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Integrability of second-order Lagrangians admitting a first-order Hamiltonian formalism

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Second-order Lagrangian densities admitting a first-order Hamiltonian formalism are studied; namely, i) necessary and sufficient conditions for the Poincaré–Cartan form of a second-order Lagrangian on an arbitrary fibred manifold $p: E \to N$ to be projectable onto J^1E are explicitly determined; ii) for each of such Lagrangians, a first-order Hamiltonian formalism is developed and a new notion of regularity is introduced; iii) the variational problems of this class defined by regular Lagrangians are proved to be involutive.

1. Introduction and preliminaries

Let $p: E \to N$ be a fibred manifold over a connected n-dimensional manifold N oriented by a volume form $v = dx^1 \wedge \cdots \wedge dx^n$, which induces a vector-bundle isomorphism

$$\phi_v^k: \bigwedge^k T_xN \to \bigwedge^{n-k} T_x^*N \tag{1}$$

for every $1 \le k \le n-1$, obtained by contracting with v, namely

$$\phi_v^k(X_1 \wedge \cdots \wedge X_k) = i_{X_1} \dots i_{X_k} v, \quad \forall X_1, \dots, X_k \in T_xN.$$

Let $p^k: J^kE \to N$ be the bundle of k-jets of local sections of p, with projections $p_l^k: J^kE \to J^lE$, $k \ge l$. Let $m = \dim E - \dim N$. Every fibred coordinate system (x^j, y^α), $1 \le j \le n$, $1 \le \alpha \le m$, for p induces

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a coordinate system (x^j, y^α_I), on the r-jet bundle, where $I = (i_1, \dots, i_n) \in \mathbb{N}^n$ is an integer multi-index of order $|I| = i_1 + \dots + i_n \leq r$, given by,

$$y^\alpha_I\left(j^r_x s\right) = \frac{\partial^{|I|}(y^\alpha \circ s)}{\partial(x^1)^{i_1} \dots \partial(x^n)^{i_n}}(x),$$

where s is a local section of p defined on a neighborhood of $x \in N$. We set $(j) = (0, \dots, 0, \overset{(j)}{1}, 0, \dots, 0) \in \mathbb{N}^n$, $(jk) = (j) + (k)$, etc., and $y^\alpha_{(j)} = y^\alpha_j$.

The Legendre form of a second-order Lagrangian density $\Lambda = Lv$ defined on $p: E \to N$, where $L \in C^\infty(J^2E)$, is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on J^3E locally given by (e.g., see [15,17,20]),

$$\omega_\Lambda = (-1)^{i-1} L^{i0}_\alpha v_i \otimes dy^\alpha + (-1)^{i-1} L^{ij}_\alpha v_i \otimes dy^\alpha_j,$$

where $v_i = dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$, and

$$L^{ij}_\alpha = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y^\alpha_{(ij)}}, \tag{2}$$

$$L^{i0}_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \frac{1}{2 - \delta_{ij}} D_j\left(\frac{\partial L}{\partial y^\alpha_{(ij)}}\right), \tag{3}$$

D_j denoting the "total derivative" with respect to the coordinate x^j, i.e.,

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|I|=0}^{\infty} \sum_{\alpha=1}^{m} y^\alpha_{I+(j)} \frac{\partial}{\partial y^\alpha_I}.$$

The Poincaré–Cartan (or P–C for short) form attached to Λ is then defined to be the ordinary n-form on J^3E given by (e.g., see [15,20]),

$$\Theta_\Lambda = \left(p^3_2\right)^* \theta^2 \wedge \omega_\Lambda + \Lambda, \tag{4}$$

where θ^1, θ^2 are the first- and second-order structure forms on J^1E, J^2E, locally given respectively in coordinates as follows (cf. [14,19]):

$$\theta^1 = \theta^\alpha \otimes \frac{\partial}{\partial y^\alpha}, \qquad \theta^2 = \theta^\alpha \otimes \frac{\partial}{\partial y^\alpha} + \theta^\alpha_h \otimes \frac{\partial}{\partial y^\alpha_h},$$

and $\theta^\alpha = dy^\alpha - y^\alpha_k dx^k$, $\theta^\alpha_i = dy^\alpha_i - y^\alpha_{(ik)} dx^k$, is the standard basis of contact 1-forms, and the exterior product of $(p^3_2)^*\theta^2$ and the Legendre form, is taken with respect to the pairing induced by duality, $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$.

The most outstanding difference with a first-order Lagrangian density is that the Legendre and Poincaré–Cartan forms associated with a second-order Lagrangian density are generally defined on J^3E, thus increasing by one the order of the Lagrangian density Λ.

For certain second-order Lagrangian densities it is known that the Poincaré–Cartan form is projectable onto J^2E; e.g., see [5]. More precisely, the Poincaré–Cartan form of a second-order Lagrangian projects onto J^2E if and only if the following system of PDEs holds (cf. [4,5]):

$$\frac{1}{2 - \delta_{ib}} \frac{\partial^2 L}{\partial y^\beta_{ac} \partial y^\alpha_{ib}} + \frac{1}{2 - \delta_{ia}} \frac{\partial^2 L}{\partial y^\beta_{bc} \partial y^\alpha_{ia}} + \frac{1}{2 - \delta_{ic}} \frac{\partial^2 L}{\partial y^\beta_{ab} \partial y^\alpha_{ic}} = 0,$$

for all indices $1 \leq a \leq b \leq c \leq n$, $\alpha, \beta = 1, \dots, m$.

Bearing these notations and preliminary results in mind, the summary of contents is as follows: In Section 2 the local characterization of second-order Lagrangians on $p: E \to N$ admitting a P–C form projectable onto J^1E is obtained, and in Section 3 a global characterization is also reached by using the properties of the fibre differential. In Section 4 the Hamiltonian function, the momenta, and the Hamilton–Cartan equations attached to each of the aforementioned Lagrangians are introduced as a consequence of a normal form for their P–C form. Section 5 deals with the notion of regularity for the class of second-order variational problems with a P–C form that projects to first-order jet bundle. Although the Hessian metric vanishes identically for the Lagrangians of such class, a suitable notion of regularity is introduced for them.

The formal integrability of the field equations for first-order Lagrangians in classical field theory has been extensively studied (e.g., see [1,3,6,8,10,13]). Finally, Section 6 is devoted to study the formal integrability of the field equations of second-order Lagrangians with projectable P–C form to first order in their Hamiltonian form. In the real analytic case, this allows one to solve the Cauchy initial value problem for this class of Lagrangians. The integrability of the Hamiltonian form of the field equations of the Einstein–Hilbert Lagrangian is studied.

2. Projecting onto J^1. Local formulation

More surprisingly, there exist second-order Lagrangians for which the associated Poincaré–Cartan form projects not only on J^2E but also on J^1E. Notably, this is the case of the Einstein–Hilbert Lagrangian in General Relativity.

