Universidad Politécnica de Madrid
Escuela Técnica Superior de Ingeniería Aeronáutica
y del Espacio

Discrete linear stability and sensitivity
analysis of fluid flows

Doctoral Thesis for
Ph.D. in Aerospace Engineering

Oliver M. F. Browne

Madrid, May 2016

Universidad Politécnica de Madrid
**Escuela Técnica Superior de Ingeniería Aeronáutica
y del Espacio**
**Departamento de Matemática Aplicada a la
Ingeniería Aeroespacial**

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Oliver M. F. Browne

Supervisors

Prof. Eusebio Valero and Dr. Esteban Ferrer

Madrid, Version: May 18, 2016

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Abstract

A discrete framework for computing the global stability and sensitivity analysis to external perturbations for any set of partial differential equations is presented. In particular, a complex-step approximation is used to achieve near analytical accuracy for the evaluation of the Jacobian matrix. Sensitivity maps for the sensitivity to base flow modifications and to a steady force are computed to identify regions of the flow field where an input could have a stabilising effect.

Four test cases are presented: (1) an analytical test case to prove the theory of the discrete framework, (2) a lid-driven cavity at low Reynolds case to show the improved accuracy in the calculation of the eigenvalues when using the complex-step approximation, (3) the 2D flow past a circular cylinder at just below the critical Reynolds number is used to validate the methodology, and finally, (4) the flow past an open cavity is presented to give an example of the discrete method applied to a convectively unstable case. The latter three (2–4) of the aforementioned cases were solved with the 2D compressible Navier–Stokes equations using a Discontinuous Galerkin Spectral Element Method. Good agreement was obtained for the validation test case, (3), with appropriate results in the literature. Furthermore, it is shown that for the calculation of the direct and adjoint eigenmodes and their sensitivity maps to external perturbations, the use of complex variables is paramount for obtaining an accurate prediction.

An analysis for stabilising the wake past an actuator disc, which represents a simple model for propellers, helicopter rotors or wind turbines is also presented. We explore the first flow bifurcation for an actuator disc and it suggests that it is associated to a Kelvin–Helmholtz type instability whose stability relies on the Reynolds number and the flow resistance applied through the disc (or actuator forcing). First, we report that decreasing the disc resistance has a similar stabilising effect to an decrease in the Reynolds number. Second, a discrete sensitivity analysis identifies two regions for suitable placement of flow control forcing, one close to the disc and one far downstream where the instability originates. Third, we show that adding a localised forcing close to the actuator provides more stabilisation than forcing far downstream. The analysis of the controlled flow fields, confirms that modifying the velocity gradient close to the actuator is more efficient to stabilise the wake than controlling the sheared flow far downstream. An interesting ap-

plication of these results is to provide guidelines for stabilising the wake of wind or tidal turbines when placed in an energy farm to minimise unsteady interactions.

Resumen

Este trabajo presenta un método discreto para el cálculo de estabilidad hidrodinámica y análisis de sensibilidad a perturbaciones externas para ecuaciones diferenciales y en particular para las ecuaciones de Navier-Stokes compresible. Se utiliza una aproximación con variable compleja para obtener una precisión analítica en la evaluación de la matriz Jacobiana. Además, mapas de sensibilidad para la sensibilidad a las modificaciones del flujo de base y a una fuerza constante permiten identificar las regiones del campo fluido donde una modificación (ej. fuerza puntual) tiene un efecto estabilizador del flujo. Se presentan cuatro casos de prueba: (1) un caso analítico para comprobar la derivación discreta, (2) una cavidad cerrada a bajo Reynolds para mostrar la mayor precisión en el cálculo de los valores propios con la aproximación de paso complejo, (3) flujo 2D en un cilindro circular para validar la metodología, y (4) flujo en un cavidad abierta, presentado para validar el método en casos de inestabilidades convectivamente inestables. Los tres últimos casos mencionados (2-4) se resolvieron con las ecuaciones de Navier-Stokes compresibles, utilizando un método Discontinuous Galerkin Spectral Element Method. Se obtuvo una buena concordancia para el caso de validación (3), cuando se comparó el nuevo método con resultados de la literatura. Además, este trabajo muestra que para el cálculo de los modos propios directos y adjuntos, así como para los mapas de sensibilidad, el uso de variables complejas es de suprema importancia para obtener una predicción precisa.

El método descrito es aplicado al análisis para la estabilización de la estela generada por un disco actuador, que representa un modelo sencillo para hélices, rotores de helicópteros o turbinas eólicas. Se explora la primera bifurcación del flujo para un disco actuador, y se sugiere que está asociada a una inestabilidad de tipo Kelvin-Helmholtz, cuya estabilidad se controla con en el número de Reynolds y en la resistencia del disco actuador (o fuerza resistente). En primer lugar, se verifica que la disminución de la resistencia del disco tiene un efecto estabilizador parecido a una disminución del Reynolds. En segundo lugar, el análisis hidrodinámico discreto identifica dos regiones para la colocación de una fuerza puntual que controle las inestabilidades, una cerca del disco y otra en una zona aguas abajo. En tercer lugar, se muestra que la inclusión de un forzamiento localizado cerca del actuador produce una estabilización más eficiente que al forzar aguas abajo. El análisis de los campos de flujo controlados confirma que modificando el gradiente de velocidad

cerca del actuador es más eficiente para estabilizar la estela. Estos resultados podrían proporcionar nuevas directrices para la estabilización de la estela de turbinas de viento o de marea cuando estén instaladas en un parque eólico y minimizar las interacciones no estacionarias entre turbinas.

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Thank you.

Oliver M. F. Browne, May 2016

Dedication

To my parents.

‘But why, some say, the Moon?... why climb the highest mountain? Why, 35 years ago, fly the Atlantic? Why does Rice play Texas? ... We choose to go to the Moon in this decade and do the other things... *not because they are easy, but because they are hard.*’

President John F. Kennedy, Rice University, 1962

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Chapter 1

Introduction

“R is a pure number; we will call it the Reynolds number.”

