Two Near-Optimal Layouts for Trusslike Bridge Structures Bearing Uniform Weight between Supports

Jaime Cervera Bravo, Carlos Vázquez Espí, and Mariano Vázquez Espí

Abstract: Although the primary objective on designing a structure is to support the external loads, the achievement of an optimal layout that reduces all costs associated with the structure is an aspect of increasing interest. The problem of finding the optimal layout for bridgelike structures subjected to a uniform load is considered. The problem is formulated following a theory on economy of frame structures, using the stress volume as the objective function and including the selection of appropriate values for statically indeterminate reactions. It is solved in a function space of finite dimension instead of using a general variational approach, obtaining near-optimal solutions. The results obtained with this profitable strategy are very close to the best layouts known to date, with differences of less than 2% for the stress volume, but with a simpler layout that can be recognized in some real bridges. This strategy could be a guide to preliminary design of bridges subject to a wide class of costs.

Introduction

To bear a uniform weight is one of the fundamental tasks of bridges and other bending structures. When these structures are designed, one of the main aims is to find optimal or at least good structural shapes accounting for all the necessary costs, following the classic rule of thermodynamics to measure efficiency [see, e.g., Clausius (1885)]. This paper will follow the formulation of the design problem stated by Maxwell (1870) and Michell (1904), known as the free loading problem after Cox (1965), or the Maxwell class of problems after Cervera Bravo (1989). Moreover, the general design problem stated by Galilei (1638) to find a structure bearing a given useful load and its own self-weight includes the Maxwell problem as the limiting case in which the structure self-weight is very small and can be neglected [see for details Cervera Bravo and Vázquez Espí (2011) and Antuña and Vázquez Espí (2012)].

The Maxwell approach is clearly different from other problem formulations for the layout optimization, as e.g. the fixed boundary approach (Cox 1965; Hemp 1973) that has received most attention by researchers in the last decades and has had some absolute optima obtained recently (Darwich et al. 2010; Pichugin et al. 2012). In a free loading problem there is no kinematic condition and loads and reactions are known. On the contrary, in a fixed boundary problem there are kinematic conditions and only the loads are known, while the reactions are determined for each feasible solution by standard structural analysis with given constitutive equations for the structural material. Both classes have a common subset of problems, those of the statically determinate kinematic conditions in the latter. And because of this fact, a controversy exists about the meaning of this distinction [e.g., Rozvany (2011) and Vázquez Espí (2013)], which was first noted by Cox (1965, p. 116–117). The fixed boundary formulation has a well-known drawback: “the reactions such as those at fixed supports, are in any case carried by some other bodies acting as structures and the true picture of the economy achieved should include their cost” (Owen 1965, p. 64) because the fixed support cost is different for each solution (Cox 1965). Recently, Rozvany and Sokol (2012) have extended the Prager-Rozvany theory trying to fill this shortcoming, but it seems that the simple case of a foundation with friction forces is not covered by the new approach (Vázquez Espí 2012). This paper is mainly interested in the free loading formulation because all costs are considered. In a general sense, this approach focuses on the notion of structural design outlined by Cross (1936), which is different from structural analysis.

The plan of exposition is as follows. First, a minimal set of definitions and theorems is introduced for a complete and clear description of the class of design problems with which the paper deals. Second, the class of problems to which the title problem belongs is described, showing the absolute optimal solutions known up to date for some problems of the class and presenting the near-optimal solutions for the title problem. Third, the process of obtaining the full-plane solution is explained in detail, and then the same method is applied for the half-plane case. In the latter Maxwell problems with given horizontal reactions at the supports are examined, showing how their solutions can be used to select a preliminary bridge layout. Finally the significance of the results is discussed.

Maxwell and Michell Approach to Structural Design

Definition 1. Maxwell problem: to find a feasible structure for a given set of known, external forces in equilibrium. Each external force must be determined in position, direction and magnitude
the number of internal forces, there

Eq. (3)]. The stress volume denomination arises from the fact that Maxwell’s theorem [Hemp 1958, p. 3, Eq. (1)].

for a Maxwell problem when the domain Ω undergoes the displacement field d (Michell 1904, p. 590). Hence, if L is defined as

\[ L = \sup_d \left[ \frac{W_d}{\Delta d} : d \in D \right] \]

\[ \leq \inf \{ V(A) : A \in S \} \]

is a minimum, and consequently from [Michell’s lemma] the volume of material in the frame M is also a minimum.” (Michell 1904, p. 591). Hence, Michell’s optimality criterion in Ω requires:

• a finite bound strictly positive Δf for the field T, and

• \( e_i^f = \Delta_f e_i^f \) holds on all members of the structure M. Because Michell does not show any proof of the existence of a pair (T, M) for every set of given external forces in equilibrium, he only considers his criterion as a sufficient condition (Michell 1904, p. 589). In spite of a sustained research effort on this subject, it has not been proven that Michell’s criterion is also a necessary condition.

