

Among these angular velocities several can be neglected:

- The polar motion is a very small motion. It can be supposed without a great impact that:

$$\vec{\omega}_{e4} \simeq \vec{0} \quad (\text{C.2})$$

- The nutation motion of the Earth is a slow motion in comparison with the typical times for the motion of ballistic missiles. As a consequence it will be considered that:

$$\vec{\omega}_{32} \simeq \vec{0} \quad (\text{C.3})$$

- The precession motion of the Earth is even slower than the nutation motion so it can also be considered that:

$$\vec{\omega}_{21} \simeq \vec{0} \quad (\text{C.4})$$

It has to be noted that the value of $\vec{\omega}_{21}$ is provided by reference [1], since it provides the inertial angular velocity of the Earth (ω), and the angular velocity of the Earth in a precessing frame ω^* . The difference between both values is the angular velocity of the precessing frame with respect to the inertial frame:

$$\begin{aligned} \omega - \omega^* = & 7292115.0 \cdot 10^{-11} - \\ & \left(7292115.8553 \cdot 10^{-11} + 4.3 \cdot 10^{-15} \cdot \frac{JD - 2451545}{36525} \right) rad/s = \\ & -0.8553 \cdot 10^{-11} + 4.3 \cdot 10^{-15} \cdot \frac{JD - 2451545}{36525} rad/s \end{aligned} \quad (\text{C.5})$$

A time of about 23000 years is required for a complete rotation with this angular velocity, so for the considered times, $\vec{\omega}_{21}$ can be neglected.

As a consequence we can approximate $\vec{\omega}_{ei}$ as:

$$\vec{\omega}_{ei} \simeq \vec{\omega}_{43} \quad (\text{C.6})$$

$\vec{\omega}_{43}$ can be expressed according to section A.3.3 as:

$$\vec{\omega}_{43} = \dot{\Lambda} \cdot \vec{k}_4 \quad (\text{C.7})$$

where $\dot{\Lambda}$ is given by equation A.15 and $\vec{k}_4$ is the vertical versor in the True Earth-Centered Earth-Fixed coordinates.

$\vec{k}_4$ can be expressed in ECEF coordinates as:

$$\vec{k}_4^e = C_4^e \cdot \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad (\text{C.8})$$

where $C_4^e = A$ is given by equation A.17.

This is:

$$\vec{\omega}_{ei}^e \simeq \vec{\omega}_{43}^e \simeq \dot{\Lambda} \begin{Bmatrix} x_p \\ -y_p \\ 1 \end{Bmatrix} \quad (\text{C.9})$$

C.2 Angular velocity of the Navigation frame w.r.t. the ECEF frame

C.2.1 Derivatives of the geodetic coordinates

The derivatives of the geodetic coordinates are required in order to compute the angular velocity of the Navigation frame with respect to the ECEF frame.

C.2.1.1 Curvature of a curve in a point

The curvature of a curve at a point is defined as the variation of the direction angle of the tangent line at that point with respect to the arc length.

Figure C.1: Curvature of a curve at a point

This is, in figure C.1 the curvature of the curve (κ) would be:

$$\kappa := \frac{d\theta}{ds} \quad (\text{C.10})$$

where the arc length of a curve given as $\vec{r} = \vec{r}(u)$ is defined as:

$$s(u) := \int_{u=a}^u \left\| \frac{d\vec{r}}{du} \right\| \cdot du \quad (\text{C.11})$$

The curvature can be obtained from the equation of the curve using the following equation (see [2]):

$$\kappa = \frac{\|\vec{r}'\|}{\|\vec{r}\|} = \frac{\|\vec{r}' \wedge \vec{r}''\|}{\|\vec{r}'\|^3} \quad (\text{C.12})$$

which will be the formula used herein to obtain the radius of curvature.

Once the curvature of a curve at a point is obtained the radius of curvature of that curve at that point can be easily computed since it is defined as:

$$\rho := \frac{1}{\kappa} \quad (\text{C.13})$$

C.2.1.2 Derivative of the geodetic latitude

If we are moving along a meridian within the WGS84 reference ellipsoid, the Z_n axis line is always perpendicular to the tangent line, so the variation of the latitude angle, that locates the Z_n axis line with respect to the equator, is the same as the variation of the direction angle of the tangent line.

Figure C.2: Tangent line in a meridian

This is:

$$\kappa_{meridian} = \frac{1}{\rho_{meridian}} = \frac{d\varphi}{ds} \quad (C.14)$$

Since in this case we are only moving along the meridian $s = s_n$ (North):

$$\kappa_{meridian} = \frac{1}{\rho_{meridian}} = \frac{d\varphi}{ds_n} = \frac{d\varphi}{dt} \cdot \frac{dt}{ds_n} = \frac{\dot{\varphi}}{v_n} \quad (C.15)$$

where v_n is the speed in the North direction with which we are moving along the meridian.

The coordinates of the points of the meridian are given by equation A.20 as a function of the reduced latitude:

$$\vec{r} = \vec{r}(X, Z) \quad \text{with} \quad \begin{cases} X = a \cdot \cos \beta \\ Z = a \cdot \sqrt{1 - e^2} \cdot \sin \beta \end{cases} \quad (C.16)$$

and the curvature at a local point of the meridian can be obtained applying equation C.12:

$$\kappa = \frac{\|\vec{r}' \wedge \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{a \cdot \sqrt{1 - e^2}}{\sqrt{(a^2 \cdot (1 - e^2 \cdot \cos^2 \beta))^3}} \quad (C.17)$$

Since the reduced latitude can be related to the geodetic latitude using equation A.22:

$$\begin{aligned}\cos \beta &= \frac{1}{\sqrt{(1 - e^2 \cdot \sin^2 \varphi)}} \cdot \cos \varphi \Rightarrow \\ \kappa &= \frac{a^2 \cdot \sqrt{1 - e^2}}{\sqrt{(a^2 \cdot (1 - e^2 \cdot \cos^2 \beta))^3}} = \frac{\sqrt{(1 - e^2 \cdot \sin^2 \varphi)^3}}{a \cdot (1 - e^2)}\end{aligned}\quad (\text{C.18})$$

As a consequence:

$$\dot{\varphi} = \kappa_{\text{meridian}} \cdot v_n = \frac{\sqrt{(1 - e^2 \cdot \sin^2 \varphi)^3}}{a \cdot (1 - e^2)} \cdot v_n \quad (\text{C.19})$$

which can be simplified using equation A.26:

$$N(\varphi) := \frac{a}{\sqrt{1 - e^2 \cdot \sin^2 \varphi}} \Rightarrow \dot{\varphi} = \frac{a^2}{1 - e^2} \cdot \frac{1}{N(\varphi)^3} \cdot v_n \quad (\text{C.20})$$

Defining:

$$M(\varphi) = N(\varphi)^3 \cdot \frac{1 - e^2}{a^2} \quad (\text{C.21})$$

we finally get:

$$\dot{\varphi} = \frac{v_n}{M(\varphi)} \quad (\text{C.22})$$

This formula (equation C.22) applies for a point located at the surface of the ellipsoid. When the height is different to zero and since the height is computed in the Z_n axis line, the radius of curvature of the trajectory has to be increased exactly in h to set the new radius of curvature.

Figure C.3: Radius of curvature in a meridian at height h

As a consequence the derivative of the geodetic latitude can be finally expressed as:

$$\dot{\varphi} = \frac{v_n}{M(\varphi) + h} \quad (\text{C.23})$$

C.2.1.3 Derivative of the geodetic longitude

If we are moving along a parallel within the WGS84 reference ellipsoid, the line in the parallel plane from the OZ_n axis to the point is always perpendicular to the tangent line, so the variation of the longitude angle, that locates this line with respect to the 0° meridian, is the same as the variation of the direction angle of the tangent line.

Figure C.4: Tangent line in a parallel

This is:

$$\kappa_{parallel} = \frac{1}{\rho_{parallel}} = \frac{d\lambda}{ds} \quad (C.24)$$

Since in this case we are only moving along the parallel $s = s_e$ (East):

$$\kappa_{parallel} = \frac{1}{\rho_{parallel}} = \frac{d\lambda}{ds_n} = \frac{d\lambda}{dt} \cdot \frac{dt}{ds_e} = \frac{\dot{\lambda}}{v_e} \quad (C.25)$$

where v_e is the speed in the East direction with which we are moving along the parallel.

The coordinates of the points of the parallel are given by equation A.27 as a function of the longitude:

$$\vec{r} = \vec{r}(x, y) \quad \text{with} \quad \begin{cases} x = (N(\varphi) + h) \cdot \cos \varphi \cdot \cos \lambda \\ y = (N(\varphi) + h) \cdot \cos \varphi \cdot \sin \lambda \end{cases} \quad (C.26)$$

and the curvature at a local point of the parallel can be obtained applying equation C.12:

$$\kappa_{parallel} = \frac{\|\vec{r}' \wedge \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{[(N(\varphi) + h) \cdot \cos \varphi]^2}{[(N(\varphi) + h) \cdot \cos \varphi]^3} = \frac{1}{(N(\varphi) + h) \cdot \cos \varphi} \quad (C.27)$$

As a consequence:

$$\dot{\lambda} = \kappa_{parallel} \cdot v_e = \frac{v_e}{(N(\varphi) + h) \cdot \cos \varphi} \quad (C.28)$$

C.2.1.4 Derivative of the altitude

In this case the derivative is very easy to obtain from the definition of the geodetic coordinates (see figure A.3):

$$\dot{h} = -v_d \quad (\text{C.29})$$

C.2.2 Expression for the angular velocity of the Navigation frame w.r.t. the ECEF frame

The angular velocity of the navigation frame with respect to the ECEF frame can be calculated as a composition of rotations in terms of the derivatives of the geodetic coordinates:

- Rotation of $\dot{\lambda}$ about the $\vec{k}_e$ axis.
- Rotation of $-\dot{\varphi}$ about the $\vec{j}_n$ axis.

This is:

$$\vec{\omega}_{ne} = \dot{\lambda} \cdot \vec{k}_e - \dot{\varphi} \cdot \vec{j}_n \quad (\text{C.30})$$

Since from equation A.35 we can express $\vec{k}_e = \cos\varphi \cdot \vec{i}_n - \sin\varphi \cdot \vec{k}_n$ this composition of rotations leads to the following angular velocity vector in the navigation frame:

$$\vec{\omega}_{ne}^n = \dot{\lambda} \cos\varphi \cdot \vec{i}_n - \dot{\varphi} \cdot \vec{j}_n - \dot{\lambda} \sin\varphi \cdot \vec{k}_n \quad (\text{C.31})$$

From section C.2.1 we have the derivatives of the geodetic latitude, longitude and height as a function of North-East-Down data:

$$\begin{aligned} \dot{\varphi} &= \frac{v_n}{M(\varphi) + h} \quad (\text{equation C.23}) \\ \dot{\lambda} &= \frac{v_e}{(N(\varphi) + h) \cdot \cos\varphi} \quad (\text{equation C.28}) \\ \dot{h} &= -v_d \quad (\text{equation C.29}) \end{aligned} \quad (\text{C.32})$$

So the following expression for the angular velocity of the Navigation frame w.r.t. the ECEF frame is obtained:

$$\vec{\omega}_{ne}^n = \begin{pmatrix} \frac{v_e}{N(\varphi) + h} \\ -\frac{v_n}{M(\varphi) + h} \\ -\frac{v_e \cdot \tan\varphi}{N(\varphi) + h} \end{pmatrix} \quad (\text{C.33})$$

C.3 Angular velocity of the Body frame w.r.t. the Navigation frame

The components of the angular velocity of the Body frame with respect to the Navigation frame will be denoted as:

$$\vec{\omega}_{bn}^b = \begin{Bmatrix} p_{bn} \\ q_{bn} \\ r_{bn} \end{Bmatrix} \quad (\text{C.34})$$

This angular velocity can be obtained taking into account the rotations indicated in section B.3:

- Rotation of $\dot{\psi}$ about the $\vec{k}_n$ axis
- Rotation of $\dot{\theta}$ about the $\vec{j}_1$ axis
- Rotation of $\dot{\phi}$ about the $\vec{i}_b$ axis

This is:

$$\vec{\omega}_{bn} = \dot{\phi} \cdot \vec{i}_b + \dot{\theta} \cdot \vec{j}_1 + \dot{\psi} \cdot \vec{k}_n \quad (\text{C.35})$$

Taking equations B.24 and B.25 into account the versors $\vec{j}_1$ and $\vec{k}_n$ can be expressed as:

$$\begin{aligned} \vec{j}_1 &= \cos \phi \vec{j}_b - \sin \phi \vec{k}_b \\ \vec{k}_n &= -\sin \theta \cdot \vec{i}_b + \sin \phi \cos \theta \vec{j}_b + \cos \phi \cos \theta \vec{k}_b \end{aligned} \quad (\text{C.36})$$

so we can finally express:

$$\begin{aligned} p_{bn} &= \dot{\phi} - \dot{\psi} \cdot \sin \theta \\ q_{bn} &= \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \\ r_{bn} &= \dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi \end{aligned} \quad (\text{C.37})$$

This is:

$$\vec{\omega}_{bn}^b = \begin{Bmatrix} \dot{\phi} - \dot{\psi} \cdot \sin \theta \\ \dot{\theta} \cos \phi + \dot{\psi} \cos \theta \sin \phi \\ \dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi \end{Bmatrix} \quad (\text{C.38})$$

The inverse equation can be obtained from equation C.38 getting:

$$\begin{aligned} \dot{\theta} &= q_{bn} \cdot \cos \phi - r_{bn} \cdot \sin \phi \\ \dot{\psi} &= (q_{bn} \cdot \sin \phi + r_{bn} \cdot \cos \phi) \cdot \frac{1}{\cos \theta} \\ \dot{\phi} &= p_{bn} + (q_{bn} \cdot \sin \phi + r_{bn} \cdot \cos \phi) \cdot \frac{\sin \theta}{\cos \theta} \end{aligned} \quad (\text{C.39})$$

Appendix C references

- [1] National Imagery and Mapping Agency, Department of Defense. *Technical Report: NIMA TR8350.2: World Geodetic System 1984. Its Definition and Relationship with Local Geodetic Systems*. National Geospatial-Intelligence Agency, amendment 1, third edition, January 2000. URL <http://earth-info.nga.mil/GandG/publications/tr8350.2/wgs84fin.pdf>. 365
- [2] J. M. Vega de Prada. Geometría Diferencial printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, October 1986. 366
- [3] M. Prieto-Alberca. Apuntes de Mecánica Racional, Cinemática y Estática printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, November 1991.
- [4] Miguel Ángel Gómez Tierno, Manuel Pérez Cortés, and César Puentes Márquez. *Mecánica del Vuelo*. Instituto Universitario de Microgravedad Ignacio da Riva (IDR/UPM), Madrid, first edition, 2009.

Appendix D

Equations of motion

We will summarize in this appendix some expressions for vectorial derivatives in moving frames and relative motion, as well as the basic dynamic equations for a particle, a system of particles with a constant mass and a rigid body. These equations will then be used to obtain the equations of motion that apply to the missiles.

In the used notation the character i indicates an inertial reference frame, whereas the character N indicates a non inertial reference frame. S indicates a generic reference frame that could be inertial or non inertial.

The expressions indicated in this appendix are used in [chapter 3](#) and [chapter 4](#).

D.1 Vectorial derivatives in moving frames

D.1.1 Coriolis theorem

The derivative of a vector can be provided with respect to different frames of reference. These derivatives can be related using the Coriolis theorem:

$$\left(\frac{d\vec{v}}{dt}\right)_a = \left(\frac{d\vec{v}}{dt}\right)_b + \vec{\omega}_{ba} \times \vec{v} \quad (\text{D.1})$$

where:

- a is a reference frame
- b is a reference frame
- $\vec{\omega}_{ba}$ is the angular velocity of the b reference frame with respect to the a reference frame.
- $\vec{v}$ is the vector whose derivatives we want to relate.

The vector for which the Coriolis theorem is applied could be the measurement of a physical magnitude with respect to a reference frame. For example it could be the velocity of a particle P with respect to the reference frame a and in this case the Coriolis theorem would apply as:

$$\left(\frac{d(\vec{v}_P)_a}{dt}\right)_a = \left(\frac{d(\vec{v}_P)_a}{dt}\right)_b + \vec{\omega}_{ba} \times (\vec{v}_P)_a \quad (\text{D.2})$$

It is important to indicate that equation D.1 is a vectorial equality that is true whatever coordinates are used to represent the vector.

The only difference is that it is very easy to obtain the time derivative of a vector whose coordinates are given in a certain reference frame with respect to that very reference frame since:

$$\left(\frac{d\vec{v}}{dt}\right)_a = \frac{dv_x^a}{dt} \cdot \vec{i}_a + \frac{dv_y^a}{dt} \cdot \vec{j}_a + \frac{dv_z^a}{dt} \cdot \vec{k}_a \quad (\text{D.3})$$

$$\left(\frac{d\vec{v}}{dt}\right)_b = \frac{dv_x^b}{dt} \cdot \vec{i}_b + \frac{dv_y^b}{dt} \cdot \vec{j}_b + \frac{dv_z^b}{dt} \cdot \vec{k}_b \quad (\text{D.4})$$

D.1.2 Properties of the angular velocity vector

We will indicate now some properties of the vector that provides the angular velocity of one reference frame with respect to another one.

Invariance of the derivative

If we apply the equation D.1 to the vector $\vec{\omega}_{ba}$ itself we would get:

$$\left(\frac{d\vec{\omega}_{ba}}{dt}\right)_a = \left(\frac{d\vec{\omega}_{ba}}{dt}\right)_b + \vec{\omega}_{ba} \times \vec{\omega}_{ba} \implies \left(\frac{d\vec{\omega}_{ba}}{dt}\right)_a = \left(\frac{d\vec{\omega}_{ba}}{dt}\right)_b \quad (\text{D.5})$$

This is, the angular velocity vector has the same derivative with respect to both reference frames.

Angular velocity vectors of 2 reference frames

We can apply equation D.1 to relate the derivatives of a vector $\vec{v}$ with respect to reference frames a and b in two ways:

$$\left(\frac{d\vec{v}}{dt}\right)_a = \left(\frac{d\vec{v}}{dt}\right)_b + \vec{\omega}_{ba} \times \vec{v} \quad (\text{D.6})$$

$$\left(\frac{d\vec{v}}{dt}\right)_b = \left(\frac{d\vec{v}}{dt}\right)_a + \vec{\omega}_{ab} \times \vec{v} \quad (\text{D.7})$$

If we substitute $\left(\frac{d\vec{v}}{dt}\right)_a$ from equation D.6 in equation D.7 we get:

$$\vec{\omega}_{ab} = -\vec{\omega}_{ba} \quad (\text{D.8})$$

Angular velocity vectors of 3 reference frames

If we apply equation D.1 for a vector $\vec{v}$ with respect to 3 reference frames a , b and c after some elaboration we get:

$$\vec{\omega}_{ac} = \vec{\omega}_{ab} + \vec{\omega}_{bc} \quad (\text{D.9})$$

D.2 Relative motion

If we have two reference frames from which a point P is observed, as indicated in figure D.1:

Figure D.1: Position of a point from 2 reference frames

we can easily relate the position of P in both reference systems as:

$$\vec{OP} = \vec{OO'} + \vec{O'P} \implies (\vec{r}_p)_S = (\vec{r}_{O'})_S + (\vec{r}_p)_{S'} \quad (\text{D.10})$$

If we now differentiate this expression with respect to reference frame S (neglecting the corrections that would be required if the Special Relativity Theory were to be applied) we get:

$$\left(\frac{d}{dt}\right)_S (\vec{r}_p)_S = \left(\frac{d}{dt}\right)_S (\vec{r}_{O'})_S + \left(\frac{d}{dt}\right)_S (\vec{r}_p)_{S'} \implies (\vec{v}_p)_S = (\vec{v}_{O'})_S + \left(\frac{d}{dt}\right)_S (\vec{r}_p)_{S'} \quad (\text{D.11})$$

Taking equation D.1 into account:

$$\left(\frac{d}{dt}\right)_S (\vec{r}_p)_{S'} = \left(\frac{d}{dt}\right)_{S'} (\vec{r}_p)_{S'} + \vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'} = \vec{v}_p^{S'} + \vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'} \quad (\text{D.12})$$

so we finally get:

$$(\vec{v}_p)_S = (\vec{v}_{O'})_S + (\vec{v}_p)_{S'} + \vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'} \quad (\text{D.13})$$

Equation D.13 allows obtaining the velocity of a point with respect to a reference frame having the velocity of that very point in a difference reference frame.

The same approach could be applied in order to obtain the acceleration vectors in both reference frames.

If we differentiate equation D.13 we get:

$$\left(\frac{d}{dt}\right)_S (\vec{v}_p)_S = \left(\frac{d}{dt}\right)_S (\vec{v}_{O'})_S + \left(\frac{d}{dt}\right)_S (\vec{v}_p)_{S'} + \left(\frac{d}{dt}\right)_S \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'}\right) \quad (\text{D.14})$$

Applying again equation D.1 we obtain:

$$\begin{aligned} \left(\frac{d}{dt}\right)_S (\vec{v}_p)_{S'} &= \left(\frac{d}{dt}\right)_{S'} (\vec{v}_p)_{S'} + \left(\vec{\Omega}_{S'S} \wedge (\vec{v}_p)_{S'}\right) = \vec{a}_p^{S'} + \left(\vec{\Omega}_{S'S} \wedge (\vec{v}_p)_{S'}\right) \\ \left(\frac{d}{dt}\right)_S (\vec{r}_p)_{S'} &= \left(\frac{d}{dt}\right)_{S'} (\vec{r}_p)_{S'} + \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'}\right) = (\vec{v}_p)_{S'} + \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'}\right) \end{aligned} \quad (\text{D.15})$$

Using these expressions in equation D.14 we get:

$$(\vec{a}_p)_S = (\vec{a}_{O'})_S + (\vec{a}_p)_{S'} + 2 \cdot \left(\vec{\Omega}_{S'S} \wedge (\vec{v}_p)_{S'}\right) + \frac{d\vec{\Omega}_{S'S}}{dt} \wedge (\vec{r}_p)_{S'} + \vec{\Omega}_{S'S} \wedge \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'}\right) \quad (\text{D.16})$$

The term $2 \cdot \left(\vec{\Omega}_{S'S} \wedge (\vec{v}_p)_{S'}\right)$ is called *Coriolis effect*.

The term $\vec{\Omega}_{S'S} \wedge \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'}\right)$ is called *centrifugal acceleration*.

D.3 Equations of motion for a single particle

D.3.1 Linear momentum

The linear momentum of a particle is defined as:

$$\vec{p} := m\vec{v} \quad (\text{D.17})$$

According to the second law of Newton:

$$\vec{F} = \left(\frac{d(\vec{p})_i}{dt} \right)_i = \left(\frac{d(m\vec{v}_p)_i}{dt} \right)_i = m(\vec{a}_p)_i \quad (\text{D.18})$$

This acceleration can be related to the one obtained in a non inertial reference frame. In this case, according to equation D.16:

$$\begin{aligned} \vec{F} &= m(\vec{a}_p)_i = \\ m &\left((\vec{a}_O)_i + (\vec{a}_p)_N + 2 \cdot \left(\vec{\Omega}_{Ni} \wedge (\vec{v}_p)_N \right) + \frac{d\vec{\Omega}_{Ni}}{dt} \wedge (\vec{r}_p)_N + \vec{\Omega}_{Ni} \wedge \left(\vec{\Omega}_{Ni} \wedge (\vec{r}_p)_N \right) \right) \end{aligned} \quad (\text{D.19})$$

where O is the origin of the non-inertial reference frame N .

Defining the *inertial force* as:

$$\vec{F}_{\text{INERTIAL}} = -m \left((\vec{a}_O)_i + 2 \cdot \left(\vec{\Omega}_{Ni} \wedge (\vec{v}_p)_N \right) + \frac{d\vec{\Omega}_{Ni}}{dt} \wedge (\vec{r}_p)_N + \vec{\Omega}_{Ni} \wedge \left(\vec{\Omega}_{Ni} \wedge (\vec{r}_p)_N \right) \right) \quad (\text{D.20})$$

it can be expressed that:

$$\vec{F} + \vec{F}_{\text{INERTIAL}} = m(\vec{a}_p)_N \quad (\text{D.21})$$

D.3.2 Angular momentum

The angular momentum of the particle with respect to a point A is defined as:

$$\vec{\mathcal{L}}_A := (\vec{r}_p - \vec{r}_A) \wedge m\vec{v}_p \quad (\text{D.22})$$

If the angular momentum is measured and differentiated with respect to a general frame S :

$$\begin{aligned} \left(\frac{d(\vec{\mathcal{L}}_A)_S}{dt} \right)_S = & \left(\frac{d(\vec{r}_p)_S}{dt} \right)_S \wedge m(\vec{v}_p)_S - \left(\frac{d(\vec{r}_A)_S}{dt} \right)_S \wedge m(\vec{v}_p)_S + (\vec{r}_p - \vec{r}_A) \wedge \left(\frac{d(m\vec{v}_p)}{dt} \right)_S = \\ & -(\vec{v}_A)_S \wedge m(\vec{v}_p)_S + (\vec{r}_p - \vec{r}_A) \wedge \left(\frac{d(m\vec{v}_p)}{dt} \right)_S \end{aligned} \quad (\text{D.23})$$

If the angular momentum is measured and differentiated with respect to an inertial reference frame, taking into account equation D.18, the derivative of the angular momentum can be expressed as:

$$\begin{aligned} \left(\frac{d(\vec{\mathcal{L}}_A)_i}{dt} \right)_i = & -(\vec{v}_A)_i \wedge m(\vec{v}_p)_i + (\vec{r}_p - \vec{r}_A) \wedge \left(\frac{d(m\vec{v}_p)_i}{dt} \right)_i = \\ & -(\vec{v}_A)_i \wedge m(\vec{v}_p)_i + (\vec{r}_p - \vec{r}_A) \wedge \vec{F} \end{aligned} \quad (\text{D.24})$$

If the angular momentum is measured and differentiated with respect to a non inertial reference frame, taking into account equation D.21 the derivative of the angular momentum can be expressed as:

$$\begin{aligned} \left(\frac{d(\vec{\mathcal{L}}_A)_N}{dt} \right)_N = & -(\vec{v}_A)_N \wedge m(\vec{v}_p)_N + (\vec{r}_p - \vec{r}_A) \wedge \left(\frac{d(m\vec{v}_p)_N}{dt} \right)_N = \\ & -(\vec{v}_A)_N \wedge m(\vec{v}_p)_N + (\vec{r}_p - \vec{r}_A) \wedge (\vec{F} + \vec{F}_{INERTIAL}) \end{aligned} \quad (\text{D.25})$$

D.4 Equations of motion for a system of particles with constant mass

D.4.1 Linear momentum

According to the expressions deduced in section D.3 for each single particle α within a system of particles with respect to a non inertial reference frame N the following equation applies:

$$\vec{F}_{\alpha\text{EXTERNAL}} + \vec{F}_{\alpha\text{INERTIAL}} + \sum_{\beta \neq \alpha} \vec{f}_{\beta\alpha} = \left(\frac{d(m_{\alpha}\vec{v}_{\alpha})_N}{dt} \right)_N \quad (\text{D.26})$$

where the global forces acting on the particle α have been decomposed in external forces ($\vec{F}_{\alpha\text{EXTERNAL}}$), inertial forces ($\vec{F}_{\alpha\text{INERTIAL}}$) and forces from other particles β of the system.