Arnold Sommerfeld, Rome, 1908 [81]

1.1 Preface

Unsteady fluid flows present themselves in innumerable areas within science and engineering. They can either be undesirable: separation of the boundary layer or flutter on an aircraft wing, turbulent flow entering a jet engine causing compressor stall or surging, strong winds passing over bridges and between buildings, a tornado or cyclone etc or desirable: vortex generators to prevent boundary layer separation, snow fences which induce turbulence in the wind as a means to force snow to be dropped on the fence, energy generation via waves and tides. However, the examples of desirable unsteadiness are unsteady by the definition $\frac{d\varphi}{dt} \neq 0$ (where φ is some parameter of the flow such as velocity, pressure, density, etc), but generally it would be desirable to have periodically steady flow rather than complete unsteadiness.

To enhance or suppress this unsteadiness we can apply inputs or perturbations to the flow. To identify the optimal locations or “sweet spots” of the flow we could conduct

experiments in a wind tunnel or computational runs and progressively try different inputs/perturbations in distinct locations. This could be a long, cumbersome and potentially expensive exercise. An alternative is to exploit linear stability and sensitivity analysis to obtain sensitivity maps which reveal the aforementioned “sweet spots”.

1.2 Linear Stability Analysis

To begin talking about flow stability it is both convenient and appropriate to start by discussing the work of British scientist Osborne Reynolds. Reynolds, one of fluid dynamics fathers and pioneers, studied the transition of fluids from a laminar regime to a turbulent regime and, along with Sir George Stokes, presented what is now considered one of the most important non-dimensional numbers in fluid dynamics, namely: Reynolds number [78]. Given by the ratio of inertial and viscous forces as

$$\text{Re} = \frac{U_\infty L}{\nu} \quad (1.1)$$

where U_∞ , L , and ν are the freestream velocity, characteristic length and kinematic viscosity respectively. Reynolds numbers not only allows the comparison of flow patterns in different fluids (ν) or configurations (U_∞, L) but also quantify the bounds of different flow regimes. For instance, in similar experiments to that of Reynolds [78], for flow in a pipe with a diameter of L : the flow is laminar in nature for $\text{Re} < 2300$ while being turbulent for $\text{Re} > 4000$. Furthermore, Reynolds number is also important for defining the boundary between stable and unstable flows. The classic example, and one which is explored later in this thesis, is flow past a circular cylinder [103]: for $\text{Re} \lesssim 46$ the flow is stable while $\text{Re} \gtrsim 48$ the flow is unstable. We define the boundary between stability/instability as the critical Reynolds number Re_{crit} . This thesis will focus on working on problems which are in the vicinity of the Re_{crit} .

This leads on nicely to how we proceed in defining stability. Within the confines of the Lyapunov stability theory [61], one can describe a solution q of a dynamical system stable if when perturbed by an infinitely small variation δ , the solution will only vary by an infinitely small quantity ε at some future time.¹ This can be written more elegantly as

$$||q(0) - q^\delta(0)|| < \delta \longrightarrow ||q(t) - q^\delta(t)|| < \varepsilon \quad (1.2)$$

Conversely, if the difference is large then the dynamical system will be characterised as unstable. Additionally, we can define when a system is asymptotically stable. If when the solution q is perturbed by an infinitely small variation δ the flow returns to the original solution then the dynamical system is asymptotically stable. We can write this as

$$||q(t) - q^\delta(t)|| = 0 \text{ when } t \rightarrow \infty \quad (1.3)$$

On top on these definitions of stability, we can also differentiate on the types of instability [44]. The two types of instability which are significant for this work are the following

- *absolute instability* - instability grows in time (Figure 1.1 a))
- *convective instability* - instability grows in space (Figure 1.1 b))

The analysis that is introduced here is a temporal analysis and leads to determining absolute instabilities. However, the spatially evolving structures can also be determined indirectly (Alizard and Robinet [2], Rodriguez [80]). We now briefly introduce some mathematical concepts of linear stability analysis (modal analysis) but these are further explored in more detail and a more probing treatment in Chapter 2. One begins linear

¹Analysis is valid for scalars and vectors.

Figure 1.1: Diagram showing an a) *absolute instability* and b) *convective instability* (figure extracted from Blackburn *et al.* [19])

stability analysis by first decomposing the solution of a dynamical system q as the sum of a steady base state (for fluids this is the base flow) $\bar{q}$ and the evolution of infinitesimal unsteady perturbations q' around the base state

$$q(x, t) = \bar{q}(x) + \epsilon q'(x, t). \quad (1.4)$$

It is possible to describe the perturbation in the frequency domain by introducing the Ansatz $q' = \hat{q}e^{\sigma t}$. By either the dynamical system (Chapter 2) being linear in nature or by some linearisation method, a linear operator is found $\mathcal{J}$. An eigenvalue problem (EVP) is subsequently defined as

$$\mathcal{J}\hat{q} = \sigma\hat{q}, \quad (1.5)$$

Note that we define a standard eigenvalue problem but could just as easily of written it as a generalised eigenvalue problem. From the EVP we extract complex eigenmodes $\hat{q}$ and a spectrum of eigenvalues $\sigma = \sigma_r + i\sigma_i$. The real part of the eigenvalue σ_r gives the growth

rate of the system while the imaginary part σ_r gives the frequency. If an eigenvalue has a positive real part $\sigma_r > 0$, then the solution q will grow in time and thus can be described as being unstable or more precisely, an *absolute instability*.

