Spaces of holomorphic functions on non-balanced domains

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ABSTRACT

This paper studies the coincidence of the $\tau_\omega$ and $\tau_3$ topologies on the space of holomorphic functions defined on an open subset $U$ of a Banach space. Dineen and Mujica proved that $\tau_\omega = \tau_3$ when $U$ is a balanced open subset of a separable Banach space with the bounded approximation property. Here, we study the $\tau_\omega = \tau_3$ problem for several types of non-balanced domains $U$.

1. Introduction

When $U$ is an open subset of a complex Banach space $E$, three topologies are usually considered on the space $H(U)$ of all holomorphic functions on $U$: the compact open topology $\tau_0$, the Nachbin topology $\tau_\omega$ and the bornological topology $\tau_3$ (the definitions are given below). It is known that $\tau_0 = \tau_\omega = \tau_3$ if $E$ is finite dimensional, while $\tau_0 < \tau_\omega \leq \tau_3$ if $E$ is infinite dimensional and several researchers have been interested in characterizing those spaces $E$ such that $\tau_\omega = \tau_3$. The first positive result on that problem was obtained by Dineen [6] in 1972. He proved that if $E$ is a Banach space with an unconditional Schauder basis and $U$ is a balanced open subset of $E$, then $\tau_\omega = \tau_3$ on $H(U)$. Soon after, Coeuré [4] proved an analogous theorem for the space $E = L^1[0,2\pi]$. Finally, in the nineties, Dineen [7, Corollary 4.18] and Mujica [10] independently obtained the most general result about the problem that we are considering:

Theorem 1 (Dineen, Mujica). If $E$ is a separable Banach space with the bounded approximation property and $U$ is a balanced open subset of $E$, then $\tau_\omega = \tau_3$ on $H(U)$.

Let us recall that $U$ is said to be balanced if $\lambda x \in U$ for all $x \in U$ and all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. We also recall that a separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a Banach space with a Schauder basis.

In this work, we obtain a result similar to Theorem 1 for some non-balanced domains:

Theorem 2. Let $E$ be a separable Banach space with the bounded approximation property and let $U$ be a balanced open subset in $E$. If $A$ is a closed bounded subset of $E$ such that $A \subset U$ and $U \setminus A$ is connected, then $\tau_\omega = \tau_3$ on $H(U \setminus A)$.
As far as we know, this theorem gives the first examples of non-balanced open domains such that $\tau_\omega = \tau_\delta$.

Let us explain the notation that appears in this work. Throughout the article, the letter $E$ always denotes a complex Banach space and $B_E(x, r)$ represents the open ball in $E$ with center $x \in E$ and radius $r > 0$. A seminorm $p$ on $H(U)$ is ported by a compact subset $K \subset U$ if for every open subset $V$ with $K \subset V \subset U$ there is a constant $C > 0$ such that

$$p(f) \leq C \sup_{x \in V} |f(x)|$$

for all $f \in H(U)$. The Nachbin topology $\tau_\omega$ is the locally convex topology on $H(U)$ defined by all the seminorms ported by the compact subsets of $U$.

The symbol $\tau_\delta$ represents the locally convex topology on $H(U)$ defined by all the seminorms $p$ on $H(U)$ with the following property: for each increasing countable open cover of $U$, $(V_n)_{n=1}^\infty$, there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that

$$p(f) \leq C \sup_{x \in V_{n_0}} |f(x)|$$

for all $f \in H(U)$. The reader is referred to the book of Mujica [11] for the main properties of holomorphic functions on infinite dimensional spaces and to book of Dineen [7] for an extensive study of the $\tau_\omega$ and the $\tau_\delta$ topologies.