If we now make a summation for all the particles:

$$\sum_{\forall \alpha} \vec{F}_{\alpha\text{EXTERNAL}} + \sum_{\forall \alpha} \vec{F}_{\alpha\text{INERTIAL}} + \sum_{\forall \alpha} \left(\sum_{\beta \neq \alpha} \vec{f}_{\beta\alpha} \right) = \sum_{\forall \alpha} \left(\frac{d(m_{\alpha}\vec{v}_{\alpha})_N}{dt} \right)_N \quad (\text{D.27})$$

Because of the third law of Newton ($\vec{f}_{\beta\alpha} = -\vec{f}_{\alpha\beta}$):

$$\sum_{\forall \alpha} \left(\sum_{\beta \neq \alpha} \vec{f}_{\beta\alpha} \right) = \vec{0} \quad (\text{D.28})$$

so the following equation of motion applies to a system of particles:

$$\sum_{\forall \alpha} \vec{F}_{\alpha\text{EXTERNAL}} + \sum_{\forall \alpha} \vec{F}_{\alpha\text{INERTIAL}} = \sum_{\forall \alpha} \left(\frac{d(m_{\alpha}\vec{v}_{\alpha})_N}{dt} \right)_N \quad (\text{D.29})$$

If an inertial reference frame is used for measuring the velocity vectors there would be no inertial forces and equation D.29 could be simplified:

$$\sum_{\forall \alpha} \vec{F}_{\alpha\text{EXTERNAL}} = \sum_{\forall \alpha} \left(\frac{d(m_{\alpha}\vec{v}_{\alpha})_i}{dt} \right)_i \quad (\text{D.30})$$

If the mass of the particles within the system remains constant with time expression D.29 can be modified into:

$$\sum_{\forall \alpha} \vec{F}_{\alpha\text{EXTERNAL}} + \sum_{\forall \alpha} \vec{F}_{\alpha\text{INERTIAL}} = \sum_{\forall \alpha} m_{\alpha} \left(\frac{d(\vec{v}_{\alpha})_N}{dt} \right)_N \quad (\text{D.31})$$

The global mass of the system is obtained as:

$$M = \sum_{\forall \alpha} m_{\alpha} \quad (\text{D.32})$$

The center of mass of the system of particles is defined as:

$$\vec{r}_{CM} := \frac{\sum_{\forall \alpha} m_{\alpha} \vec{r}_{\alpha}}{M} \quad (\text{D.33})$$

Differentiating equation D.33 we get an expression for the velocity vector of the center of mass:

$$(\vec{v}_{CM})_S = \left(\frac{d(\vec{r}_{CM})_S}{dt} \right)_S = \frac{\sum_{\forall \alpha} \left(\frac{d(m_{\alpha} \vec{r}_{\alpha})_S}{dt} \right)_S}{M} \quad (\text{D.34})$$

Since in this case the particles of the system have a constant mass with time this can be re-written as:

$$(\vec{v}_{CM})_S = \left(\frac{d(\vec{r}_{CM})_S}{dt} \right)_S = \frac{\sum_{\forall \alpha} m_{\alpha} \left(\frac{d\vec{r}_{\alpha}}{dt} \right)_S}{M} = \frac{\sum_{\forall \alpha} m_{\alpha} (\vec{v}_{\alpha})_S}{M} \quad (\text{D.35})$$

Differentiating again we get:

$$\left(\frac{d(\vec{v}_{CM})_S}{dt} \right)_S = \frac{\sum_{\forall \alpha} m_{\alpha} \left(\frac{d(\vec{v}_{\alpha})_S}{dt} \right)_S}{M} \quad (\text{D.36})$$

Using expression D.36 in equation D.31 we obtain the equation of motion that applies to a system of particles in which the mass of the particles remains constant:

$$M \left(\frac{d(\vec{v}_{CM})_N}{dt} \right)_N = \sum_{\forall \alpha} \vec{F}_{\alpha \text{EXTERNAL}} + \sum_{\forall \alpha} \vec{F}_{\alpha \text{INERTIAL}} \quad (\text{D.37})$$

If an inertial reference frame were used for measuring the vectors and their derivatives, there would be no inertial forces and equation D.37 could be simplified:

$$M \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i = \sum_{\forall \alpha} \vec{F}_{\alpha \text{EXTERNAL}} \quad (\text{D.38})$$

Sometimes the velocity vector of the center of mass $(\vec{v}_{CM})_i$ is easier to express in coordinates of a non inertial reference frame N . In this case we can use equation D.1 to obtain $\left(\frac{d(\vec{v}_{CM}^N)_i}{dt} \right)_i$ from $\left(\frac{d(\vec{v}_{CM}^N)_i}{dt} \right)_N$:

$$M \left(\frac{d(\vec{v}_{CM}^N)_i}{dt} \right)_N + \vec{\Omega}_{Ni} \wedge (\vec{v}_{CM}^N)_i = \sum_{\forall \alpha} \vec{F}_{\alpha \text{EXTERNAL}} \quad (\text{D.39})$$

D.4.2 Angular momentum

The angular momentum of a system of particles with respect to a point A is defined as:

$$\vec{\mathcal{L}}_A := \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge m_\alpha \vec{v}_\alpha \quad (\text{D.40})$$

If the selected point for computing the angular momentum of the system of particles is the center of mass of the system equation D.40 becomes:

$$\vec{\mathcal{L}}_{CM} = \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_{CM}) \wedge m_\alpha \vec{v}_\alpha \quad (\text{D.41})$$

Both angular momentums can be related:

$$\begin{aligned} (\vec{r}_\alpha - \vec{r}_A) &= (\vec{r}_\alpha - \vec{r}_{CM} + \vec{r}_{CM} - \vec{r}_A) \\ \implies \vec{\mathcal{L}}_A &= \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge m_\alpha \vec{v}_\alpha = \\ &= \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_{CM}) \wedge m_\alpha \vec{v}_\alpha + \sum_{\forall \alpha} (\vec{r}_{CM} - \vec{r}_A) \wedge m_\alpha \vec{v}_\alpha = \\ &= \vec{\mathcal{L}}_{CM} + (\vec{r}_{CM} - \vec{r}_A) \wedge \sum_{\forall \alpha} m_\alpha \vec{v}_\alpha \end{aligned} \quad (\text{D.42})$$

If we consider that the mass of the particles remains constant with time we can apply equation D.35 and obtain the following expression:

$$\vec{\mathcal{L}}_A = \vec{\mathcal{L}}_{CM} + (\vec{r}_{CM} - \vec{r}_A) \wedge M \vec{v}_{CM} \quad (\text{D.43})$$

If we consider a non inertial reference frame for measuring the vectors and differentiate the angular momentum (equation D.40) we get:

$$\begin{aligned} \left(\frac{d(\vec{\mathcal{L}}_A)_N}{dt} \right)_N &= \sum_{\forall \alpha} \left(\frac{d(\vec{r}_\alpha)_N}{dt} \right)_N \wedge m_\alpha (\vec{v}_\alpha)_N - \sum_{\forall \alpha} \left(\frac{d(\vec{r}_A)_N}{dt} \right)_N \wedge m_\alpha (\vec{v}_\alpha)_N + \\ &+ \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge \left(\frac{d(m_\alpha \vec{v}_\alpha)_N}{dt} \right)_N = \\ &= -(\vec{v}_A)_N \wedge M (\vec{v}_{CM})_N + \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge \left(\frac{d(m_\alpha \vec{v}_\alpha)_N}{dt} \right)_N \end{aligned} \quad (\text{D.44})$$

Taking equation D.26 into account this can be re-written as:

$$\begin{aligned} \left(\frac{d(\vec{\mathcal{L}}_A)_N}{dt} \right)_N &= -(\vec{v}_A)_N \wedge M (\vec{v}_{CM})_N + \\ &+ \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge (\vec{F}_{\alpha \text{EXTERNAL}} + \vec{F}_{\alpha \text{INERTIAL}} + \sum_{\beta \neq \alpha} \vec{f}_{\beta \alpha}) \end{aligned} \quad (\text{D.45})$$

The last term of equation D.45 can be obtained adding the effects of one particle to another in pairs. For each pair:

$$\sum_{\alpha,\beta} (\vec{r}_\alpha - \vec{r}_A) \wedge \sum_{\beta \neq \alpha} \vec{f}_{\beta\alpha} = (\vec{r}_\alpha - \vec{r}_A) \wedge \vec{f}_{\beta\alpha} + (\vec{r}_\beta - \vec{r}_A) \wedge \vec{f}_{\alpha\beta} \quad (\text{D.46})$$

Since according to the third law of Newton $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$:

$$\sum_{\alpha,\beta} (\vec{r}_\alpha - \vec{r}_A) \wedge \sum_{\beta \neq \alpha} \vec{f}_{\beta\alpha} = (\vec{r}_\alpha - \vec{r}_A) \wedge \vec{f}_{\beta\alpha} \quad (\text{D.47})$$

We can hypothesize that the forces that particles exert on each other is parallel to the line joining these particles (this is what is called the *strong form of the third law of Newton*, which applies to the gravitational forces).

If that is the case then this term is null:

$$\sum_{\alpha,\beta} (\vec{r}_\alpha - \vec{r}_A) \wedge \sum_{\beta \neq \alpha} \vec{f}_{\beta\alpha} = (\vec{r}_\alpha - \vec{r}_\beta) \wedge \vec{f}_{\beta\alpha} = \vec{0} \quad \text{if } (\vec{r}_\alpha - \vec{r}_\beta) \parallel \vec{f}_{\beta\alpha} \quad (\text{D.48})$$

In this case equation D.45 can be expressed as:

$$\begin{aligned} \left(\frac{d(\vec{\mathcal{L}}_A)_N}{dt} \right)_N &= -(\vec{v}_A)_N \wedge M(\vec{v}_{CM})_N + \\ &+ \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge (\vec{F}_{\alpha\text{EXTERNAL}} + \vec{F}_{\alpha\text{INERTIAL}}) \end{aligned} \quad (\text{D.49})$$

If computed with respect to the center of mass of the system equation D.49 can be expressed as:

$$\left(\frac{d(\vec{\mathcal{L}}_{CM})_N}{dt} \right)_N = \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_{CM}) \wedge (\vec{F}_{\alpha\text{EXTERNAL}} + \vec{F}_{\alpha\text{INERTIAL}}) \quad (\text{D.50})$$

If an inertial reference frame is used for measuring the vectors and their derivatives there would be no inertial forces and equation D.49 could be simplified:

$$\left(\frac{d(\vec{\mathcal{L}}_A)_i}{dt} \right)_i = -(\vec{v}_A)_i \wedge M(\vec{v}_{CM})_i + \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge \vec{F}_{\alpha\text{EXTERNAL}} \quad (\text{D.51})$$

D.5 Equations of motion for a rigid body

The equations of motion for a rigid body will be deduced herein.

D.5.1 Definition of rigid body and properties

D.5.1.1 Definition

A rigid body is an idealization of a solid body in which deformation is neglected. This is, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it.

D.5.1.2 Properties

D.5.1.2.1 Mass

The mass of a rigid body is defined in the same way as for any other system of particles:

$$M = \sum_{\forall \alpha} m_{\alpha} \quad (\text{D.52})$$

D.5.1.2.2 Inertia tensor

The spatial distribution of mass within the rigid body will be provided by the inertial tensor.

The moment of inertia of a rigid body with respect to an axis in the S reference frame is defined as:

$$I_{axis}^S := \sum_{\forall \alpha} m_{\alpha} \cdot d_{axis-\alpha}^2 \quad (\text{D.53})$$

where $d_{axis-\alpha}$ is the distance between the particle α and the axis.

The product of inertia of a rigid body with respect to 2 planes π_1 and π_2 in the S reference frame is defined as:

$$J_{\pi_1 \pi_2}^S := \sum_{\forall \alpha} m_{\alpha} \cdot d_{\pi_1-\alpha} d_{\pi_2-\alpha} \quad (\text{D.54})$$

where $d_{\pi_1-\alpha}$ is the distance between the particle α and the plane π_1 and $d_{\pi_2-\alpha}$ is the distance between the particle α and the plane π_2 .

The inertia tensor of a rigid body with respect to a point O in a frame S given by the axes Ox , Oy and Oz is defined as:

$$I_O^S := \sum_{\forall \alpha} \left(\|\vec{r}_\alpha^S\|^2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - [\vec{r}_\alpha^S] \cdot [\vec{r}_\alpha^S]^T \right) m_\alpha \quad (\text{D.55})$$

This is (neglecting for simplicity the superscript S in every coordinate):

$$\begin{aligned} I_O^S &= \sum_{\forall \alpha} \left((x_\alpha^2 + y_\alpha^2 + z_\alpha^2) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_\alpha^2 & x_\alpha \cdot y_\alpha & x_\alpha \cdot z_\alpha \\ x_\alpha \cdot y_\alpha & y_\alpha^2 & y_\alpha \cdot z_\alpha \\ x_\alpha \cdot z_\alpha & y_\alpha \cdot z_\alpha & z_\alpha^2 \end{bmatrix} \right) m_\alpha \\ &= \sum_{\forall \alpha} \left(\begin{bmatrix} y_\alpha^2 + z_\alpha^2 & -x_\alpha \cdot y_\alpha & -x_\alpha \cdot z_\alpha \\ -x_\alpha \cdot y_\alpha & x_\alpha^2 + z_\alpha^2 & -y_\alpha \cdot z_\alpha \\ -x_\alpha \cdot z_\alpha & -y_\alpha \cdot z_\alpha & x_\alpha^2 + y_\alpha^2 \end{bmatrix} \right) m_\alpha \end{aligned} \quad (\text{D.56})$$

Taking into account the previous definitions of moments and products of inertia this can also be expressed as:

$$I_O^S = \begin{bmatrix} I_x^S & -J_{xy}^S & -J_{xz}^S \\ -J_{xy}^S & I_y^S & -J_{yz}^S \\ -J_{xz}^S & -J_{yz}^S & I_z^S \end{bmatrix} \quad (\text{D.57})$$

It is possible to relate the inertia tensor of a system with respect to a point O in a frame S with the inertia tensor with respect to the center of mass CM in the same axes as S using the Huygens-Steiner theorem:

$$I_O^S = I_{CM}^S + M \cdot \begin{bmatrix} y_{CM}^2 + z_{CM}^2 & -x_{CM} \cdot y_{CM} & -x_{CM} \cdot z_{CM} \\ -x_{CM} \cdot y_{CM} & x_{CM}^2 + z_{CM}^2 & -y_{CM} \cdot z_{CM} \\ -x_{CM} \cdot z_{CM} & -y_{CM} \cdot z_{CM} & x_{CM}^2 + y_{CM}^2 \end{bmatrix} \quad (\text{D.58})$$

where (x_{CM}, y_{CM}, z_{CM}) are the coordinates of the center of mass of the rigid body from O in the S axes.

When the used axes are such that the products of inertia of the rigid body are null then these axes are called *principal axes*.

D.5.1.2.3 Angular velocity of the rigid body

If we consider a body-fixed frame S' with origin in a point A of the body then, from a reference frame S not fixed to the body the velocity vector of a point P in the body would be, according to equation D.13:

$$(\vec{v}_p)_S = (\vec{v}_A)_S + (\vec{v}_p)_{S'} + \vec{\Omega}_{S'S} \wedge (\vec{r}_p)_{S'} \quad (\text{D.59})$$

In the body-fixed frame S' the vector $\vec{AP}$ remains constant with time, this is:

$$(\vec{v}_p)_{S'} = \left(\frac{d(\vec{r}_p)_{S'}}{dt} \right)_{S'} = \left(\frac{d\vec{AP}}{dt} \right)_{S'} = \vec{0} \quad (\text{D.60})$$

so equation D.59 can be expressed as:

$$(\vec{v}_p)_S = (\vec{v}_A)_S + \vec{\Omega}_{S'S} \wedge \vec{AP} \quad (\text{D.61})$$

If we now consider another body-fixed frame S'' with origin in a point B of the body the velocity vector of the point P would be given by:

$$(\vec{v}_p)_S = (\vec{v}_B)_S + \vec{\Omega}_{S''S} \wedge \vec{BP} \quad (\text{D.62})$$

If we apply equation D.62 to the point A instead of point P :

$$(\vec{v}_A)_S = (\vec{v}_B)_S + \vec{\Omega}_{S''S} \wedge \vec{BA} \quad (\text{D.63})$$

Combining equations D.61 and D.63 we have:

$$(\vec{v}_p)_S = (\vec{v}_A)_S + \vec{\Omega}_{S'S} \wedge \vec{AP} = (\vec{v}_B)_S + \vec{\Omega}_{S''S} \wedge \vec{BA} + \vec{\Omega}_{S'S} \wedge \vec{AP} \quad (\text{D.64})$$

We can equal expressions D.62 and D.64 and obtain:

$$\vec{0} = (\vec{\Omega}_{S'S} - \vec{\Omega}_{S''S}) \wedge \vec{AP} \quad (\forall P) \implies \vec{\Omega}_{S'S} = \vec{\Omega}_{S''S} \quad (\text{D.65})$$

This means that all the body-fixed frames have the same angular velocities, which becomes as such a property of the rigid body: its angular velocity.

D.5.2 Equations of motion

In the following equations, the character I indicates an inertial reference frame, whereas the character N indicates a non inertial reference frame. S indicates a generic reference frame that could be inertial or non inertial.

D.5.2.1 Linear momentum

Equations D.37 and D.38 apply to a rigid body without any modification:

$$M \left(\frac{d(\vec{v}_{CM})_N}{dt} \right)_N = \sum_{\forall \alpha} \vec{F}_{\alpha \text{EXTERNAL}} + \sum_{\forall \alpha} \vec{F}_{\alpha \text{INERTIAL}} \quad (\text{D.66})$$

$$M \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i = \sum_{\forall \alpha} \vec{F}_{\alpha \text{EXTERNAL}} \quad (\text{D.67})$$

It is usual, however, to express the velocity vector $(\vec{v}_{CM})_i$ in coordinates of a moving frame. In this case equation D.1 has to be applied in order to obtain $\left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i$ as a function of $\left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_N$, which leads to the following equation (equation D.39):

$$M \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_N + \vec{\Omega}_{Ni} \wedge (\vec{v}_{CM})_i = \sum_{\forall \alpha} \vec{F}_{\alpha \text{EXTERNAL}}$$

D.5.2.2 Angular momentum

The angular momentum of a rigid body with respect to a point A is defined in the same way as for any system of particles.

However it is customary in the case of rigid bodies to denote it using the character $\vec{\mathcal{H}}$:

$$\vec{\mathcal{H}}_A := \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge m_\alpha \vec{v}_\alpha \quad (\text{D.68})$$

If we consider that A is a point of the body, according to equation D.61:

$$(\vec{v}_\alpha)_S = (\vec{v}_A)_S + \vec{\Omega}_{bS} \wedge (\vec{r}_\alpha - \vec{r}_A) \quad (\text{D.69})$$

where b indicates a body-fixed frame, so denoting $(\vec{r}_\alpha - \vec{r}_A) = \vec{r}'_\alpha$ we can express the angular momentum of the rigid body with respect to a point A of the body as:

$$\begin{aligned} (\vec{\mathcal{H}}_A)_S &= \sum_{\forall \alpha} \vec{r}'_\alpha \wedge m_\alpha \left\{ (\vec{v}_A)_S + \vec{\Omega}_{bS} \wedge \vec{r}'_\alpha \right\} = \\ &= \sum_{\forall \alpha} m_\alpha \vec{r}'_\alpha \wedge (\vec{v}_A)_S + \sum_{\forall \alpha} m_\alpha \vec{r}'_\alpha \wedge (\vec{\Omega}_{bS} \wedge \vec{r}'_\alpha) \end{aligned} \quad (\text{D.70})$$

This equation can be re-written according to the definition of the center of mass of the system (equation D.33) as:

$$\left(\vec{\mathcal{H}}_A\right)_S = M(\vec{r}_{CM} - \vec{r}_A) \wedge (\vec{v}_A)_S + \sum_{\forall \alpha} m_\alpha \vec{r}'_\alpha \wedge (\vec{\Omega}_{bS} \wedge \vec{r}'_\alpha) \quad (\text{D.71})$$

The second part of equation D.71 contains the term $\vec{r}'_\alpha \wedge (\vec{\Omega}_{bS} \wedge \vec{r}'_\alpha)$ which can be expressed according to properties of the cross product and dot product of vectors as:

$$\vec{r}'_\alpha \wedge (\vec{\Omega}_{bS} \wedge \vec{r}'_\alpha) = \|\vec{r}'_\alpha\|^2 \vec{\Omega}_{bS} - (\vec{r}'_\alpha \cdot \vec{\Omega}_{bS}) \cdot \vec{r}'_\alpha \quad (\text{D.72})$$

The summation of these terms leads to the following expression:

$$\begin{aligned} & \sum_{\forall \alpha} m_\alpha \vec{r}'_\alpha \wedge (\vec{\Omega}_{bS} \wedge \vec{r}'_\alpha) = \\ & \sum_{\forall \alpha} m_\alpha \left[(x'_\alpha{}^2 + y'_\alpha{}^2 + z'_\alpha{}^2) \cdot (\Omega_x \vec{i} + \Omega_y \vec{j} + \Omega_z \vec{k}) - (x'_\alpha \vec{i} + y'_\alpha \vec{j} + z'_\alpha \vec{k})(x'_\alpha \Omega_x + y'_\alpha \Omega_y + z'_\alpha \Omega_z) \right] = \\ & \sum_{\forall \alpha} m_\alpha \left([(x'_\alpha{}^2 + y'_\alpha{}^2 + z'_\alpha{}^2) \Omega_x - x'_\alpha x'_\alpha \Omega_x - x'_\alpha y'_\alpha \Omega_y - x'_\alpha z'_\alpha \Omega_z] \vec{i} + \right. \\ & \quad \left. [(x'_\alpha{}^2 + y'_\alpha{}^2 + z'_\alpha{}^2) \Omega_y - y'_\alpha x'_\alpha \Omega_x - y'_\alpha y'_\alpha \Omega_y - y'_\alpha z'_\alpha \Omega_z] \vec{j} + \right. \\ & \quad \left. [(x'_\alpha{}^2 + y'_\alpha{}^2 + z'_\alpha{}^2) \Omega_z - z'_\alpha x'_\alpha \Omega_x - z'_\alpha y'_\alpha \Omega_y - z'_\alpha z'_\alpha \Omega_z] \vec{k} \right) = \\ & \quad \begin{bmatrix} I_{x'} & -J_{x'y'} & -J_{x'z'} \\ -J_{x'y'} & I_{y'} & -J_{y'z'} \\ -J_{x'z'} & -J_{y'z'} & I_{z'} \end{bmatrix} \cdot \vec{\Omega}_{bS} = I_A^S \cdot \vec{\Omega}_{bS} \end{aligned} \quad (\text{D.73})$$

This means that the angular momentum of a rigid body with respect to a point A of the body (from equation D.71) can be expressed as:

$$\left(\vec{\mathcal{H}}_A\right)_S = M(\vec{r}_{CM} - \vec{r}_A) \wedge (\vec{v}_A)_S + I_A^S \cdot \vec{\Omega}_{bS} \quad (\text{D.74})$$

Equation D.74 can be simplified when the considered point A is not moving with respect to the used reference frame for measuring the vectors (S) or when A is the center of mass of the rigid body:

$$\begin{aligned} (\vec{v}_A)_S = \vec{0} & \implies \left(\vec{\mathcal{H}}_A\right)_S = I_A^S \cdot \vec{\Omega}_{bS} \\ \left(\vec{\mathcal{H}}_{CM}\right)_S & = I_{CM}^S \cdot \vec{\Omega}_{bS} \end{aligned} \quad (\text{D.75})$$

The equation for the variation of the angular momentum of a system of particles (equation D.51) can be applied in the case of a rigid body:

$$\left(\frac{d(\vec{\mathcal{H}}_A)_i}{dt} \right)_i = -(\vec{v}_A)_i \wedge M(\vec{v}_{CM})_i + \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge \vec{F}_{\alpha \text{EXTERNAL}} \quad (\text{D.76})$$

The momentum of the external forces with respect to a point A is defined as:

$$\vec{\mathcal{M}}_A = \sum_{\forall \alpha} (\vec{r}_\alpha - \vec{r}_A) \wedge \vec{F}_{\alpha \text{EXTERNAL}} \quad (\text{D.77})$$

As a consequence the variation of the angular momentum of the rigid body with respect to a point A is:

$$\left(\frac{d(\vec{\mathcal{H}}_A)_i}{dt} \right)_i = -(\vec{v}_A)_i \wedge M(\vec{v}_{CM})_i + \vec{\mathcal{M}}_A \quad (\text{D.78})$$

Since according to equation D.75 it is much easier to provide the angular momentum of the rigid body in a body-fixed frame (the inertial tensor would remain constant with time) it is easier to apply equation D.1 and obtain $\left(\frac{d(\vec{\mathcal{H}}_A^b)_i}{dt} \right)_i$ from $\left(\frac{d(\vec{\mathcal{H}}_A^b)_i}{dt} \right)_b$:

$$\left(\frac{d(\vec{\mathcal{H}}_A^b)_i}{dt} \right)_b + \vec{\Omega}_{bi} \wedge (\vec{\mathcal{H}}_A^b)_i = -(\vec{v}_A)_i \wedge M(\vec{v}_{CM})_i + \vec{\mathcal{M}}_A \quad (\text{D.79})$$

D.6 Equations of motion for the missile system

The previous chapters describe the equations of motion that apply to a system of particles with constant mass (chapter D.4) and the equations of motion applicable to a rigid body (chapter D.5).

However the fuel-related part of the missile is not a rigid body, since it moves away when burnt. Also, it is not a system of particles with a constant mass, since the fuel leaves this system when it is burnt.

As a consequence, the general equations of motion for the missile system have to be obtained carefully, taking the fuel leaving the nozzles into account.

D.6.1 Considered system

All the missile frame particles and all the fuel particles inside the nozzles surface in t are considered within the missile system. This includes the particles that do not move with respect to the frame of the missile (denoted as $m_{\alpha rigid}$), the fuel particles that after burnt will still be inside the nozzles surface in $t + dt$ (denoted as $m_{\alpha mobile}$ in figure D.2), and the fuel particles that are inside the nozzles surface in t but will be outside in $t + dt$ (denoted as m_{β} in figure D.2).

This missile system is represented with a dotted line in the following figure:

Figure D.2: Considered system of particles for the missile

Since the system is losing mass, its mass has to be computed each time for the particles composing the system:

$$M(t) = \sum_{\forall \alpha} m_{\alpha} + \sum_{\forall \beta} m_{\beta} \quad (\text{D.80})$$

It will be considered that each time the mass associated to the particles that travel together with the frame of the missile is much higher than the mass associated with the particles that have a relative motion with respect to it:

$$\sum_{\forall \alpha} m_{\alpha \text{rigid}}(t) \gg \sum_{\forall \alpha} m_{\alpha \text{mobile}}(t) + \sum_{\forall \beta} m_{\beta}(t) \quad (\text{D.81})$$

As a consequence:

- $\vec{r}_{CM}(t) = \frac{\sum_{\forall \alpha} m_{\alpha} \vec{r}_{\alpha}(t) + \sum_{\forall \beta} m_{\beta} \vec{r}_{\beta}(t)}{M(t)}$ varies very slowly with time as seen from the missile frame.

Note that $(\vec{r}_{CM}(t))_b = \vec{0}$ by definition since the origin of the body frame is always placed in the center of mass (see appendix chapter A.8).

- The system can be considered to be during dt a rigid body (see section D.5) and as such there will be a unique angular velocity for the missile system ($\vec{\Omega}_{bI}$).

The only difference with a normal rigid body will be that in this case the inertia tensor is, as the mass of the system, time-dependent:

$$I_{CM}^b(t) = \begin{bmatrix} I_x^b(t) & -J_{xy}^b(t) & -J_{xz}^b(t) \\ -J_{xy}^b(t) & I_y^b(t) & -J_{yz}^b(t) \\ -J_{xz}^b(t) & -J_{yz}^b(t) & I_z^b(t) \end{bmatrix} \quad (\text{D.82})$$

D.6.2 Linear momentum of the missile system

The derivative of the linear momentum ($\vec{p}$) of the particles of the missile system between t and $t + dt$ is provided by equation D.29 (see section D.4.1):

$$\begin{aligned} \sum_{\forall p} \vec{F}_{p\text{EXTERNAL}} = \sum_{\forall p} m_p \left\{ \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i + \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b + 2 \cdot \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right) + \right. \\ \left. + \frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b + \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \end{aligned} \quad (\text{D.83})$$

Since $\sum_{\forall p} m_p (\vec{r}_p)_b = M(t) \cdot (\vec{r}_{CM}(t))_b = \vec{0}$ the following terms in this equation are null:

$$\sum_{\forall p} m_p \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) = \frac{d\vec{\Omega}_{bi}}{dt} \wedge \left(\sum_{\forall p} m_p (\vec{r}_p)_b \right) = \vec{0} \quad (\text{D.84})$$

$$\sum_{\forall p} m_p \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} = \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge \sum_{\forall p} m_p (\vec{r}_p)_b \right) = \vec{0} \quad (\text{D.85})$$

Also $\sum_{\forall p} m_p = M(t)$ so that:

$$\sum_{\forall p} m_p \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i = M(t) \cdot \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i \quad (\text{D.86})$$

So equation D.83 can be re-written as:

$$\sum_{\forall p} \vec{F}_{p\text{EXTERNAL}} = M(t) \cdot \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i + \sum_{\forall p} m_p \left\{ \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b + 2 \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right) \right\} \quad (\text{D.87})$$

where:

- $\sum_{\forall p} 2m_p \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right)$ is a term due to the Coriolis acceleration.
- $\sum_{\forall p} m_p \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b$ is a term due to the relative acceleration of the particles.

D.6.2.1 Term related to the Coriolis acceleration

The Coriolis term $(\sum_{\forall p} 2m_p (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b))$ can be processed by considering the system as composed by continuous differential elements:

$$\sum_{\forall p} 2m_p (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b) = 2 \left\{ \vec{\Omega}_{bi} \wedge \left(\sum_{\forall p} m_p \cdot (\vec{v}_p)_b \right) \right\} = 2 \left\{ \vec{\Omega}_{bi} \wedge \left(\int_{System} (\vec{v}_p)_b dm \right) \right\} \quad (D.88)$$

Applying the Reynolds transport theorem (see §5.7 in reference [1]) we have:

$$\frac{d}{dt} \int_{System} (\vec{r}_p)_b dm = \frac{\partial}{\partial t} \int_{Vol} (\vec{r}_p)_b \rho_p dVol + \int_{A_{Vol}} (\vec{r}_p)_b \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} \quad (D.89)$$

where Vol is the control volume inside the nozzles surface (see figure D.2), surrounded by a surface A_{Vol} , and ρ_p is the considered density of the differential element.

It has to be noted that, since the mass remains constant for the system between t and $t + dt$, the following expression holds:

$$\frac{d}{dt} \int_{System} (\vec{r}_p)_b dm = \int_{System} \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b dm \quad (D.90)$$

Since, according to D.81, $\int_{Vol} (\vec{r}_p)_b \rho_p dVol \simeq \int_{System} (\vec{r}_p)_b dm$, then:

$$\int_{Vol} (\vec{r}_p)_b \rho_p dVol \simeq \int_{System} (\vec{r}_p)_b dm = M(t) \cdot (\vec{r}_{CM}(t))_b = \vec{0} \quad (D.91)$$

then:

$$\int_{System} \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b dm = \int_{System} (\vec{v}_p)_b dm = \int_{A_{Vol}} (\vec{r}_p)_b \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} \quad (D.92)$$

Taking into account that, according to the system definition (see figure D.2), only the particles β leave the control volume between t and dt through the nozzle surface A_e equation D.92 becomes:

$$\int_{System} (\vec{v}_p)_b dm = \int_{A_e} (\vec{r}_\beta)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \quad (D.93)$$

The mass flow parameter of the missile is defined as the rate of change of the system mass (made positive) caused by the outflow of gases through the nozzle exit plane A_e :

$$\dot{m} = -\frac{dm}{dt} = \int_{A_e} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}] dA_e \quad (D.94)$$

The mass flow center $(\vec{r}_e)_b$ is defined (see [2] (van der Ha and Janssens, 2005)) as the center of mass of the ejected mass in the exhaust surface A_e :

$$(\vec{r}_e)_b = \frac{1}{\dot{m}} \int_{A_e} \rho_\beta (\vec{r}_\beta)_b [(\vec{v}_\beta)_b \cdot \vec{n}] dA_e \quad (D.95)$$

Taking these definitions into account it can be concluded that:

$$\int_{System} (\vec{v}_p)_b dm = \int_{A_e} \rho_\beta (\vec{r}_\beta)_b [(\vec{v}_\beta)_b \cdot \vec{n}] dA_e = \dot{m} \cdot (\vec{r}_e)_b \quad (D.96)$$

so that finally we have that the Coriolis term becomes:

$$\sum_{\forall p} 2m_p \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right) = 2 \left\{ \vec{\Omega}_{bi} \wedge \left(\int_{System} (\vec{v}_p)_b dm \right) \right\} = 2 \left(\vec{\Omega}_{bi} \wedge \dot{m} \cdot (\vec{r}_e)_b \right) \quad (D.97)$$

D.6.2.2 Term related to the relative acceleration

The relative acceleration term $\left(\sum_{\forall p} m_p \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b \right)$ can also be processed by considering the system as composed by continuous differential elements:

$$\sum_{\forall p} m_p \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b = \int_{System} \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b dm \quad (D.98)$$

Applying again the Reynolds transport theorem:

$$\frac{d}{dt} \int_{System} (\vec{v}_p)_b dm = \frac{\partial}{\partial t} \int_{Vol} (\vec{v}_p)_b \rho_p dVol + \int_{A_{Vol}} (\vec{v}_p)_b \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} \quad (D.99)$$

where Vol is the considered control volume inside the nozzles surface (see figure D.2), surrounded by a surface A_{Vol} .