Linear stability analysis is now looked at from the application to fluid dynamics. The dynamics of complex aerodynamic flows such as shear layers, cavities, wakes or detached flows; present unstable behaviours when some physical dependent parameter (e.g. Reynolds or Mach number) surpasses a critical threshold. In these cases, unstable modes appear and their temporal evolution drives the system dynamics to a new equilibrium configuration, sometimes called the saturated regime. Modes can be considered as a flow structure resulting from an eigenvalue analysis. Linear stability analysis studies the appearance and evolution of these modes which are related to parameters involved in the problem [100, 89, 86]. Examples of stability analysis for different flows can be found in the literature: cavity flows [30, 20, 96, 95], boundary layers [8], circular cylinder [9, 10, 12], backward facing step [19].

The prediction of the onset of unsteadiness or the characterisation of unstable structures in complex 3D geometries or at high-Reynolds numbers is still an open problem. This study is limited to modal analysis. Non-modal analysis, a tool for the efficient description of small time-scale disturbance behaviour (Schmid [87]), although undoubtedly important is not within the scope of this work.

1.3 Sensitivity Analysis

The development and rise of sensitivity analysis has naturally marched on from linear stability analysis. Sensitivity analysis in its most basic definition deals with the uncertainty of the input/output relationship of a mathematical system but is employed here as a tool to determine how the system can be modified in order to achieve greater stability. Recent studies for this application are, [40, 60], oriented towards the investigation of the sensitivity of the unstable eigenmodes (i.e. growth rates and frequencies) to the introduction

of perturbations in the base flow or by external forces.

The pioneering work of Strykowski and Sreenivasan [93] showed, experimentally, that introducing a small object in the wake of a cylinder may delay the onset of unsteadiness, well above the critical Reynolds, for the first bifurcation of Hopf type (at $Re_{crit} \approx 47$) [10, 12]. In a more general framework, the introduction of a small perturbation in the flow: sucking or blowing, changes to the equilibrium condition, or introducing a small control object that acts as a steady force, can have a stabilising or destabilising effect [29]. Sensitivity analysis identifies the regions of the flow where introducing such a modification has the greatest effect on the stability of the flow. This could be of importance for suppressing laminar-turbulent transition or to enhance mixing (e.g. vortex generator on a wing to keep boundary layer attached) to prevent the flow from detaching, or even to provide means for acoustic control [45, 3].

The numerical study of flow sensitivity relies on the use of adjoint eigenmodes (section 2.3, [26, 59]). These modes, which are eigen-solutions of an adjoint system of equations provide receptivity information to external forcing [59, 87]. Sensitivity maps are provided by combining direct and adjoint modes. The importance of adjoint modes and sensitivity maps, together with the mathematical machinery required for the study of the sensitive flow regions to different parameters, has been studied by different authors [87, 26, 59, 44, 24]. In addition, these studies provide detailed reviews of the “state of the art” techniques in linear stability analysis and sensitivity analysis, addressing the relevance that the adjoint plays in the analysis of these problems, and also giving a comprehensive overview of the above-mentioned techniques and outlining the possibilities of linking sensitivity information with open/closed loop control. The mathematical formulation and the numerical computation of sensitivities has been detailed by Giannetti and Luchini [40], who showed that the structural sensitivity or “wavemaker” region may be obtained by combining (using a particular norm) the direct and adjoint eigenmodes. This can be considered the origin of the instability.

1.3.1 Discrete vs Continuous

Figure 1.2: Discrete vs continuous approach.

A distinction between a continuous approach and a discrete approach is made. In the continuous approach, the adjoint equations are derived from the direct equations (i.e. continuous in space), which are then linearised. Subsequently, the continuous direct and adjoint operators may be formed, and finally, these continuous operators are discretised. The discrete approach requires: first, the numerical discretisation of the direct equations which are subsequently linearised to form the direct discrete operator and the adjoint operator can then be obtained. A simple flow chart of the discrete vs continuous approach is provided in Figure 1.2.

Marquet *et al.* [64] formulated a method using a continuous approach and Lagrange multipliers to obtain equations for the sensitivity fields for the incompressible Navier-Stokes (NS) equations. In this work they derive equations for the sensitivity to base flow modifications and sensitivity to a steady force with a method of Lagrangian multipliers. They applied to this flow past a circular cylinder and found good agreement with the experimental results of Strykowski and Sreenivasan [93]. Furthermore, Meliga, Sipp and Chomaz [67] computed, also using the continuous framework, the sensitivity to base flow modifications for axisymmetric flows in the compressible regime (subsonic). Meliga *et al.* [66] also employed sensitivity analysis to identify the regions sensitive to a steady force with the aim of using a small control cylinder to alter the shedding frequency aft a

D-shaped cylinder geometry.

In more recent years the discrete approach has become more popular within the stability/sensitivity community. Chandler *et al.* [24] compared the discrete and the continuous approach for the low Mach number equations for a low density jet and showed that both methods provide very similar results. On the one hand, the continuous approach, for this particular case, showed a faster rate of convergence while requiring a less resolved mesh. On the other hand, the discrete approach was found to be more straightforward to implement. The boundary conditions for the adjoint problem are automatically applied in the discrete framework as stated by [40] whereas they need to be defined explicitly in the continuous case. Mettot *et al.* [69], independently to the author of this thesis, developed a method using a discrete framework to compute the sensitivity to base flow modifications and to steady forcing for high Reynolds number flows with turbulence models. They further applied this methodology for finding the regions to place a small control cylinder for stabilising the flow past a D-shaped cylinder in [70].

1.3.2 Computation of the Jacobian

The Jacobian (the definition of this is addressed in Chapter 2) matrix can be formed in three ways: 1) analytically, 2) automatic differentiation and 3) numerical differentiation. This thesis will focus on 3) but a short summary of the formers methodology is presented.

Firstly, analytically forming the Jacobian means symbolically computing the derivatives either by hand or using software with this capability (Maple [71], Mathematica [77]). Although being an effective technique, by hand is prone to errors and one can run into resource limitations (relatively slow) when approaching larger problems when using software [7].