**Michell’s Theorems**

Let Δ be a finite, strictly positive strain. Let D be the set of bounded, continuous displacement fields d such that the strain εd of the field d at all points and directions of the considered domain, Ω, fulfills |εd| ≤ Δd. Let S be the set of all Maxwell structures for a Maxwell problem enclosed into Ω.

**Theorem 9.** (Michell’s first theorem)

\[ V(A) \in D \times S: \frac{W_d}{\Delta d} \leq V(A) \]

where \( W_d \) is the virtual work of the given external forces of the Maxwell problem when the domain Ω undergoes the displacement field d (Michell 1904, p. 590). Hence, if L is defined as

**Corollary 6.** The Maxwell number is the difference between stress volumes of tension and compression, \( M = V^+ - V^- \) [Maxwell 1870, p. 176; Michell 1904, Eq. (1)].
condition. The best result known to date is that although a maximizer and minimizer pair for Eq. (3) always exists [Bouchitté et al. 2008, Eqs. (2.22) and (2.24) and Proposition 2.1], the minimizer of the right-hand side of Eq. (3) "may not be a Michell truss" (W. Gangbo, personal communication, 2012). Moreover, Bouchitté et al. (2008, §3.2 and Theorem 3.1) have reformulated the original problem so that only Michell trusses belong to the counterpart of S in Eq. (3), redefining the counterpart of D accordingly. Next, they have proved that the infima of both formulations are equal. Moreover, "if we could prove existence of a minimizer in (3.6) [ibidem], we will use the optimum [Radon measure] γ to construct a minimizer σ" which we know will be a Michell truss" (W. Gangbo, personal communication, 2012). Bouchitté et al. (2008, p. 1601–1602, Problem 5.1) conclude saying, "we strongly believe that our approach could be a useful tool to investigate the properties of optimal structures. However, it is still necessary to prove the existence of a minimizing measure for the new formulation." This argument refers only to Michell’s original criterion for Maxwell problems. The frequently and unfortunately equal named criteria for fixed boundary problems are different for each cost, e.g., Hemp’s criterion for minimum volume [1973, Eq. (4.8)] or Prager’s criterion for minimum weight (Srithongchai and Dewhurst 2003). As the adjective Michell is nowadays overloaded, remember that this paper uses Michell solution to refer to a Maxwell structure that fulfills Michell’s criterion (Theorem 10), i.e., an absolute optimum solution for a given Maxwell problem. However, Michell net is used in its usual sense, i.e., to refer to layouts, frames, or Maxwell structures that follow, fully or partially, the geometrical conditions derived by Michell (1904, p. 591–594) from his second theorem, even if there is no proof that they fulfill Michell’s criterion.

Definition 11. Michell number v of a structure: The dimensionless ratio between the stress volume and the product of the useful load and the size of the structure (Cervera Bravo and Vázquez Espín 2011; McConnel 1974, quantity k).

Bridge Class of Design Problems

The bridge problem is the Maxwell problem of equilibrating a uniform weight w over a horizontal length L with vertical forces (reactions) in the load line. Depending on the number S of suitable reactions and their relative distances, there are different problems, so it is preferable to refer to the bridge class of problems (Fig. 1). All have the same size L and useful load wL, so the Michell number of every solution is computed as $v = V/(wL^2)$.

For $S = 1$, the vertical reaction is equal to $wL$ acting at midspan, so there is a unique problem [Fig. 1(a)]. The Michell solutions for the full plane or the half plane are known after the Michell solution for three parallel forces (Fig. 2) [Parkes 1965, pp. 177, Fig. 102(d)].

For $S = 2$, the sum of the two vertical reactions must be equal to $wL$, but their magnitudes depend on their relative position [Fig. 1(b)]. Each pair of values $a, b$ with $L - a - b > 0$ and $a \leq b$ defines a different Maxwell problem. This case covers a fairly large subset of real bridges. No Michell solutions are known to date.