Since the system has a constant mass between t and $t + dt$:

$$\frac{d}{dt} \int_{System} (\vec{v}_p)_b dm = \int_{System} \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b dm \quad (D.100)$$

As a consequence:

$$\int_{System} \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b dm = \frac{\partial}{\partial t} \int_{Vol} (\vec{v}_p)_b \rho_p dVol + \int_{A_{Vol}} (\vec{v}_p)_b \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} \quad (D.101)$$

Taking into account that, according to the system definition (see figure D.2), only the particles β leave the system between t and dt through the nozzles surface A_e equation D.101 becomes:

$$\int_{System} \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b dm = \frac{\partial}{\partial t} \int_{Vol} (\vec{v}_p)_b \rho_p dVol + \int_{A_e} (\vec{v}_\beta)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \quad (D.102)$$

Since according to D.81 we can approximate $\int_{Vol} (\vec{v}_p)_b \rho_p dVol \simeq \int_{System} (\vec{v}_p)_b dm$ and since $\int_{System} (\vec{v}_p)_b dm = \dot{m} \cdot (\vec{r}_e)_b$ (according to equation D.96), then:

$$\int_{System} \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b dm = \left(\frac{d[\dot{m} \cdot (\vec{r}_e)_b]}{dt} \right)_b + \int_{A_e} (\vec{v}_\beta)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \quad (D.103)$$

The mean exhaust velocity $(\vec{v}_e)_b$ is defined (see [2] ([van der Ha and Janssens, 2005](#))) as the mean velocity of the ejected mass in the exhaust surface A_e :

$$(\vec{v}_e)_b = \frac{1}{\dot{m}} \int_{A_e} \rho_\beta (\vec{v}_\beta)_b [(\vec{v}_\beta)_b \cdot \vec{n}] dA_e \quad (\text{D.104})$$

Taking this definition into account it can be concluded that the term related to the relative acceleration becomes:

$$\int_{System} \left(\frac{d(\vec{v}_p)_b}{dt} \right)_b dm = \left(\frac{d[\dot{m} \cdot (\vec{r}_e)_b]}{dt} \right)_b + \dot{m}(\vec{v}_e)_b \quad (\text{D.105})$$

The term related to the relative acceleration will be included among the external forces in the final expression for the linear momentum of the missile (equation [D.106](#)) and thus it will not appear there directly.

This term will be considered as one of the components of the thrust, that accounts for the force that would be noticed having the rocket without motion nor rotation on a static bench. The components of the thrust will be detailed in chapter [4.4](#).

D.6.2.3 Final expression

Taking expression [D.97](#) into account and considering the term related to the relative acceleration of the particles ([D.105](#)) as part of the external forces, as indicated previously, the final expression for the linear momentum of the missile system in t is:

$$\sum_{\forall p} \vec{F}_{p\text{EXTERNAL}} = M(t) \cdot \left(\frac{d(\vec{v}_{CM})_i}{dt} \right)_i + 2 \left(\vec{\Omega}_{bi} \wedge \dot{m} \cdot (\vec{r}_e)_b \right) \quad (\text{D.106})$$

D.6.3 Angular momentum of the missile system

The derivative of the angular momentum of the particles of the missile system with respect to its center of mass ($\vec{\mathcal{H}}_{CM}$) between t and $t + dt$ is provided by equation D.50 (see section D.4.2):

$$\begin{aligned} \sum_{\forall p} (\vec{r}_p)_b \wedge \vec{F}_{p\text{EXTERNAL}} &= \sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_{CM})_i + \sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_p)_b + \\ &+ \sum_{\forall p} (\vec{r}_p)_b \wedge 2m_p \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right) + \sum_{\forall p} (\vec{r}_p)_b \wedge m_p \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) + \\ &+ \sum_{\forall p} (\vec{r}_p)_b \wedge m_p \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \end{aligned} \quad (\text{D.107})$$

Since $\sum_{\forall p} m_p (\vec{r}_p)_b = M(t) \cdot (\vec{r}_{CM}(t))_b = \vec{0}$ the following term in equation D.107 is null:

$$\sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_{CM})_i = \sum_{\forall p} m_p (\vec{r}_p)_b \wedge (\vec{a}_{CM})_i = \vec{0} \quad (\text{D.108})$$

so equation D.107 can be re-written as:

$$\begin{aligned} \sum_{\forall p} (\vec{r}_p)_b \wedge \vec{F}_{p\text{EXTERNAL}} &= \sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_p)_b + \\ &+ \sum_{\forall p} (\vec{r}_p)_b \wedge 2m_p \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right) + \sum_{\forall p} (\vec{r}_p)_b \wedge m_p \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) + \\ &+ \sum_{\forall p} (\vec{r}_p)_b \wedge m_p \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \end{aligned} \quad (\text{D.109})$$

where:

- $\sum_{\forall p} (\vec{r}_p)_b \wedge m_p \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\}$ is a term due to the centrifugal acceleration.
- $\sum_{\forall p} (\vec{r}_p)_b \wedge m_p \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right)$ is a term due to the angular acceleration.
- $\sum_{\forall p} (\vec{r}_p)_b \wedge 2m_p \left(\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b \right)$ is a term due to the Coriolis acceleration.
- $\sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_p)_b$ is a term due to the relative acceleration of the particles.

D.6.3.1 Term related to the centrifugal acceleration

The term related to the centrifugal acceleration can be processed using the Jacobi identity:

$$\vec{a} \wedge (\vec{b} \wedge \vec{c}) + \vec{b} \wedge (\vec{c} \wedge \vec{a}) + \vec{c} \wedge (\vec{a} \wedge \vec{b}) = \vec{0} \quad (\text{D.110})$$

Applying this expression to the product $(\vec{r})_b \wedge \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \right\}$ we get:

$$(\vec{r})_b \wedge \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \right\} + \vec{\Omega}_{bi} \wedge \left\{ \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \wedge (\vec{r})_b \right\} = \vec{0} \quad (\text{D.111})$$

so:

$$(\vec{r})_b \wedge \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \right\} = \vec{\Omega}_{bi} \wedge \left\{ (\vec{r})_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \right\} \quad (\text{D.112})$$

As a consequence the term related to the centrifugal acceleration becomes:

$$\sum_{\forall p} m_p (\vec{r}_p)_b \wedge \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} = \vec{\Omega}_{bi} \wedge \sum_{\forall p} m_p \left\{ (\vec{r})_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \right\} \quad (\text{D.113})$$

Using again the hypothesis [D.81](#) that allows considering the system to be each time a rigid body with a time-dependent inertia tensor, and considering expression [D.73](#) for rigid bodies we get:

$$\sum_{\forall p} m_p \left\{ (\vec{r})_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r})_b \right) \right\} = I_{CM}^b(t) \cdot \vec{\Omega}_{bi} \quad (\text{D.114})$$

so that the centrifugal acceleration term becomes:

$$\sum_{\forall p} m_p (\vec{r}_p)_b \wedge \left\{ \vec{\Omega}_{bi} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} = \vec{\Omega}_{bi} \wedge I_{CM}^b(t) \cdot \vec{\Omega}_{bi} \quad (\text{D.115})$$

D.6.3.2 Term related to the angular acceleration

The term due to the angular acceleration $(\sum_{\forall p} m_p (\vec{r}_p)_b \wedge \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right))$ is similar to expression [D.114](#) so this term can be written as:

$$\sum_{\forall p} m_p (\vec{r}_p)_b \wedge \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) = I_{CM}^b(t) \cdot \frac{d\vec{\Omega}_{bi}}{dt} \quad (\text{D.116})$$

In this equation we are also considering the hypothesis [D.81](#): we will neglect the inertia terms related to the particles with a relative motion with respect to the missile frame in comparison with the inertia of the particles that travel together with the missile frame. With this hypothesis the system will be considered to be each time a rigid body with a time-dependent inertia tensor.

D.6.3.3 Term related to the Coriolis acceleration

The term $\sum_{\forall p} (\vec{r}_p)_b \wedge 2m_p (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b)$ is related to the Coriolis acceleration.

The following derivative expression holds:

$$\begin{aligned} \left(\frac{d \left\{ (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b) \right\}}{dt} \right)_b &= \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b) + (\vec{r}_p)_b \wedge \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) + \\ &+ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \right) \end{aligned} \quad (\text{D.117})$$

Since, according to the Jacobi identity:

$$\left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b) = (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \right) + \vec{\Omega}_{bi} \wedge \left(\left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \wedge (\vec{r}_p)_b \right) \quad (\text{D.118})$$

expression D.117 can be re-written as:

$$\begin{aligned} \left(\frac{d \left\{ (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b) \right\}}{dt} \right)_b &= (\vec{r}_p)_b \wedge \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) + 2 \cdot (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \right) + \\ &+ \vec{\Omega}_{bi} \wedge \left(\left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \wedge (\vec{r}_p)_b \right) \end{aligned} \quad (\text{D.119})$$

As a consequence the term related to the Coriolis acceleration becomes:

$$\begin{aligned} \sum_{\forall p} 2m_p (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b) &= \sum_{\forall p} m_p \left(\frac{d \left\{ (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b) \right\}}{dt} \right)_b + \\ &- \sum_{\forall p} m_p (\vec{r}_p)_b \wedge \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) - \sum_{\forall p} m_p \vec{\Omega}_{bi} \wedge \left(\left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \wedge (\vec{r}_p)_b \right) \end{aligned} \quad (\text{D.120})$$

The following term has already been identified before:

$$\sum_{\forall p} m_p \left\{ (\vec{r}_p)_b \wedge \left(\frac{d\vec{\Omega}_{bi}}{dt} \wedge (\vec{r}_p)_b \right) \right\} = I_{CM}^b(t) \cdot \frac{d\vec{\Omega}_{bi}}{dt} \quad (\text{see equation D.116}) \quad (\text{D.121})$$

so the term related to the Coriolis acceleration is:

$$\begin{aligned} \sum_{\forall p} 2m_p (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b) &= \sum_{\forall p} m_p \left(\frac{d \left\{ (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b) \right\}}{dt} \right)_b + \\ &- I_{CM}^b \cdot \frac{d\vec{\Omega}_{bi}}{dt} - \vec{\Omega}_{bi} \wedge \sum_{\forall p} m_p \left(\left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \wedge (\vec{r}_p)_b \right) \end{aligned} \quad (\text{D.122})$$

Considering the system as composed by continuous differential elements:

$$\sum_{\forall p} m_p \left(\frac{d \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\}}{dt} \right)_b = \int_{System} \left(\frac{d \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\}}{dt} \right)_b dm \quad (D.123)$$

If we apply now the Reynolds transport theorem we get:

$$\begin{aligned} \frac{d}{dt} \int_{System} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} dm &= \frac{\partial}{\partial t} \int_{Vol} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \rho_p dVol \\ &+ \int_{A_{Vol}} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} \end{aligned} \quad (D.124)$$

where Vol is the control volume inside the nozzles surface (see figure D.2), surrounded by a surface A_{Vol} , and ρ_p is the considered density of the differential element.

Since the mass of the system remains constant between t and $t + dt$, the following expression holds:

$$\frac{d}{dt} \int_{System} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} dm = \int_{System} \left(\frac{d \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\}}{dt} \right)_b dm \quad (D.125)$$

Taking into account that, according to the system definition (see figure D.2), only the particles β leave the control volume between t and dt through the nozzles surface A_e :

$$\begin{aligned} \int_{A_{Vol}} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} &= \\ \int_{A_e} \left\{ (\vec{r}_\beta)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_\beta)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \end{aligned} \quad (D.126)$$

On the other hand, according to the hypothesis D.81:

$$\begin{aligned} \int_{Vol} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \rho_p dVol &\simeq \\ \int_{System} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} dm \end{aligned} \quad (D.127)$$

so taking equation D.114 into account:

$$\begin{aligned} \int_{Vol} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} \rho_p dVol &\simeq \\ \int_{System} \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\} dm &= I_{CM}^b(t) \cdot \vec{\Omega}_{bi} \end{aligned} \quad (D.128)$$

and we finally get:

$$\int_{System} \left(\frac{d \left\{ (\vec{r}_p)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_p)_b \right) \right\}}{dt} \right)_b dm = \left(\frac{d(I_{CM}^b(t) \cdot \vec{\Omega}_{bi})}{dt} \right)_b + \int_{A_e} \left\{ (\vec{r}_\beta)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_\beta)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \quad (D.129)$$

The term $\int_{A_e} \left\{ (\vec{r}_\beta)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_\beta)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e$ is usually called "jet-damping" because it has been shown (see [3]) to have attenuating effects on the angular rates in some types of rocket systems.

Considering that in A_e :

$$(\vec{r}_\beta)_b = (\vec{r}_e)_b + \vec{v} \quad (D.130)$$

the term related to the jet damping can be expressed as:

$$\begin{aligned} & \int_{A_e} \left\{ (\vec{r}_\beta)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_\beta)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \\ & \int_{A_e} \left\{ ((\vec{r}_e)_b + \vec{v}) \wedge \left(\vec{\Omega}_{bi} \wedge ((\vec{r}_e)_b + \vec{v}) \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \\ & \quad \textcircled{1} \int_{A_e} \left\{ (\vec{r}_e)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e + \\ & \quad \textcircled{2} \int_{A_e} \left\{ (\vec{r}_e)_b \wedge \left(\vec{\Omega}_{bi} \wedge \vec{v} \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e + \\ & \quad \textcircled{3} \int_{A_e} \left\{ \vec{v} \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e + \\ & \quad \textcircled{4} \int_{A_e} \left\{ \vec{v} \wedge \left(\vec{\Omega}_{bi} \wedge \vec{v} \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \end{aligned} \quad (D.131)$$

Taking the definition of the mass flow parameter (equation D.94) into account the term $\textcircled{1}$ in equation D.131 can be expressed as:

$$\int_{A_e} \left\{ (\vec{r}_e)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \dot{m} \left\{ (\vec{r}_e)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b \right) \right\} \quad (D.132)$$

The terms $\textcircled{2}$ and $\textcircled{3}$ in equation D.131 are null since from the definition of $(\vec{r}_e)_b$ (see equation D.95):

$$\int_{A_e} \vec{v} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \vec{0} \quad (D.133)$$

The term $\textcircled{4}$, on the other hand, can be ignored since for the particles in A_e , $\|\vec{v}\| \ll \|(\vec{r}_e)_b\|$.

This means that the term related to the jet damping can be expressed as:

$$\begin{aligned} & \int_{A_e} \left\{ (\vec{r}_\beta)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_\beta)_b \right) \right\} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \simeq \\ & \quad \dot{m} \left\{ (\vec{r}_e)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b \right) \right\} \end{aligned} \quad (D.134)$$

As a consequence the term related to the Coriolis acceleration becomes:

$$\begin{aligned} \sum_{\forall p} 2m_p (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b) &= \left(\frac{d(I_{CM}^b(t) \cdot \vec{\Omega}_{bi})}{dt} \right)_b + \dot{m} \left\{ (\vec{r}_e)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b) \right\} + \\ &\quad - I_{CM}^b \cdot \frac{d\vec{\Omega}_{bi}}{dt} + \vec{\Omega}_{bi} \wedge \sum_{\forall p} m_p \left\{ (\vec{r}_p)_b \wedge \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \right\} \end{aligned} \quad (\text{D.135})$$

Since only some particles ($m_{\alpha_{mobile}}$ and m_β) move with respect to the missile frame (see D.2), and the center of mass of the missile system moves very slowly with respect to the frame (see D.81), $(\vec{v}_p)_b$ can be neglected for most of the particles. For the moving particles, the velocity is mostly directed in the direction to the exhaust. As a consequence, for most of the moving particles, $(\vec{r}_p)_b \parallel (\vec{v}_p)_b$. This means that the following term can be neglected:

$$\sum_{\forall p} m_p \left\{ (\vec{r}_p)_b \wedge \left(\frac{d(\vec{r}_p)_b}{dt} \right)_b \right\} \simeq \vec{0} \quad (\text{D.136})$$

and the term related to the Coriolis acceleration can be expressed as:

$$\begin{aligned} \sum_{\forall p} 2m_p (\vec{r}_p)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{v}_p)_b) &= \left(\frac{dI_{CM}^b(t)}{dt} \right)_b \cdot \vec{\Omega}_{bi} + \\ &\quad + \dot{m} \left\{ (\vec{r}_e)_b \wedge (\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b) \right\} \end{aligned} \quad (\text{D.137})$$

D.6.3.4 Term related to the relative acceleration

The term $\sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_p)_b$ is related to the relative acceleration of the particles.

Considering the system as composed by continuous differential elements this term becomes:

$$\sum_{\forall p} (\vec{r}_p)_b \wedge m_p (\vec{a}_p)_b = \int_{System} (\vec{r}_p)_b \wedge (\vec{a}_p)_b dm \quad (D.138)$$

Applying the Reynolds transport theorem:

$$\begin{aligned} \frac{d}{dt} \int_{System} (\vec{r}_p)_b \wedge (\vec{v}_p)_b dm &= \frac{\partial}{\partial t} \int_{Vol} (\vec{r}_p)_b \wedge (\vec{v}_p)_b \rho_p dVol + \\ &+ \int_{A_{Vol}} (\vec{r}_p)_b \wedge (\vec{v}_p)_b \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} \end{aligned} \quad (D.139)$$

where Vol is the control volume inside the nozzles surface (see figure D.2), surrounded by a surface A_{Vol} , and ρ_p is the considered density of the differential element.

Since the mass of the system remains constant between t and $t + dt$, the following expression holds:

$$\begin{aligned} \frac{d}{dt} \int_{System} ((\vec{r}_p)_b \wedge (\vec{v}_p)_b) dm &= \int_{System} \left(\frac{d((\vec{r}_p)_b \wedge (\vec{v}_p)_b)}{dt} \right)_b dm = \\ &= \int_{System} ((\vec{r}_p)_b \wedge (\vec{a}_p)_b) dm \end{aligned} \quad (D.140)$$

On the other hand using the hypothesis D.81 we can approximate :

$$\int_{Vol} (\vec{r}_p)_b \wedge (\vec{v}_p)_b \rho_p dVol \simeq \int_{System} (\vec{r}_p)_b \wedge (\vec{v}_p)_b dm \quad (D.141)$$

and with the same assumptions as in D.136:

$$\int_{System} (\vec{r}_p)_b \wedge (\vec{v}_p)_b dm \simeq \vec{0} \quad (D.142)$$

Taking into account that, according to the system definition (see figure D.2), only the particles β leave the system between t and dt through the exhaust surface A_e :

$$\int_{A_{Vol}} (\vec{r}_p)_b \wedge (\vec{v}_p)_b \rho_p [(\vec{v}_p)_b \cdot \vec{n}_{A_{Vol}}] dA_{Vol} = \int_{A_e} (\vec{r}_\beta)_b \wedge (\vec{v}_\beta)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \quad (D.143)$$

As a consequence the relative acceleration term becomes:

$$\int_{System} (\vec{r}_p)_b \wedge (\vec{a}_p)_b dm = \int_{A_e} (\vec{r}_\beta)_b \wedge (\vec{v}_\beta)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e \quad (D.144)$$

If we consider that in A_e :

$$\begin{aligned}(\vec{r}_\beta)_b &= (\vec{r}_e)_b + \vec{v} \\ (\vec{v}_\beta)_b &= (\vec{v}_e)_b + \vec{\eta}\end{aligned}\tag{D.145}$$

then:

$$\begin{aligned}& \int_{A_e} (\vec{r}_\beta)_b \wedge (\vec{v}_\beta)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \\& \textcircled{1} \int_{A_e} (\vec{r}_e)_b \wedge (\vec{v}_e)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e + \\& \textcircled{2} \int_{A_e} (\vec{r}_e)_b \wedge (\vec{\eta}) \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e + \\& \textcircled{3} \int_{A_e} \vec{v} \wedge (\vec{v}_e)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e + \\& \textcircled{4} \int_{A_e} \vec{v} \wedge \vec{\eta} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e\end{aligned}\tag{D.146}$$

Taking the definition of the mass flow parameter (equation D.94) into account the term $\textcircled{1}$ in equation D.146 can be expressed as:

$$\int_{A_e} (\vec{r}_e)_b \wedge (\vec{v}_e)_b \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \dot{m}[(\vec{r}_e)_b \wedge (\vec{v}_e)_b]\tag{D.147}$$

The terms $\textcircled{2}$ and $\textcircled{3}$ in equation D.131 are null since from the definition of $(\vec{r}_e)_b$ (see equation D.95) and $(\vec{v}_e)_b$ (see equation D.104):

$$\int_{A_e} \vec{v} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \vec{0}\tag{D.148}$$

$$\int_{A_e} \vec{\eta} \rho_\beta [(\vec{v}_\beta)_b \cdot \vec{n}_{A_e}] dA_e = \vec{0}\tag{D.149}$$

The term $\textcircled{4}$ in D.146 can be ignored since for the particles in A_e $\|\vec{v}\| \ll \|(\vec{r}_e)_b\|$ and $\|\vec{\eta}\| \ll \|(\vec{v}_e)_b\|$.

This means that the term related to the relative acceleration can be expressed as:

$$\int_{System} (\vec{r}_p)_b \wedge (\vec{a}_p)_b dm = \dot{m}[(\vec{r}_e)_b \wedge (\vec{v}_e)_b]\tag{D.150}$$

D.6.3.5 Final expression

Taking expressions D.115, D.116, D.137 and D.150 into account the final equation for the angular momentum of the missile system in t is:

$$\begin{aligned}\sum_{\forall p} (\vec{r}_p)_b \wedge \vec{F}_{p\text{EXTERNAL}} &= \vec{\Omega}_{bi} \wedge I_{CM}^b(t) \cdot \vec{\Omega}_{bi} + I_{CM}^b(t) \cdot \frac{d\vec{\Omega}_{bi}}{dt} + \\&+ \left(\frac{dI_{CM}^b(t)}{dt} \right)_b \cdot \vec{\Omega}_{bi} + \dot{m} \left\{ (\vec{r}_e)_b \wedge \left(\vec{\Omega}_{bi} \wedge (\vec{r}_e)_b \right) \right\} + \\&+ \dot{m}[(\vec{r}_e)_b \wedge (\vec{v}_e)_b]\end{aligned}\tag{D.151}$$

Appendix D references

- [1] Irving H. Shames. *Mecánica de Fluidos*. McGraw-Hill, Santa Fé de Bogotá, tercera edition, 1995. [393](#)
- [2] Josef van der Ha and Frank L. Janssens. Jet-damping and misalignment effects during solid-rocket-motor burn. *Journal of Guidance, Control and Dynamics*, 28(3), May-June 2005. doi: 10.2514/1.3852. [393](#), [395](#)
- [3] Fidelis O. Eke. *Technical Report NASA/CRE-1998-208246: Dynamics of Variable Mass Systems*. NASA, January 1998. [400](#)
- [4] R.A. Collins and W. A. Miller. *Several Models of Six Degree of Freedom Equations of Motion for a Ballistic Missile*. *Chrysler Report No. AA-TN-10-61*. *Air Force Technical Note No. BSD-TDR-62-22*. Air Force Ballistic Systems Division, Inglewood, California, March 1962.
- [5] M. Prieto-Alberca. *Apuntes de Mecánica Racional, Cinemática y Estática* printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, November 1991.
- [6] M. Prieto-Alberca. *Curso de Mecánica Racional, Dinámica* printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, November 1991.
- [7] Rafael Ramis-Abril. *Guiones de Mecánica I* printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, September 2004.
- [8] Miguel Ángel Gómez Tierno, Manuel Pérez Cortés, and César Puentes Márquez. *Mecánica del Vuelo*. Instituto Universitario de Microgravedad Ignacio da Riva (IDR/UPM), Madrid, first edition, 2009.
- [9] Pedro Sanz Aránguez. *Resúmenes de Misiles y Vehículos Espaciales* printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, 1996.
- [10] Yunus A. Çengel and John M. Cimbala. *Fluid Mechanics, Fundamentals and Applications*. McGraw Hill, New York, September 2006.

Appendix E

Gravitational potential

The expressions for the gravitational potential are detailed herein.

The expressions indicated in this appendix are used in [chapter 4](#) and [chapter 7](#).

E.1 Gravitational potential

Newton's theory of gravitation (1687) states that two masses attract each other with an attractive force given by the following formula:

$$\vec{F}_{1 \text{ over } 2} = -G \frac{m_1 \cdot m_2}{\|\vec{r}_{12}\|^2} \cdot \frac{\vec{r}_{12}}{\|\vec{r}_{12}\|} = -\vec{F}_{2 \text{ over } 1} \quad (\text{E.1})$$

If we want to know the effect of m on any arbitrary mass m_2 in any point of space then we can take into account the gravitational attraction ($\vec{a}$) of m which in this case would be a vector field given by:

$$\vec{a} = -G \frac{m}{\|\vec{r}\|^2} \cdot \frac{\vec{r}}{\|\vec{r}\|} \quad (\text{E.2})$$

being $\vec{r}$ a position vector with origin in the position of m .

This attraction field is measured in *Gal* where $1 \text{ Gal} = 10^{-2} \text{ m/s}^2 = 1 \text{ cm/s}^2$.

The gravitational attraction field $\vec{a}$ is a conservative field, since it is curl-free:

$$\nabla \times \vec{a} = \vec{0} \quad (\text{E.3})$$

Since the curl of any gradient field is always zero ($\nabla \times \nabla F = \vec{0} \quad \forall F$), the field $\vec{a}$ can be written as a gradient of a scalar field, called gravitational potential, V , which would satisfy:

$$V = \int_A^B \vec{a} \cdot d\vec{r} \quad \forall (\text{path A-B}), \text{ since the field is conservative} \quad (\text{E.4})$$

$$\vec{a} = \nabla V = \begin{Bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial V}{\partial z} \end{Bmatrix}$$

In this case (a single point mass) the gravitational potential would be given by:

$$V = G \frac{m}{r} \quad (\text{E.5})$$

with $r = \|\vec{r}\|$

The superposition principle applies to the gravitational field. This is, the gravitational potential of a system of masses would be given by:

$$V = \sum_{i=1}^N V_i = G \frac{m_1}{r_1} + G \frac{m_2}{r_2} + \dots + G \frac{m_N}{r_N} = G \sum_{i=1}^N \frac{m_i}{r_i} \quad (\text{E.6})$$

and the gravitational attraction would be given by:

$$\vec{a} = \nabla V = \sum_{i=1}^N \nabla V_i = -G \sum_{i=1}^N \frac{m_i}{r_i^3} \cdot \vec{r}_i \quad (\text{E.7})$$

This superposition principle can be easily applied to a continuous system having:

$$V = G \iiint_{\Omega} \frac{\rho(x, y, z)}{r} \, dxdydz \quad (\text{E.8})$$

and the gravitational attraction would be given by:

$$\vec{a} = \nabla V = -G \iiint_{\Omega} \frac{\rho(x, y, z)}{r^3} \cdot \vec{r} \, dxdydz \quad (\text{E.9})$$

The Gauss divergence theorem indicates that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface:

$$\iiint_{Volume} \nabla \cdot \vec{a} \, dVolume = \iint_S \vec{a} \cdot d\vec{S} \quad (\text{E.10})$$

Intuitively, this theorem states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

Applying this theorem for an infinitesimal volume outside the Earth we get:

$$\iiint_{Volume} \nabla \cdot \vec{a} \, dVolume = \vec{0} \implies \nabla \cdot \vec{a} = \vec{0} \quad (\text{E.11})$$

If we combine the zero curl property of the gravitational field and the null divergence (outside the Earth) we get:

$$\begin{aligned} \nabla \times \vec{a} = \vec{0} &\implies \vec{a} = \nabla V \\ \nabla \cdot \vec{a} = \vec{0} &\implies \nabla \cdot \nabla V = \Delta V = 0 \end{aligned} \quad (\text{E.12})$$

Equation E.12 indicates that the gravitational potential V has to fulfill the Laplace equation.

Any function satisfying the Laplace equation is called *harmonic*. This is, the gravitational potential V is *harmonic*.

If we use spherical coordinates:

- r = radius
- φ' = geocentric latitude
- λ = longitude

for the Laplace equation and then apply separation of variables we get the following 4 base functions satisfying Laplace's equation:

$$\left\{ \begin{array}{c} r^{-(\ell+1)} \\ r^{\ell} \end{array} \right\} \cdot P_{\ell m}(\sin \varphi') \cdot \left\{ \begin{array}{c} \cos m\lambda \\ \sin m\lambda \end{array} \right\} \quad (\text{E.13})$$

where $P_{\ell m}(\sin \varphi')$ are the associated Legendre functions of the first kind.

These functions are called *solid spherical harmonics*. *Harmonics* because they solve Laplace's equation, *spherical* because they have spherical coordinates as argument and *solid* because they are functions in three dimensions.

Any solution of the Laplace's equation will be given as a summation of these 4 base functions multiplied by coefficients.

If we leave the radial part out we have the so-called *surface spherical harmonics*:

$$Y_{\ell m}(\varphi', \lambda) = P_{\ell m}(\sin \varphi') \cdot \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \quad (\text{E.14})$$

where:

- ℓ is the spherical harmonic degree
- m is the spherical harmonic order ($m \leq \ell$)

Imposing the regularity condition:

$$\lim_{r \rightarrow \infty} V(r) = 0 \quad (\text{E.15})$$

the amplifying terms (r^ℓ) disappear.