An alternative to symbolically computing the derivatives is automatic differentiation. Automatic differentiation successively applies the chain rule compute the numerical value of a derivative without ever explicitly having the symbolic derivative [7, 32]. While also

being a powerful tool, for the applications considered here a continuous approach must be employed if the system is non-linear to enable its use.

Lastly, we introduce numerical differentiation and why it is selected for computing the matrices in this thesis. Numerical differentiation [48, 5] is robust in that once it is implemented, modifying the governing equations requires little, if any, adjustment to the users source code.² Furthermore, numerical differentiation is based on perturbing the non-linear fluxes (see Chapter 2) and as such, the users solver can be treated as a black box which has the fluxes as an input and the Jacobian as an output. However, the drawback is the selection of the perturbation parameter must be carefully chosen as to not being too small as to introduce round-off error but not big enough for truncation error to become significant. This is overcome with the implementation of the complex-step derivative approximation [63, 62, 55, 65]. The use of this method enables wider range of acceptable values for the perturbation parameter and after a finite value machine precision accuracy is achieved (see Chapter 3).

1.4 Layout for Thesis

In this thesis, a methodology to perform global stability and sensitivity analysis using a fully discrete approach is presented. To the authors knowledge, this is the first such work to compute the Jacobian using complex variables for a fluid dynamics problem. Furthermore, this is the first sensitivity analysis study to use a high order spectral element method for the numerics. The discrete approach can be implemented into any kind of CFD solver, e.g. laminar or turbulent, compressible or incompressible. It allows for computing the direct and adjoint modes of the flow, either stable or not, and the sensitivity formulation to these modes. Henceforth, this thesis extends the discrete approaches of [24, 69, 70], while also closely following the mathematical formulation of the continuous approach of [64] to retrieve the sensitivity fields for a steady force and to base flow modifications. We hereby extend the methodology of [69, 70] by using a first order

²Source code for computing the Jacobian.

complex-step approximation for the Jacobian and first order for the computation of the sensitivity matrix. It will be shown that the method of the complex-step approximation is advantageous in that it will obtain a higher degree of accuracy (near analytical) with a lower cost of computational time.

1.5 Organisation of the Thesis

The thesis is organised into the following chapters:

- Chapter 2: The mathematical formulation of the discrete approach for linear stability and sensitivity analysis is presented. The derivation of the equations for computing the sensitivity to base flow modifications and the sensitivity to a steady force are also defined. Also detailed is the way in which the discrete adjoint matrix is computed and the method of Lagrangian multipliers employed for the sensitivity analysis.
- Chapter 3: The numerical evaluation of the matrices is shown in this chapter. In particular the complex-step derivative approximation is presented for computing the Jacobian matrix. It is shown that this method is superior to the classical standard finite difference method. Also shown is the method for computing the sensitivity matrix. An analytical case and the lid driven cavity case are given to show the validity of the method and implementation.
- Chapter 4: This chapter presents the 2D compressible Navier-Stokes equations and the discretisation methods. The discontinuous Galerkin spectral element method DGSEM is detailed for the spatial discretisation while the 3rd order Runge-Kutta method and Newton iteration are shown for the temporal discretisation. Also presented is the shift and invert Arnoldi method for computing the eigenvectors and eigenvalues.

- Chapter 5: The methodology of previous chapters is validated with two cases: 1) flow past a circular cylinder at $Re = 45$ and 2) flow past an open cavity at $Re = 1400$. The results are compared with other results in the linear stability and sensitivity analysis literature.
- Chapter 6: We apply the methodology to the case of an actuator, with this being used to model a wind turbine. Regions in the flow field are identified where perturbations can be applied to stabilise the flow. This is shown by recomputing the eigenvalues.
- Chapter 7: The conclusions of the thesis and avenues for future work are presented.

1.6 Publications and Conferences

1.6.1 Journal Papers

O. M. F. Browne, E. Ferrer, and E. Valero. Sensitivity analysis to control the wake unsteadiness of an actuator disc. *Under consideration*.

O. M. F. Browne, G. Rubio, E. Ferrer, and E. Valero. Sensitivity analysis to unsteady perturbations of complex flows: a discrete approach. *International Journal for Numerical Methods in Fluids*, 76(12):1088–1110, 2014.

M. Chavez, **O. M. F. Browne**, and E. Valero. Numerical and parametric study of MVGs on a UAV geometry in subsonic flow. *Under consideration*.

1.6.2 Conferences

O. M. F. Browne, G. Rubio, E. Ferrer, and E. Valero. Optimisation of Complex Flows with Discrete Sensitivity Analysis. *13th US National Congress on Computational Mechanics*, San Diego, July 2015.

O. M. F. Browne, G. Rubio, E. Ferrer, and E. Valero. Discrete sensitivity analysis of a NACA0015 aerofoil. *ECCOMAS 2014: 11th World Congress on Computational Mechanics (WCCM XI)*, Barcelona, July 2014.

O. M. F. Browne, G. Rubio, E. Ferrer, and E. Valero. Discrete sensitivity analysis for cylinder flows. *International Workshop for Numerical Prediction of Detached Flows*, Madrid, October 2013.

1.7 ANADE: Advances in Numerical and Analytical tools for DEtached flow

The research leading to these results has received funding from the European Union Seventh Framework Programme: [FP7/2007-2013] under grant agreement number PITN-GA-2011-289428. The work was conducted within the project ANADE. ANADE is an Initial Training Network (ITN) project funded under the European Commission' Seventh Framework Programme (FP7) within People work programme (Marie Curie actions).

Chapter 2

Mathematical Formulation

2.1 Introduction

This chapter will lay out the mathematical theory and framework by which the linear stability and sensitivity analysis will be conducted in this thesis. The formulation will purposely be kept in a purely general setting in an effort to demonstrate that it can be applied to any set of partial differential equations which can be written in the form of a time derivative and spatial operator. The formulation will start with the time dependent governing equations and then proceed in a discrete manner until the sensitivity fields are retrieved. This framework work presented here is also presented, by the author, in Browne *et al.* [22, 21].