As is well known (e.g., see [7, (1.3)], [17, 2.1]), $p^r_{r-1}: J^rE \to J^{r-1}E$ admits an affine bundle structure modelled over the vector bundle

$$W^r = \left(p^{r-1}\right)^* S^rT^*N \otimes \left(p_0^{r-1}\right)^* V(p) \to J^{r-1}E. \tag{5}$$

Proposition 2.1. *(Cf. [11, Theorem 1].) The Poincaré–Cartan form of a Lagrangian $L \in C^\infty(J^2E)$ projects onto J^1E if and only if L is an affine function with respect to the affine structure of $p^2_1: J^2E \to J^1E$, i.e.,*

$$L = L^{ij}_\alpha y^\alpha_{(ij)} + L_0, \quad L^{ji}_\alpha = L^{ij}_\alpha \in C^\infty\left(J^1E\right),\ L_0 \in C^\infty\left(J^1E\right), \tag{6}$$

and the following equations hold:

$$0 = 2\frac{\partial L^{hi}_\beta}{\partial y^\alpha_a} - \frac{\partial L^{ai}_\alpha}{\partial y^\beta_h} - \frac{\partial L^{ah}_\alpha}{\partial y^\beta_i}, \quad a, h, i = 1, \dots, n,\ \alpha, \beta = 1, \dots, m. \tag{7}$$

Proof. By writing the formula (4) in coordinates, we obtain

$$\Theta_\Lambda = (-1)^{i-1}\left(L^{i0}_\alpha dy^\alpha + L^{ih}_\alpha dy^\alpha_h\right) \wedge v_i + \left(L - y^\alpha_i L^{i0}_\alpha - y^\alpha_{(hi)} L^{ih}_\alpha\right)v. \tag{8}$$

Hence Θ_Λ projects to J^1E if and only if,

$$L^{i0}_\alpha \in C^\infty\left(J^1E\right), \tag{9}$$

$$L^{ih}_\alpha \in C^\infty\left(J^1E\right), \tag{10}$$

$$L - y^\alpha_i L^{i0}_\alpha - y^\alpha_{(hi)} L^{ih}_\alpha \in C^\infty\left(J^1E\right). \tag{11}$$

Taking the condition (10) into account, by integrating Eq. (2) we conclude that L is an affine function on J^1E, as in the formula (6) in the statement. Hence the condition (11) holds identically, as replacing (6) into (11), we have

$$L - y_i^\alpha L_\alpha^{i0} - y_{(hi)}^\alpha L_\alpha^{ih} = L_0 - y_i^\alpha L_\alpha^{i0},$$

which belongs to $C^\infty(J^1E)$ by virtue of (9). Moreover, from (3) we have

$$\begin{aligned} L_\alpha^{a0} &= \frac{\partial L}{\partial y_a^\alpha} - D_j\left(L_\alpha^{aj}\right) \\ &= \frac{\partial L_0}{\partial y_a^\alpha} - \frac{\partial L_\alpha^{aj}}{\partial x^j} - y_j^\beta \frac{\partial L_\alpha^{aj}}{\partial y^\beta} + \left(\frac{\partial L_\beta^{ih}}{\partial y_a^\alpha} - \frac{\partial L_\alpha^{ai}}{\partial y_h^\beta}\right) y_{(hi)}^\beta. \end{aligned}$$

Accordingly, the condition (9) holds if and only if the equations in the statement hold. □

Remark 2.1. Eqs. (7) hold if and only if the following ones hold:

$$\frac{\partial L_\beta^{ih}}{\partial y_a^\alpha} = \frac{\partial L_\alpha^{ia}}{\partial y_h^\beta}, \quad a, h, i = 1, \dots, n, \; \alpha, \beta = 1, \dots, m. \tag{12}$$

Below, the meaning of Eqs. (12) is discussed from a variational point of view.

The Euler–Lagrange (or E–L for short) operator of an arbitrary second-order Lagrangian can be written in terms of the coefficients of the P–C form (see the formulas (2), (3)) as follows:

$$\begin{aligned} \mathcal{E}_\alpha(L) &= \sum_{i \le j} D_i D_j \left(\frac{\partial L}{\partial y_{(ij)}^\alpha}\right) - D_i\left(\frac{\partial L}{\partial y_i^\alpha}\right) + \frac{\partial L}{\partial y^\alpha} \\ &= \frac{\partial L}{\partial y^\alpha} - D_i\left(L_\alpha^{i0}\right), \quad 1 \le \alpha \le m. \end{aligned}$$

As is readily checked, the Euler–Lagrangian equations for an affine second-order Lagrangian L, given as in the formula (6), are of third order. Besides this fact, we obtain

Proposition 2.2. *The Euler–Lagrange equations of an affine second-order Lagrangian are of second order if and only if Eqs. (12) hold.*

Proof. As a computation shows, the third-order part of $\mathcal{E}_\alpha(L)$ is given by,

$$y_{(hij)}^\beta \left(\frac{\partial L_\alpha^{ij}}{\partial y_h^\beta} - \frac{\partial L_\beta^{ij}}{\partial y_h^\alpha}\right). \qquad \square$$

3. Projecting onto J^1. Global formulation

Taking the affine structure of the projection $p_{r-1}^r : J^rE \to J^{r-1}E$ into account, we obtain a natural isomorphism of vector bundles,

$$I^r : \left(p_{r-1}^r\right)^* W^r = \left(p^r\right)^* S^r T^* N \otimes \left(p_0^r\right)^* V(p) \xrightarrow{\cong} V\left(p_{r-1}^r\right), \tag{13}$$

where the vector bundle W^r is defined in (5).

Given a vector bundle $W \to N$, there exists a unique antiderivation of degree $+1$, $d_{E/N} : \Gamma(E, \bigwedge^r V^*(p) \otimes p^*W) \to \Gamma(E, \bigwedge^{r+1} V^*(p) \otimes p^*W)$—called the fibre differential (e.g., see [7, (1.9)])—such that, $d_{E/N}(fp^*\xi) = df|_{V(p)} \otimes \xi$, for all $f \in C^\infty(E)$ and all $\xi \in \Gamma(E, W)$. (In the previous paragraph, the relevant fact is that the vector bundle $W \to N$ is defined over the base manifold N, and not over the fibred manifold E.)

In what follows we are mainly concerned with the fibre derivative d_{J^1E/J^0E}, which will simply be denoted by d_{10} for the sake of simplicity.

A Lagrangian $L \in C^\infty(J^2E)$ is an affine function with respect to the affine structure of $p^2_1\colon J^2E \to J^1E$ if and only if there exists a—necessarily unique—linear form $w_L\colon W^2 \to \mathbb{R}$ such that,

$$L(\tau + j^2_x s) = w_L(\tau) + L(j^2_x s), \quad \forall \tau \in S^2T^*_xN \otimes V_{s(x)}(p),\ \forall j^2_x s \in J^2E.$$

By using the natural identification $(W^2)^* \cong (p^1)^*S^2TN \otimes (p^1_0)^*V^*(p)$, the linear form w_L defines a section of the vector bundle $(p^1)^*S^2TN \otimes (p^1_0)^*V^*(p) \to J^1E$. If L is locally given by the formula (6), then

$$w_L = L^{hi}_\alpha \frac{\partial}{\partial x^h} \odot \frac{\partial}{\partial x^i} \otimes dy^\alpha|_{V(p)},$$

where the symbol $\odot$ denotes symmetric product.