We will also impose a Dirichlet boundary condition on a reference surface, this is:

$$V(R) = V_0. \quad (\text{E.16})$$

which leads to the following expression for the gravitational potential:

$$V(r, \varphi', \lambda) = V_0 \sum_{\ell=0}^{\ell_{max}} \left(\frac{R}{r} \right)^{\ell+1} \sum_{m=0}^{\ell} P_{\ell m}(\sin \varphi') \cdot (u_{\ell m} \cos m\lambda + v_{\ell m} \sin m\lambda) \quad (\text{E.17})$$

It is customary, though, to express the gravitational potential using the following expression:

$$V(r, \varphi', \lambda) = \frac{GM}{R} \sum_{\ell=0}^{\ell_{max}} \left(\frac{R}{r} \right)^{\ell+1} \sum_{m=0}^{\ell} \bar{P}_{\ell m}(\sin \varphi') \cdot (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda) \quad (\text{E.18})$$

or

$$V(r, \varphi', \lambda) = \frac{GM}{r} \sum_{\ell=0}^{\ell_{max}} \left(\frac{R}{r} \right)^{\ell} \sum_{m=0}^{\ell} \bar{P}_{\ell m}(\sin \varphi') \cdot (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda) \quad (\text{E.19})$$

where:

- r, φ', λ = geocentric coordinates of the point
(radius, latitude, longitude)
- R = reference radius
- GM = product of the gravitational constant and the mass of the Earth considered in the model
- ℓ = degree of the spherical harmonic terms
- ℓ_{max} = maximum degree considered in the model
- m = order of the spherical harmonic terms
- $\bar{P}_{\ell m}$ = fully normalized Legendre functions
- $\bar{C}_{\ell m}, \bar{S}_{\ell m}$ = Stokes' coefficients (fully normalized)

The product $GM \times Y_{00}$ represents the gravitational constant times the mass of the Earth associated with the model. As a consequence, Y_{00} is defined as 1 to preserve the meaning of GM itself.

The degree 1 harmonic coefficients ($C_{10}, S_{10}, C_{11}, S_{11}, C_{12}, S_{12}$) are related to the geocenter coordinates and are zero if the origin of the coordinates system coincides with the geocenter (as it is the case with WGS84).

Taking this into account the gravitational potential in the WGS84 ECEF coordinates frame can be expressed as (using a as the reference radius R):

$$V(r, \varphi', \lambda) = \frac{GM}{r} \left[1 + \sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r} \right)^\ell \sum_{m=0}^{\ell} \bar{P}_{\ell m}(\sin \varphi') \cdot (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda) \right] \quad (E.20)$$

This equation can also be expressed as:

$$V(r, \varphi', \lambda) = \frac{GM}{r} \left[1 + \sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r} \right)^\ell \sum_{m=0}^{\ell} \bar{P}_{\ell 0} \bar{C}_{\ell 0}(\sin \varphi') + \sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r} \right)^\ell \sum_{m=1}^{\ell} \bar{P}_{\ell m}(\sin \varphi') \cdot (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda) \right] \quad (E.21)$$

where the solution is divided in two types of expressions:

- zonal terms: $\bar{P}_{\ell 0} \bar{C}_{\ell 0}(\sin \varphi')/r^\ell$
- tesseral terms: $\bar{P}_{\ell m}(\sin \varphi') \cdot (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda)/r^\ell$

It is also customary when expression E.21 is used to denote the Stokes' coefficients (fully normalized) $\bar{C}_{\ell 0}$ as J_ℓ

r can be easily computed from the geodetic coordinates obtaining first the ECEF coordinates:

$$\left. \begin{aligned} x &= (N(\varphi) + h) \cdot \cos \varphi \cdot \cos \lambda \\ y &= (N(\varphi) + h) \cdot \cos \varphi \cdot \sin \lambda \\ z &= (N(\varphi) \cdot (1 - e^2) + h) \cdot \sin \varphi \end{aligned} \right\} \implies r = \sqrt{x^2 + y^2 + z^2} \quad (E.22)$$

The relationship between geocentric latitude and geodetic latitude can be obtained from figure A.5:

$$\tan \varphi' = \tan \frac{z}{\sqrt{x^2 + y^2}} \text{ with } \implies \tan \varphi' = \frac{N(\varphi)(1 - e^2) + h}{N(\varphi) + h} \cdot \tan \varphi \quad (E.23)$$

When $h = 0$ this equation leads to the simplified expression:

$$\tan \varphi' = (1 - e^2) \cdot \tan \varphi \quad (\text{only for } h=0) \quad (E.24)$$

The Stokes's coefficients are provided by a gravitational model. The most recent one is EGM2008 ([1]). This model provides the Stokes's coefficients for a 'zero tide' system and a 'tide free' system in a file with rows formatted in the following way:

$$\ell, m, \bar{C}_{\ell m}, \bar{S}_{\ell m}, \sigma(\bar{C}_{\ell m}), \sigma(\bar{S}_{\ell m})$$

EGM2008 is a model complete up to degree (ℓ) and order (m) 2159. It also contains additional spherical harmonic coefficients extending to degree 2190 and order 2159.

The scaling factors used by EGM2008 are:

$$GM = 3986004.415 \cdot 10^8 m^3/s^2$$

$$a = 6378136.3m$$

The first coefficients for the 'zero tide' system is shown as an example in table E.1.

Table E.1: EGM2008 gravitational model coefficients ('zero tide') for $\ell \leq 3$

ℓ	m	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	$\sigma(\bar{C}_{\ell m})$	$\sigma(\bar{S}_{\ell m})$
2	0	-0.48416931D-03	0	0.74812394D-11	0
2	1	-0.20661550D-09	0.13844138D-08	0.70637815D-11	0.73483472D-11
2	2	0.24393835D-05	-0.14002737D-05	0.72302317D-11	0.74258169D-11
3	0	0.957161D-06	0	0.573143D-11	0
3	1	0.203046D-05	0.248200D-06	0.572663D-11	0.597669D-11
3	2	0.904787D-06	-0.619005D-06	0.637477D-11	0.640183D-11
3	3	0.721321D-06	0.141434D-05	0.602913D-11	0.602831D-11

Once the coefficients are known the only missing part in order to compute E.20 are the fully normalized Legendre functions ($\bar{P}_{\ell m}$). These functions are defined as:

$$\bar{P}_{\ell m}(\sin \varphi') = \left[\frac{(\ell - m)!(2\ell + 1)k}{(\ell + m)!} \right]^{1/2} P_{\ell m}(\sin \varphi') \quad (\text{E.25})$$

where:

$$k = 1 \quad \text{if } \ell = 0$$

$$k = 2 \quad \text{if } m \neq 0 \quad (\text{E.26})$$

being $P_{\ell m}(t)$ the associated Legendre function, defined as:

$$P_{\ell m}(t) = (1 - t^2)^{m/2} \frac{d^m [P_{\ell}(t)]}{dt^m} \quad (\text{E.27})$$

where $P_{\ell}(t)$ is the Legendre polynomial defined as:

$$P_{\ell}(t) = \frac{1}{2^{\ell} \ell!} \frac{d^{\ell} (t^2 - 1)^{\ell}}{dt^{\ell}} \quad (\text{E.28})$$

The computation of the fully normalized Legendre functions by performing these operations is very difficult. It is much easier to obtain them with the following recursive procedure:

$$\begin{aligned}\bar{P}_{00}(\sin \varphi') &= 1 \\ \bar{P}_{\ell\ell}(\sin \varphi') &= W_{mm} \cos \varphi' \bar{P}_{\ell-1,\ell-1}(\sin \varphi') \\ \bar{P}_{\ell m}(\sin \varphi') &= W_{\ell m} \left[\sin \varphi' \bar{P}_{\ell-1,m}(\sin \varphi') - W_{\ell-1,m}^{-1} \bar{P}_{\ell-2,m}(\sin \varphi') \right]\end{aligned}\tag{E.29}$$

with:

$$\begin{aligned}\bar{P}_{\ell-1,m} &= \bar{P}_{\ell-1,\ell-1} = 0 \quad \text{if } \ell < 1 \\ \bar{P}_{\ell-2,m} &= 0 \quad \text{if } \ell < 2 \\ W_{11} &= \sqrt{3} \quad W_{mm} = \sqrt{\frac{2m+1}{2m}} \quad W_{\ell m} = \sqrt{\frac{(2\ell+1)(2\ell-1)}{(\ell+m)(\ell-m)}}\end{aligned}\tag{E.30}$$

The derivatives of the Legendre functions can be computed according to [2] (Bosch, 2000) with the following equations:

$$\begin{aligned}\frac{dP_{00}}{d\theta} &= 0 \quad \frac{dP_{\ell\ell}}{d\theta} = \ell P_{\ell,\ell-1} \\ \text{if } m = 0 &\implies \frac{dP_{\ell 0}}{d\theta} = -P_{\ell 1} \\ \text{if } m \neq 0 &\implies \frac{dP_{\ell m}}{d\theta} = \frac{1}{2} [(\ell+m)(\ell-m+1)P_{\ell,m-1} - P_{\ell,m+1}]\end{aligned}\tag{E.31}$$

where θ is the geocentric colatitude ($\frac{\pi}{2} - \varphi'$) so:

$$\frac{dP_{\ell m}}{d\varphi'} = -\frac{dP_{\ell m}}{d\theta}\tag{E.32}$$

The attraction force related to this potential V will be given by the gradient of the gravitational potential computed in spherical coordinates:

$$\vec{a} = \nabla V = \frac{\partial V}{\partial r} \vec{u}_r + \frac{1}{r} \frac{\partial V}{\partial \varphi'} \vec{u}_{\varphi'} + \frac{1}{r \cos \varphi'} \frac{\partial V}{\partial \lambda} \vec{u}_{\lambda}\tag{E.33}$$

where:

$$\begin{aligned}\frac{\partial V}{\partial r} &= -\frac{GM}{r^2} \left[1 + \sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r}\right)^{\ell} (\ell+1) \sum_{m=0}^{\ell} \bar{P}_{\ell m}(\sin \varphi') (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda) \right] \\ \frac{\partial V}{\partial \varphi'} &= \frac{GM}{r} \left[\sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} \frac{\partial \bar{P}_{\ell m}(\sin \varphi')}{\partial \varphi'} \cdot (\bar{C}_{\ell m} \cos m\lambda + \bar{S}_{\ell m} \sin m\lambda) \right] \\ \frac{\partial V}{\partial \lambda} &= \frac{GM}{r} \left[\sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r}\right)^{\ell} \sum_{m=0}^{\ell} \bar{P}_{\ell m}(\sin \varphi') \cdot (-m \bar{C}_{\ell m} \sin m\lambda + m \bar{S}_{\ell m} \cos m\lambda) \right] \\ \vec{u}_r &= \frac{x}{\|x^2 + y^2 + z^2\|} \cdot \vec{i}_{ECEF} + \frac{y}{\|x^2 + y^2 + z^2\|} \cdot \vec{j}_{ECEF} + \frac{z}{\|x^2 + y^2 + z^2\|} \cdot \vec{k}_{ECEF} \\ \vec{u}_{\varphi'} &= -\sin \varphi' \cdot (\cos \lambda \cdot \vec{i}_{ECEF} + \sin \lambda \cdot \vec{j}_{ECEF}) + \cos \varphi' \cdot \vec{k}_{ECEF} \\ \vec{u}_{\lambda} &= -\sin \lambda \cdot \vec{i}_{ECEF} + \cos \lambda \cdot \vec{j}_{ECEF}\end{aligned}\tag{E.34}$$

E.2 Gravity potential

The acceleration of a particle P in a reference frame S can be obtained from the acceleration in a reference frame S' using equation D.16:

$$(\vec{a}_P)_S = (\vec{a}_{O'})_S + (\vec{a}_P)_{S'} + 2 \cdot \left(\vec{\Omega}_{S'S} \wedge (\vec{v}_P)_{S'} \right) + \frac{d\vec{\Omega}_{S'S}}{dt} \wedge (\vec{r}_P)_{S'} + \vec{\Omega}_{S'S} \wedge \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_P)_{S'} \right)$$

where the term $\vec{\Omega}_{S'S} \wedge \left(\vec{\Omega}_{S'S} \wedge (\vec{r}_P)_{S'} \right)$ is called *centrifugal acceleration*.

If we apply the curl operator ($\nabla \times$) to the centrifugal acceleration field in the Earth ($\vec{\omega} \wedge (\vec{\omega} \wedge \vec{r})$) we get a zero vector. This means that the centrifugal acceleration is conservative and as such a corresponding centrifugal potential exists.

This centrifugal acceleration potential is given by:

$$\Phi(x, y, z) = \frac{1}{2} \omega^2 d_z^2 \quad (\text{E.35})$$

where $\vec{\omega}$ is the angular velocity of the Earth and $d_z = \sqrt{x^2 + y^2}$ is the distance to the rotational axis z .

Taking into account the terms that relate the acceleration vectors in 2 frames (equation D.16), if we consider a constant angular velocity for the Earth ($\frac{d\vec{\omega}}{dt} = \vec{0}$) the static force per mass unit experienced by any mass in a position given by $\vec{r}$ would be given by the gravitational attraction plus the centrifugal acceleration. As a consequence, it is conventional to add all effects and consider a gravity potential (W) composed by the gravitational potential and the centrifugal acceleration potential:

$$W = V + \Phi \quad (\text{E.36})$$

In physical geodesy the gravity potential W is usually divided in two terms:

- U = normal potential (W_0)

This is the theoretical potential related to a rotating body with a perfect ellipsoidal shape, the same angular velocity of the Earth and the same global mass. This potential is set by the the following hypotheses:

- It is rotationally symmetric (zonal)
 - It has equatorial symmetry
 - It is constant on the ellipsoid (the Earth ellipsoid is an equipotential surface)
 - T = disturbing potential (δW)
- The disturbing potential includes the gravitational terms that have to be added to the normal potential in order to obtain the real gravity field.

In the same way that the gravity potential, the normal potential includes the centrifugal potential:

$$U = U_a + \Phi \quad (\text{E.37})$$

whereas the disturbing potential (T) does not.

Having these potential fields the following definitions are made:

- Gravity field vector

$$\vec{g} := \nabla W = \nabla U + \nabla T = \nabla W_0 + \nabla \delta W \quad (\text{E.38})$$

This is the static force per mass unit experienced at any point.

It includes the gravitational attraction plus the centrifugal acceleration.

- Normal gravity vector

$$\vec{\gamma} := \nabla U \quad (\text{E.39})$$

This is the static force per mass unit experienced at any point over a perfect ellipsoid with the same angular velocity of the Earth and the same global mass.

It includes the gravitational attraction plus the centrifugal acceleration.

- Gravity disturbance vector

It is defined as:

$$\vec{\delta g} := \nabla T = \nabla W - \nabla U = \vec{g} - \vec{\gamma} \quad (\text{E.40})$$

- Gravity disturbance

It is not defined as the modulus of the gravity disturbance vector, but as:

$$\delta g := \|\nabla W\| - \|\nabla U\| = \|\vec{g}\| - \|\vec{\gamma}\| \neq \|\vec{\delta g}\| \quad (\text{E.41})$$

Appendix E references

- [1] Nikolaus K. Pavlis, Simon A. Holmes, Steve C. Kenyon, and John K. Factor. An Earth Gravitational Model to Degree 2160: EGM2008. *General Assembly of the European Geosciences Union*, April 2008. [410](#)
- [2] W. Bosch. On the computation of derivatives of Legendre Functions. *Physics and Chemistry of the Earth (Part A)*, 25:665–659, 2000. doi: 10.1016/S1464-1895(00)00101-0. [411](#)
- [3] Franz Barthelmes. *Definition of Functionals of the Geopotential and Their Calculation from Spherical Harmonic Models*. Helmholtz-Zentrum Postdam, Deutsches GeoForschungsZentrum, Postdam, Germany, second edition, January 2013. doi: 10.2312/GFZ.b103-0902-26. URL <http://icgem.gfz-potsdam.de/ICGEM/theory/str-0902-revised.pdf>.
- [4] Nico Sneeuw. *Physical Geodesy*. Institute of Geodesy, Universität Stuttgart, Stuttgart, Germany, first edition, June 2006. URL http://www.uni-stuttgart.de/gi/education/BSC/19840_12_Physikalische_Geodäsie/LNErdm.pdf.

Appendix F

Orbital Motion Problems

This appendix indicates the orbital motion problems of which solutions will be required within the missiles simulator and briefly provides the algorithms used for their solution. The expressions indicated in this appendix are used in [chapter 7](#).

F.1 Introduction to orbital motion

This section provides in a very simple way the basic concepts for the representation of the orbit in space, and the most relevant equations that are the base for the orbital motion problems whose solution is required within this thesis.

F.1.1 Orbital elements

The position of a point mass orbiting a central body with an elliptic orbit is given by 6 parameters, usually known as *orbital elements*:

- Elements that provide the shape and size of the orbit:
 - a = semimajor axis. This is the distance from the center of the orbit to the apoapsis (furthest point of the trajectory to the central body)
 - e = eccentricity. This parameter indicates how much the orbit is elongated compared to a circle.
- Elements that provide the orientation of the orbital plane in which the ellipse is embedded:
 - i = inclination. This is the vertical tilt of the ellipse with respect to the reference plane (Oxy plane), measured at the ascending node (point where the orbit passes upward through the reference plane).
 - Ω = Longitude of the ascending node. This is the angle measured in the plane of reference (from the Ox axis) to the ascending node.
- Elements that provide the position of point mass within the orbit:
 - ω = argument of periapsis. This angle defines the orientation of the motion in the orbital plane. It is the angle measured from the ascending node to the periapsis (closest approach to the central body).
 - ν = True anomaly. It is the angle in the plane of the ellipse between the periapsis and the position of the point mass at a given time.

These orbital elements can be observed in the following picture:

Figure F.1: Orbital elements

F.1.2 Basic equations

The basic equations for orbital motion will be derived herein. Because of its simplicity the derivation indicated in [1] has been used.

As a starting point we consider a point mass in the presence of a central gravitational field so the following equation of motion applies:

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^3} \cdot \vec{r} \quad (\text{F.1})$$

From the derivative of $\vec{r} \wedge \vec{v}$ using for $\frac{d\vec{v}}{dt}$ the previous equation we obtain the following expression for the angular momentum:

$$\vec{h} = \vec{r} \wedge \vec{v} = \vec{c} \text{ (constant)} \quad (\text{F.2})$$

Note that using polar coordinates this leads to:

$$h = r^2 \cdot \frac{d\nu}{dt} \quad (\text{F.3})$$

which is Kepler's second law of planetary motion being ν the swept angle in polar coordinates.

From $\frac{d\vec{v}}{dt} \wedge \vec{h}$ we get that $\frac{d}{dt}(\vec{v} \wedge \vec{h}) = \mu \frac{d}{dt} \left(\frac{\vec{r}}{r} \right)$ which can be integrated leading to a constant vector (the eccentricity vector):

$$\mu \vec{e} = \vec{v} \wedge \vec{h} - \frac{\mu}{r} \cdot \vec{r} \quad (\text{F.4})$$

Operating on the previous expression F.4 to obtain the modulus of the eccentricity vector we get:

$$e^2 = \vec{e} \cdot \vec{e} \Rightarrow 1 - e^2 = \frac{h^2}{\mu} \cdot \left(\frac{2}{r} - \frac{v^2}{\mu} \right) \quad (\text{F.5})$$

The first factor in this equation is known as *the parameter*:

$$p = \frac{h^2}{\mu} \quad (\text{F.6})$$

Since *the parameter* is constant and the eccentricity vector is also constant the second factor of the equation must also be constant so we can define:

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu} \right)^{-1} \quad (\text{F.7})$$

From equations F.5 and F.7 we can deduce that:

$$p = a \cdot (1 - e^2) \quad (\text{F.8})$$

Equation F.7 can also be expressed in terms of energy as:

$$\frac{1}{2}v^2 - \frac{\mu}{r} = \text{constant} = -\frac{\mu}{2a} \quad (\text{F.9})$$

where $\frac{1}{2}v^2$ is the kinetic energy and $-\frac{\mu}{r}$ is the potential energy.

This latter equation (F.9) can also be expressed as:

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \quad (\text{F.10})$$

which is called the *vis-viva* integral.

Applying the scalar product of the eccentricity vector (equation F.4) and $\vec{r}$ we get:

$$\mu \vec{e} \cdot \vec{r} = (\vec{v} \wedge \vec{h}) \cdot \vec{r} - \frac{\mu}{r} \cdot \vec{r} \cdot \vec{r} \quad (\text{F.11})$$

Since $\vec{r} \cdot (\vec{v} \wedge \vec{h}) = \vec{h} \cdot (\vec{r} \wedge \vec{v}) = h^2$, this equation becomes:

$$\vec{e} \cdot \vec{r} = p - r \quad (\text{F.12})$$

If we define ν as the angle between the vectors $\vec{e}$ and $\vec{r}$, equation F.12 becomes the equation of the orbit in polar coordinates:

$$r = \frac{p}{1 + e \cos \nu} \quad (\text{F.13})$$

It can be observed that with this definition ν is in fact the true anomaly angle as defined in section F.1.1.

If we now consider rectangular Cartesian coordinates with the Ox axis in the direction of the $\vec{e}$ vector and the origin placed such that $x = \vec{e} \cdot \vec{r}$ we get the following equation for the orbit (if $e \neq 1$):

$$r = p - e \cdot x \implies \frac{(x + ea)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 \quad (\text{F.14})$$

For $e < 1$ this is the equation of an ellipse, which is Kepler's first law of motion, being a the semimajor axis. The case $e < 1$ is the case of interest herein since $e > 1$ leads to unbounded trajectories.

The semiminor axis of the ellipse would be given according to equation F.14 by:

$$b^2 = a^2(1 - e^2) \quad (\text{F.15})$$

Taking into account Kepler's second law deduced after equation F.2 and denoting the period of the orbit by P we have:

$$h = \frac{2\pi ab}{P} \quad (\text{F.16})$$

where πab is the area of the ellipse.

Substituting the values for b (equation F.15), h (equation F.6) and p (equation F.8) in this equation we get:

$$P = 2\pi\sqrt{\frac{a^3}{\mu}} \quad (\text{F.17})$$

which is Kepler's third law of motion.

The period of the orbit is usually given by the variable *mean motion* defined as:

$$n := \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}} \quad (\text{F.18})$$

which leads to the following formulation for Kepler's third law of motion:

$$\mu = n^2 a^3 \quad (\text{F.19})$$

The orbit will be completely set defining the position of the point mass for some particular instant of time. This can be done by combining Kepler's second law (equation F.3) and the equation of the orbit (equation F.13), obtaining:

$$\sqrt{\frac{\mu}{p^3}} dt = \frac{d\nu}{(1 + e \cos \nu)^2} \quad (\text{F.20})$$

This equation provides the relationship between the true anomaly ν and the time t . The required integration constant is usually chosen as the time τ in which the point mass is at the pericenter, this is the *time of pericenter passage*.

In the case of an ellipse it is common to use the eccentric anomaly angle E instead of the true anomaly ν , where the relationship between both angles can be observed in the following figure:

Figure F.2: Eccentric anomaly and true anomaly

In terms of E the equation of the ellipse can be expressed as:

$$\begin{aligned} x' &= a \cos E \\ y' &= b \sin E \end{aligned} \quad (\text{F.21})$$

where now the coordinates x' and y' are given with respect to the center of the ellipse and not with respect to a focus, as in equation F.14. Using these coordinates the equation of the orbit would be:

$$r = a - e \cdot x' \implies \frac{(x')^2}{a^2} + \frac{y'^2}{a^2(1-e^2)} = 1 \quad (\text{F.22})$$

which can be expressed as:

$$r = a(1 - e \cos E) \quad (\text{F.23})$$

Comparing equation F.23 with the polar equation of the ellipse (F.13) we obtain the identities:

$$\cos \nu = \frac{\cos E - e}{1 - e \cos E} \quad \cos E = \frac{e + \cos \nu}{1 + e \cos \nu} \quad (\text{F.24})$$

Since $y = b \sin E = a\sqrt{1-e^2} \sin E = r \sin \nu$ we also obtain the expressions:

$$\sin \nu = \frac{\sqrt{1-e^2} \sin E}{1 - e \cos E} \quad \sin E = \frac{\sqrt{1-e^2} \sin \nu}{1 + e \cos \nu} \quad (\text{F.25})$$

After some manipulations on equations F.24 and F.25 we get the following expression:

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \quad (\text{F.26})$$

which is a very useful relation between ν and E since $\frac{\nu}{2}$ and $\frac{E}{2}$ are always in the same quadrant. Differentiating equation F.26 and taking previous equations into account we obtain:

$$r d\nu = b \cdot dE \quad (\text{F.27})$$

From the law of areas (equation F.3) we get:

$$h \cdot dt = r^2 d\nu = b \cdot r \cdot dE = b \cdot a(1 - e \cos E) dE \quad (\text{F.28})$$

which can be integrated as:

$$M = \int_{\nu=0}^{\nu} \frac{r^2}{ab} d\nu = \int_{t=\tau}^t \frac{h}{ab} dt = \int_{E=0}^E dE - e \int_{E=0}^E \cos E dE \implies M = E - e \sin E \quad (\text{F.29})$$

which is called *Kepler's equation*.

The quantity M , called the *mean anomaly*, can be obtained from equations F.15 and F.6 as:

$$M = \int_{\nu=0}^{\nu} \frac{r^2}{ab} d\nu = \int_{t=\tau}^t \frac{h}{ab} dt = \int_{t=\tau}^t \sqrt{\frac{\mu}{a^3}} dt = \sqrt{\frac{\mu}{a^3}} (t - \tau) = n(t - \tau) \quad (\text{F.30})$$

being n the mean motion (equation F.18) and τ the time of passage through the pericenter.

F.1.3 Orbit determination

We deal here with the problem of determining an orbit (this is, its orbital elements) having as input data the position $\vec{r}$ and velocity vector $\vec{v}$ at a certain point and given time t .

This problem can be easily solved using the following expressions:

- The first step is computing the angular momentum at the given point:

$$\begin{aligned}\vec{h} &= \vec{r} \times \vec{v} \\ h &= \|\vec{h}\|\end{aligned}\tag{F.31}$$

- The semimajor axis is obtained from the vis-viva equation (equation F.7):

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{\mu}}\tag{F.32}$$

- We compute the eccentricity vector with equation F.4:

$$\begin{aligned}\vec{e} &= \left(v^2 - \frac{\mu}{r}\right) \cdot \vec{r} - \frac{(\vec{r} \cdot \vec{v})\vec{v}}{\mu} \\ e &= \|\vec{e}\|\end{aligned}\tag{F.33}$$

- The inclination of the orbit is directly obtained from the angular momentum vector:

$$\begin{aligned}\cos i &= \frac{\vec{h}}{h} \\ i &= \arccos(\cos i)\end{aligned}\tag{F.34}$$

- We compute a vector within the orbit plane perpendicular to the Oz axis:

$$\vec{n} = \vec{k} \times \vec{h}\tag{F.35}$$

- The longitude of the ascending node would be obtained as the angle between the Ox axis and $\vec{n}$:

$$\Omega = \arccos\left(\frac{\vec{n}|_x}{\|\vec{n}\|}\right)\tag{F.36}$$

where the following condition is added:

$$\text{if } \vec{n}|_y < 0 \implies \Omega = 2\pi - \Omega\tag{F.37}$$

- The argument of periapsis would be given by the angle between $\vec{n}$ and $\vec{e}$:

$$\omega = \arccos\left(\frac{\vec{n} \cdot \vec{e}}{\|\vec{n}\|e}\right)\tag{F.38}$$

where the following condition is added:

$$\text{if } \vec{e}|_z < 0 \implies \omega = 2\pi - \omega\tag{F.39}$$

- Since the true anomaly is the angle between the $\vec{e}$ and $\vec{r}$:

$$\nu = \arccos \left(\frac{\vec{r} \cdot \vec{e}}{\|\vec{r}\|e} \right) \quad (\text{F.40})$$

where the following condition is added:

$$\text{if } \vec{r} \cdot \vec{v} < 0 \implies \nu = 2\pi - \nu \quad (\text{F.41})$$

- The eccentric anomaly is obtained from equations F.23 and F.26:

$$\begin{aligned} \cos E &= \frac{a - r}{ae} \\ \tan \frac{E}{2} &= \sqrt{\frac{1-e}{1+e}} \cdot \tan \frac{\nu}{2} \implies \sin E = \sin \left\{ 2 \cdot \operatorname{atan} \left(\tan \frac{E}{2} \right) \right\} \\ &\implies E = \operatorname{atan} \left(\frac{\sin E}{\cos E} \right) \end{aligned} \quad (\text{F.42})$$

- The mean motion is computed from its definition (equation F.18):

$$n = \sqrt{\frac{\mu}{a^3}} \quad (\text{F.43})$$

- The perigee passing time is obtained from Kepler's equation (equation F.29) and the definition of the mean anomaly (equation F.30):

$$\tau = t - \frac{E - e \sin E}{n} \quad (\text{F.44})$$

F.2 Considered Orbital Motion Problems

There are two major problems in astrodynamics:

- The Kepler problem

The Kepler problem consists in finding a future state (position and velocity) in an orbit at a given time knowing the initial state (position and velocity) for a point mass within a central gravitational field.

- The Lambert problem

The Lambert problem consists in finding the trajectory that connects 2 given position vectors in a given transfer time for a point mass within a central gravitational field.

The solutions applied for these problems in the simulator will be briefly detailed hereafter.

F.2.1 Kepler's Problem

F.2.1.1 Formulation

As previously indicated, the Kepler problem consists in finding a future state (position and velocity) in an orbit at a given time knowing the initial state (position and velocity) for a point mass within a central gravitational field.

The first part of the problem consists in finding the orbital elements from the initial state of the missile, which will be done using the orbit determination equations given in section F.1.3.

Once the orbital elements are obtained we solve Kepler's equation (equation F.29) in order to obtain $E(t)$.