For $S = 3$, there are four degrees of freedom to define a Maxwell problem: the positions of supports, a, b, c, and the magnitude of one of the reactions, [Fig. 1(c)]. The number of degrees of freedom to define a Maxwell problem will increase with S. Therefore, each value of S represents a new set of Maxwell problems.

Solutions for the $S=2$, $a=b=0$ Problem

The results for the case $S = 2$, with $a=b=0$, i.e., with the reactions at the edges of L, are presented [Fig. 1(d)]. The traditional solutions follow from parabolic curves and vertical hangers (label P), with a horizontal tie in the half-plane case (Table 1). On the contrary, the investigation is around a basic layout that consists of a nonparabolic arch with oblique hangers (AOH), with two versions for both full and half plane [Table 1 and Figs. 3(c) and 3]. They follow from a layout family whose first solution was obtained by W. J. Supple and published by Hemp (1973, p. 21) and later proposed by McConnel (1974, Figs. 5 and 6, p. 889) as simpler approximations of his own proposed Michell nets (p. 897). Although McConnel only worked with trusses of finite numbers of nodes—with linear programming (LP) and non-LP algorithms—he predicted with an extrapolation method the value 0.75779 for $v = V/wL^2$ in the full-plane case, which agrees with the result in this paper, 0.75800, from numerical variational calculus. McConnel predicts slightly better values for his proposed Michell nets (label M), 0.75490 in the full-plane case [Table 1 and Figs. 3(a and d)].

![Fig. 2. Michell solutions for one support: (a) half-plane solution; (b) full-plane solution](image)

![Table 1. Solutions for $S = 2$ and $a = b = 0$](table)

<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>Label</th>
<th>$v = V/(wL^2)$</th>
<th>$H/L$</th>
</tr>
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<td>Ph</td>
<td>1.155</td>
<td>0.433</td>
</tr>
<tr>
<td></td>
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<td>0.98468</td>
<td>0.44191</td>
</tr>
<tr>
<td></td>
<td>SA,h</td>
<td>0.97431</td>
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</tr>
<tr>
<td></td>
<td>M,h</td>
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<td></td>
</tr>
<tr>
<td>Full plane</td>
<td>Pf</td>
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<td>0.612</td>
</tr>
<tr>
<td></td>
<td>AOH, f</td>
<td>0.75800</td>
<td>0.63345</td>
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<tr>
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<tr>
<td></td>
<td>M,f</td>
<td>0.75490</td>
<td></td>
</tr>
</tbody>
</table>

Note: AOH = arch and oblique hangers; f = full-plane case; h = half-plane case; M = McConnel estimate (no actual layout here); P = parabolic arch and vertical hangers; SA = layout from simulated annealing search.
Hemp (1974) presents a Michell net for the fixed boundary version of the upper half plane problem (with two fixed supports), corresponding to $v = 0.78887$, while McConnel predicts 0.78787, and recently Pichugin et al. (2012) predicted 0.78791, so the predicted values by McConnel can be accepted as sound estimates for stress volume optima in all cases and are thus bolded in Table 1 (label M).

There is also a search with the simulated annealing (SA) algorithm [e.g., Vázquez Espí and Vázquez Espí (1997)] [Table 1 and Figs. 3(b and e)] (SA trusses data—node coordinates and bar nodes—available at http://habitat.aq.upm.es/gi/mve/dt/). As in the algorithms used by McConnel, the SA solutions are of discrete nature. The original uniform weight $w$ is represented by $M - 1$ inner point loads of magnitude $wL/M$ uniformly distributed along the span, with $M \in [15, 40]$. For each considered $\Omega$, there is a search for symmetrical and nonsymmetrical solutions with a number of nodes $N$ in the interval $[M + 2, p \cdot M]$ where $p \in \{3, 4, 5, 6, 8\}$. Although the upper bound for $N$ could seem small (e.g., 320), as no ground structure is used so the positions of this small, variable number of nodes and their connectivity change during the search to explore the whole space of feasible solutions without any other restriction.

In both domains, the SA solutions slightly improve the stress volume of the AOH layouts. The best SA,h solution is similar to the Michell net proposed by McConnel and its stress volume is slightly smaller than McConnel’s optimum estimate (Table 1). The main difference between AOH and the better SA layouts occurs near the supports, where the SA solutions show Michell nets, strongly suggesting that Michell solutions could exist in both cases, although only in the half-plane case can improvement of the stress volume of the AOH layout up to approximately 1% be ensured. The solutions obtained by J. Hernando García (personal communication, 2011) using LP techniques have been at the authors’ disposal, and his best results confirm McConnel’s estimates, the presence of oblique hangers and Michell nets near supports, and the probable existence of Michell solutions.