If $\iota^2\colon (W^2)^* \to (p^1)^* \otimes^2 TN \otimes (p^1_0)^*V^*(p)$ is the natural embedding, then we consider the section

$$w'_L = \frac{1}{2}\left(\tilde{I}^1 \circ \iota^2 \circ w_L\right)\colon J^1E \to \left(p^1\right)^*TN \otimes V^*\left(p^1_0\right) \tag{14}$$

obtained by composing the following mappings:

$$\begin{aligned} J^1E &\xrightarrow{w_L} \left(p^1\right)^*S^2TN \otimes \left(p^1_0\right)^*V^*(p) \\ &= \left(W^2\right)^* \xrightarrow{\iota^2} \left(p^1\right)^* \otimes^2 TN \otimes \left(p^1_0\right)^*V^*(p) \\ &= \left(p^1\right)^*TN \otimes \left[\left(p^1\right)^*TN \otimes \left(p^1_0\right)^*V^*(p)\right] \xrightarrow{\tilde{I}^1} \left(p^1\right)^*TN \otimes V^*\left(p^1_0\right), \end{aligned}$$

where $\tilde{I}^1 = 1_{(p^1)^*TN} \otimes ((I^1)^*)^{-1}$ is the isomorphism deduced from (13) for $r = 1$. As $I^1(dx^a \otimes \partial/\partial y^\alpha) = \partial/\partial y^\alpha_a$, dually we obtain $(I^1)^*(d_{10}y^\alpha_a) = \partial/\partial x^a \otimes dy^\alpha|_{V(p)}$.

Hence

$$w'_L = L^{hi}_\alpha d_{10}\left(y^\alpha_h\right) \otimes \frac{\partial}{\partial x^i}.$$

Remark 3.1. Eqs. (12) simply means that for every index h the form $\eta^h = L^{hi}_\alpha dy^\alpha_i$ is d_{10}-closed, namely $d_{10}\eta^h = 0$. Hence, there exist functions $L^i \in C^\infty(J^1E)$ such that locally,

$$\left.\begin{aligned} &\text{(i)} \quad L^{ih}_\alpha = \frac{\partial L^i}{\partial y^\alpha_h}, \\ &\text{(ii)} \quad \frac{\partial L^h}{\partial y^\alpha_i} = \frac{\partial L^i}{\partial y^\alpha_h}, \end{aligned}\right\} \quad 1 \le \alpha \le m,\ h, i = 1, \ldots, n, \tag{15}$$

Eqs. (ii) above being a consequence of the symmetry $L^{hi}_\alpha = L^{ih}_\alpha$. Moreover, from these formulas it follows the existence of functions $L'^\alpha \in C^\infty(J^1E)$ such that,

$$L^h = \frac{\partial L'^\alpha}{\partial y^\alpha_h}, \quad 1 \le \alpha \le m,\ h, i = 1, \ldots, n.$$

More exactly, letting $W = TN$ in the definition of the fibre differential above, recalling that the Poincaré lemma also holds for fibre differentiation (e.g., see [16]) and taking account of the fact that the fibres of $p^1_0\colon J^1E \to E$ are simply connected as they are diffeomorphic to $\mathbb{R}^{mn}$, the following global characterization of second-order variational with a P–C form projecting onto J^1E, is obtained:

Proposition 3.1. *The Poincaré–Cartan form of a Lagrangian $L \in C^\infty(J^2E)$ projects onto J^1E if and only if L is an affine function with respect to the affine structure of $p_1^2\colon J^2E \to J^1E$ and the TN-valued 1-form w'_L defined in the formula (14) is d_{10}-closed.*

In this case, for every global (smooth) section $\sigma\colon E \to J^1E$ of p_0^1, there exists a unique globally defined section $w_L^\sigma \in \Gamma(J^1E,(p^1)^*TN)$ such that,

$$d_{10}\big(w_L^\sigma\big) = w'_L,$$
$$w_L^\sigma\big(\sigma(e)\big) = 0, \quad \forall e \in E.$$

Remark 3.2. A general procedure to obtain global sections $\sigma\colon E \to J^1E$ of p_0^1 is to use Ehresmann (or non-linear) connections, i.e., to use a differential 1-form γ on E taking values in the vertical sub-bundle $V(p)$ such that $\gamma(X) = X$, $\forall X \in V(p)$; hence, locally (cf. [18]), $\gamma = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y_j^\alpha}$, $\gamma_j^\alpha \in C^\infty(E)$. The vertical differential of a section $s\colon U \to E$ (defined on a neighborhood U of $x \in N$) at $e = s(x)$ is defined to be the linear mapping $(d^v s)_e\colon T_eE \to V_e(p)$, $(d^v s)_e X = X - s_* p_*(X)$, $\forall X \in T_eE$. We claim that for every $e \in E$, there exists a unique $j_x^1 s \in J^1E$ such that, i) $s(x) = e$, where $x = p(e)$, and ii) $(d^v s)_e = \gamma_e$. In fact, one has $(\partial(y^\alpha \circ s)/\partial x^j)(x) = -\gamma_j^\alpha(e)$, and the section σ^γ attached to γ is defined by, $\sigma^\gamma(e) = j_x^1 s$.

4. Hamiltonian formalism

In the usual (i.e., first-order) calculus of variations, a section s is an extremal of the Lagrangian density Λ on J^1E if and only if it satisfies the so-called "Hamilton–Cartan equations" (or H–C for short; e.g., see [7, (3.8)], [5, (1)]), namely, if and only if the following equation holds: $(j^1s)^*(i_X d\Theta_\Lambda) = 0$ for every p^1-vertical vector field X on J^1E.