2. Hartogs’s Theorem in Banach spaces

Our study about the $\tau_\omega = \tau_\delta$ problem on $H(U\setminus A)$ strongly depends on the existence of holomorphic extension from $U\setminus A$ to $U$ and that has led us to consider generalizations of the classical Hartogs’s Theorem. This result states that if $U$ is an open subset of $\mathbb{C}^n$ with $n \geq 2$, $K$ is a compact subset of $U$ and $U\setminus K$ is connected, then for every $f \in H(U\setminus K)$ there is $\tilde{f} \in H(U)$ such that $\tilde{f} = f$ on $U\setminus K$. In 1968, Alexander [1, p. 48] proved the same result when $U$ is a bounded domain in a Banach space $E$ with dimension $\dim(E) \geq 2$. Later, in 1985, Mujica [11, Theorem 22.6] proved it when $E$ is a separable Hilbert space; in this last case, $U$ can be unbounded. In 1970, Ramis [12, pp. 26–27] presented a general version of the theorem for all open subsets $U$ and all Banach spaces $E$ with $\dim(E) \geq 2$. However, Ramis’s proof is very brief and it seems that it could contain some gaps. Because of that and for the sake of completeness, we prove here a particular version of Hartogs’s Theorem that will be used in the study of the $\tau_\omega$ and $\tau_\delta$ topologies. We will need the following two results.

Theorem 3. If $n \in \mathbb{N}$ and $M = \mathbb{R}^n$ or $M = \mathbb{C}^n$, then every balanced open subset of $M$ is homeomorphic to $M$.

Proof. The proof in the real case appears in Berger [2, Theorem 11.3.6.1]. The complex case can be deduced if $\mathbb{C}^n$ is topologically identified with $\mathbb{R}^{2n}$. ⊓⊔

Theorem 4 (Zorn). Let $U$ be a connected open subset of a Banach space $E$. A function $f : U \to \mathbb{C}$ is holomorphic on $U$ if both $f$ is continuous at a point of $U$ and $f|_{M \cap U}$ is holomorphic on $M \cap U$ for every finite dimensional subspace $M \subset E$.

Proof. See Dineen [7, Example 3.8b]. ⊓⊔

Theorem 5. Let $U$ be a balanced open subset of a Banach space $E$ with $\dim(E) \geq 2$. Let $A$ be a closed bounded subset of $E$ such that $A \subset U$ and $U\setminus A$ is connected. If $f \in H(U\setminus A)$, then there is a unique $\tilde{f} \in H(U)$ such that $\tilde{f} = f$ on $U\setminus A$. 

Proof. See Dineen [7, Example 3.8b]. ⊓⊔
Proof. Let $M$ be any subspace of $E$ with $2 \leq \dim(M) < \infty$. The set

$$V = M \cap U$$

is balanced and open in $M$. The set

$$A' = M \cap A$$

is closed and bounded in $M$, so $A'$ is a compact subset of $V$. By Theorem 3, there is a homeomorphism $\varphi : V \to M$. Hence $\varphi(A')$ is a compact subset of $M$. If $R = \max_{x \in A'} \| \varphi(x) \|$, then $K = \varphi^{-1}(\overline{B_M}(0, R))$ is compact and

$$A' \subset K \subset V.$$

Since $M \setminus \overline{B_M}(0, R)$ is connected, $V \setminus K$ is connected as well. By Hartogs’s Theorem for finite dimensional spaces, there is $g_M \in H(V)$ such that $g_M = f$ on $V \setminus K$. We define

$$\tilde{f} = g_M \quad \text{on } V.$$

The definition of $\tilde{f}$ does not depend on the choice of $M$. Indeed, let $M$ and $N$ be subspaces of $E$ with $2 \leq \dim(M) < \infty$ and $2 \leq \dim(N) < \infty$. Let $L = M + N$. As we have seen previously, there are compact subsets $K_1, K_2$ and $K_3$ such that

$$M \cap A \subset K_1 \subset V_1 = M \cap U,$$
$$N \cap A \subset K_2 \subset V_2 = N \cap U,$$
$$L \cap A \subset K_3 \subset V_3 = L \cap U.$$