Having $E(t)$ the position vector is easily obtained from the equation of the orbit (equation F.23):

$$r = a(1 - e \cos E(t)) \quad (\text{F.45})$$

and taking into account the definition of the orbital elements (see chapter 4.11 in reference [2]):

$$\begin{aligned} x &= r \cdot \cos(\Omega) \cos(\omega + \nu) - \sin(\Omega) \cdot \sin \omega + \nu \cdot \cos i \\ y &= r \cdot \sin(\Omega) \cos(\omega + \nu) + \cos(\Omega) \cdot \sin \omega + \nu \cdot \cos i \\ z &= r \cdot \sin(\omega + \nu) \cdot \sin i \end{aligned} \quad (\text{F.46})$$

The velocity vector can be obtained by computing first the angular momentum vector from the orbital elements:

$$p = a \cdot (1 - e^2) \implies h = \sqrt{p \cdot \mu} \quad (\text{F.47})$$

If we now write the velocity vector in the radial component and the component perpendicular to the radius:

$$\vec{v} = v_r \cdot \vec{u}_r + v_\perp \cdot \vec{u}_\perp = \dot{r} \vec{u}_r + r \dot{\nu} \vec{u}_\perp \quad (\text{F.48})$$

where:

$$\begin{aligned} \vec{u}_r &= \frac{\vec{r}}{r} \\ \vec{u}_\perp &= \frac{\vec{h} \times \vec{u}_r}{h} \end{aligned} \quad (\text{F.49})$$

since $\vec{h} = \vec{r} \wedge \vec{v}$ (equation F.2) then:

$$v_\perp = \frac{h}{r} \quad (\text{F.50})$$

v_r is finally obtained from equations F.20 and F.13:

$$v_r = \frac{\mu \cdot e \cdot \sin \nu}{h} \quad (\text{F.51})$$

so the final velocity vector is given by:

$$\vec{v} = v_r \cdot \vec{u}_r + v_\perp \cdot \vec{u}_\perp \quad (\text{F.52})$$

F.2.1.2 Kepler's equation

Taking into account the formulation given to Kepler's problem the only difficult step is solving Kepler's equation. This is, finding E having as inputs M and e in the following equation:

$$M = E - e \sin E \quad (\text{F.53})$$

Kepler's equation is the most famous of all transcendental equations and has motivated a number of significant developments in mathematics like Lagrange's expansion theorem, Bessel functions, Fourier series, some aspects of complex function theory and various techniques of numerical analysis (see [1]).

In the missile simulator we will use a simple Newton's method since this method provides enough accuracy with a fast convergence:

Starting value:

$$\begin{aligned} M < \pi &\implies E = M + \frac{e}{2} \\ M > \pi &\implies E = M - \frac{e}{2} \end{aligned} \quad (\text{F.54})$$

Iteration loop:

$$\begin{aligned} F &= E - e \sin E - M \\ \frac{dF}{dE} &= 1 - e \cos E \\ E_{k+1} &= E - \frac{F}{\frac{dF}{dE}} \end{aligned} \quad (\text{F.55})$$

where the loop is exited when the expected tolerance is achieved.

F.2.1.3 Kepler's problem in this thesis

Within this thesis the implemented algorithm will be noted as f_{Kepler} .

The inputs of this function will always be a position vector and a velocity vector ($\vec{r}$ and $\vec{v}$) in the inertial reference frame at the present time t , and the increment of time Δt from the present instant for which the future state is required.

The considered output will be the position vector and velocity vector at $t + \Delta t$:

$$[\vec{r}(t + \Delta t), \vec{v}(t + \Delta t)] = f_{Kepler}(\vec{r}(t), \vec{v}(t), \Delta t) \quad (\text{F.56})$$

F.2.2 Lambert's Problem

F.2.2.1 Lambert theorem

The formulation of Lambert's problem is a derivation of Lambert theorem (1761) that states that:

"In elliptic motion under a Newtonian law the time occupied in describing any arc depends only on the major axis, the sum of distances from the center of forces to the initial and final points and the length of the chord joining these points, so that if these three elements are given the time is determinate, whatever the form of the ellipse."

This is, Lambert theorem states that:

$$time(i - f)|_{\mu} = f(a, \|\vec{r}_i\| + \|\vec{r}_f\|, c) \quad (F.57)$$

From the basic equations of orbital motion (section F.1) this theorem is easy to demonstrate (see reference [3]):

- If we have 2 points of the same orbit then from equation F.23:

$$r_1 + r_2 = a[2 - e(\cos E_1 + \cos E_2)] = 2a(1 - e \cos G \cos g) \quad (F.58)$$

where:

$$\begin{aligned} G &:= \frac{E_1 + E_2}{2} \\ g &:= \frac{E_2 - E_1}{2} \end{aligned} \quad (F.59)$$

Defining:

$$e \cos G := \cos j \quad (F.60)$$

then expression F.58 can be written as:

$$r_1 + r_2 = 2a(1 - \cos j \cos g) \quad (F.61)$$

- The coordinates of these 2 points of the same orbit would be given by:

P_1 :

$$\begin{aligned} - x_1 &= a \cos E_1 \\ - y_1 &= b \sin E_1 = a\sqrt{1 - e^2} \sin E_1 \end{aligned}$$

P_2 :

$$\begin{aligned} - x_2 &= a \cos E_2 \\ - y_2 &= b \sin E_2 = a\sqrt{1 - e^2} \sin E_2 \end{aligned}$$

so the chord P_1P_2 would be:

$$\begin{aligned} P_1P_2 = c &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \implies \\ c^2 &= a^2(\cos E_2 - \cos E_1)^2 + a^2(1 - e^2)(\sin E_2 - \sin E_1)^2 = \\ &= 4a^2 \sin^2 G \sin^2 g + 4a^2(1 - e^2) \cos^2 G \sin^2 g = 4a^2 \sin^2 g(1 - \cos^2 j) \\ &\implies c = 2a \sin g \sin j \end{aligned} \quad (F.62)$$

- If we now define the angles:

$$\begin{aligned}\epsilon &= j + g \\ \delta &= j - g\end{aligned}\tag{F.63}$$

we can get the following expressions combining equations F.61 and F.62:

$$\begin{aligned}r_1 + r_2 + c &= 2a(1 - \cos j \cos g) + 2a \sin g \sin j = 2a(1 - \cos \epsilon) = 4a \sin^2 \frac{\epsilon}{2} \\ r_1 + r_2 - c &= 2a(1 - \cos j \cos g) - 2a \sin g \sin j = 2a(1 - \cos \delta) = 4a \sin^2 \frac{\delta}{2}\end{aligned}\tag{F.64}$$

- From equation F.29 the time of flight between points 1 and 2 would be given by:

$$\begin{aligned}n(t_2 - t_1) &= (E_2 - E_1) - e(\sin E_2 - \sin E_1) = 2(g - \cos j \sin g) = \\ &= (\epsilon - \delta) - (\sin \epsilon - \sin \delta)\end{aligned}\tag{F.65}$$

It is easy to observe from equation F.65 that the time of flight between points 1 and 2 ($t_2 - t_1$) is only a function of n ($= \sqrt{\frac{\mu}{a^3}}$ according to equation F.18), ϵ and δ .

Since according to equation F.64 ϵ and δ only depend on $r_1 + r_2 + c$ and $r_1 + r_2 - c$, the theorem is demonstrated.

F.2.2.2 Consequence of Lambert theorem

The implication of Lambert's theorem is that if the initial and final position vectors are given there will always exist a unique trajectory able to join those points in a desired time, that could be obtained by modifying the semimajor axis of the ellipse.

This means that, when the geometry of the radius vectors is fixed, there is only one free parameter left that wholly defines the transfer time between P1 and P2.

As a consequence, once a suitable parametrization of the orbits passing through P1 and P2 is obtained, it is possible to rephrase Lambert's problem in terms of the evaluation of the parameter value, such that the corresponding orbit is characterized by a transfer time that matches exactly the prescribed one.

F.2.2.3 Solutions of Lambert's problem

Lambert's problem is one of the most extensively studied problems in celestial mechanics. Several famous solutions have been provided along history, starting with the one given by Carl Friedrich Gauss in his book *Theoria Motus*.

The problem remained during centuries as just a mathematical development without real applications until the Space Age in the 1950's required its solution in a fast and accurate way. The most relevant development in this sense was given by Richard H. Battin (see reference [1]). Chapter 7 analyses modern solutions and indicates the one applied within this thesis.

F.2.2.4 Lambert's problem in this thesis

Within this thesis the implemented algorithm will be noted as $f_{Lambert}$.

The inputs of this function will always be the 2 position vectors ($\vec{r}_1$ and $\vec{r}_2$) in the inertial reference frame, and the time of flight between them.

The considered output will be the required velocity vector at t_1 , given with flight path angles and the modulus of the velocity vector, or given directly in a vector expression:

$$\begin{aligned} [\vec{v}_1] &= f_{Lambert}(\vec{r}_1, \vec{r}_2, tof) \\ \text{or} & \\ [||\vec{v}_1||, \gamma_z, \gamma_y] &= f_{Lambert}(\vec{r}_1, \vec{r}_2, tof) \end{aligned} \tag{F.66}$$

Appendix F references

- [1] Richard H. Battin. *An Introduction to the Mathematics and Methods of Astrodynamics*. American Institute of Aeronautics and Astronautics Inc., Alexander Bell Drive, Reston, VA, revised edition edition, 1999. [417](#), [424](#), [427](#)
- [2] Archie E. Roy. *Orbital motion*. Adam Hilger Ltd., Bristol, Great Britain, second edition, 1982. [423](#)
- [3] George M. Siouris. *Missile Guidance and Control Systems*. Springer-Verlag New York, Inc., New York, 2004. [425](#)
- [4] M.A. Gómez-Tierno. Apuntes de Determinación de Órbitas printed by the Escuela Técnica Superior de Ingenieros Aeronáuticos, 2003.
- [5] James F. Jordan. *Technical Report 32-521. The Application of Lambert's Theorem to the Solution of Interplanetary Transfer Problems*. Jet Propulsion Laboratory. California Institute of Technology, Pasadena, California, February 1964.

Appendix G

Missile parameters

This chapter summarizes the different parameters used for the simulation of the ICBM and the interceptor missile required according to the equations described in [chapter 3](#) and [chapter 4](#).

It has to be noted that many of the parameters collected here are not openly disclosed or scattered among many different sources whose reliability is not confirmed. As a consequence a great effort has been put in reviewing the global consistency of the chosen parameters, so that even without being confirmed they are at least coherent with each other.

The expressions indicated in this appendix are used in [chapter 4](#) and [chapter 9](#).

G.1 ICBM parameters

The parameters of the LGM-30G Minuteman III Intercontinental Ballistic Missile that are required for its simulation are detailed herein.

G.1.1 Motors parameters

G.1.1.1 Stage 1

Name of the motor = Thiokol M-55E1

Figure G.1: Thiokol M-55 motor
(photo from reference [1])

Type of motor = Solid fuel motor

Fuel composition = Ammonium perchlorate (AP) oxidizer and Aluminum powder TP-H1011 or TP-H1043 (propellant) (according to [2] and [3])

Burning time = 60 s (according to [4] and [5])

Number of nozzles = 4

$A_e \simeq 4 \times 0.2290 \text{ m}^2$ (estimated from photographs)

E_0 = Vacuum thrust = 892,000 N (according to [2] and [5])

p_e = Considered pressure of the gas at the exit of the nozzle = 101,235 Pa
(we suppose that it is an adapted nozzle at sea level).

$(v_e)_{\text{stage 1}}$ can be obtained from the equation for the thrust (equation 4.47):

$$(v_e)_{\text{stage 1}} = \frac{E_0 - p_e \cdot A_e}{(\dot{M})_{\text{stage 1}}} \quad (\text{G.1})$$

($\dot{M}$ is given in section G.1.3.3)

G.1.1.2 Stage 2

Name of the motor = Aerojet General SR19-AJ-1

Figure G.2: Aerojet SR19-AJ-1 motor
(photo from reference [1])

Type of motor = Solid fuel motor

Fuel composition = Ammonium perchlorate (AP) oxidizer and Aluminum powder ANB-3066 (propellant) (according to [3] and [6] (Winter and James, 1995))

Burning time stage 2 = 66 s (according to [4] and [5])

Number of nozzles = 1

A_e = exhaust nozzle surface $\simeq 1.3478 \text{ m}^2$ (estimated from photographs)

E_0 = Vacuum thrust = 267.000 N (according to [4])

As a hypothesis and since it is basically the same fuel we will consider:

$$(v_e)_{\text{stage 2}} = (v_e)_{\text{stage 1}} \quad (\text{G.2})$$

This allows obtaining p_e from the equation for the thrust (equation 4.47):

$$p_e = \frac{E_0 - (\dot{M})_{\text{stage 2}} \cdot (v_e)_{\text{stage 2}}}{A_e} \quad (\text{G.3})$$

($\dot{M}$ is given in section G.1.3.6)

G.1.1.3 Stage 3

Name of the motor = Aerojet-Thiokol SR73-AJ-TC-1

Figure G.3: Aerojet-Thiokol SR73-AJ-TC-1 motor
(photo from reference [1])

Type of motor = Solid fuel motor

Fuel composition = Unknown oxidizer and Aluminum powder ANB-3066 (propellant) (according to [2] and [3])

Burning time stage 3 = 61 s (according to [4] and [5])

Number of nozzles = 1

A_e = exhaust nozzle surface $\simeq 0.7854 \text{ m}^2$ (estimated from photographs)

E_0 = Vacuum thrust = 152,000 N (according to [5])

As a hypothesis and since it is basically the same fuel we will consider:

$$(v_e)_{\text{stage 3}} = (v_e)_{\text{stage 1}} \quad (\text{G.4})$$

This allows obtaining p_e from the equation for the thrust (equation 4.47):

$$p_e = \frac{E_0 - (\dot{M})_{\text{stage 3}} \cdot (v_e)_{\text{stage 3}}}{A_e} \quad (\text{G.5})$$

($\dot{M}$ is given in section G.1.3.9)

G.1.1.4 Propulsion System Rocket Engine

The Minuteman III ICBM has a post-boost liquid rocket engine with the purpose of increasing the overall range of the missile and controlling the attitude of the Post-Boost Control System preparing the reentry vehicle for the reentry phase.

Name of the motor = Rocketdyne RS-14

Figure G.4: Rocketdyne RS-14 motor
(photo from reference [4])

Type of motor = Restartable liquid fuel motor

Fuel composition = Nitrogen tetroxide (N_2O_4) and Monomethyl hydrazine (MMH: $CH_3(NH)NH_2$) (according to [7]).

Burning time Post-Boost = Intermittent

Number of nozzles = 1 principal + 10 for attitude control (according to [3] and [8])

$(\dot{M})_{PSRE} = 0.466 \text{ kg/s}$ (according to [9])

A_e = exhaust nozzle surface = 0.2 m^2 (estimated from photographs)

E_0 = Vacuum thrust = 1400 N (according to [7])

p_e = Considered pressure of the gas at the exit of the nozzle = 5.98 MPa (according to [10])

$(v_e)_{PSRE}$ can be obtained from the equation for the thrust (equation 4.47):

$$(v_e)_{PSRE} = \frac{E_0 - p_e \cdot A_e}{(\dot{M})_{PSRE}} \quad (G.6)$$

G.1.2 Control parameters

G.1.2.1 Control parameters in stage 1

G.1.2.1.1 Geometry of the controls

We consider (estimated from pictures) that the mass flow centers of the 4 nozzles in stage 1 are placed in the following coordinates with respect to the base of the missile:

$$\text{Nozzle 1: } (\vec{r}_{e1})_{\text{from base}} = (0.0, 0.0, -0.53)$$

$$\text{Nozzle 2: } (\vec{r}_{e2})_{\text{from base}} = (0.0, 0.53, 0.0)$$

$$\text{Nozzle 3: } (\vec{r}_{e3})_{\text{from base}} = (0.0, 0.0, 0.53)$$

$$\text{Nozzle 4: } (\vec{r}_{e4})_{\text{from base}} = (0.0, -0.53, 0.0)$$

G.1.2.1.2 Time parameters

We consider 3 seconds without any control after stage 1 ignition:

$$\text{Time without control}_{\text{stage 1}} = 3 \text{ s}$$

This has been set as a supposition considering enough time for the missile to leave the silo.

We consider that the yaw/roll controls are activated 6 seconds after stage 1 ignition:

$$\text{Time without yaw/roll control}_{\text{stage 1}} = 6 \text{ s}$$

This has been set as a supposition considering that the missile starts controlling in pitch and controls in yaw/roll later, according to reference [11].

We consider (purely supposition) that the initial kick of the gravity turn manoeuvre as considered in section 7.1.1.2 lasts 20 seconds:

$$\text{Time for initial kick}_{\text{stage 1}} = 20 \text{ s}$$

G.1.2.2 Control parameters in stage 2

G.1.2.2.1 Geometry of the controls

We consider that the mass flow center of the nozzle in stage 2 is placed in the following coordinates with respect to the base of the initial missile (middle of the end of the stage 1 section):

$$\text{Nozzle: } (\vec{r}_e)_{\text{from base}} = (7.08, 0.0, 0.0)$$

We consider (as a supposition) that the freon ejection points in stage 2 in the body frame are placed in the following coordinates:

$$\text{Ejection point 1: } (y_{ej1})_b = 0.0 \quad (z_{ej1})_b = -0.3$$

$$\text{Ejection point 2: } (y_{ej2})_b = 0.3 \quad (z_{ej2})_b = 0.0$$

$$\text{Ejection point 3: } (y_{ej3})_b = 0.0 \quad (z_{ej3})_b = 0.3$$

$$\text{Ejection point 4: } (y_{ej4})_b = -0.3 \quad (z_{ej4})_b = 0.0$$

We consider that the lever arm for the roll control in stage 2 (this is, the distance between the opposite roll control nozzles) equals the stage 2 diameter:

$$\text{roll arm}_{\text{stage 2}} = 1.31 \text{ m}$$

G.1.2.2.2 Time parameters

We consider 0.5 seconds after the stage 1 has been dropped before stage 2 motor is ignited:

$$\text{Time without ignition}_{\text{stage 2}} = 0.5 \text{ s}$$

This has been estimated after analysing existing times between stages in other launchers.

G.1.2.3 Control parameters in stage 3

G.1.2.3.1 Geometry of the controls

We consider that the mass flow center of the nozzle in stage 3 is placed in the following coordinates with respect to the base of the initial missile (middle of the end of the stage 2 section):

$$\text{Nozzle: } (\vec{r}_e)_{\text{from base}} = (11.35, 0.0, 0.0)$$

We consider (as a supposition) that the strontium perchlorate ejection points in stage 2 in the body frame are placed in the following coordinates:

$$\text{Ejection point 1: } (y_{ej1})_b = 0.0 \quad (z_{ej1})_b = -0.2$$

$$\text{Ejection point 2: } (y_{ej2})_b = 0.2 \quad (z_{ej2})_b = 0.0$$

$$\text{Ejection point 3: } (y_{ej2})_b = 0.0 \quad (z_{ej3})_b = 0.2$$

$$\text{Ejection point 4: } (y_{ej2})_b = -0.2 \quad (z_{ej4})_b = 0.0$$

We consider that the lever arm for the roll control in stage 3 (this is, the distance between the opposite roll control nozzles) equals the stage 3 diameter:

$$\text{roll arm}_{\text{stage 2}} = 1.31 \text{ m}$$

G.1.2.3.2 Time parameters

We consider 0.5 seconds after the stage 2 has been dropped before stage 3 motor is ignited:

$$\text{Time without ignition}_{\text{stage 3}} = 0.5 \text{ s}$$

This has been estimated after analysing existing times between stages in other launchers.

Even though the maximum time for the stage 3 actuation is known (see section G.1.1.3), the actual time that the stage 3 is working can be inferior in order to modify the range of the missile, so the time the stage 3 is operating will be an output of the guidance module (see Part III).

G.1.2.4 Control parameters in the post-boost phase

G.1.2.4.1 Geometry of the controls

In this case we only consider in the simulation the main nozzle used to increase the missile range, whose mass flow center we place in the middle of the beginning of the post-boost control system:

$$\text{Nozzle: } (\vec{r}_e)_{\text{from base}} = (14.06, 0.0, 0.0)$$

There are other 10 nozzles for attitude control (according to [3] and [8]), but since the simulation of this attitude change using the PSRE motor is very complex and meaningless for the results of the simulation, we will just consider in this phase that the control variables are the control moments $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_z$ used to provide the reentry vehicle with the desired attitude.

G.1.2.4.2 Time parameters

We consider 0.5 seconds after the stage 3 has been dropped before the post-boost motor is ignited:

Time without ignition_{post-boost} = 0.5 s

This has been estimated after analysing existing times between stages in other launchers.

The time required for the PSRE actuation depends on the increase of range required for the missile so it will be an output of the guidance module (see Part III). In any case the maximum time considered is set as 80 s, and it is considered that the actuation of the PSRE for attitude control lasts another 80 seconds. This is justified by the indications in [3] where it is said that "the fourth stage adds a portion of controlled flight that is nearly equal in time to the combined first three stages".

We consider that the reentry vehicle is released 1 second after the PSRE stops working for adapting the attitude of the reentry vehicle:

Time for RV release after PSRE is stopped for attitude control = 1 s

This has been set as a supposition taking into account that the sooner the RV is released the more time the missile travels with a very low infrared firm and radar cross section, so the hardest it is for any antimissile system to destroy it.

G.1.2.5 Control parameters in the reentry vehicle

The reentry vehicle has no control mechanisms. It has a stable trajectory because it descends into the atmosphere with a high spin rate, but this trajectory is an uncontrolled free fall and the missile can only choose in this phase the height at which it detonates its warhead according to the initially chosen targeting options.

We simulate the spin of the reentry vehicle by applying a moment of 1 $N \cdot m$ in the x_b axis during 2 seconds:

$$\mathcal{M}_x = 1 \text{ N} \cdot \text{m} \quad (\text{G.7})$$

The value applied to $\mathcal{M}_x$ and the time in which it is applied have been estimated. The missile uses a hot gas system to create this spin, but details about this procedure have not been found.

G.1.3 Missile components

The global layout of the Minuteman III ICBM can be observed in figure G.5.

For the different big sections depicted in G.5 an estimation of the mass, position vector of the center of mass of each component with respect to the base of the missile, and inertia tensor with respect to the center of mass of that component will be provided.

These estimations have been generated from parameters provided by many different sources so that the global resulting parameters match the ones selected as valid from the bibliography about the complete missile ([2], [4], [12] and [13])

Within this analysis, the effect of all the small components has been neglected and several approximations are made in order to assign mass and inertial properties to the considered sections (for example the motor structure section includes the motorcase, the burnout, the nozzles and electronical systems in this section).

Figure G.5: Exploded view of the Minuteman III missile (picture from Wikimedia Commons)

G.1.3.1 Skirt section A6950

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$L_{\text{Skirt}} = 0.97 \text{ m (estimated from pictures)}$$

$$R_{\text{Skirt}} = 0.84 \text{ m (according to [2])}$$

$$th_{\text{Skirt}} = 0.007 \text{ m (estimated)}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm skirt}})_{\text{from base}} = (0.48, 0.0, 0.0) \text{ m}$$

The mass in a hollow cylinder geometry is provided by the following formula:

$$M = \rho_{\text{material}} \cdot L \cdot \pi [R_{\text{external}}^2 - (R_{\text{external}} - th)^2] \quad (\text{G.8})$$

which in this case leads to:

$$\rho_{\text{Titanium}} = 4420 \text{ kg/m}^3 \implies M_{\text{skirt}} \simeq 157.7 \text{ kg (supposing Titanium)}$$

The moments of inertia of a hollow cylinder with respect to its center of mass are given by these formulae:

$$\begin{aligned} I_{xx} &= \frac{1}{2} \pi \rho_{\text{material}} \cdot L \cdot [R_{\text{external}}^4 - (R_{\text{external}} - th)^4] \\ I_{yy} &= \frac{1}{12} \pi \rho_{\text{material}} \cdot L \cdot (3 \cdot [R_{\text{external}}^4 - (R_{\text{external}} - th)^4] + L^2 \cdot [R_{\text{external}}^2 - (R_{\text{external}} - th)^2]) \\ I_{zz} &= \frac{1}{12} \pi \rho_{\text{material}} \cdot L \cdot (3 \cdot [R_{\text{external}}^4 - (R_{\text{external}} - th)^4] + L^2 \cdot [R_{\text{external}}^2 - (R_{\text{external}} - th)^2]) \end{aligned} \quad (\text{G.9})$$

which in this case provides the following results:

$$I_{\text{skirt}} = \begin{bmatrix} 110.34 & 0 & 0 \\ 0 & 67.53 & 0 \\ 0 & 0 & 67.53 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Titanium)}$$

G.1.3.2 Stage 1 motor structure

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$L_{\text{Stage 1 motor structure}} = 7.08 \text{ m (estimated from pictures)}$$

$$R_{\text{Stage 1 motor structure}} = 0.84 \text{ m (according to [2])}$$

$$th_{\text{Stage 1 motor structure}} = 0.007 \text{ m (estimated)}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 1 structure}})_{\text{from base}} = (3.54, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\rho_{\text{Steel}} = 7861 \text{ kg/m}^3 \implies M_{\text{stage 1 structure}} \simeq 2077.0 \text{ kg (D6AC steel according to [2])}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{stage 1 structure}} = \begin{bmatrix} 1453.19 & 0 & 0 \\ 0 & 9399.38 & 0 \\ 0 & 0 & 9399.38 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (steel according to [2])}$$

G.1.3.3 Stage 1 motor fuel

We will consider a radial burn as indicated in [3]. This is, a hollow cylinder fuel distribution. This geometry is defined by the following length and minimum and maximum fuel radii:

$$\begin{aligned} L_{\text{stage 1}} &= 5.53 \text{ m (estimated from pictures)} \\ R_{\text{minimum stage 1 fuel}} &= 0.1259 \text{ m (estimated)} \\ R_{\text{maximum stage 1 fuel}} &= 0.8329 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):
 $(\vec{r}_{\text{cm stage 1 fuel}})_{\text{from base}} = (4.02, 0.0, 0.0) \text{ m}$

The mass properties are in this case:

$$\begin{aligned} \rho_{\text{stage 1 fuel}} &= 1760.0 \text{ kg/m}^3 \text{ (according to [14])} \\ (M_0)_{\text{stage 1 fuel}} &= 20716.0 \text{ kg (according to [2])} \\ \dot{M}_{\text{stage 1 fuel}} &\simeq \frac{(M_0)_{\text{stage 1 fuel}}}{\text{burning time stage 1}} = 345.27 \text{ kg/s} \end{aligned}$$

so the mass of stage 1 fuel as a function of time since stage 1 ignition can be computed with the following expression:

$$M_{\text{stage 1 fuel}}(t) = (M_0 - \dot{M}_{\text{stage 1 fuel}} \cdot t_{\text{since stage 1 ignition}}) \text{ kg} \quad (\text{G.10})$$

From the equation for the mass in a hollow cylinder (equation G.8) we can obtain the internal fuel radius as a function of time since stage 1 ignition:

$$R_{\text{internal stage 1 fuel}}(t) = \sqrt{R_{\text{minimum stage 1 fuel}}^2 + \frac{\dot{M}_{\text{stage 1 fuel}} \cdot t_{\text{since stage 1 ignition}}}{\pi \cdot \rho_{\text{stage 1 fuel}} \cdot L_{\text{stage 1}}}} \quad (\text{G.11})$$

We can also derive this latter equation (G.11) to obtain $(\dot{R})_{\text{internal stage 1 fuel}}(t)$:

$$(\dot{R})_{\text{internal stage 1 fuel}}(t) = \frac{1}{2} \cdot \frac{\dot{M}_{\text{stage 1 fuel}}}{\pi \cdot \rho_{\text{stage 1 fuel}} \cdot L_{\text{stage 1}}} \cdot \frac{1}{R_{\text{internal stage 1 fuel}}(t)} \quad (\text{G.12})$$

It is easy to obtain the moments of inertia as a function of time from the supposition of a hollow cylinder fuel distribution (equation G.9) having $R_{\text{internal stage 1 fuel}}(t)$:

$$\begin{aligned}
 (I_{xx})_{\text{stage 1 fuel}}(t) &= \frac{1}{2}\pi\rho \cdot L \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) \\
 (I_{yy})_{\text{stage 1 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 (I_{zz})_{\text{stage 1 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 &\quad (G.13) \\
 I_{\text{stage 1 fuel}} &= \begin{bmatrix} (I_{xx})_{\text{stage 1 fuel}}(t) & 0 & 0 \\ 0 & (I_{yy})_{\text{stage 1 fuel}}(t) & 0 \\ 0 & 0 & (I_{zz})_{\text{stage 1 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

In the same way it is easy to obtain the derivatives of the moments of inertia as a function of time by deriving the previous expressions having $(\dot{R})_{\text{internal stage 1 fuel}}(t)$:

$$\begin{aligned}
 (\dot{I}_{xx})_{\text{stage 1 fuel}}(t) &= -\frac{1}{2}\pi\rho \cdot L \cdot (4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t)) \\
 (\dot{I}_{yy})_{\text{stage 1 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t)] \\
 &\quad (G.14) \\
 (\dot{I}_{zz})_{\text{stage 1 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t)] \\
 (\dot{I})_{\text{stage 1 fuel}} &= \begin{bmatrix} (\dot{I}_{xx})_{\text{stage 1 fuel}}(t) & 0 & 0 \\ 0 & (\dot{I}_{yy})_{\text{stage 1 fuel}}(t) & 0 \\ 0 & 0 & (\dot{I}_{zz})_{\text{stage 1 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

G.1.3.4 INSTG I-II section A6750

A hollow cylinder geometry is considered with an external radius equal to the medium radius of the real section (which has in fact a conical shape).