If $\Lambda = Lv$ is an arbitrary second-order Lagrangian density on E, then the following formula can be proved (e.g., see [15]):

$$d\Theta_\Lambda = \mathcal{E}_\alpha(L)\theta^\alpha \wedge v + \eta_{n+1}(L), \tag{16}$$

where $\eta_{n+1}(L) = (-1)^i \eta_2^i(L) \wedge v_i$ and $\eta_2^i(L)$ is the 2-contact 2-form given by,

$$\begin{aligned}\eta_2^i(L) &= \frac{\partial L_\alpha^{i0}}{\partial y^\beta}\theta^\alpha \wedge \theta^\beta + \left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta} - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}\right)\theta^\alpha \wedge \theta_j^\beta \\ &\quad + \sum_{j\le k}\frac{\partial L_\alpha^{i0}}{\partial y_{(jk)}^\beta}\theta^\alpha \wedge \theta_{(jk)}^\beta + \sum_{i\le k\le l}\frac{\partial L_\alpha^{i0}}{\partial y_{(jkl)}^\beta}\theta^\alpha \wedge \theta_{(jkl)}^\beta \\ &\quad + \frac{\partial L_\alpha^{ij}}{\partial y_k^\beta}\theta_j^\alpha \wedge \theta_k^\beta + \sum_{k\le l}\frac{\partial L_\alpha^{ij}}{\partial y_{(kl)}^\beta}\theta_j^\alpha \wedge \theta_{(kl)}^\beta.\end{aligned}$$

From the formula (16) it follows that the H–C equations also characterize critical sections for a second-order density Λ; i.e., s is an extremal for Λ if and only if, $(j^3s)^*(i_X d\Theta_\Lambda) = 0$ for every p^3-vertical vector field X on J^3E.

Remark 4.1. If the P–C form of a second-order density Λ projects onto J^1E, then its H–C equations have the same formal expression of a first-order density (see the formula (20) below), although there is no first-order density having Θ_Λ as its P–C form. In fact, the P–C form of a first-order Lagrangian density $\tilde{\Lambda} = \tilde{L}v$, $\tilde{L} \in C^\infty(J^1E)$, is given by,

$$\Theta_{\tilde{\Lambda}} = (-1)^{i-1}\frac{\partial \tilde{L}}{\partial y_i^\alpha}dy^\alpha \wedge v_i + \tilde{H}v, \quad \tilde{H} = \tilde{L} - \frac{\partial \tilde{L}}{\partial y_i^\alpha}y_i^\alpha. \tag{17}$$

If $\Theta_\Lambda = \Theta_{\tilde{\Lambda}}$, then comparing (17) to (8), the following three equations are obtained:

$$1)\quad L_\alpha^{ih} = 0, \qquad 2)\quad L_0 - y_i^\alpha L_\alpha^{i0} = \tilde{L} - \frac{\partial \tilde{L}}{\partial y_i^\alpha} y_i^\alpha, \qquad 3)\quad L_\alpha^{i0} = \frac{\partial \tilde{L}}{\partial y_i^\alpha}.$$

From (6) and 1) it follows $L = L_0$; hence L is of first order. Moreover, taking (3) into account, the formulas 2) and 3) above are respectively rewritten as,

$$L_0 - \tilde{L} = y_i^\alpha \frac{\partial (L_0 - \tilde{L})}{\partial y_i^\alpha}, \qquad \frac{\partial (L_0 - \tilde{L})}{\partial y_i^\alpha} = 0.$$

Hence $\tilde{L} = L$.

Theorem 4.1. *If L is a second-order Lagrangian on E whose Poincaré–Cartan form projects onto J^1E, then letting*

$$\begin{aligned} p_\alpha^i &= L_\alpha^{i0} - \frac{\partial L^i}{\partial y^\alpha}, \quad 1 \le \alpha \le m,\ 1 \le i \le n, \\ H &= L_0 - y_i^\alpha L_\alpha^{i0} - \frac{\partial L^i}{\partial x^i}, \end{aligned} \tag{18}$$

where the functions L^i are defined by the formulas (15)-(i), one obtains,

$$d\Theta_\Lambda = (-1)^{i-1} dp_\alpha^i \wedge dy^\alpha \wedge v_i + dH \wedge v. \tag{19}$$

Furthermore, if the linear forms $d_{10}(p_\alpha^i): V(p_0^1) \to \mathbb{R}$, $1 \le \alpha \le m$, $1 \le i \le n$, are linearly independent, then a section $s: N \to E$ is an extremal for Λ if and only if it satisfies the following equations:

$$\begin{cases} 0 = \dfrac{\partial (p_\alpha^i \circ j^1 s)}{\partial x^i} - \dfrac{\partial H}{\partial y^\alpha} \circ j^1 s, & 1 \le \alpha \le m, \\ 0 = \dfrac{\partial (y^\alpha \circ s)}{\partial x^i} + \dfrac{\partial H}{\partial p_\alpha^i} \circ j^1 s, & 1 \le \alpha \le m,\ 1 \le i \le n. \end{cases} \tag{20}$$

Proof. Taking the formulas (8), (6) and the conditions (9)–(11) into account, and recalling the formulas (12), one deduces

$$\Theta_\Lambda = (-1)^{i-1}\left(L_\alpha^{i0} - \frac{\partial L^i}{\partial y^\alpha}\right) dy^\alpha \wedge v_i + (-1)^{i-1} dL^i \wedge v_i + \left(L_0 - y_i^\alpha L_\alpha^{i0} - \frac{\partial L^i}{\partial x^i}\right) v.$$

Hence,

$$\begin{aligned} d\Theta_\Lambda = {} & (-1)^{i-1} d\left(L_\alpha^{i0} - \frac{\partial L^i}{\partial y^\alpha}\right) \wedge dy^\alpha \wedge v_i + \frac{\partial^2 L^i}{\partial x^i \partial y^\alpha} dy^\alpha \wedge v \\ & + (-1)^{i-1} \frac{\partial^2 L^i}{\partial x^i \partial y^\alpha} dx^i \wedge dy^\alpha \wedge v_i + (-1)^{i-1} \frac{\partial^2 L^i}{\partial y^\beta \partial y^\alpha} dy^\beta \wedge dy^\alpha \wedge v_i \\ & + d\left(L_0 - y_i^\alpha L_\alpha^{i0} - \frac{\partial L^i}{\partial x^i}\right) \wedge v. \end{aligned}$$

Letting

$$p_\alpha^i = L_\alpha^{i0} - \frac{\partial L^i}{\partial y^\alpha}, \qquad H = L_0 - y_i^\alpha L_\alpha^{i0} - \frac{\partial L^i}{\partial x^i},$$

and recalling that

$$\frac{\partial^2 L^i}{\partial y^\alpha \partial y^\beta} dy^\beta \wedge dy^\alpha = 0,$$

the formula (19) is obtained.

The forms $d_{10}(p_\alpha^i)$ are linearly independent if and only if the Jacobian of the functions $(p_\alpha^i)_{1\le i\le n, 1\le \alpha\le m}$ with respect to the variables $(y_j^\beta)_{1\le\beta\le m, 1\le j\le n}$ does not vanish. In this case, the functions $(x^h, y^\alpha, p_\beta^i)$ constitute a coordinate system on J^1E as

$$\frac{\partial(x^h, y^\alpha, p_\beta^i)}{\partial(x^j, y^\gamma, y_k^\delta)} = \frac{\partial(p_\beta^i)}{\partial(y_k^\delta)} \neq 0.$$

By contracting Eq. (19) with $\partial/\partial y^\alpha$ and $\partial/\partial p_\alpha^i$, and pulling the result back along $j^1 s$, we obtain the first and second equations in (20), respectively. $\square$

5. Regularity

As is well known (e.g., see [7]), if the Hessian metric Hess(L) of a first-order density $\Lambda = Lv$ is non-singular, then every section $s^1: N \to J^1E$ of the projection $p^1: J^1E \to N$ that satisfies the P–C equation for Λ is holonomic; i.e., s^1 coincides with the 1-jet extension of the section $s = p_0^1 \circ s^1$ of the projection p. Namely, $(s^1)^*(i_X d\Theta_\Lambda) = 0$ for every p^1-vertical vector field X on J^1E, implies $s^1 = j^1 s$.