There are also holomorphic functions $g_M \in H(V_1), g_N \in H(V_2)$ and $g_L \in H(V_3)$ such that

$$g_M = f \quad \text{on } V_1 \setminus K_1,$$
$$g_N = f \quad \text{on } V_2 \setminus K_2,$$
$$g_L = f \quad \text{on } V_3 \setminus K_3.$$

The set $M \cap K_3$ is contained in $V_1$:

$$M \cap K_3 \subset M \cap V_3 = M \cap L \cap U = M \cap U = V_1.$$ 

Then $K_1 \cup (M \cap K_3)$ is a compact subset of $V_1$, so $V_1 \setminus (K_1 \cup (M \cap K_3)) \neq \emptyset$. Hence

$$g_L = f = g_M \quad \text{on } V_1 \setminus (K_1 \cup (M \cap K_3)) \neq \emptyset.$$

By the Identity Theorem, we have that $g_L = g_M$ on $V_1$. Similarly, $g_L = g_N$ on $V_2$. Therefore, if $x \in M \cap N \cap U = V_1 \cap V_2$, then

$$g_M(x) = g_L(x) = g_N(x).$$

This proves that the definition of $\tilde{f}$ does not depend on the choice of the finite dimensional subspace $M$. 
By the definition of \( \tilde{f} \), if \( M \) is a subspace of \( E \) such that \( 2 \leq \dim(M) < \infty \), then \( \tilde{f}|_{M \cap U} = g_M \in H(M \cap U) \).

By Theorem 4, in order to show that \( f \) is holomorphic on \( U \), we only have to prove that \( \tilde{f} \) is continuous at a point of \( U \). Let \( r > 0 \) be such that \( A \subset \overline{B}_E(0, r) \) and let \( e \in E \) with \( \|e\| = 1 \). The set

\[
\{ t \geq 0 : te \in A \cup \{0\} \}
\]
is closed and non-empty in \([0, r]\). Therefore, there exists

\[
\alpha = \max\{ t \geq 0 : te \in A \cup \{0\} \}.
\]

Let

\[
\beta = \sup\{ t > 0 : te \in U \}
\]

(\( \beta \) could be infinite). Since \( U \) is open, it follows that \( \beta e \notin U \). As \( \alpha e \in A \cup \{0\} \subset U \), we deduce that \( \alpha < \beta \). Since \( U \) is balanced, if \( \alpha < t < \beta \), then \( te \in U \backslash A \), so there is \( r_t > 0 \) such that \( B_E(te, r_t) \subset U \backslash A \). The set

\[
W = \bigcup_{\alpha < t < \beta} B_E(te, r_t)
\]
is open and it is contained in \( U \backslash A \).

Let \( x \in W \). If \( M \) is a two dimensional subspace of \( E \) such that \( x, e \in M \), \( V = M \cap U \) and \( A' = M \cap A \), we have seen previously that there is a compact subset \( K \subset V \) and there is \( g_M \in H(V) \) such that \( A' \subset K \subset V \) and \( g_M = f \) on \( V \backslash K \). Let

\[
W' = \bigcup_{\alpha < t < \beta} B_M(te, r_t).
\]

Then

\[
W' = M \cap W \subset M \cap (U \backslash A) = V \backslash A'.
\]

Let us remark that \( W' \) is connected because it is the union of the segment \( \{te : \alpha < t < \beta\} \) with the balls \( B_M(te, r_t) \) centered at the points of the segment.