2.2 Discrete Linear Stability Analysis

In what follows, we consider a discretisation of a system of partial differential equations (PDE) over a computational mesh. Geometry, boundary conditions and external forcing can also be considered as part of the mathematical problem. The method is independent of the particular numerical scheme used to discretise the spatial and temporal derivatives of the equations. The governing equations are posed as a time varying system

$$\mathcal{M} \frac{d\mathbf{q}}{dt} = \tilde{\mathcal{F}}(\mathbf{q}), \quad (2.1)$$

where $\mathbf{q}$ is the state vector (of size n) holding the conservative variables and $\tilde{\mathcal{F}}(\mathbf{q})$ is a spatial operator which defines the numerical fluxes obtained by the discretisation of the continuous PDE on the computational mesh, and $\mathcal{M}$ is the mass matrix associated to a spatial discretisation selected for the discrete approach.³ The size of $\tilde{\mathcal{F}}(\mathbf{q})$ will be dependent on the number of equations and the number of computational mesh points. For simplicity the spatial operator is redefined as $\mathcal{F}(\mathbf{q}) = \mathcal{M}^{-1} \tilde{\mathcal{F}}(\mathbf{q})$.

Linear stability analysis is charged with determining whether a small disturbance on some base state will lead to the system becoming unstable. Hence, the state vector is decomposed into a steady base state (with the premise that such a state exists) and an unsteady infinitesimal perturbation (disturbance) such that

$$\mathbf{q}(\mathbf{x}, t) = \bar{\mathbf{q}}(\mathbf{x}) + \epsilon \mathbf{q}'(\mathbf{x}, t), \quad (2.2)$$

where $\epsilon \ll 1$. Inserting this into Equation (2.1), the following is obtained

$$\frac{d\bar{\mathbf{q}}}{dt} + \frac{d\mathbf{q}'}{dt} = \mathcal{F}(\bar{\mathbf{q}} + \mathbf{q}'). \quad (2.3)$$

The base state is steady so the first time derivative can be dropped. In general, the spatial operator $\mathcal{F}$ will be non-linear and to linearise it, $\mathcal{F}$ is expanded with a Taylor series expansion around the base state

³Mass matrix computation is considered in Chapter 4.

$$\mathcal{F}(\bar{\mathbf{q}} + \mathbf{q}') = \mathcal{F}(\bar{\mathbf{q}}) + \left. \frac{\partial \mathcal{F}(\mathbf{q})}{\partial \mathbf{q}} \right|_{\bar{\mathbf{q}}} \mathbf{q}' + \mathcal{O}(\mathbf{q}'^2). \quad (2.4)$$

Many branches of science and mathematics require the steady solution to a time varying system and $\bar{\mathbf{q}}$ is found by solving for

$$\mathcal{F}(\bar{\mathbf{q}}) = \mathbf{0}. \quad (2.5)$$

By again considering Equation (2.4), neglecting the higher order terms, writing $\mathcal{J}(\bar{\mathbf{q}}) = \left. \frac{\partial \mathcal{F}(\mathbf{q})}{\partial \mathbf{q}} \right|_{\bar{\mathbf{q}}}$ and combining this expansion with Equation (2.3), we obtain

$$\frac{d\mathbf{q}'}{dt} = \mathcal{J}(\bar{\mathbf{q}})\mathbf{q}', \quad (2.6)$$

where $\mathcal{J}(\bar{\mathbf{q}})$ is the Jacobian matrix whose spectrum contains the complex eigenvalues of the system $\sigma = \sigma_r + i\sigma_i$. Equation (2.6) may be advanced in time to simulate the growth or decay of perturbation upon $\bar{\mathbf{q}}$ [11, 34]. The perturbations are solved in the frequency domain by characterising them as normal modes, $\mathbf{q}' = \hat{\mathbf{q}}e^{\sigma t}$, where $\hat{\mathbf{q}}$ is an eigenmode of the system. This results in a standard eigenvalue problem

$$\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}} = \sigma\hat{\mathbf{q}}; \text{ where } \frac{d\mathbf{q}'}{dt} = \sigma\hat{\mathbf{q}}e^{\sigma t} \quad (2.7)$$

The real part of the eigenvalue σ_r gives the growth rate of the system while the imaginary part σ_i gives the frequency. For the convention defined here, if one such eigenvalue has a real part with a positive value ($\sigma_r > 0$) then the system is said to be absolutely unstable. Numerical strategies for obtaining the Jacobian matrix and extracting the

eigenvalues/eigenmodes are given in Chapter 4.

2.3 Discrete Adjoint

In the previous subsection the direct eigenvalue problem was formed. From this we obtain information about how the system behaves subject to infinitesimal perturbations applied to the steady-state. Adjoint analysis allows one to also obtain information about the receptive and sensitive regions (or degrees of freedom) of the system.

We proceed by choosing an L_2 volume inner product⁴ that is defined for a pair of continuous vectors, $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$, is defined as

$$\langle \tilde{\mathbf{c}}, \tilde{\mathbf{d}} \rangle_{\Omega} = \int_{\Omega} \tilde{\mathbf{c}}^* \tilde{\mathbf{d}} \, d\Omega = \mathbf{c}^* \mathcal{M} \mathbf{d}, \quad (2.8)$$

where $\bullet^*$ denotes the conjugate transpose, $\mathbf{c}$ and $\mathbf{d}$ are the discretised version of $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$, in the computational domain, Ω , and as stated at the beginning of this chapter, $\mathcal{M}$ is the mass matrix.