In the case of a second-order density with a P–C form projecting onto J^1E, the following result holds:

Proposition 5.1. *If $\Lambda = Lv$ is a second-order Lagrangian on E such that,*

(i) *Its Poincaré–Cartan form Θ_Λ projects onto J^1E.*
(ii) *The linear forms $d_{10}(p_\alpha^i): V(p_0^1) \to \mathbb{R}$, $1 \le \alpha \le m$, $1 \le i \le n$, where the functions p_α^i are introduced in (18), are linearly independent,*

then every solution to its H–C equations, is holonomic.

Proof. If $s^1: N \to J^1E$ is a solution to the H–C equations for Λ, then letting $X = \partial/\partial y_h^\beta$ one obtains

$$\begin{aligned}(s^1)^*(i_{\partial/\partial y_h^\beta} d\Theta) &= (-1)^{i-1}\left(\frac{\partial L_\alpha^{i0}}{\partial y_h^\beta} \circ s^1 - \frac{\partial L_\beta^{ih}}{\partial y^\alpha} \circ s^1\right)\left((s^1)^*\theta^\alpha\right) \wedge v_i \\ &= \left(\frac{\partial L_\alpha^{i0}}{\partial y_h^\beta} \circ s^1 - \frac{\partial L_\beta^{ih}}{\partial y^\alpha} \circ s^1\right)\left(\frac{\partial s^\alpha}{\partial x^i} - (s^1)_i^\alpha\right) v,\end{aligned}$$

where $s^\alpha = y^\alpha \circ s^1$, $(s^1)_i^\alpha = y_i^\alpha \circ s^1$, and one concludes by simply remarking that,

$$\frac{\partial p_\alpha^i}{\partial y_h^\beta} = \frac{\partial L_\alpha^{i0}}{\partial y_h^\beta} - \frac{\partial L_\beta^{ih}}{\partial y^\alpha}. \qquad \square \tag{21}$$

Lemma 5.2. *If $\Lambda = Lv$ is a second-order density on E whose P–C form projects onto J^1E, then $i_Y(d\Theta_\Lambda) \in T_{s(x)}^*E \wedge (p^1)^* \bigwedge^{n-1} T_x^*N$ for all $Y \in V_{j_x^1 s}(p_0^1)$.*

Proof. In the present case, the formula (16) becomes,

$$d\Theta_\Lambda = \mathcal{E}_\alpha(L)\theta^\alpha \wedge v + (-1)^i \left\{ \frac{\partial L_\alpha^{i0}}{\partial y^\beta}\theta^\alpha \wedge \theta^\beta + \left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta} - \frac{\partial L_\beta^{ij}}{\partial y^\alpha} \right)\theta^\alpha \wedge \theta_j^\beta \right\} \wedge v_i,$$

as

$$\frac{\partial L_\alpha^{i0}}{\partial y_{(jk)}^\beta} = 0, \qquad \frac{\partial L_\alpha^{i0}}{\partial y_{(jkl)}^\beta} = 0, \qquad \frac{\partial L_\alpha^{ij}}{\partial y_{(kl)}^\beta} = 0,$$

$$\frac{\partial L_\alpha^{ij}}{\partial y_k^\beta}\theta_j^\alpha \wedge \theta_k^\beta = \frac{\partial^2 L^i}{\partial y_j^\alpha \partial y_k^\beta}\theta_j^\alpha \wedge \theta_k^\beta = 0.$$

Letting $Y = (\partial/\partial y_j^\beta)_{j_x^1 s}$, we thus obtain

$$i_Y(d\Theta_\Lambda) = (-1)^{i-1}\left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s) \right)(\theta^\alpha \wedge v_i)_{j_x^1 s}, \tag{22}$$

and the result follows. □

As $p_0^1 \colon J^1E \to E$ is an affine bundle modelled over $W^1 = p^*(T^*N) \otimes V(p)$ (cf. (5)), there is a canonical isomorphism $I \colon (p_0^1)^* W^1 \xrightarrow{\cong} V(p_0^1)$ locally given by $I(j_x^1 s, (dx^i)_x \otimes (\partial/\partial y^\alpha)_{s(x)}) = (\partial/\partial y_i^\alpha)_{j_x^1 s}$.

According to the previous lemma, we can define a bilinear form

$$\begin{cases} b_\Lambda \colon (p_0^1)^* W^1 \times_{J^1E} (p_0^1)^* W^1 \to \mathbb{R}, \\ b_\Lambda(j_x^1 s; w_0 \otimes Y_0, w_1 \otimes Y_1) = \langle w_0, (\phi_v^1)^{-1}(i_{Y_0} i_Y(d\Theta_\Lambda)) \rangle, \\ \quad w_a \in T_x^* N, Y_a \in V_{s(x)}(p), a = 0, 1;\ Y = I(j_x^1 s, w_1 \otimes Y_1), \end{cases} \tag{23}$$

where ϕ_v^1 is the isomorphism defined in (1) for $k = 1$. If $w_0 = (dx^i)_x$ and $Y_0 = (\partial/\partial y^\alpha)_{s(x)}$, then from the formula (22) one readily obtains,

$$i_{Y_0} i_Y(d\Theta_\Lambda) = (-1)^{i-1}\left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s) \right)(v_i)_x,$$

$$\langle w_0, (\phi_v^1)^{-1}(i_{Y_0} i_Y(d\Theta_\Lambda)) \rangle = \frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s).$$

In other words,

$$b_\Lambda\left(j_x^1 s; (dx^i)_x \otimes \left(\frac{\partial}{\partial y^\alpha}\right)_{s(x)}, (dx^j)_x \otimes \left(\frac{\partial}{\partial y^\beta}\right)_{s(x)} \right) = \frac{\partial L_\alpha^{i0}}{\partial y_j^\beta}(j_x^1 s) - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}(j_x^1 s).$$

Hence, the next result follows:

Corollary 5.3. *Let Λ be a second-order density on E whose P–C form projects onto J^1E. If the bilinear form defined in (23) is non-singular, then every solution to the H–C equations for Λ is holonomic.*