As \( K \) is a compact subset of \( U \) and \( \beta e \notin U \), there is \( t_0 < \beta \) such that \( t_0 e \notin K \), that is, \( W' \backslash K \neq \emptyset \). The functions \( f|_{V \backslash A'} \) and \( g_M \) are holomorphic on \( V \backslash A' \) and \( g_M = f \) on \( V \backslash K \), so \( g_M = f \) on \( W' \backslash K \). By the Identity Theorem, \( g_M = f \) on \( W' \). Since \( x \in W \cap M = W' \), it follows that

\[
\tilde{f}(x) = g_M(x) = f(x).
\]

Therefore, \( \tilde{f}(x) = f(x) \) for every \( x \in W \), which implies the continuity of \( \tilde{f} \) on \( W \). By Theorem 4, \( \tilde{f} \) is holomorphic on \( U \).

The functions \( \tilde{f} \) and \( f \) are holomorphic on \( U \backslash A \) and it was proved that \( \tilde{f} = f \) on \( W \). By hypothesis, \( U \backslash A \) is connected, so \( \tilde{f} = f \) on \( U \backslash A \). The uniqueness of \( \tilde{f} \) is a consequence of \( U \) being connected. \( \square \)

**Corollary 6.** Let \( U \) be an open subset of a Banach space \( E \) with \( \dim(E) \geq 2 \). Let \( A \) be closed bounded subset of \( E \) with the following property: there is a balanced open subset \( V \) in \( E \) such that \( A \subset V \subset U \) and \( V \backslash A \) is connected. Then every holomorphic function on \( U \backslash A \) has a unique holomorphic extension to \( U \).
Proof. By Theorem 5, if \( f \in H(U \setminus A) \), then there is \( \tilde{f} \in H(V) \) such that \( \tilde{f} = f \) on \( V \setminus A \). Thus, the function

\[
g = \begin{cases} 
  f & \text{on } U \setminus A, \\
  \tilde{f} & \text{on } V
\end{cases}
\]

is well defined and is holomorphic on \( U \). Moreover, if \( h \) is another holomorphic function on \( U \) such that \( h = f \) on \( U \setminus A \), then \( h = f = \tilde{f} \) on \( V \setminus A \). As \( V \) is connected, \( h = \tilde{f} \) on \( V \). Therefore, \( h = g \) on \( U \).

3. The \( \tau_\omega \) and the \( \tau_\delta \) topologies on \( H(U) \)

In order to study the coincidence of the \( \tau_\omega \) and the \( \tau_\delta \) topologies on \( H(U) \), we apply several already known theorems about holomorphic extensions, the envelope of holomorphy and the spectrum of \( H(U) \). Let us recall the definition of these notions. If \( \tau \) is a topology on \( H(U) \), the spectrum of \( (H(U), \tau) \), denoted by \( \text{Spec}(H(U), \tau) \), is the set of all non-zero multiplicative linear functions from \( H(U) \) into \( \mathbb{C} \) which are \( \tau \)-continuous.

A Riemann domain over a Banach space \( E \) is a pair \((X, \pi)\) where \( X \) is a Hausdorff topological space and \( \pi : X \to E \) is a local homeomorphism. That means that for every \( x \in X \) there is an open set \( \Omega \) in \( X \) such that \( x \in \Omega \), \( \pi(\Omega) \) is open in \( E \) and \( \pi|_\Omega : \Omega \to \pi(\Omega) \) is a homeomorphism. A function \( f : X \to \mathbb{C} \) is said to be holomorphic if \( f \circ (\pi|_\Omega)^{-1} \in H(\pi(\Omega)) \) for every open subset \( \Omega \subset X \) such that \( \pi|_\Omega : \Omega \to \pi(\Omega) \) is a homeomorphism.

Let \( U \) be an open subset of \( E \) and let \((X, \pi)\) be a Riemann domain over \( E \). A continuous mapping \( \varphi : U \to X \) is said to be a holomorphic extension of \( U \) if \( \pi(\varphi(x)) = x \) for all \( x \in U \) and for every \( f \in H(U) \) there is an unique \( g \in H(X) \) such that \( g \circ \varphi = f \). The envelope of holomorphy of \( U \) is a Riemann domain \((\mathcal{E}(U), \pi)\) over \( E \) with the following properties:

(a) There is a holomorphic extension \( \varphi : U \to \mathcal{E}(U) \).