By definition, matrix, $\mathcal{A}$, and its discrete adjoint, $\mathcal{A}^+$, can be related in the following expression

$$\langle \mathcal{A} \tilde{\mathbf{c}}, \tilde{\mathbf{d}} \rangle_{\Omega} = \langle \tilde{\mathbf{c}}, \mathcal{A}^+ \tilde{\mathbf{d}} \rangle_{\Omega}. \quad (2.9)$$

Considering the definition of the inner product defined in Eq. (2.8), we obtain

⁴historically L_2 has been used but other norms have also been studied (Schmid [1])

$$\begin{aligned}
(\mathcal{A}\mathbf{c})^* \mathcal{M}\mathbf{d} &= \mathbf{c}^* \mathcal{M}\mathcal{A}^+ \mathbf{d}, \\
\mathbf{c}^* \mathcal{A}^* \mathcal{M}\mathbf{d} &= \mathbf{c}^* \mathcal{M}\mathcal{A}^+ \mathbf{d}, \\
\mathcal{A}^* \mathcal{M} &= \mathcal{M}\mathcal{A}^+, \\
\mathcal{M}^{-1} \mathcal{A}^* \mathcal{M} &= \mathcal{A}^+,
\end{aligned} \tag{2.10}$$

an expression which relates the conjugate transpose of the Jacobian to its adjoint.

Therefore the adjoint eigenvalue problem can be derived and is given by

$$\mathcal{J}^+(\bar{\mathbf{q}})\hat{\mathbf{q}}^+ = \sigma^* \hat{\mathbf{q}}^+. \tag{2.11}$$

Where according to the definition of the adjoint Eq. (2.10), $\mathcal{J}^+(\bar{\mathbf{q}})$ is the adjoint of the Jacobian and $\hat{\mathbf{q}}^+$ is the adjoint eigenmode.

2.4 Discrete Sensitivity Analysis

The interest for sensitivity analysis, is how an eigenvalue will change, if at all, with some arbitrary modification to the system. Rather than an arbitrary modification, a framework is presented whereby sensitivity fields are retrieved that indicate the degrees of freedom of the system that should be modified in order to shift the eigenvalue in a particular way. This could be to stabilise or conversely destabilise the system by shifting σ_r or to change the frequency of the eigenvalue by modifying σ_i . For the application of this framework in later chapters of this thesis, we will focus on stabilising the system by analysing the most unstable eigenvalue of the system. The aim will be to defer the transition to instability of the system, with Figure 2.1 illustrating an example of this aim. As the system passes the critical value of a parameter (Ma, Re, angle of attack etc.) the eigenvalues cross

the imaginary axis. As seen in Figure 2.1, by introducing a new critical point for the parameter, by way of some modification to the system, the range of obtaining a stable system increases.

Figure 2.1: Figure showing an example eigenvalue spectrum: a) with a pair of complex unstable eigenvalues, b) one example of the desired effect that the sensitivity analysis will have on the unstable eigenvalues. (unstable ●, stable ●)

In this work, we consider the sensitivity of the eigenvalue subject to a modification of the base state, $\delta\bar{\mathbf{q}}$, or steady forcing, $\delta\mathbf{q}_f$. If $\Delta\sigma$ is defined as the variation of the eigenvalue subjected to external perturbations. The following relation holds

$$\Delta_{\bar{\mathbf{q}}}\sigma = \langle \nabla_{\bar{\mathbf{q}}}\sigma, \delta\bar{\mathbf{q}} \rangle, \quad (2.12)$$

$$\Delta_{\mathbf{q}_f}\sigma = \langle \nabla_{\mathbf{q}_f}\sigma, \delta\mathbf{q}_f \rangle. \quad (2.13)$$

The resulting sensitivity fields $\nabla_{\bar{\mathbf{q}}}\sigma$, $\nabla_{\mathbf{q}_f}\sigma$, can be obtained following the method of Lagrangian multipliers, [87], which is aimed at maximising the change in the variable, σ , subject to some constraints. This approach is described by Marquet *et al.* [64] to derive the equations for the sensitivity fields in a continuous approach. Mettrot *et al.* [69, 70] present a discrete method similar to the one presented here, developed independently of this work Browne *et al.* [22, 21]. Mettrot *et al.* derive the same equations as this work but

with an alternate approach; namely, they apply perturbations to their governing equations as opposed the Lagrangian approach used herein.

2.4.1 Lagrangian Multipliers

The method of Lagrangian multipliers is employed as an optimisation strategy to find extrema of a cost functional α subject to one or multiple constraints. With a constraint of $\beta(\mathbf{x}) = 0$ the Lagrangian can be defined in the following manner

$$\mathcal{L}(\mathbf{x}, \gamma^+) = \alpha + \langle \gamma^+, \beta(\mathbf{x}) \rangle \quad (2.14)$$

such that γ^+ is a Lagrangian multiplier. Differentiating with respect to the control variable $\mathbf{x}$ we obtain

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} \delta \mathbf{x} = \langle \gamma^+, \frac{\partial \beta}{\partial \mathbf{x}} \delta \mathbf{x} \rangle \quad (2.15)$$

Using the definition of the inner product in Equation (2.8) and (2.9), and given that the equation must hold for an variation $\delta \mathbf{x}$ we arrive at

$$\nabla_{\mathbf{x}} \mathcal{L} = \left(\frac{\partial \beta}{\partial \mathbf{x}} \right)^+ \gamma^+ \quad (2.16)$$

At stationary points $\nabla_{\mathbf{x}} \mathcal{L}$ reduces to zero [1]. This equation now gives a relation of how to maximise α by modifying the control variable $\mathbf{x}$.

2.4.2 Sensitivity to a Steady Force

The discrete mathematical formulation of the sensitivity of a given eigenvalue (eigenmode) to the introduction of a steady force in the flow $\nabla_{\mathbf{q}_f}\sigma$, is derived herein.