This geometry is defined by the following length, external radius and wall thickness:

$$L_{\text{INSTG I-II}} = 1.36 \text{ m (estimated from pictures)}$$

$$R_{\text{INSTG I-II}} = (0.84 \text{ m} - 0.655 \text{ m})/2 = 0.75 \text{ m (according to [2])}$$

$$th_{\text{INSTG I-II}} = 0.007 \text{ m (estimated)}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm INSTG I-II}})_{\text{from base}} = (7.47, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\rho_{\text{Titanium}} = 4420 \text{ kg/m}^3$$

$$M_{\text{INSTG I-II}} \simeq 196.3 \text{ kg (supposing Titanium)}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{INSTG I-II}} = \begin{bmatrix} 108.69 & 0 & 0 \\ 0 & 76.56 & 0 \\ 0 & 0 & 76.56 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Titanium)}$$

G.1.3.5 Stage 2 motor structure

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$L_{\text{Stage 2 structure}} = 3.88 \text{ m (estimated from pictures)}$$

$$R_{\text{Stage 2 structure}} = 0.655 \text{ m (according to [2])}$$

$$th_{\text{Stage 2 structure}} = 0.012 \text{ m (estimated)}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 2 structure}})_{\text{from base}} = (9.41, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\rho_{\text{Titanium}} = 4420 \text{ kg/m}^3$$

$$M_{\text{stage 2 structure}} = 827.0 \text{ kg (6AL-4V Titanium alloy according to [2])}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{stage 2 structure}} = \begin{bmatrix} 348.45 & 0 & 0 \\ 0 & 1211.04 & 0 \\ 0 & 0 & 1211.04 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

(Titanium according to [2])

G.1.3.6 Stage 2 motor fuel

We will consider a radial burn as indicated in [3]. This is, a hollow cylinder fuel distribution. This geometry is defined by the following length and minimum and maximum fuel radii:

$$\begin{aligned} L_{\text{stage 2}} &= 2.91 \text{ m (estimated from pictures)} \\ R_{\text{minimum stage 2 fuel}} &= 0.167 \text{ m (estimated)} \\ R_{\text{maximum stage 2 fuel}} &= 0.643 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 2 fuel}})_{\text{from base}} = (9.60, 0.0, 0.0) \text{ m}$$

The mass properties are in this case:

$$\begin{aligned} \rho_{\text{stage 2 fuel}} &= 1760.0 \text{ kg/m}^3 \text{ (according to [14])} \\ (M_0)_{\text{stage 2 fuel}} &= 6205.0 \text{ kg (according to [2])} \\ \dot{M}_{\text{stage 2 fuel}} &\simeq \frac{M_{0 \text{ stage 2 fuel}}}{\text{burning time stage 2}} = 94.02 \text{ kg/s} \end{aligned}$$

so the mass of stage 2 fuel as a function of time since stage 2 ignition can be computed with the following expression:

$$M_{\text{stage 2 fuel}}(t) = (M_0 - \dot{M}_{\text{stage 2 fuel}} \cdot t_{\text{since stage 2 ignition}}) \text{ kg} \quad (\text{G.15})$$

From the equation for the mass in a hollow cylinder (equation G.9) we can obtain the internal fuel radius as a function of time since stage 2 ignition:

$$R_{\text{internal stage 2 fuel}}(t) = \sqrt{R_{\text{minimum stage 2 fuel}}^2 + \frac{\dot{M}_{\text{stage 2 fuel}} \cdot t_{\text{since stage 2 ignition}}}{\pi \cdot \rho_{\text{stage 2 fuel}} \cdot L_{\text{stage 2}}}} \quad (\text{G.16})$$

We can also derive this latter equation (G.16) to obtain $(\dot{R})_{\text{internal stage 2 fuel}}(t)$:

$$(\dot{R})_{\text{internal stage 2 fuel}}(t) = \frac{1}{2} \cdot \frac{\dot{M}_{\text{stage 2 fuel}}}{\pi \cdot \rho_{\text{stage 2 fuel}} \cdot L_{\text{stage 2}}} \cdot \frac{1}{R_{\text{internal stage 2 fuel}}(t)} \quad (\text{G.17})$$

It is easy to obtain the moments of inertia as a function of time from the supposition of a hollow cylinder fuel distribution (equation G.9) having $R_{\text{internal stage 2 fuel}}(t)$:

$$\begin{aligned}
 (I_{xx})_{\text{stage 2 fuel}}(t) &= \frac{1}{2}\pi\rho \cdot L \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) \\
 (I_{yy})_{\text{stage 2 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 (I_{zz})_{\text{stage 2 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 &\quad (G.18) \\
 I_{\text{stage 2 fuel}} &= \begin{bmatrix} (I_{xx})_{\text{stage 2 fuel}}(t) & 0 & 0 \\ 0 & (I_{yy})_{\text{stage 2 fuel}}(t) & 0 \\ 0 & 0 & (I_{zz})_{\text{stage 2 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

In the same way it is easy to obtain the derivatives of the moments of inertia as a function of time by deriving the previous expressions having $(\dot{R})_{\text{internal stage 2 fuel}}(t)$:

$$\begin{aligned}
 (\dot{I}_{xx})_{\text{stage 2 fuel}}(t) &= -\frac{1}{2}\pi\rho \cdot L \cdot (4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t)) \\
 (\dot{I}_{yy})_{\text{stage 2 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t)] \\
 &\quad (G.19) \\
 (\dot{I}_{zz})_{\text{stage 2 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t)] \\
 (\dot{I})_{\text{stage 2 fuel}} &= \begin{bmatrix} (\dot{I}_{xx})_{\text{stage 2 fuel}}(t) & 0 & 0 \\ 0 & (\dot{I}_{yy})_{\text{stage 2 fuel}}(t) & 0 \\ 0 & 0 & (\dot{I}_{zz})_{\text{stage 2 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

G.1.3.7 INSTG II-III section A6560

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{INSTG II-III}} &= 1.36 \text{ m (estimated from pictures)} \\ R_{\text{INSTG II-III}} &= 0.655 \text{ m (according to [2])} \\ th_{\text{INSTG II-III}} &= 0.007 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm INSTG II-III}})_{\text{from base}} = (11.73, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Titanium}} &= 4420 \text{ kg/m}^3 \\ M_{\text{INSTG II-III}} &= 171.94 \text{ kg (supposing Titanium)} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{INSTG II-III}} = \begin{bmatrix} 72.98 & 0 & 0 \\ 0 & 62.90 & 0 \\ 0 & 0 & 62.90 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Titanium)}$$

G.1.3.8 Stage 3 motor structure

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{Stage 3 structure}} &= 2.52 \text{ m (estimated from pictures)} \\ R_{\text{Stage 3 structure}} &= 0.655 \text{ m (according to [2])} \\ th_{\text{Stage 3 structure}} &= 0.011 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 3 structure}})_{\text{from base}} = (12.80, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Fiberglass}} &= 2580 \text{ kg/m}^3 \\ M_{\text{stage 3 structure}} &= 292.0 \text{ kg (S-901 fiberglass according to [2])} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{stage 3 structure}} = \begin{bmatrix} 123.19 & 0 & 0 \\ 0 & 216.26 & 0 \\ 0 & 0 & 216.26 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (fiberglass according to [2])}$$

G.1.3.9 Stage 3 motor fuel

We will consider a radial burn. This is, a hollow cylinder fuel distribution. This geometry is defined by the following length and minimum and maximum fuel radii:

$$\begin{aligned} L_{\text{stage 3}} &= 1.65 \text{ m (estimated from pictures)} \\ R_{\text{minimum stage 3 fuel}} &= 0.228 \text{ m (estimated)} \\ R_{\text{maximum stage 3 fuel}} &= 0.644 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 3 fuel}})_{\text{from base}} = (13.24, 0.0, 0.0) \text{ m}$$

The mass properties are in this case:

$$\begin{aligned} \rho_{\text{stage 3 fuel}} &= 1760.0 \text{ kg/m}^3 \text{ (according to [14])} \\ (M_0)_{\text{stage 3 fuel}} &= 3308.0 \text{ kg (according to [2])} \\ \dot{M}_{\text{stage 3 fuel}} &\simeq \frac{M_{0\text{stage 3 fuel}}}{\text{burning time stage 3}} = 54.23 \text{ kg/s} \end{aligned}$$

so the mass of stage 3 fuel as a function of time since stage 3 ignition can be computed with the following expression:

$$M_{\text{stage 3 fuel}}(t) = (M_0 - \dot{M}_{\text{stage 3 fuel}} \cdot t_{\text{since stage 3 ignition}}) \text{ kg} \quad (\text{G.20})$$

From the equation for the mass in a hollow cylinder (equation G.9) we can obtain the internal fuel radius as a function of time since stage 3 ignition:

$$R_{\text{internal stage 3 fuel}}(t) = \sqrt{R_{\text{minimum stage 3 fuel}}^2 + \frac{\dot{M}_{\text{stage 3 fuel}} \cdot t_{\text{since stage 3 ignition}}}{\pi \cdot \rho_{\text{stage 3 fuel}} \cdot L_{\text{stage 3}}}} \quad (\text{G.21})$$

We can also derive this latter equation (G.21) to obtain $(\dot{R})_{\text{internal stage 3 fuel}}(t)$:

$$(\dot{R})_{\text{internal stage 3 fuel}}(t) = \frac{1}{2} \cdot \frac{\dot{M}_{\text{stage 3 fuel}}}{\pi \cdot \rho_{\text{stage 3 fuel}} \cdot L_{\text{stage 3}}} \cdot \frac{1}{R_{\text{internal stage 3 fuel}}(t)} \quad (\text{G.22})$$

It is easy to obtain the moments of inertia as a function of time from the supposition of a hollow cylinder fuel distribution (equation G.9) having $R_{\text{internal stage 3 fuel}}(t)$:

$$\begin{aligned}
 (I_{xx})_{\text{stage 3 fuel}}(t) &= \frac{1}{2}\pi\rho \cdot L \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) \\
 (I_{yy})_{\text{stage 3 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 (I_{zz})_{\text{stage 3 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 &\quad (G.23) \\
 I_{\text{stage 3 fuel}} &= \begin{bmatrix} (I_{xx})_{\text{stage 3 fuel}}(t) & 0 & 0 \\ 0 & (I_{yy})_{\text{stage 3 fuel}}(t) & 0 \\ 0 & 0 & (I_{zz})_{\text{stage 3 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

In the same way it is easy to obtain the derivatives of the moments of inertia as a function of time by deriving the previous expressions having $(\dot{R})_{\text{internal stage 3 fuel}}(t)$:

$$\begin{aligned}
 (\dot{I}_{xx})_{\text{stage 3 fuel}}(t) &= -\frac{1}{2}\pi\rho \cdot L \cdot (4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t)) \\
 (\dot{I}_{yy})_{\text{stage 3 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t)] \\
 &\quad (G.24) \\
 (\dot{I}_{zz})_{\text{stage 3 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t)] \\
 (\dot{I})_{\text{stage 3 fuel}} &= \begin{bmatrix} (\dot{I}_{xx})_{\text{stage 3 fuel}}(t) & 0 & 0 \\ 0 & (\dot{I}_{yy})_{\text{stage 3 fuel}}(t) & 0 \\ 0 & 0 & (\dot{I}_{zz})_{\text{stage 3 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

G.1.3.10 Post-Boost Control System

The Post-Boost Control System includes a propulsion system rocket (PSRE, see section G.1.1.4), a battery and a missile guidance set. We will also include in this section the deployment module on which the reentry vehicles are placed.

A solid cylinder geometry is considered. This geometry is defined by the following length and external radius:

$$L_{\text{PBCS}} = 1.75 \text{ m (estimated from pictures)}$$

$$R_{\text{PBCS}} = 0.655 \text{ m (according to [2])}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm PBCS}})_{\text{from base}} = (14.93, 0.0, 0.0) \text{ m}$$

The following mass is considered:

$$M_{\text{PBCS}} = 345.7 \text{ kg}$$

The PSRE alone weighs 271 kg according to [3]. Another 50 kg are considered for the liquid fuel. As a simplification it will be placed in the center of mass of the section. The rest of the mass has been assigned so that the global mass of the missile matches the data in the sources (35400 kg with 3 RV according to [7]) and [4], which leads to 34760 kg with 1 RV).

The moments of inertia of a solid cylinder (supposing a uniform distribution of mass) with respect to its center of mass are given by the following equations:

$$\begin{aligned} I_{xx} &= \frac{1}{2} M \cdot R_{\text{max}}^2 \\ I_{yy} &= \frac{1}{12} M \cdot (3 \cdot R_{\text{max}}^2 + L^2) \\ I_{zz} &= \frac{1}{12} M \cdot (3 \cdot R_{\text{max}}^2 + L^2) \end{aligned} \tag{G.25}$$

In this case this leads to:

$$I_{\text{PBCS}} = \begin{bmatrix} 74.17 & 0 & 0 \\ 0 & 124.86 & 0 \\ 0 & 0 & 124.86 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

As a simplification, we will not decrease the moments of inertia even when the PSRE is used.

G.1.3.11 Reentry Vehicle

The Minuteman III ICBM allows up to 3 MK-12A or 1 MK-21 reentry vehicles (RVs), but for the sake of simplicity only the case when 1 reentry vehicle is carried by the missile is taken into account.

Figure G.6: MK12-A reentry vehicles over the deployment module
(photo from the National Museum of the U.S. Air Force)

A solid cone geometry is considered. This geometry is defined by the following length and maximum radius:

$$L_{RV} = 1.81 \text{ m (according to [15] for MK-12A)}$$

$$(R_{max})_{RV} = 0.27 \text{ m (according to [15] for MK-12A)}$$

Supposing a uniform distribution of mass we have (center of mass located at $1/4$ of the length of the cone):

$$(\vec{r}_{cm RV})_{\text{from base}} = (16.26, 0.0, 0.0) \text{ m}$$

The following mass is considered:

$$M_{RV} = 320.0 \text{ kg (according to [15] for MK-12A)}$$

The moments of inertia of a solid cone (supposing a uniform distribution of mass) with respect to its center of mass are given by the following equations:

$$\begin{aligned} I_{xx} &= \frac{3}{10} M \cdot R_{max}^2 \\ I_{yy} &= \frac{3}{5} M \cdot \left(\frac{R_{max}^2}{4} + L^2 \right) \\ I_{zz} &= \frac{3}{5} M \cdot \left(\frac{R_{max}^2}{4} + L^2 \right) \end{aligned} \tag{G.26}$$

which leads to the following moments of inertia for the RV:

$$I_{RV} = \begin{bmatrix} 7.02 & 0 & 0 \\ 0 & 632.52 & 0 \\ 0 & 0 & 632.52 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

G.1.3.12 Shroud assembly

A hollow cone geometry is considered as an approximation. This geometry is defined by the following length, maximum radius and wall thickness:

$$\begin{aligned} L_{\text{Shroud}} &= 3.39 \text{ m (estimated from pictures)} \\ (R_{\text{max}})_{\text{Shroud}} &= 0.655 \text{ m (according to [2])} \\ th_{\text{Shroud}} &= 0.007 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass located at 1/4 of the length of the cone):

$$(\vec{r}_{\text{cm Shroud}})_{\text{from base}} = (15.69, 0.0, 0.0) \text{ m}$$

The mass can be obtained by the following equation:

$$\frac{1}{3} \rho_{\text{material}} \cdot L \cdot \pi [R_{\text{max}}^2 - (R_{\text{max}} - th)^2] \quad (\text{G.27})$$

which in this case leads to:

$$\begin{aligned} \rho_{\text{Titanium}} &= 4420 \text{ kg/m}^3 \\ M_{\text{Shroud}} &= 143.28 \text{ kg (supposing Titanium)} \end{aligned}$$

The moments of inertia are provided by the following equations:

$$\begin{aligned} I_{xx} &= \frac{1}{10} \cdot \rho_{\text{material}} \cdot L \cdot \pi [R_{\text{max}}^4 - (R_{\text{max}} - th)^4] \\ I_{yy} &= \frac{1}{5} \cdot \rho_{\text{material}} \cdot L \cdot \pi \left[\frac{1}{4} (R_{\text{max}}^4 - (R_{\text{max}} - th)^4) + L^2 \cdot (R_{\text{max}}^2 - (R_{\text{max}} - th)^2) \right] \\ I_{zz} &= \frac{1}{5} \cdot \rho_{\text{material}} \cdot L \cdot \pi \left[\frac{1}{4} (R_{\text{max}}^4 - (R_{\text{max}} - th)^4) + L^2 \cdot (R_{\text{max}}^2 - (R_{\text{max}} - th)^2) \right] \end{aligned} \quad (\text{G.28})$$

which in this case leads to:

$$I_{\text{Shroud}} = \begin{bmatrix} 36.49 & 0 & 0 \\ 0 & 1008.47 & 0 \\ 0 & 0 & 1008.47 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Titanium)}$$

G.1.4 Missile characteristics per stage

G.1.4.1 Missile with stage 1 active

In this case all the components detailed in [G.1.3](#) are included in the missile:

- Skirt section A9650
- Stage 1 motor structure
- Stage 1 motor fuel
This is the item whose properties (mass and inertia tensor) vary with time for the missile in this stage
- INSTG I-II section A6750
- Stage 2 motor structure
- Stage 2 motor fuel
This fuel remains constant with time for the missile in this stage
- INSTG II-III section A6560
- Stage 3 motor structure
- Stage 3 motor fuel
This fuel remains constant with time for the missile in this stage
- Post-Boost Control System
- Reentry vehicle (in this simulation we only consider the case with a single reentry vehicle: SRV)
- Shroud

Taking all the components into account we can obtain the global characteristics for the missile in this phase:

Length = 18.23 m.

Maximum radius = 0.84 m

The mass can be obtained as a function of the time since liftoff by adding the mass of all the sections and taking into account the variation of mass of the fuel of the stage 1:

$$M(t) = (34760.0 - 345.27 \cdot t_{\text{since stage 1 ignition}}) \text{ kg} \quad (\text{G.29})$$

The position of the center of mass of the missile with respect to the base of the missile will be obtained by applying the general equation for obtaining the center of mass (equation D.33) to the set of components:

$$x_{\text{cm missile stage 1 active}}(t) = \sum_{\forall \text{ sections}} \frac{(x_{\text{cm}})_{\text{other sections}} \cdot M_{\text{other sections}} + (x_{\text{cm}})_{\text{stage 1 fuel}} \cdot M_{\text{stage 1 fuel}}(t)}{M_{\text{other sections}} + M_{\text{stage 1 fuel}}(t)} \quad (\text{G.30})$$

so the global position vector from the original base of the missile will be:

$$(\vec{r}_{\text{cm missile stage 1 active}})_{\text{from base}} = (x_{\text{cm missile stage 1 active}}(t), 0.0, 0.0) \text{ m}$$

The moments of inertia with respect to the center of mass can be obtained easily for each $t_{\text{since stage 1 ignition}}$ once the moments of inertia of each section have been obtained by applying the Steiner theorem (equation D.58):

$$\begin{aligned} I_{xx} &= \sum_{\forall \text{ section}} (I_{xx})_{\text{section}} \\ I_{yy} &= \sum_{\forall \text{ section}} (I_{yy})_{\text{section}} + M_{\text{section}} \cdot (x_{\text{cm missile stage 1 active}} - x_{\text{cm section}})^2 \\ I_{zz} &= \sum_{\forall \text{ section}} (I_{zz})_{\text{section}} + M_{\text{section}} \cdot (x_{\text{cm missile stage 1 active}} - x_{\text{cm section}})^2 \end{aligned} \quad (\text{G.31})$$

The derivatives of the moments of inertia will be obtained by deriving the previous equations:

$$\begin{aligned} \dot{I}_{xx} &= (\dot{I}_{xx})_{\text{stage 1 fuel}}(t) \\ \dot{I}_{yy} &\simeq (\dot{I}_{yy})_{\text{stage 1 fuel}}(t) + \dot{M}_{\text{stage 1 fuel}} \cdot (x_{\text{cm missile}} - x_{\text{stage 1 fuel}})^2 \\ \dot{I}_{zz} &\simeq (\dot{I}_{zz})_{\text{stage 1 fuel}}(t) + \dot{M}_{\text{stage 1 fuel}} \cdot (x_{\text{cm missile}} - x_{\text{stage 1 fuel}})^2 \end{aligned} \quad (\text{G.32})$$

Note that $\dot{x}_{\text{stage 1 fuel}} = 0$ according to section G.1.3.6 and that we consider $\dot{x}_{\text{cm missile}} \simeq 0$ according to hypothesis D.81.

G.1.4.2 Missile with stage 2 active

In this case some sections have been detached from the missile and only the following components among the ones detailed in G.1.3 form part of the missile:

- Stage 2 motor structure
- Stage 2 motor fuel
- INSTG II-III section A6560
- Stage 3 motor structure
- Stage 3 motor fuel

This fuel remains constant with time for the missile in this stage

- Post-Boost Control System
- Reentry vehicle
- Shroud

Taking the previous components into account we can obtain the global characteristics for the missile in this phase:

Length = 10.76 m.

Maximum radius = 0.655 m

The mass can be obtained as a function of the time since stage 2 ignition by adding the mass of all the sections and taking into account the variation of mass of the fuel of the stage 2:

$$M(t) = (11612.96 - 94.02 \cdot t_{\text{since stage 1 ignition}}) \text{ kg} \quad (\text{G.33})$$

The position of the center of mass of the missile with respect to the base of the initial missile will be obtained with equation G.30 (changing "stage 1" by "stage 2").

The moments of inertia of the missile with respect to its center of mass will be obtained with equation G.31 (changing "stage 1" by "stage 2").

In the same way the derivatives of the moments of inertia will be obtained with equation G.32 (changing "stage 1" by "stage 2"), taking into account that $\dot{x}_{\text{stage 2 fuel}} = 0$ according to section G.1.3.6 and that we consider $\dot{x}_{\text{cm missile}} \simeq 0$ according to hypothesis D.81.

G.1.4.3 Missile with stage 3 active

In this case more sections have been detached from the missile, including the shroud (see reference [2]), and only the following components among the ones detailed in G.1.3 remain:

- Stage 3 motor structure
- Stage 3 motor fuel

This is the item whose properties (mass and inertia tensor) vary with time for the missile in this stage
- Post-Boost Control System
- Reentry vehicle (in this simulation we only consider the case with a single reentry vehicle: SRV)

Taking the previous components into account we can obtain the global characteristics for the missile in this phase:

Length = 6.07 m.

Maximum radius = 0.655 m

The mass can be obtained as a function of the time since stage 3 ignition by adding the mass of all the sections and taking into account the variation of mass of the fuel of the stage 3:

$$M(t) = (4265.74 - 54.23 \cdot t_{\text{since stage 3 ignition}}) \text{ kg} \quad (\text{G.34})$$

The position of the center of mass of the missile with respect to the base of the initial missile will be obtained with equation G.30 (changing "stage 1" by "stage 3").

The moments of inertia of the missile with respect to its center of mass will be obtained with equation G.31 (changing "stage 1" by "stage 3").

In the same way the derivatives of the moments of inertia will be obtained with equation G.32 (changing "stage 1" by "stage 3"), taking into account that $\dot{x}_{\text{stage 3 fuel}} = 0$ according to section G.1.3.9 and that we consider $\dot{x}_{\text{cm missile}} \simeq 0$ according to hypothesis D.81.

G.1.4.4 Missile in the post-boost phase

In this case only a few sections among the ones detailed in G.1.3 remain:

- Post-Boost Control System (including the deployment module)
- Reentry vehicle

The global characteristics for the missile in this phase are:

Length = 3.56 m.

Maximum radius = 0.655 m

When the PSRE is active the Post-Boost Control System reduces its mass with time, but since the PSRE is a restartable motor it is not possible to provide an easy expression for the mass of the missile as a function of time. As a simplification we will consider a linear function with time for the mass of the missile in this phase related only to the usage of this motor for increasing the missile range:

$$M(t) = 665.74 - 0.466 \cdot t_{\text{since post-boost liftoff}} \text{ kg} \quad (\text{G.35})$$

and we will neglect the reduction of the fuel mass because of the usage of the PSRE for changing the attitude of the missile.

The position of the center of mass of the missile with respect to the base of the initial missile will be obtained with equation G.30 (changing "stage 1" by "post-boost").

The moments of inertia of the missile with respect to its center of mass are constant in this case according to the simplifications indicated in section G.1.3.10.

They can be easily computed using equation G.31 (changing "stage 1" by "post-boost"), obtaining:

$$I_{xx} = 81.19 \text{ kg}$$

$$I_{yy} = 1051.3 \text{ kg}$$

$$I_{zz} = 1051.3 \text{ kg}$$

G.1.4.5 Reentry vehicle

In this case only the reentry vehicle constitutes the missile in the last part of its trajectory towards its target.

The characteristics of the missile in this phase have already been indicated in section G.1.3.11.

G.2 Interceptor parameters

The parameters of the Ground-Based Interceptor (GBI) missile that are required for its simulation are detailed herein.

G.2.1 Motors parameters

G.2.1.1 Stage 1

Name of the motor = Orion-50-SXLG

Figure G.7: Orion 50 SXLG rocket
(photo from reference [16])

Type of motor = Solid fuel motor

Fuel composition = Hydroxyl-terminated poly-butadiene (HTPB) polymer, 19% aluminum (according to [16])

Number of nozzles = 1

$A_e = 0.6567 \text{ m}^2$ (according to [16])

The curve for the vacuum thrust with time is provided by the manufacturer ([16]):

Figure G.8: Orion 50 SXLG rocket vacuum thrust versus time (graph from reference [16])

but we will use average values, also provided by the manufacturer in [16]:

Average burning time = 69.0 s

E_0 = Average vacuum thrust = 583,211 N

p_e = Considered pressure of the gas at the exit of the nozzle = 101,235 Pa
(we suppose that it is an adapted nozzle at sea level).

$(v_e)_{\text{stage 1}}$ can be obtained from the equation for the thrust (equation 4.47):

$$(v_e)_{\text{stage 1}} = \frac{E_0 - p_e \cdot A_e}{(\dot{M})_{\text{stage 1}}} \quad (\text{G.36})$$

($\dot{M}$ is given in section G.2.3.3)

G.2.1.2 Stage 2

Name of the motor = Orion-50-XL

Figure G.9: Orion 50 XL rocket
(photo from reference [16])

Type of motor = Solid fuel motor

Fuel composition = Hydroxyl-terminated poly-butadiene (HTPB) polymer, 19% aluminum (according to [16])

Number of nozzles = 1

$A_e = 0.5823 \text{ m}^2$ (according to [16])

The curve for the vacuum thrust with time is provided by the manufacturer but we will use average values, also provided by the manufacturer in [16]:

Average burning time = 71.0 s

E_0 = Average vacuum thrust = 157,854 N

As a hypothesis and since it is basically the same fuel we will consider:

$$(v_e)_{\text{stage 2}} = (v_e)_{\text{stage 1}} \quad (\text{G.37})$$

This allows obtaining p_e from the equation for the thrust (equation 4.47):

$$p_e = \frac{E_0 - (\dot{M})_{\text{stage 2}} \cdot (v_e)_{\text{stage 2}}}{A_e} \quad (\text{G.38})$$

($\dot{M}$ is given in section G.2.3.6)

G.2.1.3 Stage 3

Name of the motor = Orion-38

Figure G.10: Orion-38 rocket
(photo from reference [16])

Type of motor = Solid fuel motor

Fuel composition = Hydroxyl-terminated poly-butadiene (HTPB) polymer, 19% aluminum (according to [16])

Number of nozzles = 1

$A_e = 0.2171 \text{ m}^2$ (according to [16])

The curve for the vacuum thrust with time is provided by the manufacturer but we will use average values, also provided by the manufacturer in [16]:

Average burning time = 66.8 s

$E_0 = \text{Average vacuum thrust} = 32,681 \text{ N}$

As a hypothesis and since it is basically the same fuel we will consider:

$$(v_e)_{\text{stage 3}} = (v_e)_{\text{stage 1}} \quad (\text{G.39})$$

This allows obtaining p_e from the equation for the thrust (equation 4.47):

$$p_e = \frac{E_0 - (\dot{M})_{\text{stage 3}} \cdot (v_e)_{\text{stage 3}}}{A_e} \quad (\text{G.40})$$

($\dot{M}$ is given in section G.2.3.9)

G.2.1.4 EKV

Name of the motor = Unknown

Figure G.11: Exoatmospheric Kill Vehicle
(picture from reference [17])

Type of motor = Liquid fuel motor

Fuel composition = Unknown

Number of nozzles = 4

$(\dot{M})_{\text{EKV}} = \frac{\text{Thrust}}{1400} \cdot 0.466 \text{ kg/s}$ (supposed as proportional to the requested thrust so that when the nominal thrust of the RS-14 Rocketdyne motor used in the post-boost phase of the ICBM the same $\dot{M}$ as in the Rocketdyne RS-14 is consumed)

$A_e \simeq 4 \times 0.0079 \text{ m}^2$ (estimated from photographs)

Burning time EKV = intermittent

E_0 : A maximum value of 2000 N is considered (as a hypothesis taking into account the value of the thrust in the Rocketdyne RS-14 motor used in the post-boost phase of the ICBM: 1400 N)

p_e = Considered pressure of the gas at the exit of the nozzle = 5.98 MPa (supposed as equal to p_e in the Rocketdyne RS-14 motor)

$(v_e)_{\text{EKV}}$ can be obtained from the equation for the thrust (equation 4.47):

$$(v_e)_{\text{EKV}} = \frac{E_0 - p_e \cdot A_e}{(\dot{M})_{\text{EKV}}} \quad (\text{G.41})$$

G.2.2 Control parameters

G.2.2.1 Control parameters in stage 1

G.2.2.1.1 Geometry of the controls

We consider that the mass flow center of the nozzle in stage 1 is placed in the following coordinates with respect to the base of the missile:

$$\text{Nozzle: } (\vec{r}_e)_{\text{from base}} = (0.0, 0.0, 0.0)$$

G.2.2.1.2 Time parameters

We consider 3 seconds without any control after stage 1 ignition:

$$\text{Time without control}_{\text{stage 1}} = 3 \text{ s}$$

This has been set as a supposition considering enough time for the missile to leave the silo.

We consider (purely supposition) that the initial kick of the gravity turn manoeuvre as considered in section 7.1.1.2 lasts 20 seconds:

$$\text{Time for initial kick}_{\text{stage 1}} = 20 \text{ s}$$

G.2.2.2 Control parameters in stage 2

G.2.2.2.1 Geometry of the controls

We consider that the mass flow center of the nozzle in stage 2 is placed in the following coordinates with respect to the base of the initial missile (middle of the end of the stage 1 section):

$$\text{Nozzle: } (\vec{r}_e)_{\text{from base}} = (9.46, 0.0, 0.0)$$

G.2.2.2.2 Time parameters

We consider 0.5 seconds after the stage 1 has been dropped before stage 2 motor is ignited:

$$\text{Time without ignition}_{\text{stage 2}} = 0.5 \text{ s}$$

This has been estimated after analysing existing times between stages in other launchers.