Proposition 5.4. *The bilinear form b_Λ defined in (23) is symmetric.*

Proof. By using the formula (21) it suffices to prove the existence of a function $\bar{L}$ such that

$$p_\alpha^i = \frac{\partial \bar{L}}{\partial y_i^\alpha},$$

as in this case the matrix of b_Λ coincides with the Hessian matrix of $\bar{L}$ with respect to the variables y_i^α. In fact, if $\bar{L}$ is defined by the following expression:

$$\bar{L} = L_0 - \frac{\partial L^i}{\partial x^i} - y_i^\alpha \frac{\partial L^i}{\partial y^\alpha},$$

where the functions L^i are determined by the formula (15)-(i), then, from the formulas (3) and (18), one obtains

$$\begin{aligned} p_\alpha^i &= \frac{\partial L_0}{\partial y_i^\alpha} - \frac{\partial^2 L^j}{\partial y_i^\alpha \partial x^j} - y_j^\beta \frac{\partial^2 L^j}{\partial y_i^\alpha \partial y^\beta} - \frac{\partial L^i}{\partial y^\alpha} \\ &= \frac{\partial}{\partial y_i^\alpha}\left(L_0 - \frac{\partial L^j}{\partial x^j} - y_j^\beta \frac{\partial L^j}{\partial y^\beta}\right). \qquad \square \end{aligned}$$

6. Involutivity

Let $p' : E' \to N$ be the bundle defined as follows: $p' = p^1$, $E' = J^1 E$.

In this section we study the involutivity of the first-order differential system $R_L^1 \subset J^1 E'$ defined by the solutions to the H–C equations of a density $\Lambda = Lv$ satisfying all the conditions in Theorem 4.1; namely, the fibre of R_L^1 over $x \in N$ is the set of elements $j_x^1 s^1 \in J^1 E'$ such that, $(s^1)^*(i_X d\Theta_\Lambda)(x) = 0$ for every p'-vertical vector field $X \in \mathfrak{X}(E')$. According to Proposition 5.1, for every (local) solution s^1 to R_L^1 one has $s^1 = j^1 s$, where the section of $p: E \to N$ defined by $s = p_0^1 \circ s^1$ is called the zero-order section attached to s^1.

If (x^i, y^α) is a fibred coordinate system for the submersion p, then from the proof of Theorem 4.1 it follows that $(x^i, y^\alpha, p_\alpha^i)$ is a fibred coordinate system for p'. Let us denote by $(x^i, y^\alpha, p_\alpha^i; y_{,j}^\alpha, p_{\alpha,j}^i)$ the induced coordinate system on $J^1 E'$; namely,

$$\begin{aligned} y_{,j}^\alpha (j_x^1 s^1) &= \frac{\partial (y^\alpha \circ p_0^1 \circ s^1)}{\partial x^j}(x), \\ p_{\alpha,j}^i (j_x^1 s^1) &= \frac{\partial (p_\alpha^i \circ s^1)}{\partial x^j}(x). \end{aligned}$$

Similarly, we set $y_{,jk}^\alpha (j_x^2 s^1) = \frac{\partial^2 (y^\alpha \circ p_0^1 \circ s^1)}{\partial x^j \partial x^k}(x)$, $p_{\alpha,jk}^i (j_x^2 s^1) = \frac{\partial^2 (p_\alpha^i \circ s^1)}{\partial x^j \partial x^k}(x)$, etc.

The functions $(x^i, y^\alpha, p_\alpha^i; y_{,j}^\alpha, p_{\alpha,j}^i; y_{,jk}^\alpha, p_{\alpha,jk}^i)$, $j \le k$, are a system of coordinates in $J^2 E'$.

Locally, one has $R_L^1 = \ker \varphi_L$, where

$$\varphi_L : J^1 E' \to B_N = (p')^* \bigoplus^{m+mn} (\wedge^n T^* N)$$

is the morphism of fibred manifolds over E' defined by,

$$\varphi_L (j_x^1 s^1) = \left(s^1(x); \left(\varphi_L^\alpha (j_x^1 s^1) v\right)_{1 \le \alpha \le m}, \left(\left(\varphi_\alpha^i\right)_L (j_x^1 s^1) v\right)_{1 \le \alpha \le m, 1 \le i \le n}\right),$$

with $\varphi_L^\alpha (j_x^1 s^1) v = (s^1)^* (i_{\partial/\partial y^\alpha} d\Theta_\Lambda)(x)$, $(\varphi_\alpha^i)_L (j_x^1 s^1) v = (s^1)^* (i_{\partial/\partial p_\alpha^i} d\Theta_\Lambda)(x)$, or equivalently, in local coordinates,

$$\varphi_L^\alpha = -p^i_{\alpha,i} + H_{y^\alpha}, \quad 1 \le \alpha \le m, \tag{24}$$

$$\left(\varphi^i_\alpha\right)_L = y^\alpha_{,i} + H_{p^i_\alpha}, \quad 1 \le \alpha \le m,\ 1 \le i \le n, \tag{25}$$

H being the Hamiltonian function of the Lagrangian L.

The first prolongation of φ_L is defined by,

$$\begin{aligned}&\operatorname{prol}_1(\varphi_L): J^2\left(E'\right) \to J^1 B_N,\\ &\operatorname{prol}_1(\varphi_L)\left(j^2_x s^1\right) = j^1_x\left(\varphi \circ j^1 s^1\right).\end{aligned}$$

Let u^α and u^i_α be the standard coordinates induced by the volume form in the direct summands $\bigoplus^m(\bigwedge^n T^*N)$ and $\bigoplus^{mn}(\bigwedge^n T^*N)$ of B_N respectively, i.e.,

$$\begin{aligned}w &= \left(w^1_n, \dots, w^m_n\right) \in \bigoplus^m \bigwedge^n T^*_x N,\\ w^\alpha_n &= u^\alpha(w) v_x, \quad 1 \le \alpha \le m,\\ w' &= \left(\left(w^1_1\right)_n, \dots, \left(w^1_m\right)_n, \dots, \left(w^i_1\right)_n, \dots, \left(w^i_m\right)_n, \dots, \left(w^n_1\right)_n, \dots, \left(w^n_m\right)_n\right)\\ &\in \bigoplus^{mn} \bigwedge^n T^*_x N,\\ \left(w^i_\alpha\right)_n &= u^i_\alpha\left(w'\right) v_x, \quad 1 \le \alpha \le m,\ 1 \le i \le n.\end{aligned}$$

As $u^\alpha \circ \varphi_L = \varphi^\alpha_L$, $u^i_\alpha \circ \varphi_L = (\varphi^i_\alpha)_L$, in coordinates one has,

$$\begin{aligned}u^\alpha_r \circ \operatorname{prol}_1(\varphi_L) &= -p^i_{\alpha,ir} + H_{x^r y^\alpha} + y^\beta_{,r} H_{y^\alpha y^\beta} + p^j_{\beta,r} H_{y^\alpha p^j_\beta},\\ u^i_{\alpha,r} \circ \operatorname{prol}_1(\varphi_L) &= y^\alpha_{,ir} + H_{x^r p^i_\alpha} + y^\beta_{,r} H_{y^\beta p^i_\alpha} + p^j_{\beta,r} H_{p^i_\alpha p^j_\beta}.\end{aligned}$$