(b) If \((X_1, \pi_1)\) is another Riemann domain and \( \varphi_1 : U \to X_1 \) is also a holomorphic extension, then there is a continuous mapping \( \psi : X_1 \to \mathcal{E}(U) \) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
U & \xrightarrow{\varphi} & \mathcal{E}(U) \\
\downarrow{\varphi_1} & & \downarrow{\psi} \\
X_1 & \xrightarrow{\psi} & \mathcal{E}(U)
\end{array}
\]

The envelope of holomorphy of \( U \) always exists and is unique up to isomorphism (see Mujica [11, Theorem 56.4]). The following theorems relate holomorphic extensions to the spectrum of \( H(U) \).

**Theorem 7** (Mujica, Schottenloher). If \( E \) is a separable Banach space with the bounded approximation property and \( U \) is a connected open subset of \( E \), then \( \mathcal{E}(U) = \text{Spec}(H(U), \tau_0) \).

**Proof.** See Mujica [11, Corollary 58.10].

**Theorem 8.** Let \( U \) and \( \bar{U} \) be connected open subsets of a Banach space such that \( U \subset \bar{U} \). Let us assume that for every \( f \in H(U) \) there is \( \tilde{f} \in H(\bar{U}) \) with \( \tilde{f} = f \) on \( U \).
1. The mapping

\[ T : f \in (H(U), \tau) \mapsto \tilde{f} \in (H(\tilde{U}), \tau) \]

is a topological isomorphism for \( \tau = \tau_\delta \).

2. If \( \tilde{U} \subset \text{Spec}(H(U), \tau_\omega) \), then the mapping \( T \) is also a topological isomorphism for \( \tau = \tau_\omega \).

**Proof.** The first statement is due to Coeuré [3] in the case of separable Banach spaces and to Hirschowitz [8] in the general case. The second statement is due to Dineen [6]. \( \square \)

Let us mention that Theorem 8 does not hold in general if \( \tilde{U} \) is not contained in \( \text{Spec}(H(U), \tau_\omega) \). Josefson in [9] gave an example of a Banach space \( E \) and open subsets \( U \subset \tilde{U} \subset E \) such that the mapping \( T \) is not continuous for \( \tau = \tau_\omega \).

**Theorem 9.** Let \( E \) be a separable Banach space with the bounded approximation property. Let \( U \) and \( \tilde{U} \) be connected open subsets of \( E \) such that \( U \subset \tilde{U} \) and suppose that for every \( f \in H(U) \) there is \( \tilde{f} \in H(\tilde{U}) \) with \( \tilde{f} = f \) on \( U \). Then \( \tau_\omega = \tau_\delta \) on \( H(U) \) if and only if \( \tau_\omega = \tau_\delta \) on \( H(\tilde{U}) \).

**Proof.** Let \( (\mathcal{E}(U), \pi) \) be the envelope of holomorphy of \( U \). By hypothesis, the inclusion \( U \to \tilde{U} \) is a holomorphic extension. Hence there is a continuous mapping \( \psi : \tilde{U} \to \mathcal{E}(U) \) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
U & \xrightarrow{\varphi} & \mathcal{E}(U) \\
\downarrow{\psi} & & \downarrow{\psi} \\
\tilde{U} & & E \\
\end{array}
\]

The mappings from \( U \) into \( \tilde{U} \) and from \( \tilde{U} \) into \( E \) are the inclusions. If \( x, y \in \tilde{U} \) and \( \psi(x) = \psi(y) \), then

\[ x = \pi(\psi(x)) = \pi(\psi(y)) = y. \]