From the steady solution of Eq. (2.1), we consider a forcing term $\mathbf{q}_f$ inserted to RHS

$$\mathcal{F}(\bar{\mathbf{q}}) = \mathbf{q}_f, \quad (2.17)$$

but one which still yields a steady solution. One would like to obtain information about the flow field, where a perturbation would give the greatest shift in the eigenvalues (see Figure 2.1). Thus, it makes sense to use the method of Lagrangian multipliers, where we maximise the change in the cost function (eigenvalue $\Delta\sigma$) subject to some physical constraints. The Lagrangian functional reads

$$\mathcal{L}(\bar{\mathbf{q}}, \sigma, \hat{\mathbf{q}}, \mathbf{q}_f, \bar{\mathbf{q}}^+, \hat{\mathbf{q}}^+) = \sigma + \langle \hat{\mathbf{q}}^+, \mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}} - \sigma\hat{\mathbf{q}} \rangle - \langle \bar{\mathbf{q}}^+, \mathcal{F}(\bar{\mathbf{q}}) - \mathbf{q}_f \rangle, \quad (2.18)$$

where the complex Lagrangian multipliers, $\hat{\mathbf{q}}^+$ and $\bar{\mathbf{q}}^+$, are the adjoint eigenmodes and adjoint base flow, respectively. The constraints are the direct eigenvalue problem, Eq. (2.7), and the steady state equations with a forcing term Equation (2.17), Eq. (2.17). The state variable is now $(\bar{\mathbf{q}}, \sigma, \hat{\mathbf{q}})$ while $\mathbf{q}_f$ is the control variable. The derivative of Equation (2.18) with respect to the adjoint variables, provides the direct eigenvalue problem and the steady state equations are recovered. Considering the definition of an inner product given by Equation (2.8) and the adjoint given by Equation (2.9), differentiation with respect to state variable gives

$$\frac{\partial \mathcal{L}}{\partial \hat{\mathbf{q}}} \delta \hat{\mathbf{q}} = \langle \hat{\mathbf{q}}^+, (\mathcal{J}(\bar{\mathbf{q}}) - \sigma \mathcal{I}) \delta \hat{\mathbf{q}} \rangle = 0, \quad (2.19)$$

$$= \langle (\mathcal{J}(\bar{\mathbf{q}}) - \sigma \mathcal{I})^+ \hat{\mathbf{q}}^+, \delta \hat{\mathbf{q}} \rangle = 0, \quad (2.20)$$

$$\mathcal{J}^+(\bar{\mathbf{q}}) \hat{\mathbf{q}}^+ = \sigma \hat{\mathbf{q}}^+, \quad (2.21)$$

$$\frac{\partial \mathcal{L}}{\partial \sigma} \delta \sigma = \delta \sigma + \langle \hat{\mathbf{q}}^+, -\delta \sigma \mathcal{I} \hat{\mathbf{q}} \rangle = 0, \quad (2.22)$$

$$= \delta \sigma - \delta \sigma \langle \hat{\mathbf{q}}^+, \mathcal{I} \hat{\mathbf{q}} \rangle = 0, \quad (2.23)$$

$$\langle \hat{\mathbf{q}}^+, \hat{\mathbf{q}} \rangle = 1, \quad (2.24)$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}} = \langle \hat{\mathbf{q}}^+, \frac{\partial (\mathcal{J}(\bar{\mathbf{q}}) \hat{\mathbf{q}})}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}} \rangle - \langle \bar{\mathbf{q}}^+, \mathcal{J}(\bar{\mathbf{q}}) \delta \bar{\mathbf{q}} \rangle = 0, \quad (2.25)$$

$$\langle \mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \hat{\mathbf{q}}^+, \delta \bar{\mathbf{q}} \rangle - \langle \mathcal{J}^+(\bar{\mathbf{q}}) \bar{\mathbf{q}}^+, \delta \bar{\mathbf{q}} \rangle = 0, \quad (2.26)$$

$$\mathcal{J}^+(\bar{\mathbf{q}}) \bar{\mathbf{q}}^+ = \mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \hat{\mathbf{q}}^+, \quad (2.27)$$

where $\mathcal{I}$ is the identity matrix and $\mathcal{B}$ is named the sensitivity matrix. Three equations are recovered from this method: the adjoint eigenvalue problem, Eq. (2.21), a normalisation condition, Eq. (2.24), and an equation relating the adjoint eigenmode with the adjoint base flow, Eq. (2.27). The normalisation condition is applied as follows

$$\hat{\mathbf{q}}_{normalised}^+ = \frac{\hat{\mathbf{q}}^+}{\langle \hat{\mathbf{q}}^+, \hat{\mathbf{q}} \rangle} = \frac{\hat{\mathbf{q}}^+}{\hat{\mathbf{q}}^{+*} \mathcal{M} \hat{\mathbf{q}}}. \quad (2.28)$$

A check can be done with

$$\left\langle \frac{\hat{\mathbf{q}}^+}{\langle \hat{\mathbf{q}}^+, \hat{\mathbf{q}} \rangle^*}, \hat{\mathbf{q}} \right\rangle = \frac{\hat{\mathbf{q}}^+}{(\hat{\mathbf{q}}^{+*} \mathcal{M} \hat{\mathbf{q}})^*} \mathcal{M} \hat{\mathbf{q}} = \frac{\hat{\mathbf{q}}^+}{\hat{\mathbf{q}}^* \mathcal{M}^* \hat{\mathbf{q}}^+} \mathcal{M} \hat{\mathbf{q}} = 1 + \imath 0, \quad (2.29)$$

to see if the normalisation has been applied correctly.