The mapping $\varphi_L: J^1E' \to B_N$ is quasi-linear as there exists a vector-bundle morphism $\sigma_L = \sigma(\varphi_L)$: $(p')^*T^*N \otimes V(p') \to B_N$ (the symbol of φ_L) such that,

$$\begin{aligned}&\varphi_L\left(\chi_1 + j^1_x s^1\right) = \sigma_L(\chi_1) + \varphi_L\left(j^1_x s^1\right),\\ &\forall \chi_1 \in T^*_x N \otimes V_{e'}\left(p'\right),\ \forall j^1_x s^1 \in J^1 E', s^1(x) = e'.\end{aligned} \tag{26}$$

In fact, if $\chi_1 = (dx^j)_x \otimes \{t^\alpha_j(\chi_1)(\partial/\partial y^\alpha)_{e'} + t^\alpha_{ij}(\chi_1)(\partial/\partial p^i_\alpha)_{e'}\}$, then the element $j^1_x \bar{s}^1 = \chi_1 + j^1_x s^1$ is given by $\bar{s}^1(x) = e' = s^1(x)$, and

$$\begin{aligned}y^\alpha_{,j}\left(j^1_x \bar{s}^1\right) &= t^\alpha_j(\chi_1) + y^\alpha_{,j}\left(j^1_x s^1\right),\\ p^i_{\alpha,j}\left(j^1_x \bar{s}^1\right) &= t^\alpha_{ij}(\chi_1) + p^i_{\alpha,j}\left(j^1_x s^1\right).\end{aligned} \tag{27}$$

As $H_{y^\alpha}, H_{p^i_\alpha} \in C^\infty(E')$ and $j^1_x s^1$ and $j^1_x \bar{s}^1$ have the same coordinates on E' according to the first two equations in (27), one has $H_{y^\alpha}(j^1_x \bar{s}^1) = H_{y^\alpha}(j^1_x s^1)$, $H_{p^i_\alpha}(j^1_x \bar{s}^1) = H_{p^i_\alpha}(j^1_x s^1)$. From (24), (25), (26), and (27) one respectively obtains

$$\begin{aligned}\left(u^\beta \circ \sigma_L\right)(\chi_1) &= -\sum_{h=1}^n t^\beta_{hh}(\chi_1),\\ \left(u^a_\beta \circ \sigma_L\right)(\chi_1) &= t^\beta_a(\chi_1).\end{aligned}$$

Hence

$$\begin{aligned} g_1 &= \ker \sigma_L \\ &= \left\{ \chi_1 \in T_x^* N \otimes V_{e'}(p') : \sum_{h=1}^{n} t_{hh}^{\beta}(\chi_1) = 0, t_a^{\beta}(\chi_1) = 0, 1 \le \beta \le m, 1 \le a \le n \right\}. \end{aligned}$$

As φ_L is quasi-linear, according to [2, IX, Proposition 2.6], the l-th prolongation $\sigma_l(\varphi_L)$ of $\sigma_L = \sigma(\varphi_L)$ coincides with the symbol of the l-th prolongation $\mathrm{prol}_l(\varphi_L)$. By using this fact, below we compute

$$\sigma_1(\varphi_L) \colon (p')^* S^2 T^* N \otimes V(p') \to (p')^* T^* N \otimes B_N.$$

If $\chi_2 = \sum_{j \le k} (dx^j)_x \odot (dx^k)_x \otimes \{ t_{jk}^{\alpha}(\chi_2)(\partial/\partial y^{\alpha})_{e'} + t_{ijk}^{\alpha}(\chi_2)(\partial/\partial p_{\alpha}^i)_{e'} \}$ is a valued 2-tensor in $S^2 T_x^* N \otimes V_{e'}(p')$, then $j_x^2 \bar{s}^1 = \chi_2 + j_x^2 s^1$ is given as follows: $j_x^1 \bar{s}^1 = j_x^1 s^1$ and

$$\begin{aligned} \frac{\partial^2 (y^{\alpha} \circ p_0^1 \circ \bar{s}^1)}{\partial x^j \partial x^k}(x) &= (1 + \delta_{jk}) t_{jk}^{\alpha}(\chi_2) + \frac{\partial^2 (y^{\alpha} \circ p_0^1 \circ s^1)}{\partial x^j \partial x^k}(x), \\ \frac{\partial^2 (p_{\alpha}^i \circ \bar{s}^1)}{\partial x^j \partial x^k}(x) &= (1 + \delta_{jk}) t_{ijk}^{\alpha}(\chi_2) + \frac{\partial^2 (p_{\alpha}^i \circ s^1)}{\partial x^j \partial x^k}(x). \end{aligned}$$

Hence $\sigma_1(\varphi_L)(\chi_2) = \mathrm{prol}_1(\varphi_L)(j_x^2 \bar{s}^1) - \mathrm{prol}_1(\varphi_L)(j_x^2 s^1)$,

$$\begin{aligned} \big(u_r^{\beta} \circ \sigma_1(\varphi_L)\big)(\chi_2) &= -\sum_{a=1}^{n} (1 + \delta_{ar}) t_{aar}^{\beta}(\chi_2), \\ \big(u_{\beta,r}^{i} \circ \sigma_1(\varphi_L)\big)(\chi_2) &= (1 + \delta_{ir}) t_{ir}^{\beta}(\chi_2). \end{aligned}$$

Therefore

$$\begin{aligned} g_2 &= \ker \sigma_1(\varphi_L) \\ &= \left\{ \chi_2 \in S^2 T_x^* N \otimes V_{e'}(p') : \sum_{a=1}^{n} (1 + \delta_{ar}) t_{aar}^{\beta}(\chi_2) = 0, t_{ir}^{\beta}(\chi_2) = 0, 1 \le \beta \le m, 1 \le i \le r \le n \right\}. \end{aligned}$$

Let $(X_1, \dots, X_n)$ be a basis of $T_x N$, with dual basis $(w^1, \dots, w^n)$ and let $S^k T_{x,\{X_1,\dots,X_j\}}^*$ be the subspace of $S^k T_x^*$ generated by the symmetric products $w^{i_1} \odot \cdots \odot w^{i_k}$, with $j + 1 \le i_1 \le \ldots \le i_k \le n$. Following [2, IX, Section 2, p. 409], for every $e' \in E'$ with $p'(e') = x$, and $k = 1, 2$, we set

$$g_{k,e',\{X_1,\dots,X_j\}} = g_{k,e'} \cap \big(S^k T_{x,\{X_1,\dots,X_j\}}^* \otimes V_{e'}(p') \big).$$

The basis $(X_1, \dots, X_n)$ is said to be quasi-regular for g_1 at e' if the following equality holds: $\dim g_{2,e'} = \dim g_{1,e'} + \sum_{j=1}^{n-1} \dim g_{1,e',\{X_1,\dots,X_j\}}$.