This shows that \( \psi \) is injective. As \( \psi \) is continuous and its inverse \( \psi^{-1} = \pi|_{\psi(\tilde{U})} \) is also continuous, \( \tilde{U} \) can be identified with a subset of \( \mathcal{E}(U) \). By Theorem 7, \( \mathcal{E}(U) = \text{Spec}(H(U), \tau_0) \) and then

\[ \tilde{U} \subset \mathcal{E}(U) = \text{Spec}(H(U), \tau_0) \subset \text{Spec}(H(U), \tau_\omega). \]

By Theorem 8, the mapping

\[ f \in (H(U), \tau) \mapsto \tilde{f} \in (H(\tilde{U}), \tau) \]

is a topological isomorphism for \( \tau = \tau_\omega \) and \( \tau = \tau_\delta \). Therefore, \( \tau_\omega = \tau_\delta \) on \( H(U) \) if and only if \( \tau_\omega = \tau_\delta \) on \( H(\tilde{U}) \). \( \square \)

**Corollary 10.** Let \( E \) be a separable Banach space with the bounded approximation property and let \( U \) be a balanced open subset of \( E \). If \( A \) is a closed bounded subset of \( E \) such that \( A \subset U \) and \( U \setminus A \) is connected, then \( \tau_\omega = \tau_\delta \) on \( H(U \setminus A) \).

**Proof.** If \( \dim(E) < \infty \), then \( \tau_0 = \tau_\omega = \tau_\delta \) on \( H(U \setminus A) \). If \( \dim(E) = \infty \), the result can be directly deduced from Theorems 1, 5 and 9. \( \square \)
The following theorem does not require the space $E$ to be separable or to have the bounded approximation property.

**Theorem 11.** Let $A$ be a closed bounded subset of a Banach space $E$ such that $E \setminus A$ is connected. Then $\tau_\omega = \tau_\delta$ on $H(E)$ if and only if $\tau_\omega = \tau_\delta$ on $H(E \setminus A)$.

**Proof.** If $\dim(E) < \infty$, then $\tau_0 = \tau_\omega = \tau_\delta$ on $H(E \setminus A)$ and $\tau_0 = \tau_\omega = \tau_\delta$ on $H(E)$. Hence we can assume that $\dim(E) = \infty$. By Theorem 5, each $f \in H(E \setminus A)$ has an extension $\tilde{f} \in H(E)$. By Theorem 8, the mapping

$$ T : f \in (H(E \setminus A), \tau) \mapsto \tilde{f} \in (H(E), \tau) $$

is a topological isomorphism for $\tau = \tau_\delta$. We will prove that $T$ is also a topological isomorphism for $\tau = \tau_\omega$.

Let us suppose that $p$ be a continuous seminorm on $(H(E), \tau_\omega)$, ported by a compact subset $K \subset E$. By hypothesis, there is $r > 0$ such that $A \subset B_E(0, r)$. Let

$$ K_1 = \{x \in K: \|x\| \geq r\}, $$

$$ K_2 = \{x \in K: \|x\| \leq r\}. $$

Note that $K_1$ and $K_2$ are compact subsets of $E$, that $K = K_1 \cup K_2$ and

$$ K_1 \subset E \setminus B_E(0, r) \subset E \setminus A. $$

We choose $e \in E$ such that $\|e\| = 1$ and define the following compact subset of $E$:

$$ \tilde{K}_2 = \{x + \lambda e: x \in K_2, \lambda \in \mathbb{C} \text{ and } |\lambda| = 2r\}. $$

If $x + \lambda e \in \tilde{K}_2$, then

$$ \|x + \lambda e\| \geq \|\lambda e\| - \|x\| \geq 2r - r = r. $$

This implies that

$$ \tilde{K}_2 \subset E \setminus B_E(0, r) \subset E \setminus A. $$

Therefore, $K_1 \cup \tilde{K}_2$ is a compact subset of $E \setminus A$.