**AOH Full-Plane Layout**

Consider a quarter of the layout [Figs. 3(f) and 4]. At the horizontal position $\chi L$, with $\chi \in [0, 1/2]$, the direction and length of the hanger are, respectively, $\alpha(\chi)$ and $c(\chi)$. Its upper extreme defines a point of the arch of coordinates

$$X = L(x, y)^T = [Lx + c \sin(\alpha), Lc \cos(\alpha)]^T = L(x + y \tan \alpha, y)^T$$

With the notation $t(\chi) = \tan \alpha(\chi)$, $(\cdot)' = d(\cdot)/dx$, the geometric conditions are

$$X = L(x,y)^T = L(\chi + yt, y)^T, \quad X' = L(x', y')^T = L[1 + (yt)', y']^T$$

The equilibrium condition at the bottom of the hanger defines its internal force, $P = (P_h, P_v)^T$ with $P_h = wt/2$, and $P_v = w/2$. The shape of the arch, $X(\chi)$, is determined by its internal force, $(N_h, N_v)^T$

$$\frac{N_h}{N_v} = \frac{dx}{dy} = \frac{x'}{y'}$$

The equilibrium of the joint of the hanger and the arch determines the variation of the internal force of the latter

$$dN_h = -P_h Ld\chi, dN_v = P_v Ld\chi$$

Now, using the rotational equilibrium condition of one-half of the structure at the vertical axis of symmetry

![Fig. 4. Quarter of the AOH layout](image)
where \( h = 2y(0) \) is the inverse of the global slenderness \( \lambda = L/H \).

By integrating Eq. (7)

\[
N_h(x) = N_h(0) - \int_0^x P_h L du = \frac{wL}{2} \left[ \frac{\lambda}{4} - \int_0^x t(u) du \right],
\]

\[
N_v(x) = \int_0^x P_v L du = \frac{wL}{2} x,
\]

where \( y(x) \) is the inverse of the global slenderness \( \lambda = L/H \).

Now compute the stress volume of one quadrant, \( V_c \). Because \( V = \int |e| ds \) and \( |e| \) and \( ds \) are the modulus of parallel vectors, its product can be computed in terms of their horizontal and vertical components, hence \( V = \int |e_x| |x| dx + \int |e_y| |y| dy = V^x + V^y \).

Furthermore, each quantity can be decomposed in hangers \((V_{c}^{x}, V_{c}^{y})\) and arch contributions \((V_{c}^{x}, V_{c}^{y})\) according to

\[
V_c = V_{c}^{x} + V_{c}^{y} + V_{c}^{x} + V_{c}^{y},
\]

where \( y(x) \) changes sign, because it follows from Eq. (6) that \( x' \) does.

The absolute value operator is not needed because all integrands are positive for \( \chi \in [0, 1/2] \). For example, if the thrust component \( N_h \) changes sign, because \( y'(x) < 0 \) it follows from Eq. (6) that \( x' \) does as well, so that the product \( N_h x' \) will be positive everywhere. The arch terms can be integrated by parts, taking into account that \( N_h \) will be positive everywhere. The integrals can be obtained from the corresponding terms \((V_{c}^{x}, V_{c}^{y})\) and arch contributions \((V_{c}^{x}, V_{c}^{y})\) according to

\[
V_c = V_{c}^{x} + V_{c}^{y} + V_{c}^{x} + V_{c}^{y},
\]

\[
V_{c}^{x} = \frac{wL^2}{2} \int_0^{1/2} t y dx,
\]

\[
V_{c}^{y} = \frac{wL^2}{2} \int_0^{1/2} y dx,
\]

\[
V_{c}^{x} = \frac{wL^2}{2} \int_0^{1/2} \chi dx,
\]

\[
V_{c}^{y} = \frac{wL^2}{2} \int_0^{1/2} \chi (-y') dx.
\]

The rest of the terms can be obtained from the corresponding terms of \( V_c \) in Eqs. (10) or (11), but multiplying by two and substituting \( \lambda/4 \) by \( \lambda/8 \). Therefore, with \( \tau = \int_0^x t(u) du - \lambda/8 \), the half-plane geometry is the solution of

\[
V_{c}^{x} = \int_0^{1/2} |t(\chi)| \chi dx = \int_0^{\chi \tau \text{opt}} N_h \chi dx - \int_0^{1/2} N_h \chi dx
\]