As $\dim(S^2 T_x^* N \otimes V_{e'}(p')) = \frac{1}{2} n(n+1)(m + nm)$, from the previous calculations, we conclude $\dim g_{1,e'} = m(n^2 - 1)$ and $\dim g_{2,e'} = \frac{1}{2} mn(n^2 + n - 2)$. Hence $\dim g_{2,e'} - \dim g_{1,e'} = \frac{1}{2} m(n-1)(n^2 - 2)$. Moreover, as a computation shows, one has $\dim g_{1,e',\{X_1,\dots,X_j\}} = m(n^2 - jn - 1)$. Therefore

$$\sum_{j=1}^{n-1} \dim g_{1,e',(X_1,\dots,X_j)} = \frac{1}{2} m(n-1)\big(n^2 - 2\big) = \dim g_{2,e'} - \dim g_{1,e'},$$

thus proving that R_L^1 is involutive. We can thus apply [2, Theorem 3.3] (also see [8,12]) obtaining the following:

Theorem 6.1. *If $\Lambda = Lv$ is a second-order density on $p\colon E \to N$ whose P–C form projects onto J^1E and satisfies the regularity condition of Corollary 5.3, then the first-order partial differential system R_L^1 defined by the solutions to H–C equations of Λ is involutive. If both $p\colon E \to N$ and Λ are of class C^ω, then given a point $\xi \in (R_L^1)_{x_0}$, there exists a section s of p of class C^ω defined on an open neighborhood U of x_0 in N such that, i) $j_{x_0}^1(j^1s) = \xi$, and ii) $j_x^1(j^1s) \in R_L^1$ for every $x \in U$.*

Remark 6.1. Analytically, Theorem 6.1 means the following: Given scalars λ^α, λ_α^i, λ_j^α, $\lambda_{\alpha,j}^i$ such that, $0 = \lambda_{\alpha,i}^i - H_{y^\alpha}(e'_0)$, $0 = \lambda_i^\alpha + H_{p_\alpha^i}(e'_0)$, where $e'_0 \in E'$ is the point over x_0 with coordinates $y^\alpha(e') = \lambda^\alpha$, $p_\alpha^i(e') = \lambda_\alpha^i$, then there exists a solution $s^1\colon U \to E'$ to the H–C equations defined on a neighborhood of x_0 such that, $y^\alpha(j_{x_0}^1 s^1) = \lambda^\alpha$, $p_\alpha^i(j_{x_0}^1 s^1) = \lambda_\alpha^i$, $y_{,j}^\alpha(j_{x_0}^1 s^1) = \lambda_j^\alpha$, $p_{\alpha,j}^i(j_{x_0}^1 s^1) = \lambda_{\alpha,j}^i$.

6.1. The basic example: Einstein–Hilbert Lagrangian

Let $p_M\colon M = M(N) \to N$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n$. Every coordinate system $(x^i)_{i=1}^n$ on an open domain $U \subseteq N$ induces a coordinate system (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where the functions $y_{jk} = y_{kj}$ are defined by,

$$\begin{aligned} g_x &= y_{ij}(g_x)\left(dx^i\right)_x \otimes \left(dx^j\right)_x \\ &= \sum_{i \le j} \frac{1}{1+\delta_{ij}} y_{ij}(g_x)\left(dx^i\right)_x \odot \left(dx^j\right)_x, \quad \forall g_x \in (p_M)^{-1}(U). \end{aligned}$$

Following the notations in [9], the Ricci tensor field attached to the symmetric connection Γ is given by $S^\Gamma(X,Y) = \operatorname{trace}(Z \mapsto R^\Gamma(Z,X)Y)$, where R^Γ denotes the curvature tensor field of the covariant derivative ∇^Γ associated to Γ on the tangent bundle; hence $S^\Gamma = (R^\Gamma)_{jl} dx^l \otimes dx^j$, where $(R^\Gamma)_{jl} = (R^\Gamma)_{jkl}^k$, and $(R^\Gamma)_{jkl}^i = \partial\Gamma_{jl}^i/\partial x^k - \partial\Gamma_{jk}^i/\partial x^l + \Gamma_{jl}^m\Gamma_{km}^i - \Gamma_{jk}^m\Gamma_{lm}^i$.

The E–H Lagrangian density is given by

$$(\Lambda_{EH})_{j_x^2 g} = g^{ij}(x)\left(R^g\right)_{ihj}^h(x) v_g(x) = L_{EH}\left(j_x^2 g\right) v_x,$$

where v is the standard volume form, R^g is the curvature tensor of the Levi-Civita connection Γ^g of the metric g, and v_g denotes the Riemannian volume form attached to g; i.e., in coordinates, $v_g = \sqrt{|\det((g_{ab})_{a,b=1}^n)|}v$. Hence,

$$L_{EH} \circ j^2 g = (\rho \circ g)\left(y^{ij} \circ g\right)\left(R^g\right)_{ihj}^h, \quad \rho = \sqrt{\left|\det\left((y_{ab})_{a,b=1}^n\right)\right|}.$$

The local expression for L_{EH} is readily seen to be

$$L_{EH} = \rho\left(y^{ac}y^{bd} - y^{ab}y^{cd}\right)y_{ab,cd} + L_0,$$

where

$$\begin{aligned} L_0 = \frac{\rho}{2} \sum_{r \le s} \sum_{k \le l} \frac{1}{(1+\delta_{kl})(1+\delta_{rs})} &\left(2y^{rs}\left(y^{ki}y^{jl} + y^{li}y^{jk}\right) - 2y^{kl}y^{sr}y^{ji}\right. \\ &+ 2y^{kl}\left(y^{jr}y^{si} + y^{js}y^{ri}\right) + 3y^{ij}\left(y^{kr}y^{ls} + y^{ks}y^{lr}\right) \\ &- y^{ir}\left(y^{ks}y^{jl} + y^{ls}y^{jk}\right) - y^{is}\left(y^{kr}y^{jl} + y^{lr}y^{jk}\right) \\ &\left.- 2y^{ki}\left(y^{sl}y^{jr} + y^{rl}y^{js}\right) - 2y^{li}\left(y^{sk}y^{jr} + y^{rk}y^{js}\right)\right)y_{kl,i}y_{rs,j}. \end{aligned}$$

Hence L_{EH} is an affine function, which is proved to have a P–C form projectable onto J^1M and to be regular in the sense of Corollary 5.3. Therefore, we can apply Theorem 6.1 to this particular Lagrangian in order to obtain the following existence theorem to Einstein's field equations in the vacuum:

Theorem 6.2. *Given symmetric scalars $\gamma^i_{jk} = \gamma^i_{kj}$, $i, j, k = 1, \dots, n$, there exists a Ricci-flat (pseudo-)Riemannian metric of signature (n^-, n^+) defined on a neighborhood of $x_0 \in N$ such that, $g_{ij}(x_0) = \delta_{ij}$, $(\Gamma^g)^i_{jk}(x_0) = \gamma^i_{jk}$, for all i, j, k.*

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