G.2.2.3 Control parameters in stage 3

G.2.2.3.1 Geometry of the controls

We consider that the mass flow center of the nozzle in stage 3 is placed in the following coordinates with respect to the base of the initial missile (middle of the end of the stage 2 section):

$$\text{Nozzle: } (\vec{r}_e)_{\text{from base}} = (12.53, 0.0, 0.0)$$

G.2.2.3.2 Time parameters

We consider 0.5 seconds after the stage 2 has been dropped before stage 3 motor is ignited:

$$\text{Time without ignition}_{\text{stage 3}} = 0.5 \text{ s}$$

This has been estimated after analysing existing times between stages in other launchers.

G.2.2.4 Control parameters in the EKV

G.2.2.4.1 Geometry of the controls

While the missile is composed by the guidance module and the EKV there are no active controls in the missile. Once the guidance module has been deployed there are 4 divert thrusters and 2 attitude control systems in the EKV.

We consider (estimated from pictures like figure 4.7) that the mass flow centers of the 4 nozzles of the divert thrusters are placed in the following coordinates with respect to the base of the initial missile:

$$\text{Nozzle 1: } (\vec{r}_{e1})_{\text{from base}} = (15.27, 0.0, -0.6)$$

$$\text{Nozzle 2: } (\vec{r}_{e2})_{\text{from base}} = (15.27, 0.6, 0.0)$$

$$\text{Nozzle 3: } (\vec{r}_{e3})_{\text{from base}} = (15.27, 0.0, 0.6)$$

$$\text{Nozzle 4: } (\vec{r}_{e4})_{\text{from base}} = (15.27, -0.6, 0.0)$$

There is not much available information about the attitude control systems so we will just consider in this phase that the control variables are the control moments $\mathcal{M}_x$, $\mathcal{M}_y$ and $\mathcal{M}_z$ used to provide the EKV with the desired attitude.

G.2.2.4.2 Time parameters

We consider 1 second after the stage 3 has been dropped before the guidance module is detached and the EKV motors can be ignited:

$$\text{Time without ignition}_{\text{EKV}} = 1 \text{ s}$$

This has been set as a supposition taking into account that the sooner the EKV is released the more time it has to modify its trajectory in order to hit the incoming ICBM.

G.2.3 Missile components

The global layout of the GBI missile can be observed in figure G.12.

For the different big sections depicted in G.12 an estimation of the mass, position vector of the center of mass of each component with respect to the base of the missile, and inertia tensor with respect to the center of mass of that component will be provided.

Within this analysis, the effect of all the small components has been neglected and several approximations are made in order to assign mass and inertial properties to the considered sections (for example the motor structure section includes the motorcase, the burnout, the nozzles and electronical systems in this section).

Figure G.12: Exploded view of the GBI missile
(picture from [19])

G.2.3.1 Skirt section

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{Skirt}} &= 1.12 \text{ m (estimated from pictures)} \\ R_{\text{Skirt}} &= 0.635 \text{ m (according to [16])} \\ th_{\text{Skirt}} &= 0.03 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm skirt}})_{\text{from base}} = (0.56, 0.0, 0.0) \text{ m}$$

The mass in a hollow cylinder geometry is provided by equation G.8 which in this case leads to:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{skirt}} &\simeq 354.2 \text{ kg (supposing Alluminium)} \end{aligned}$$

The moments of inertia of a hollow cylinder with respect to its center of mass are given by equations G.9 which in this case provide the following results:

$$I_{\text{skirt}} = \begin{bmatrix} 136.22 & 0 & 0 \\ 0 & 105.29 & 0 \\ 0 & 0 & 105.29 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Alluminium)}$$

G.2.3.2 Stage 1 motor structure

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{Stage 1 motor structure}} &= 8.34 \text{ m (estimated from pictures)} \\ R_{\text{Stage 1 motor structure}} &= 0.635 \text{ m (according to [16])} \\ th_{\text{Stage 1 motor structure}} &= 0.012 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 1 structure}})_{\text{from base}} = (5.29, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{stage 1 structure}} &= 1080.0 \text{ kg (according to [16])} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{stage 1 structure}} = \begin{bmatrix} 427.24 & 0 & 0 \\ 0 & 6470.06 & 0 \\ 0 & 0 & 6470.06 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (Alluminium according to [16])}$$

G.2.3.3 Stage 1 motor fuel

We will consider a radial burn. This is, a hollow cylinder fuel distribution. This geometry is defined by the following length and minimum and maximum fuel radii:

$$\begin{aligned} L_{\text{stage 1}} &= 8.34 \text{ m (estimated from pictures)} \\ R_{\text{minimum stage 1 fuel}} &= 0.2635 \text{ m (estimated)} \\ R_{\text{maximum stage 1 fuel}} &= 0.6229 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 1 fuel}})_{\text{from base}} = (5.29, 0.0, 0.0) \text{ m}$$

The mass properties are in this case:

$$\begin{aligned} \rho_{\text{stage 1 fuel}} &= 1801.9 \text{ kg/m}^3 \text{ (according to [20])} \\ (M_0)_{\text{stage 1 fuel}} &= 15034 \text{ kg (according to [16])} \\ \dot{M}_{\text{stage 1 fuel}} &\simeq \frac{(M_0)_{\text{stage 1 fuel}}}{\text{burning time stage 1}} = 217.88 \text{ kg/s} \end{aligned}$$

so the mass of stage 1 fuel as a function of time since stage 1 ignition can be computed with the following expression:

$$M_{\text{stage 1 fuel}}(t) = (M_0 - \dot{M}_{\text{stage 1 fuel}} \cdot t_{\text{since stage 1 ignition}}) \text{ kg} \quad (\text{G.42})$$

From the equation for the mass in a hollow cylinder (equation G.8) we can obtain the internal fuel radius as a function of time since stage 1 ignition:

$$R_{\text{internal stage 1 fuel}}(t) = \sqrt{R_{\text{minimum stage 1 fuel}}^2 + \frac{\dot{M}_{\text{stage 1 fuel}} \cdot t_{\text{since stage 1 ignition}}}{\pi \cdot \rho_{\text{stage 1 fuel}} \cdot L_{\text{stage 1}}}} \quad (\text{G.43})$$

We can also derive this latter equation (G.43) to obtain $(\dot{R})_{\text{internal stage 1 fuel}}(t)$:

$$(\dot{R})_{\text{internal stage 1 fuel}}(t) = \frac{1}{2} \cdot \frac{\dot{M}_{\text{stage 1 fuel}}}{\pi \cdot \rho_{\text{stage 1 fuel}} \cdot L_{\text{stage 1}}} \cdot \frac{1}{R_{\text{internal stage 1 fuel}}(t)} \quad (\text{G.44})$$

It is easy to obtain the moments of inertia as a function of time from the supposition of a hollow cylinder fuel distribution (equation G.9) having $R_{\text{internal stage 1 fuel}}(t)$:

$$\begin{aligned}
 (I_{xx})_{\text{stage 1 fuel}}(t) &= \frac{1}{2}\pi\rho \cdot L \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) \\
 (I_{yy})_{\text{stage 1 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 (I_{zz})_{\text{stage 1 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 &\quad (G.45) \\
 I_{\text{stage 1 fuel}} &= \begin{bmatrix} (I_{xx})_{\text{stage 1 fuel}}(t) & 0 & 0 \\ 0 & (I_{yy})_{\text{stage 1 fuel}}(t) & 0 \\ 0 & 0 & (I_{zz})_{\text{stage 1 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

In the same way it is easy to obtain the derivatives of the moments of inertia as a function of time by deriving the previous expressions having $(\dot{R})_{\text{internal stage 1 fuel}}(t)$:

$$\begin{aligned}
 (\dot{I}_{xx})_{\text{stage 1 fuel}}(t) &= -\frac{1}{2}\pi\rho \cdot L \cdot (4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t)) \\
 (\dot{I}_{yy})_{\text{stage 1 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t)] \\
 &\quad (G.46) \\
 (\dot{I}_{zz})_{\text{stage 1 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 1 fuel}}(t)] \\
 (\dot{I})_{\text{stage 1 fuel}} &= \begin{bmatrix} (\dot{I}_{xx})_{\text{stage 1 fuel}}(t) & 0 & 0 \\ 0 & (\dot{I}_{yy})_{\text{stage 1 fuel}}(t) & 0 \\ 0 & 0 & (\dot{I}_{zz})_{\text{stage 1 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

G.2.3.4 S1/S2 Interstage

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{S1/S2 Interstage}} &= 0.70 \text{ m (estimated from pictures)} \\ R_{\text{S1/S2 Interstage}} &= 0.635 \text{ m (according to [16])} \\ th_{\text{S1/S2 Interstage}} &= 0.03 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm S1/S2 Interstage}})_{\text{from base}} = (9.81, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{S1/S2 Interstage}} &\simeq 220.16 \text{ kg (supposing Alluminium)} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{S1/S2 Interstage}} = \begin{bmatrix} 84.68 & 0 & 0 \\ 0 & 51.27 & 0 \\ 0 & 0 & 51.27 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Alluminium)}$$

G.2.3.5 Stage 2 motor structure

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{Stage 2 structure}} &= 2.37 \text{ m (estimated from pictures)} \\ R_{\text{Stage 2 structure}} &= 0.635 \text{ m (according to [16])} \\ th_{\text{Stage 2 structure}} &= 0.015 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 2 structure}})_{\text{from base}} = (11.34, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{stage 2 structure}} &= 391 \text{ kg (according to [16])} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{stage 2 structure}} = \begin{bmatrix} 931.88 & 0 & 0 \\ 0 & 2301.97 & 0 \\ 0 & 0 & 2301.97 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

(Alluminium according to [16])

G.2.3.6 Stage 2 motor fuel

We will consider a radial burn. This is, a hollow cylinder fuel distribution. This geometry is defined by the following length and minimum and maximum fuel radii:

$$\begin{aligned} L_{\text{stage 2}} &= 2.37 \text{ m (estimated from pictures)} \\ R_{\text{minimum stage 2 fuel}} &= 0.304 \text{ m (estimated)} \\ R_{\text{maximum stage 2 fuel}} &= 0.620 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 2 fuel}})_{\text{from base}} = (11.34, 0.0, 0.0) \text{ m}$$

The mass properties are in this case:

$$\begin{aligned} \rho_{\text{stage 1 fuel}} &= 1801.9 \text{ kg/m}^3 \text{ (according to [20])} \\ (M_0)_{\text{stage 2 fuel}} &= 3915.0 \text{ kg (according to [16])} \\ \dot{M}_{\text{stage 2 fuel}} &\simeq \frac{M_{0\text{stage 2 fuel}}}{\text{burning time stage 2}} = 55.14 \text{ kg/s} \end{aligned}$$

so the mass of stage 2 fuel as a function of time since stage 2 ignition can be computed with the following expression:

$$M_{\text{stage 2 fuel}}(t) = (M_0 - \dot{M}_{\text{stage 2 fuel}} \cdot t_{\text{since stage 2 ignition}}) \text{ kg} \quad (\text{G.47})$$

From the equation for the mass in a hollow cylinder (equation G.9) we can obtain the internal fuel radius as a function of time since stage 2 ignition:

$$R_{\text{internal stage 2 fuel}}(t) = \sqrt{R_{\text{minimum stage 2 fuel}}^2 + \frac{\dot{M}_{\text{stage 2 fuel}} \cdot t_{\text{since stage 2 ignition}}}{\pi \cdot \rho_{\text{stage 2 fuel}} \cdot L_{\text{stage 2}}}} \quad (\text{G.48})$$

We can also derive this latter equation (G.48) to obtain $(\dot{R})_{\text{internal stage 2 fuel}}(t)$:

$$(\dot{R})_{\text{internal stage 2 fuel}}(t) = \frac{1}{2} \cdot \frac{\dot{M}_{\text{stage 2 fuel}}}{\pi \cdot \rho_{\text{stage 2 fuel}} \cdot L_{\text{stage 2}}} \cdot \frac{1}{R_{\text{internal stage 2 fuel}}(t)} \quad (\text{G.49})$$

It is easy to obtain the moments of inertia as a function of time from the supposition of a hollow cylinder fuel distribution (equation G.9) having $R_{\text{internal stage 2 fuel}}(t)$:

$$\begin{aligned}
 (I_{xx})_{\text{stage 2 fuel}}(t) &= \frac{1}{2}\pi\rho \cdot L \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) \\
 (I_{yy})_{\text{stage 2 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 (I_{zz})_{\text{stage 2 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 &\quad (G.50) \\
 I_{\text{stage 2 fuel}} &= \begin{bmatrix} (I_{xx})_{\text{stage 2 fuel}}(t) & 0 & 0 \\ 0 & (I_{yy})_{\text{stage 2 fuel}}(t) & 0 \\ 0 & 0 & (I_{zz})_{\text{stage 2 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

In the same way it is easy to obtain the derivatives of the moments of inertia as a function of time by deriving the previous expressions having $(\dot{R})_{\text{internal stage 2 fuel}}(t)$:

$$\begin{aligned}
 (\dot{I}_{xx})_{\text{stage 2 fuel}}(t) &= -\frac{1}{2}\pi\rho \cdot L \cdot (4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t)) \\
 (\dot{I}_{yy})_{\text{stage 2 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t)] \\
 &\quad (G.51) \\
 (\dot{I}_{zz})_{\text{stage 2 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 2 fuel}}(t)] \\
 (\dot{I})_{\text{stage 2 fuel}} &= \begin{bmatrix} (\dot{I}_{xx})_{\text{stage 2 fuel}}(t) & 0 & 0 \\ 0 & (\dot{I}_{yy})_{\text{stage 2 fuel}}(t) & 0 \\ 0 & 0 & (\dot{I}_{zz})_{\text{stage 2 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

G.2.3.7 S2/S3 Interstage

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{S2/S3 Interstage}} &= 0.4 \text{ m (estimated from pictures)} \\ R_{\text{S2/S3 Interstage}} &= 0.635 \text{ m (according to [16])} \\ th_{\text{S2/S3 Interstage}} &= 0.03 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm S2/S3 Interstage}})_{\text{from base}} = (12.73, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{S2/S3 Interstage}} &= 126.85 \text{ kg (supposing Alluminium)} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{S2/S3 Interstage}} = \begin{bmatrix} 48.79 & 0 & 0 \\ 0 & 26.10 & 0 \\ 0 & 0 & 26.10 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Alluminium)}$$

G.2.3.8 Stage 3 motor structure

A hollow cylinder geometry is considered. This geometry is defined by the following length, external radius and wall thickness:

$$\begin{aligned} L_{\text{Stage 3 structure}} &= 0.938 \text{ m (estimated from pictures)} \\ R_{\text{Stage 3 structure}} &= 0.635 \text{ m (according to [16])} \\ th_{\text{Stage 3 structure}} &= 0.010 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 3 structure}})_{\text{from base}} = (12.93, 0.0, 0.0) \text{ m}$$

This geometry leads (applying equation G.8) to the following mass:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{stage 3 structure}} &= 103.0 \text{ kg (according to [16])} \end{aligned}$$

and the following moments of inertia with respect to its center of mass (applying equation G.9):

$$I_{\text{stage 3 structure}} = \begin{bmatrix} 40.87 & 0 & 0 \\ 0 & 27.98 & 0 \\ 0 & 0 & 27.98 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (Alluminium according to [16])}$$

G.2.3.9 Stage 3 motor fuel

We will consider a radial burn. This is, a hollow cylinder fuel distribution. This geometry is defined by the following length and minimum and maximum fuel radii:

$$\begin{aligned} L_{\text{stage 3}} &= 0.938 \text{ m (estimated from pictures)} \\ R_{\text{minimum stage 3 fuel}} &= 0.495 \text{ m (estimated)} \\ R_{\text{maximum stage 3 fuel}} &= 0.625 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm stage 3 fuel}})_{\text{from base}} = (12.93, 0.0, 0.0) \text{ m}$$

The mass properties are in this case:

$$\begin{aligned} \rho_{\text{stage 1 fuel}} &= 1801.9 \text{ kg/m}^3 \text{ (according to [20])} \\ (M_0)_{\text{stage 3 fuel}} &= 770.0 \text{ kg (according to [16])} \\ \dot{M}_{\text{stage 3 fuel}} &\simeq \frac{M_{0\text{stage 3 fuel}}}{\text{burning time stage 3}} = 11.53 \text{ kg/s} \end{aligned}$$

so the mass of stage 3 fuel as a function of time since stage 3 ignition can be computed with the following expression:

$$M_{\text{stage 3 fuel}}(t) = (M_0 - \dot{M}_{\text{stage 3 fuel}} \cdot t_{\text{since stage 3 ignition}}) \text{ kg} \quad (\text{G.52})$$

From the equation for the mass in a hollow cylinder (equation G.9) we can obtain the internal fuel radius as a function of time since stage 3 ignition:

$$R_{\text{internal stage 3 fuel}}(t) = \sqrt{R_{\text{minimum stage 3 fuel}}^2 + \frac{\dot{M}_{\text{stage 3 fuel}} \cdot t_{\text{since stage 3 ignition}}}{\pi \cdot \rho_{\text{stage 3 fuel}} \cdot L_{\text{stage 3}}}} \quad (\text{G.53})$$

We can also derive this latter equation (G.53) to obtain $(\dot{R})_{\text{internal stage 3 fuel}}(t)$:

$$(\dot{R})_{\text{internal stage 3 fuel}}(t) = \frac{1}{2} \cdot \frac{\dot{M}_{\text{stage 3 fuel}}}{\pi \cdot \rho_{\text{stage 3 fuel}} \cdot L_{\text{stage 3}}} \cdot \frac{1}{R_{\text{internal stage 3 fuel}}(t)} \quad (\text{G.54})$$

It is easy to obtain the moments of inertia as a function of time from the supposition of a hollow cylinder fuel distribution (equation G.9) having $R_{\text{internal stage 3 fuel}}(t)$:

$$\begin{aligned}
 (I_{xx})_{\text{stage 3 fuel}}(t) &= \frac{1}{2}\pi\rho \cdot L \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) \\
 (I_{yy})_{\text{stage 3 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 (I_{zz})_{\text{stage 3 fuel}}(t) &= \frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot (R_{\text{maximum}}^4 - R_{\text{internal}}^4(t)) + L^2 \cdot (R_{\text{maximum}}^2 - R_{\text{internal}}^2(t))] \\
 &\quad (G.55) \\
 I_{\text{stage 3 fuel}} &= \begin{bmatrix} (I_{xx})_{\text{stage 3 fuel}}(t) & 0 & 0 \\ 0 & (I_{yy})_{\text{stage 3 fuel}}(t) & 0 \\ 0 & 0 & (I_{zz})_{\text{stage 3 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

In the same way it is easy to obtain the derivatives of the moments of inertia as a function of time by deriving the previous expressions having $(\dot{R})_{\text{internal stage 3 fuel}}(t)$:

$$\begin{aligned}
 (\dot{I}_{xx})_{\text{stage 3 fuel}}(t) &= -\frac{1}{2}\pi\rho \cdot L \cdot (4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t)) \\
 (\dot{I}_{yy})_{\text{stage 3 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t)] \\
 &\quad (G.56) \\
 (\dot{I}_{zz})_{\text{stage 3 fuel}}(t) &= -\frac{1}{12}\pi\rho \cdot L \cdot [3 \cdot 4 \cdot R_{\text{internal}}^3(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t) + \\
 &\quad + L^2 \cdot 2 \cdot R_{\text{internal}}(t) \cdot (\dot{R})_{\text{internal stage 3 fuel}}(t)] \\
 (\dot{I})_{\text{stage 3 fuel}} &= \begin{bmatrix} (\dot{I}_{xx})_{\text{stage 3 fuel}}(t) & 0 & 0 \\ 0 & (\dot{I}_{yy})_{\text{stage 3 fuel}}(t) & 0 \\ 0 & 0 & (\dot{I}_{zz})_{\text{stage 3 fuel}}(t) \end{bmatrix}
 \end{aligned}$$

G.2.3.10 Guidance module and deployment module

The missile includes a guidance module. We will also include in this section the deployment module on which the EKV is placed.

A solid cylinder geometry is considered. This geometry is defined by the following length and external radius:

$$L_{\text{GM}} = 1.0 \text{ m (estimated from pictures)}$$

$$R_{\text{GM}} = 0.635 \text{ m (according to [16])}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm GM}})_{\text{from base}} = (14.37, 0.0, 0.0) \text{ m}$$

The following mass is considered:

$$M_{\text{GM}} = 445.2 \text{ kg}$$

This mass has been assigned so that the global mass of the missile matches the data in the sources (22,700 kg according to [4]).

The moments of inertia of a solid cylinder (supposing a uniform distribution of mass) with respect to its center of mass are given by the following equations:

$$\begin{aligned} I_{xx} &= \frac{1}{2} M \cdot R_{\text{max}}^2 \\ I_{yy} &= \frac{1}{12} M \cdot (3 \cdot R_{\text{max}}^2 + L^2) \\ I_{zz} &= \frac{1}{12} M \cdot (3 \cdot R_{\text{max}}^2 + L^2) \end{aligned} \tag{G.57}$$

In this case this leads to:

$$I_{\text{GM}} = \begin{bmatrix} 89.76 & 0 & 0 \\ 0 & 81.98 & 0 \\ 0 & 0 & 81.98 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

G.2.3.11 EKV

The mission of the GBI missile is to place the Exoatmospheric Kill Vehicle (EKV) on the right path so that it can track and hit the ICBM.

A solid cylinder geometry is considered. This geometry is defined by the following length and external radius:

$$L_{\text{EKV}} = 1.4 \text{ m (according to [18])}$$

$$R_{\text{EKV}} = 0.635 \text{ m (according to [16])}$$

Supposing a uniform distribution of mass we have (center of mass in the middle of the section):

$$(\vec{r}_{\text{cm EKV}})_{\text{from base}} = (15.57, 0.0, 0.0) \text{ m}$$

The following mass is considered:

$$M_{\text{EKV}} = 64.0 \text{ kg (according to [18])}$$

The moments of inertia of a solid cylinder (supposing a uniform distribution of mass) with respect to its center of mass are given by the following equations:

$$\begin{aligned} I_{xx} &= \frac{1}{2} M \cdot R_{max}^2 \\ I_{yy} &= \frac{1}{12} M \cdot (3 \cdot R_{max}^2 + L^2) \\ I_{zz} &= \frac{1}{12} M \cdot (3 \cdot R_{max}^2 + L^2) \end{aligned} \tag{G.58}$$

In this case this leads to:

$$I_{\text{EKV}} = \begin{bmatrix} 2.88 & 0 & 0 \\ 0 & 11.89 & 0 \\ 0 & 0 & 11.89 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

G.2.3.12 Shroud assembly

A hollow cone geometry is considered as an approximation. This geometry is defined by the following length, maximum radius and wall thickness:

$$\begin{aligned} L_{\text{Shroud}} &= 3.39 \text{ m (estimated from pictures)} \\ (R_{\text{max}})_{\text{Shroud}} &= 0.655 \text{ m (according to [2])} \\ th_{\text{Shroud}} &= 0.007 \text{ m (estimated)} \end{aligned}$$

Supposing a uniform distribution of mass we have (center of mass located at 1/4 of the length of the cone):

$$(\vec{r}_{\text{cm Shroud}})_{\text{from base}} = (14.98, 0.0, 0.0) \text{ m}$$

The mass can be obtained by the following equation:

$$\frac{1}{3} \rho_{\text{material}} \cdot L \cdot \pi [R_{\text{max}}^2 - (R_{\text{max}} - th)^2] \quad (\text{G.59})$$

which in this case leads to:

$$\begin{aligned} \rho_{\text{Alluminium}} &= 2700 \text{ kg/m}^3 \\ M_{\text{Shroud}} &= 196.61 \text{ kg (supposing Alluminium)} \end{aligned}$$

The moments of inertia are provided by the following equations:

$$\begin{aligned} I_{xx} &= \frac{1}{10} \cdot \rho_{\text{material}} \cdot L \cdot \pi [R_{\text{max}}^4 - (R_{\text{max}} - th)^4] \\ I_{yy} &= \frac{1}{5} \cdot \rho_{\text{material}} \cdot L \cdot \pi \left[\frac{1}{4} (R_{\text{max}}^4 - (R_{\text{max}} - th)^4) + L^2 \cdot (R_{\text{max}}^2 - (R_{\text{max}} - th)^2) \right] \\ I_{zz} &= \frac{1}{5} \cdot \rho_{\text{material}} \cdot L \cdot \pi \left[\frac{1}{4} (R_{\text{max}}^4 - (R_{\text{max}} - th)^4) + L^2 \cdot (R_{\text{max}}^2 - (R_{\text{max}} - th)^2) \right] \end{aligned} \quad (\text{G.60})$$

which in this case leads to:

$$I_{\text{Shroud}} = \begin{bmatrix} 58.98 & 0 & 0 \\ 0 & 935.03 & 0 \\ 0 & 0 & 935.03 \end{bmatrix} \text{ kg} \cdot \text{m}^2 \text{ (supposing Alluminium)}$$

G.2.4 Missile characteristics per stage

G.2.4.1 Missile with stage 1 active

In this case all the components detailed in G.2.3 are included in the missile:

- Skirt section
- Stage 1 motor structure
- Stage 1 motor fuel

This is the item whose properties (mass and inertia tensor) vary with time for the missile in this stage

- S1/S1 Interstage
- Stage 2 motor structure
- Stage 2 motor fuel

This fuel remains constant with time for the missile in this stage

- S2/S3 Interstage
- Stage 3 motor structure
- Stage 3 motor fuel

This fuel remains constant with time for the missile in this stage

- Guidance Module (including the deployment module)
- EKV
- Shroud

Taking all the components into account we can obtain the global characteristics for the missile in this phase:

Length = 16.8 m.

Maximum radius = 0.635 m

The mass can be obtained as a function of the time since liftoff by adding the mass of all the sections and taking into account the variation of mass of the fuel of the stage 1:

$$M(t) = (22700 - 217.88 \cdot t_{\text{since stage 1 ignition}}) \text{ kg} \quad (\text{G.61})$$

The position of the center of mass of the missile with respect to the base of the missile will be obtained by applying the general equation for obtaining the center of mass (equation D.33) to the set of components:

$$x_{\text{cm missile stage 1 active}}(t) = \sum_{\forall \text{ sections}} \frac{(x_{\text{cm}})_{\text{other sections}} \cdot M_{\text{other sections}} + (x_{\text{cm}})_{\text{stage 1 fuel}} \cdot M_{\text{stage 1 fuel}}(t)}{M_{\text{other sections}} + M_{\text{stage 1 fuel}}(t)} \quad (\text{G.62})$$

so the global position vector from the original base of the missile will be:

$$(\vec{r}_{\text{cm missile stage 1 active}})_{\text{from base}} = (x_{\text{cm missile stage 1 active}}(t), 0.0, 0.0) \text{ m}$$

The moments of inertia with respect to the center of mass can be obtained easily for each $t_{\text{since stage 1 ignition}}$ once the moments of inertia of each section have been obtained by applying the Steiner theorem (equation D.58):

$$\begin{aligned} I_{xx} &= \sum_{\forall \text{ section}} (I_{xx})_{\text{section}} \\ I_{yy} &= \sum_{\forall \text{ section}} (I_{yy})_{\text{section}} + M_{\text{section}} \cdot (x_{\text{cm missile stage 1 active}} - x_{\text{cm section}})^2 \\ I_{zz} &= \sum_{\forall \text{ section}} (I_{zz})_{\text{section}} + M_{\text{section}} \cdot (x_{\text{cm missile stage 1 active}} - x_{\text{cm section}})^2 \end{aligned} \quad (\text{G.63})$$

The derivatives of the moments of inertia will be obtained by deriving the previous equations:

$$\begin{aligned} \dot{I}_{xx} &= (\dot{I}_{xx})_{\text{stage 1 fuel}}(t) \\ \dot{I}_{yy} &\simeq (\dot{I}_{yy})_{\text{stage 1 fuel}}(t) + \dot{M}_{\text{stage 1 fuel}} \cdot (x_{\text{cm missile}} - x_{\text{stage 1 fuel}})^2 \\ \dot{I}_{zz} &\simeq (\dot{I}_{zz})_{\text{stage 1 fuel}}(t) + \dot{M}_{\text{stage 1 fuel}} \cdot (x_{\text{cm missile}} - x_{\text{stage 1 fuel}})^2 \end{aligned} \quad (\text{G.64})$$

Note that $\dot{x}_{\text{stage 1 fuel}} = 0$ according to section G.2.3.6 and that we consider $\dot{x}_{\text{cm missile}} \simeq 0$ according to hypothesis D.81.

G.2.4.2 Missile with stage 2 active

In this case some sections have been detached from the missile and only the following components among the ones detailed in G.2.3 form part of the missile:

- Stage 2 motor structure
- Stage 2 motor fuel
- S2/S3 Interstage
- Stage 3 motor structure
- Stage 3 motor fuel

This fuel remains constant with time for the missile in this stage

- Guidance Module (including the deployment module)
- EKV
- Shroud

Taking the previous components into account we can obtain the global characteristics for the missile in this phase:

Length = 6.64 m.

Maximum radius = 0.635 m

The mass can be obtained as a function of the time since stage 2 ignition by adding the mass of all the sections and taking into account the variation of mass of the fuel of the stage 2:

$$M(t) = (6011.68 - 55.14 \cdot t_{\text{since stage 2 ignition}}) \text{ kg} \quad (\text{G.65})$$

The position of the center of mass of the missile with respect to the base of the initial missile will be obtained with equation G.62 (changing "stage 1" by "stage 2").

The moments of inertia of the missile with respect to its center of mass will be obtained with equation G.63 (changing "stage 1" by "stage 2").

In the same way the derivatives of the moments of inertia will be obtained with equation G.64 (changing "stage 1" by "stage 2"), taking into account that $\dot{x}_{\text{stage 2 fuel}} = 0$ according to section G.2.3.6 and that we consider $\dot{x}_{\text{cm missile}} \simeq 0$ according to hypothesis D.81.