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\]

Some polynomial solutions \( \tau_n(\chi) \) are shown in Table 3. The coefficients \( z_i \) are determined as previously, but now a new equation must be introduced for \( \chi \tau \text{opt} \), \( \mathcal{O}V/\partial \chi \text{opt} = 0 \). However, note that

\[
\frac{a(0.5)}{(deg)}
\]

<table>
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<th>( n )</th>
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<td>0</td>
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<tr>
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<td>3</td>
<td>0.29466</td>
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<tr>
<td>4</td>
<td>0.28735</td>
</tr>
</tbody>
</table>

**Table 2.** Polynomial Solutions \( \tau_n \) for the Full-Plane Domain
that in the case of a constant function, \( \tau_0 \), \( \chi_{cr} = 1/2 \) because \( \tau_0^2(\chi) = 0 \), that is, the hangers are vertical and \( T(\chi) \) is constant.

### Extension to Cases with Horizontal Reactions

New Maxwell problems can be obtained from the latter simply by adding a pair of opposite horizontal reactions in the edges. If these reactions were free of cost, it is clear that they could reduce the stress volume of the tie and hence \( V \). This is precisely the case of friction forces between foundations and ground (Bow 1873, p. 69–71). Let the horizontal reaction magnitude be given as a fraction \( \phi \) of the vertical one, \( R = \phi wL/2 \), hence \( M = -\phi wL^2/2 \) from Definition 3. Now the internal force in the tie is \( T(\chi) = N_h(\chi) - R \) and its stress volume is given by

\[
V_h = \int_0^{1/2} |T(\chi)| d\chi = \int_0^{\chi_{cr}} N_h L d\chi - \int_{\chi_{cr}}^{1/2} N_h L d\chi + \left( \frac{1}{2} \chi_{cr} \right) \phi wL^2
\]

where \( N_h \) is given by Eq. (17). Therefore, with \( \tau = \int_0^{1/2} (\phi u) d\chi - \lambda/8 \), the AOH half-plane geometry with horizontal reactions is the solution of

\[
V_{oh}^{\phi \text{opt}}(\phi) = 4wL^2 \min_{\tau_{\chi_{cr}}} \left[ \int_0^{1/2} \left( 1 - 4\chi^2 \right) \tau^2 d\chi - \int_{\chi_{cr}}^{1/2} \tau d\chi \right]
\]

\[+ \left( \frac{1}{2} \phi \left( \frac{1}{4} - \chi_{cr} \right) \right) \]

\[(21)\]

A first inquiry is about the optimal \( \phi \), i.e., to solve the problem

\[
\min_{\phi} V_{oh}^{\phi \text{opt}}(\phi)
\]

\[(22)\]

The optimal \( \phi \) is found to be 0.738 with \( v_{\text{opt}} = 0.801 \).

These values can be compared with those of the solution of Hemp (1974) for the fixed boundary version of the problem because in both cases the optimal horizontal reaction is searched. This Michell net corresponds to \( \phi = 0.739 \) and \( v_{\text{opt}} = 0.789 \), i.e., the solution of this paper will have 1.5% more geometrical volume for equal tension and compression allowable stresses according to Eq. (1).

The relationship between \( \phi \) and the friction coefficient \( \mu \) depends on the weight of the foundation \( P \). From the Coulomb theory, \( R = \mu(P + wL/2) \)

\[
(\phi(P, \mu) = \frac{R}{wL/2} = \mu \left( 1 + \frac{P}{wL/2} \right)
\]

\[(23)\]

Only with a soil with more strength than the structural material can \( P \approx 0 \) be expected. Hence, it will generally be \( \phi > \mu \). The soil properties and the foundation shape will require a minimum foundation weight \( P_{\text{min}} \) to bear the vertical reaction, \( wL/2 \), so it can count with a free-of-cost fraction \( \phi(P_{\text{min}}, \mu) \). For these cases in which this value is smaller than 0.738, the solution can be obtained from Eq. (21) (Table 4 and Fig. 5).