We will prove that the seminorm

$$ f \in H(E \setminus A) \mapsto p \circ T(f) = p(\tilde{f}) $$

is ported by $K_1 \cup \tilde{K}_2$. Let $V$ be an open subset of $E$ such that

$$ K_1 \cup \tilde{K}_2 \subset V \subset E \setminus A. $$

Then there is an open neighborhood of zero $W \subset E$ such that

$$ (K_1 \cup \tilde{K}_2) + W \subset V. $$

As $K + W$ is an open neighborhood of $K$ and $p$ is ported by $K$, there exists $C > 0$ such that

$$ p(g) \leq C \sup_{z \in K + W} |g(z)| = C \sup_{z \in (K_1 \cup \tilde{K}_2) + W} |g(z)| \quad (1) $$
for all \( g \in H(E) \). As \( K_1 + W \) is contained in \( V \),
\[
\sup_{z \in K_1 + W} |g(z)| \leq \sup_{z \in V} |g(z)|
\]
for all \( g \in H(E) \).

Now we are going to prove that also
\[
\sup_{z \in K_2 + W} |g(z)| \leq \sup_{z \in V} |g(z)|
\]
for all \( g \in H(E) \). Let \( g \in H(E) \), \( x \in K_2 \) and \( y \in W \). The function of one complex variable
\[
h(\lambda) = g(x + y + \lambda e)
\]
is holomorphic on \( C \). By the Maximum Modulus Theorem,
\[
|g(x + y)| = |h(0)| \leq \sup_{|\lambda| = 2r} |h(\lambda)| = \sup_{|\lambda| = 2r} |g(x + y + \lambda e)|
\]
\[
\leq \sup_{z \in K_2 + W} |g(z)| \leq \sup_{z \in V} |g(z)|.
\]
This holds for all \( x \in K_2 \) and all \( y \in W \), so
\[
\sup_{z \in K_2 + W} |g(z)| \leq \sup_{z \in V} |g(z)|.
\]
By the inequalities (1), (2) and (3),
\[
p(g) \leq C \sup_{z \in V} |g(z)|
\]
for all \( g \in H(E) \).

Given \( f \in H(E \setminus A) \), let \( \tilde{f} \in H(E) \) such that \( \tilde{f} = f \) on \( E \setminus A \). Then
\[
p \circ T(f) = p(\tilde{f}) \leq C \sup_{z \in V} |\tilde{f}(z)| = C \sup_{z \in V} |f(z)|.
\]
This proves that the seminorm
\[
f \in H(E \setminus A) \mapsto p \circ T(f)
\]
is ported by \( K_1 \cup \overline{K}_2 \), so \( p \circ T \) is continuous on \( (H(E \setminus A), \tau_\omega) \). That holds for every continuous seminorm \( p \) on \( (H(E), \tau_\omega) \). Therefore, the mapping \( T \) is continuous for \( \tau = \tau_\omega \).

Finally, it is easy to check that the inverse
\[
T^{-1} : g \in (H(E), \tau_\omega) \mapsto g|_{E \setminus A} \in (H(E \setminus A), \tau_\omega)
\]
is continuous. Indeed, if \( q \) is a seminorm on \( H(E \setminus A) \) ported by a compact subset \( K' \subset E \setminus A \), then \( q \circ T^{-1} \) is also ported by \( K' \). Thus, \( T \) is a topological isomorphism for \( \tau = \tau_\omega \) and for \( \tau = \tau_\delta \). Therefore, \( \tau_\omega = \tau_\delta \) on \( H(E \setminus A) \) if and only if \( \tau_\omega = \tau_\delta \) on \( H(E) \).

**Corollary 12.** If \( A \) is a closed bounded subset of \( \ell_\infty \) such that \( \ell_\infty \setminus A \) is connected, then \( \tau_\omega \neq \tau_\delta \) on \( H(\ell_\infty \setminus A) \).

**Proof.** Dineen [5] proved that \( \tau_\omega \neq \tau_\delta \) on \( H(\ell_\infty) \) and then the result follows from Theorem 11. \( \square \)
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References