G.2.4.3 Missile with stage 3 active

In this case more sections have been detached from the missile, including the shroud, and only the following components among the ones detailed in G.2.3 remain:

- Stage 3 motor structure
- Stage 3 motor fuel

This is the item whose properties (mass and inertia tensor) vary with time for the missile in this stage

- Guidance Module (including the deployment module)
- EKV

Taking the previous components into account we can obtain the global characteristics for the missile in this phase:

Length = 3.87 m.

Maximum radius = 0.635 m

The mass can be obtained as a function of the time since stage 3 ignition by adding the mass of all the sections and taking into account the variation of mass of the fuel of the stage 3:

$$M(t) = (1382.22 - 11.53 \cdot t_{\text{since stage 3 ignition}}) \text{ kg} \quad (\text{G.66})$$

The position of the center of mass of the missile with respect to the base of the initial missile will be obtained with equation G.62 (changing "stage 1" by "stage 3").

The moments of inertia of the missile with respect to its center of mass will be obtained with equation G.63 (changing "stage 1" by "stage 3").

In the same way the derivatives of the moments of inertia will be obtained with equation G.64 (changing "stage 1" by "stage 3"), taking into account that $\dot{x}_{\text{stage 3 fuel}} = 0$ according to section G.2.3.9 and that we consider $\dot{x}_{\text{cm missile}} \simeq 0$ according to hypothesis D.81.

G.2.4.4 Missile in the post-boost phase

In this case only a few sections among the ones detailed in [G.1.3](#) remain:

- Guidance Module (including the deployment module)
- EKV

The global characteristics for the missile in this phase are:

Length = 2.4 m.

Maximum radius = 0.635 m

While the Guidance Module is attached to the EKV the mass of the missile remains constant:

$$M = 509.22 \text{ kg} \quad (\text{G.67})$$

The position of the center of mass of the missile with respect to the base of the initial missile also remains constant:

$$(\vec{r}_{\text{cm RV}})_{\text{from base}} = (14.52, 0.0, 0.0) \text{ m}$$

The moments of inertia of the missile with respect to its center of mass are also constant in this case:

$$I_{xx} = 92.64 \text{ kg}$$

$$I_{yy} = 174.39 \text{ kg}$$

$$I_{zz} = 174.39 \text{ kg}$$

G.2.4.5 EKV

In this case only the EKV constitutes the missile in the last part of its trajectory towards its target. The characteristics of the missile in this phase have already been indicated in section [G.2.3.11](#).

Appendix G references

- [1] Norbert Brüggel. Spaceroockets 2. http://www.b14643.de/Spaceroockets_2/index.htm. [web page accessed on 24/09/2013]. 430, 431, 432
- [2] ICBM Prime Team, Prime - 19378, TRW Systems. *Minuteman Weapon System History and Description*. ICBM System Program Office, Hill Air Force Base Utah, second edition, July 2001. 430, 432, 436, 438, 439, 441, 442, 444, 445, 447, 449, 453, 474
- [3] David P. Blanks, Anthony M. Logue, Stephen J. Skotte, Douglas M. Bruce, Ralph A. Sandfry, and Michael L. Zywiec. A Two-Stage Intercontinental Ballistic Missile (ICBM) Design Optimization Study and Life Cycle Cost Analysis. Master's thesis, Faculty of the School of Engineering of the Air Force Institute of Technology Air University, Wright-Patterson Air Force Base, Ohio, December 1992. 430, 431, 432, 433, 435, 436, 439, 442, 447
- [4] Mark Wade. Encyclopedia astronautica. <http://www.astronautix.com>. [web page accessed on 24/09/2013]. 430, 431, 432, 433, 436, 447, 472
- [5] Jonathan McDowell. Jonathan's Space Report Launch Vehicle Database, 2011 Edition. <http://planet4589.org/space/lvdb/>, 2011. [web page accessed on 24/09/2013]. 430, 431, 432
- [6] Frank H. Winter and George S. James. Highlights of 50 years of Aerojet, a pioneering American rocket company, 1942-1992. *Acta Astronautica*, 35(9-11):677-698, 1995. doi: 10.1016/0094-5765(95)00029-Y. 431
- [7] Nuclear Weapon Archive. The Minuteman III ICBM. <http://nuclearweaponarchive.org/Usa/Weapons/Mmiii.html>, October 1997. [web page accessed on 21/09/2013]. 433, 447
- [8] Aerojet Rocketdyne. Minuteman III PSRE. <http://www.rocket.com/minuteman-iii-psre>, 2013. [web page accessed on 28/09/2013]. 433, 435
- [9] Scott Schneeweis. Artifact: Rocket engine, RS-2101C, liquid propellant, Mars Viking Orbiter, Orbit Insertion Engine. http://www.spaceaholic.com/index.php/Detail/Object/Show/object_id/18. [web page accessed on 02/11/2014]. 433
- [10] L. Cutrone, M. Ihme, and M. Herrmann. Modeling of high-pressure mixing and combustion in liquid rocket injectors. *Center for Turbulence Research. Proceedings of the 2006 Summer Program*, pages 269-281, 2006. 433
- [11] 341st Operations Group. Operation of the Minuteman III ICBM. <https://www.youtube.com/watch?v=PJ9tgSgx3PY>, March 2008. [web page accessed on 04/08/2014]. 434
- [12] National Park Service. Minuteman and the next generation (1960s - present). http://www.nps.gov/history/history/online_books/mimi/hrs1-3c.htm, November 2003. [web page accessed on 22/09/2013]. 436
- [13] National Museum of the US Air Force. Minuteman Timeline. <http://www.nationalmuseum.af.mil/factsheets/factsheet.asp?id=13569>. [web page accessed on 22/09/2013]. 436

-
- [14] Delft University of Technology. Solid rocket propellants and their properties. <http://www.lr.tudelft.nl/en/organisation/departments/space-engineering/space-systems-engineering/expertise-areas/space-propulsion/design-of-elements/rocket-propellants/solids/>, September 2004. [web page accessed on 02/11/2014]. 439, 442, 445
- [15] Carey Sublette. The W-78 Warhead. <http://nuclearweaponarchive.org/Usa/Weapons/W78.html>, September 2001. [web page accessed on 02/11/2014]. 448
- [16] ATK. ATK Space Propulsion Products Catalog. <http://cms.atk.com/SiteCollectionDocuments/ProductsAndServices/ATK-Motor-Catalog-2012.pdf>, September 2012. 455, 456, 457, 458, 463, 464, 466, 467, 469, 470, 472, 473
- [17] Jim Mallard. Web home of Jim Mallard. <http://www.jimmallard3.com>. [web page accessed on 02/01/2015]. 459
- [18] Missile Defense Advocacy Alliance. Ground Based Interceptor Exoatmospheric Kill Vehicle. <http://missiledefenseadvocacy.org/missile-defense-systems/u-s-deployed-intercept-systems/ground-based-midcourse-defense/ground-based-interceptor-exoatmospheric-kill-vehicle/>. [web page accessed on 08/11/2013]. 473
- [19] Randolph R. Stone. DODIG-2014-111, Exoatmospheric Kill Vehicle Quality Assurance and Reliability Assessment - Part A. <http://www.dodig.mil/pubs/documents/DODIG-2014-111.pdf>, September 2014. [web page accessed on 02/01/2015]. 462
- [20] Stephen P. Prince and Douglas W. Banning. *AL/EQ-TR-1994-0042. Launch Vehicle Abort Source Strength Model. Volume II: Source Characterization*. Armstrong Laboratory. Air Force Material Command. Tyndall Air Force Base, Florida, March 1995. 464, 467, 470

This page intentionally left blank.

Symbols

α	Angle of attack of the missile
β	Sideslip angle of the missile
γ	Flight path angle of the missile (angle of the tangent to the trajectory)
γ_y	Horizontal flight path angle of the missile
γ_z	Vertical flight path angle of the missile (angle of the tangent to the trajectory with the local vertical angle)
θ	Pitch angle of the missile
λ	Longitude of the missile
λ	Deflection angle of a nozzle with respect to its axis (control variable for the missile)
μ	Angle of deflection of a nozzle in the plane of the nozzle (control variable for the missile)
ξ	Angle of the thrust vector with respect to the inertial reference frame
ξ	Damping ratio of a second order system
ρ	Air density at the present height
ϕ	Roll angle of the missile
φ	Latitude of the missile
ψ	Yaw angle of the missile
ω	Angular frequency of a second order system
A	Aerodynamic axial force acting on the missile (force in the Ox_b axis)
A	Generic representation of the state matrix of a linearized system: $\dot{\overrightarrow{\Delta X}}(t) = A \cdot \overrightarrow{\Delta X}(t) + B \cdot \overrightarrow{\Delta u}(t)$
A	Transformation matrix between True Earth-Centered Earth-Fixed coordinates and Mean Earth-Centered Earth-Fixed coordinates (CTS). This matrix is given in equation A.17

A_1	First component of the vector $\vec{A} = (I_{CM}^b) \cdot (\dot{p}, \dot{q}, \dot{r})^T$
A_2	Second component of the vector $\vec{A} = (I_{CM}^b) \cdot (\dot{p}, \dot{q}, \dot{r})^T$
A_3	Third component of the vector $\vec{A} = (I_{CM}^b) \cdot (\dot{p}, \dot{q}, \dot{r})^T$
A_e	Surface area at the exit of the nozzle
B	Generic representation of the control matrix of a linearized system: $\dot{\vec{\Delta X}}(t) = A \cdot \vec{\Delta X}(t) + B \cdot \vec{\Delta u}(t)$
B	Transformation matrix between Mean True Earth-Centered Inertial of Date coordinates and True Earth-Centered Earth-Fixed coordinates. This matrix is given in equation A.8
b	Body reference frame
C	Transformation matrix between Mean Earth-Centered Inertial of Date coordinates and Mean True Earth-Centered Inertial of Date coordinates. This matrix is given in equation A.4
C_A	Aerodynamic coefficient related to the Axial force (force in the Ox_b axis)
C_D	Aerodynamic coefficient related to the Drag force
C_L	Aerodynamic coefficient related to the Lift force
C_{LL}	Aerodynamic coefficient related to the rolling moment (moment in the Ox_b axis)
C_{LN}	Aerodynamic coefficient related to the yawing moment (moment in the Oz_b axis)
C_M	Aerodynamic coefficient related to the pitching moment (moment in the Oy_b axis)
CM	Center of mass
C_N	Aerodynamic coefficient related to the Normal force (force in the Oz_b axis)
C_Y	Aerodynamic coefficient related to the Lateral force (force in the Oy_b axis)
C_a^b	Transformation matrix that allows changing from reference frame a to reference frame b : $\vec{v}^b = C_a^b \cdot \vec{v}^a$
C_b^i	Transformation matrix from the body reference frame to the inertial reference frame. This matrix is given in equation 3.17
C_b^n	Transformation matrix from the body reference frame to the navigation reference frame. $C_b^n = (C_n^b)^T$
C_e^i	Transformation matrix from the ECEF reference frame to the ECI reference frame = $[A \cdot B \cdot C \cdot D]^T$
C_e^n	Transformation matrix from the ECEF reference frame to the navigation reference frame. This matrix is given in equation A.35

C_i^b	Transformation matrix from the inertial reference frame to the body reference frame. $C_i^b = (C_b^i)^T$
C_i^e	Transformation matrix from the ECI reference frame to the ECEF reference frame = $[A \cdot B \cdot C \cdot D]$
C_n^b	Transformation matrix from the navigation reference frame to the body reference frame. This matrix is given in equation A.37
C_n^e	Transformation matrix from the navigation reference frame to the ECEF reference frame. $C_n^e = (C_e^n)^T$
D	Drag force acting on the missile
D	Transformation matrix between CIS coordinates and Mean Earth-Centered Inertial of Date coordinates. This matrix is given in equation A.1
E	Ellipsoidal coordinates
e	ECEF reference frame
F_x	Component in the Ox_b axis of the external forces acting on the missile
F_y	Component in the Oy_b axis of the external forces acting on the missile
F_z	Component in the Oz_b axis of the external forces acting on the missile
G	Reference system used for the integration of the thrust in the Iterative Guidance Mode (IGM)
i	ECI reference frame
I_x	Moment of inertia of the missile with respect to the Ox_b axis
I_y	Moment of inertia of the missile with respect to the Oy_b axis
I_z	Moment of inertia of the missile with respect to the Oz_b axis
J	Functional to be minimized within an optimal control strategy
J_{xy}	Product of inertia of the missile with respect to the planes $x_b = 0$ and $y_b = 0$
J_{xz}	Product of inertia of the missile with respect to the planes $x_b = 0$ and $z_b = 0$
J_{yz}	Product of inertia of the missile with respect to the planes $y_b = 0$ and $z_b = 0$
K_D	Differential gain in the PID control scheme
K_I	Integral gain in the PID control scheme
K_P	Proportional gain in the PID control scheme
K_{DL}	Adjustment gain in the PID control scheme

$\mathcal{L}$	Component in the Ox_b axis of the momentum of the external forces acting on the missile with respect to the center of mass
L	Lift force acting on the missile
$\mathcal{M}$	Component in the Oy_b axis of the momentum of the external forces acting on the missile with respect to the center of mass
M	Mass of the missile
$\mathcal{N}$	Component in the Oz_b axis of the momentum of the external forces acting on the missile with respect to the center of mass
N	Aerodynamic normal force acting on the missile (force in the Oz_b axis)
n	Navigation reference frame
N_{nz}	Number of nozzles
p	Component in the Ox_b axis of the angular velocity vector of the missile
p_0	Atmospheric pressure at the present height
p_e	Internal pressure in the nozzle of the missile
q	Component in the Oy_b axis of the angular velocity vector of the missile
q_0	Scalar component of the rotation quaternion from the inertial to the body frame (q_{bi})
q_1	First vector component of the rotation quaternion from the inertial to the body frame (q_{bi})
q_2	Second vector component of the rotation quaternion from the inertial to the body frame (q_{bi})
q_2	Third vector component of the rotation quaternion from the inertial to the body frame (q_{bi})
q_{bi}	Rotation quaternion from the inertial to the body reference frame
q_{bn}	Rotation quaternion from the navigation to the body reference frame
R	Rectangular coordinates
r	Component in the Oz_b axis of the angular velocity vector of the missile
$\mathcal{R}_A$	Region of absolute stability of a numerical method
S	Spherical coordinates
T	Air temperature at the present height
T	Disturbing potential that has to be added to the U potential to reach the real W gravity potential

t	Time
t_{go}	Remaining time for the missile to reach its target
U	Gravity potential of a perfect ellipsoidal body
u	Component in the Ox_b axis of the velocity vector of the center of mass of the missile $((\vec{v}_{CM}^b)_i)$
V	Gravitational potential
v	Component in the Oy_b axis of the velocity vector of the center of mass of the missile $((\vec{v}_{CM}^b)_i)$
V_c	Closing velocity (modulus of the relative velocity between the missile and its target)
v_d	Component in the Down axis of the velocity vector of the center of mass of the missile $((\vec{v}_{CM}^n)_e)$
v_e	Component in the East axis of the velocity vector of the center of mass of the missile $((\vec{v}_{CM}^n)_e)$
v_n	Component in the North axis of the velocity vector of the center of mass of the missile $((\vec{v}_{CM}^n)_e)$
W	Gravity potential (gravitational potential plus a potential that takes into account the centrifugal acceleration)
w	Component in the Oz_b axis of the velocity vector of the center of mass of the missile $((\vec{v}_{CM}^b)_i)$
x	Component in the Ox_b axis of the position of the center of mass of the missile $((\vec{r}_{CM}^i)_i)$
Y	Aerodynamic lateral force acting on the missile (force in the Oy_b axis)
y	Component in the Oy_b axis of the position of the center of mass of the missile $((\vec{r}_{CM}^i)_i)$
z	Component in the Oz_b axis of the position of the center of mass of the missile $((\vec{r}_{CM}^i)_i)$
$\vec{\gamma}$	Normal gravity field (related to a perfectly ellipsoidal Earth)
$\vec{\lambda}_F$	Vector in the direction of the thrust
$\vec{\lambda}_v$	Unit vector in the direction of the velocity to be gained $(\vec{v}_g)$
$\vec{\xi}$	Unit thrust vector
$\vec{\Omega}_{bi}^b$	Angular velocity vector of the missile w.r.t. the inertial reference frame
$\vec{\omega}_{ei}^e$	Angular velocity vector of the ECEF reference frame with respect to the ECI reference frame. It is given by equation C.9
$\vec{\omega}_{ne}^n$	Angular velocity vector of the Navigation reference frame with respect to the ECEF reference frame. It is given by equation C.33

$\vec{E}$	Thrust force acting on the missile
$\vec{g}$	Gravitational acceleration acting on the missile
I_{CM}^b	Tensor of inertia in the b reference frame with respect to the center of mass of the missile
$\vec{\mathcal{M}}_{CM}^b$	Momentum of the external forces acting on the missile with respect to the center of mass
$\dot{m}_k$	Mass flow through the nozzle k
N'	Gain to be used in the proportional navigation algorithms
$\vec{n}_e$	Normal vector at the nozzle pointing outwards
$\mathcal{O}(\Delta t^{q+1})$	Order of a numerical method (q in this case)
$\{(\vec{r}_e^b)_b\}_k$	Position vector of the mass flow center k
$(\vec{r}_{CM}^i)_i$	Position of the center of mass of the missile relative to the inertial reference frame with coordinates given in the inertial reference frame
T^{n+1}	Local truncation error in the step n of a numerical method
$\vec{u}$	Generic representation of the control vector of a system
$\vec{u}_F$	Unit thrust vector
u_{ek}	Component in the Ox_b axis of the velocity vector of the mass flow center k
v_{ek}	Component in the Oy_b axis of the velocity vector of the mass flow center k
w_{ek}	Component in the Oz_b axis of the velocity vector of the mass flow center k
$(\vec{v}_{CM}^b)_i$	Velocity of the center of mass of the missile relative to the inertial reference frame with coordinates given in the body reference frame
$(\vec{v}_{CM}^n)_e$	Velocity of the center of mass of the missile relative to the ECEF reference frame with coordinates given in the navigation reference frame
$\{(\vec{v}_e^b)_b\}_k$	Velocity vector of the mass flow center k
$\vec{v}_c$	Correlated velocity (Velocity required to reach the target in the desired time from the present position)
$\vec{v}_g$	Velocity to be gained (difference between the present missile velocity and the required from a Lambert's problem solver to reach the target in the desired time)
$\vec{X}$	Generic representation of the state vector of a system
x_{ek}	Component in the Ox_b axis of the position vector of the mass flow center k
y_{ek}	Component in the Oy_b axis of the position vector of the mass flow center k
z_{ek}	Component in the Oz_b axis of the position vector of the mass flow center k

Definitions

Atmospheric ascent guidance. An ascent guidance algorithm used during the atmospheric part of the ascent.

Boost phase. Phase of the trajectory of a missile before the motor of the last stage is stopped.

Boost-Phase interception. An ICBM interception strategy in which the interception is intended while the ICBM is in its boost phase.

Close-loop guidance. Guidance algorithm that takes into account the actual flight dynamics of the vehicle to generate the commands.

Closing velocity. Modulus of the relative velocity between the missile and its target.

Correlated velocity. Velocity required to reach the target in the desired time from the present position (provided by a Lambert's problem solver).

Cutoff. Instant when the motor of the last stage of the missile is stopped (secondary motors could be active from this point, if a post-boost phase is considered).

Delta guidance. An open-loop ascent guidance algorithm in which a reference path computed on ground is used to generate the guidance commands. Delta guidance is explained in section [8.3.1](#).

Direct orbit. Orbit in which the direction of motion is in the direction of the rotational motion of the Earth (to the East).

Exoatmospheric ascent guidance. An ascent guidance algorithm used after the atmospheric part of the ascent when the aerodynamic forces acting on the missile can be neglected.

Gravity turn. Manoeuvre customarily used in the atmospheric part of the ascent guidance in which after an initial kick the missile follows a zero lift trajectory (null angle of attack). Gravity turn is detailed in section [7.1.1.2](#).

Iterative Guidance Mode (IGM guidance). An ascent guidance algorithm based on Linear Tangent Guidance (LTG) in which an integration of the thrust of the missile is performed in order to generate guidance commands. IGM guidance is explained in section [8.3.5.1](#).

Kármán line. Line that customarily represents the boundary between the Earth’s atmosphere and the outer space, and that is placed at an altitude of 100 km above the sea level.

Kepler’s problem. Problem that consists in finding a future state (position and velocity) in an orbit at a given time knowing the initial state (position and velocity) for a point mass within a central gravitational field. Kepler’s problem is detailed in section [F.2.1](#).

Lambert guidance. An ascent guidance algorithm in which the velocity required to reach the target is computed each point and directly used to generate guidance commands. Lambert guidance is explained in section [8.3.3](#).

Lambert’s problem. Orbital boundary value problem that consists in finding the velocity vectors $\vec{v}_1(t_1)$ and $\vec{v}_2(t_2)$ knowing the position vectors $\vec{r}_1(t_1)$ and $\vec{r}_2(t_2)$ and the time of flight $(t_2 - t_1)$ between these 2 points. Lambert’s problem is detailed in section [F.2.2](#).

Line of sight. Line that connects the present position of the missile with the present position of its target.

Linear Tangent Guidance (LTG). An ascent guidance algorithm based on applying a bilinear tangent law for the thrust direction. LTG is explained in section [8.3.5](#).

Midcourse interception. An ICBM interception strategy in which the interception is intended while the ICBM is in its midcourse phase.

Open-loop guidance. Guidance algorithm that does not take into account the actual flight dynamics of the vehicle when computing the commands, but uses a scheduled set of commands.

Optimal guidance Guidance algorithms based on the solution of an optimization problem for some variables of the missile state. The optimal guidance algorithms developed within this thesis are detailed in chapter [10](#).

Path-adaptive guidance. An open-loop ascent guidance algorithm in which a set of reference paths computed on ground are used to generate the guidance commands. Path-adaptive guidance is detailed in section [8.3.2](#).

Post-boost phase. Phase of the trajectory of a missile after the motor of the last stage is stopped, when secondary motors are active from this point.

Powered Explicit Guidance (PEG). An ascent guidance algorithm based on Linear Tangent Guidance (LTG) in which a vector integration of the thrust of the missile is performed in order to generate guidance commands. PEG guidance is explained in section [8.3.5.2](#).

Predictive guidance. Terminal guidance algorithm in which the estimated Zero Error Miss is minimized. Predictive guidance is detailed in section [9.3.4](#).

Proportional navigation. Terminal guidance algorithm in which the lead angle is changed proportionally to the angular rate of the line of sight to the target. Different types of proportional navigation are detailed in sections [9.3.1](#) and [9.3.2](#).

Q guidance. An ascent guidance algorithm in which a Q matrix is used as an intermediate way to generate guidance commands. Q guidance is explained in section [8.3.4](#).

Retrograde orbit. Orbit in which the direction of motion is in the opposite direction of the rotational motion of the Earth (to the West).

Velocity to be gained. Difference between the present missile velocity and the required from a Lambert's problem solver to reach the target in the desired time.

Zero error Miss (ZEM). Estimated distance at the interception between the missile and its target if no command is executed.

This page intentionally left blank.

Abbreviations

AAD Advanced Air Defence (India).

AB Adams Bashforth numerical method.

ABM Anti-Ballistic Missile.

AIAA The American Institute of Aeronautics and Astronautics.

AM Adams Moulton numerical method.

AOA Angle of Attack.

AOS Angle of sideslip.

AP Ammonium perchlorate.

APS American Physical Society.

ATO Abort to orbit (abort manoeuvre for the Space Shuttle).

AWE Atomic Weapons Establishment (United Kingdom).

BIH Bureau International de l'Heure (International Time Bureau).

BMD Ballistic Missile Defence.

BMDS Ballistic Missile Defense System (U.S.).

BRBM Battlefield range ballistic missile.

BV Boost Vehicle (part of the GBI missile).

C.P. Center of pressure.

C2BMC Command and Control, Battle Management, and Communications (part of the BMDS).

CADAC Computer Aided Design of Aerospace Concepts (simulation testbed created by the USAF and the University of Florida).

CBO Congressional Budget Office (U.S.).

CE Capability Enhancement (for versions of the EKV).

- CEP** Celestial Ephemeris Pole.
- CIS** Conventional Inertial reference System.
- CMD** C++ Model Developer (framework created by the U.S. Army).
- CNES** Centre National d'Études Spatiales (France).
- COTS** Commercial Off-the-Shelf.
- CRS** Congressional Research Service (U.S.).
- CTS** Conventional Terrestrial System.
- DACS** Divert and Attitude Control System (part of the EKV).
- DATCOM** Data Compendium.
- DF** Dong Feng missile (China).
- DODIG** Department of Defense Inspector General (U.S.).
- DoF** Degrees of Freedom.
- doi** digital object identifier.
- DRDL** Defence Research and Development Laboratory (India).
- ECEF** Earth-Centered Earth-Fixed reference frame.
- ECI** Earth-Centered Inertial reference frame.
- EGM2008** Earth Gravitational Model 2008.
- EIS** European Interceptor Site (part of the GMD).
- EKV** Exo-atmospheric Kill Vehicle (part of the GBI missile).
- EO** Electro-optical sensor.
- EPAA** European Phased Adaptive Approach (part of the GMD).
- ESA** European Space Agency.
- ESDU** Engineering Sciences Data Unit.
- FAS** Federation of American Scientists.
- FOBS** Fractional Orbital Bombardment System.
- GBI** Ground-Based Interceptor missile.
- GFC** Ground Support & Fire Control (part of the BMDS).

- GIRD** Group for the Study of Reactive Motion, Группа изучения реактивного движения.
- GLONASS** Global Navigation Satellite System (Russia), Глобальная навигационная спутниковая система.
- GMD** Ground-Based Midcourse Defense (U.S.).
- GMST** Greenwich Mean Sidereal Time.
- GNSS** Global Navigation Satellite Systems.
- GPS** Global Positioning System (U.S.).
- H.O.T.** Higher Order Terms.
- HMI** Human-Machine Interface.
- HTPB** Hydroxyl-terminated poly-butadiene.
- IAU** International Astronomical Union.
- ICBM** Intercontinental ballistic missile. In the context of this thesis refers to both ICBMs and SLBMs.
- IEEE** Institute of Electrical and Electronics Engineers.
- IERS** International Earth Rotation Reference System.
- IGM** Iterative Guidance Mode.
- IGMDP** Integrated Guided Missile Development Programme (India).
- INF** Intermediate-Range Nuclear Forces Treaty.
- IR** Infrared.
- IRBM** Intermediate-range ballistic missile.
- IRM** IERS Reference Meridian.
- IRP** IERS Reference Pole.
- ISA** International Standard Atmosphere.
- ISS** International Space Station.
- ISSCAA** International Symposium on Systems and Control in Aeronautics and Astronautics (IEEE symposium).
- IVP** Initial Value Problem.
- JED** Julian Ephemeris Day.

KKV Kinetic Kill Vehicle.

LA Los Angeles.

LOS Line of Sight.

LQ Linear quadratic.

LQR Linear quadratic regulator.

LQT Linear quadratic tracking.

LTG Linear Tangent Guidance.

MDA Missile Defense Agency (U.S.).

MIRV Multiple Independently targetable Reentry vehicle.

MRBM Medium-range ballistic missile.

MSIC Missile and Space Intelligence Center (U.S.).

NASA National Air and Space Administration (U.S.).

NASIC National Air and Space Intelligence Center (U.S.).

NATO North Atlantic Treaty Organization.

NED North-East-Down navigation reference frame.

NEP Nuclear Electromagnetic Pulse.

NFIRE Near Field Infrared Experiment satellite (part of the BMDS).

NIMA National Imagery and Mapping Agency (U.S.).

NLP Non linear programming.

NMD National Missile Defense (U.S.).

NORAD North American Aerospace Defense Command.

NTI Nuclear Threat Initiative.

NYC New York City.

OBV Orbital Boost Vehicle (part of the GBI missile).

ODE Ordinary differential equations.

OSC Orbital Sciences Corporation.

OST Outer Space Treaty.

PAC Patriot Advanced Capability missile.

PBCS Post-Boost Control System (part of the LGM-30G Minuteman III ICBM).

PECE Predictor-Corrector numerical method where the predictor is an AB method and the corrector an AM method.

PEG Powered Explicit Guidance.

PhD Doctor of Philosophy.

PIP Predicted Intercept Point.

PK Probability of Kill.

PLV Payload Launch Vehicle (launcher used in some GBI missile tests).

PN Proportional Navigation.

PSRE Propulsion System Rocket Engine (part of the LGM-30G Minuteman III ICBM).

PyKEP Python Keplerian toolbox (scientific library developed by the ESA).

RAF Royal Air Force (U.K.).

RTLS Return to launch site (abort manoeuvre for the Space Shuttle).

RV Reentry Vehicle.

SALT Strategic Arms Limitation Talks.

SBX Sea-Based X-Band radar (part of the BMDS).

SDI Strategic Defense Initiative (U.S.).

SDRE State Dependent Riccati Equation.

SLBM Submarine-launched ballistic missile.

SLS Space Launch System (for the Space Shuttle).

SORT Strategic Offensive Reductions Treaty.

SOSE System of Systems Engineering (IEEE International Conference).

SRBM Short-range ballistic missile.

SRP Solar radiation pressure.

START Strategic Arms Reduction Treaty.

STSS Space Tracking and Surveillance System (part of the BMDS).

TAL Transoceanic abort landing (abort manoeuvre for the Space Shuttle).

TBM Theatre ballistic missile.

THAAD Theater High Altitude Area Defense (part of the BMDS).

TLV Target Launch Vehicle (target for some GBI missile tests).

TOF Time of Flight.

TVC Thrust Vector Control.

U.K. United Kingdom.

U.S. United States.

U.S.S.R. Union of Soviet Socialist Republics.

UEWR Upgraded Early Warning Radars (part of the BMDS).

URL Uniform Resource Locator.

USA United States of America.

USAF United States Air Force.

USNO United States Naval Observatory.

UT1 Mean solar time.

UTC Coordinated Universal Time.

WGS84 World Geodetic System 1984.

WWII World War II.

ZEM Zero Error Miss.

This page intentionally left blank.