Furthermore, depending on the costs of foundations and structure materials, it could be convenient to invest in \( P \) to increase \( \phi \). Let \( c_c, c_s, c_\text{cr} \) be the weight-specific costs of foundation, tension, and compression materials, respectively. The unitary costs of the stress volume are \( k_+ = c_c \gamma_+, f_+ = c_s / \gamma_+ \), and \( k_- = c_c \gamma_- / f_- = c_- / \gamma_- \), where \( \gamma = f / \gamma \) is the structural scope, a characteristic length of the structural material. Then the total cost for optimal AOH geometries with given \( P \) is

\[
C(P) = 2c_cP + \frac{1}{2} [(k_+ + k_-) \cdot V_{oh}^{\phi \text{opt}}(\phi) + (k_+ - k_-) \cdot \mathcal{M}(\phi)]
\]

\[(24)\]

and the minimum condition is

---

**Table 3. Polynomial Solutions \( \tau_n \) for the Half-Plane Domain**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( v )</th>
<th>( h )</th>
<th>( \chi_{cr} )</th>
<th>( \alpha(\chi_{cr}) )</th>
<th>( \alpha(0.5) )</th>
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<td>1.08521</td>
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</table>

**Table 4. Polynomial Solutions \( \tau_n \) for the Half-Plane Domain with Horizontal Reactions with \( n = 4 \)**

<table>
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<th>( v )</th>
<th>( h )</th>
<th>( \chi_{cr} )</th>
<th>( \alpha(\chi_{cr}) )</th>
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<td>0.49063</td>
<td>-2.33128</td>
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</tbody>
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**Fig. 5.** AOH layouts for the half-plane domain with horizontal reactions: (a) \( \phi = 0.25 \); (b) \( \phi = 0.50 \); (c) \( \phi = 0.738 \)
subject to \( P \geq P_{\text{min}} \). From Eq. (23), \( d\phi / dP = 2\mu L \). As \( M = -\omega w_L^2/2 \), \( dM/d\phi = -wL^2/2 \). A simple but accurate approximation to Eq. (21) is

\[
V_{\text{opt}}^\text{AOH} (\phi) \approx (0.9853 - 0.4653\phi + 0.2908\phi^2)wL^2
\]

From Eqs. (25) and (26), the optimum \( \phi \) for \( C \) can be obtained as

\[
\phi_{\text{opt}}^C \approx -0.05966c_+ - 3.439c_+ - 6.66c_+ - 6.66c_+ \mu > 0.738
\]

and from Eq. (23) the optimum \( P \) is

\[
P_{\text{opt}}^P \approx \frac{1}{2} wL \left[ \frac{1.660 - \mu}{\mu} c_+ + \frac{1}{2} - \frac{0.05966 + \mu}{\mu} c_+ - \frac{1}{2} - 3.439c_+ \right]
\]

Conclusions

The stress volume of the traditional parabolic arch with vertical hanger solution (\( \tau_0 \) in Table 3) is approximately 17.3\% greater than that of the solution with the layout proposed here, and the latter is approximately 1\% greater that the best solution known to date, the solution obtained with simulated annealing in Table 1. Accounting with normal values of friction coefficient, the Michell number of the proposed layout can be reduced up to 0.85, approximately 86\% of the amount of the nonfrictional case, and only approximately 12.6\% greater than the absolute optimum estimate computed by McConnell for the full-plane case, which can be considered an absolute lower limit. The layouts proposed in this paper are simpler than the proposed Michell nets by McConnell or Hemp, or that of the better solutions obtained with simulated annealing. From a practical point of view, this difference on shape is more meaningful than the small stress volume difference. Therefore, it seems reasonable to speak of near-optimal layouts.

There are some real bridges whose layouts resemble the AOH ones, at least in the use of oblique hangers [e.g., the Apollo bridge in Bratislava (Gabler 2006)]. The named network arch layout has a close relationship with the layouts proposed here (Tveit 2007). From a theoretical point of view, Picugia et al. (2012, Fig. 2) report oblique hangers in the best solutions they obtain for the fixed boundary version of the problem. A simple but an accurate approximation to Eq. (21) is

\[
V_{\text{opt}}^\text{AOH} (\phi) \approx (0.9853 - 0.4653\phi + 0.2908\phi^2)wL^2
\]

\[
\frac{dC}{dP} = 2\phi + \frac{1}{2} \left[ (k_+ + k_-) - \frac{dV_{\text{opt}}^\text{AOH} (\phi)}{d\phi} \right] + (k_+ - k_-) \frac{dM (\phi)}{d\phi} \frac{d\phi}{dP} = 0
\]