The gradient of the Lagrangian with respect to a steady force $\mathbf{q}_f$ is

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}_f} \delta \mathbf{q}_f = \langle \bar{\mathbf{q}}^+, \delta \mathbf{q}_f \rangle. \quad (2.30)$$

It was shown by [1] that the gradient of the objective functional $\nabla \sigma$ and the gradient of the Lagrangian $\nabla \mathcal{L}$ coincide and are equal. Therefore, it can be written

$$\frac{\partial \sigma}{\partial \mathbf{q}_f} \delta \mathbf{q}_f = \frac{\partial \mathcal{L}}{\partial \mathbf{q}_f} \delta \mathbf{q}_f. \quad (2.31)$$

providing the sensitivity to a steady force as

$$\nabla_{\mathbf{q}_f} \sigma = \bar{\mathbf{q}}^+. \quad (2.32)$$

2.4.3 Sensitivity to Base Flow Modifications

The same analysis is now applied to obtain the sensitivity to base flow modifications $\nabla_{\bar{\mathbf{q}}} \sigma$. Lagrangian functional for sensitivity of an eigenvalue σ subject to base flow modifications reads

$$\mathcal{L}(\bar{\mathbf{q}}, \sigma, \hat{\mathbf{q}}, \hat{\mathbf{q}}^+) = \sigma - \langle \hat{\mathbf{q}}^+, \mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}} - \sigma\hat{\mathbf{q}} \rangle. \quad (2.33)$$

Differentiating the Lagrangian functional with respect to $\bar{\mathbf{q}}$ we obtain;

$$\frac{\partial \mathcal{L}}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}} = -\langle \hat{\mathbf{q}}^+, \mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \delta \bar{\mathbf{q}} \rangle, \quad (2.34)$$

$$= -\langle \mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \hat{\mathbf{q}}^+, \delta \bar{\mathbf{q}} \rangle. \quad (2.35)$$

In the same fashion as Equation (2.31), it can be written

$$\frac{\partial \sigma}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}} = \frac{\partial \mathcal{L}}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}}. \quad (2.36)$$

Hence, the gradient of the sensitivity to base flow modifications is given as

$$\nabla_{\bar{\mathbf{q}}} \sigma = -\mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \hat{\mathbf{q}}^+. \quad (2.37)$$

2.4.4 Sensitivity to Geometric Modifications

A framework for computing the sensitivity to base flow modifications $\nabla_{\bar{\mathbf{q}}}\sigma$ and to a steady force $\nabla_{\mathbf{q}_f}\sigma$ was presented in previous subsections. This subsection will extend this framework and proceed in a similar manner to derive the sensitivity to geometry modifications $\nabla_{\mathbf{w}}\sigma$. $\mathbf{w}$ is a formal parameter of our surface geometry, for example: the coordinates or splines defining the surface geometry. The effect of perturbing the surface geometry by an infinitesimally small amount will be studied. This can be viewed as describing surface roughness or the junction where two separate components meet (e.g. different components on the nacelle of an engine or where the wing and the flaps/aileron meet). As for $\nabla_{\bar{\mathbf{q}}}\sigma$ and to a steady force $\nabla_{\mathbf{q}_f}\sigma$, the cost function will be the eigenvalue. This section just addresses the methodology without presenting results.

2.4.4.1 Mathematical Formulation

The problem is set up in the same way

$$\frac{d\mathbf{q}(\mathbf{w})}{dt} = \mathcal{F}(\mathbf{q}(\mathbf{w}), \mathbf{w}), \quad (2.38)$$

where $\mathbf{w}$ is a formal parameter of our surface geometry. $\bar{\mathbf{q}}$ and $\mathbf{w}$ have dimensions n and m respectively ($n \gg m$). Again we want to find a solution of $\mathbf{q}$ and $\mathbf{w}$ which satisfies

$$\mathcal{F}(\bar{\mathbf{q}}(\mathbf{w}), \mathbf{w}) = \mathbf{0}, \quad (2.39)$$

2.4.4.2 Inner Products and Adjoint

Firstly, inner products for the volume, Ω , and the surface, S , for a pair of continuous vectors, $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{d}}$, is defined as

$$\langle \tilde{\mathbf{c}}, \tilde{\mathbf{d}} \rangle_{\Omega} = \int_{\Omega} \tilde{\mathbf{c}}_{\Omega}^* \tilde{\mathbf{d}}_{\Omega} \, d\Omega = \mathbf{c}_{\Omega}^* \mathcal{M}_{\Omega} \mathbf{d}_{\Omega}, \quad (2.40)$$

$$\{\tilde{\mathbf{c}}, \tilde{\mathbf{d}}\}_S = \int_S \tilde{\mathbf{c}}_S^* \tilde{\mathbf{d}}_S \, dS = \mathbf{c}_S^* \mathcal{M}_S \mathbf{d}_S. \quad (2.41)$$

where $\mathcal{M}$ is the mass matrix associated to a spatial discretisation selected for the discrete approach.

A square matrix, $\mathcal{A}$, and its discrete adjoint, $\mathcal{A}^+$, can be related by the identity defined in Section 2.3, to give

$$\mathcal{M}_{\Omega}^{-1} \mathcal{A}^* \mathcal{M}_{\Omega} = \mathcal{A}^+, \quad (2.42)$$

A rectangular matrix, $\mathcal{A}$, and its discrete adjoint, $\mathcal{A}^+$, can be related in the following expression [41]

$$\langle \mathcal{A} \tilde{\mathbf{c}}_S, \tilde{\mathbf{d}}_{\Omega} \rangle_{\Omega} = \{\tilde{\mathbf{c}}_S, \mathcal{A}^+ \tilde{\mathbf{d}}_{\Omega}\}_S. \quad (2.43)$$

Considering the definition of the inner products defined in Eq. (2.40) and (2.41), we obtain

$$\begin{aligned} (\mathcal{A} \mathbf{c}_S)^* \mathcal{M}_{\Omega} \mathbf{d}_{\Omega} &= \mathbf{c}_S^* \mathcal{M}_S \mathcal{A}^+ \mathbf{d}_{\Omega}, \\ \mathbf{c}_S^* \mathcal{A}^* \mathcal{M}_{\Omega} \mathbf{d}_{\Omega} &= \mathbf{c}_S^* \mathcal{M}_S \mathcal{A}^+ \mathbf{d}_{\Omega}, \\ \mathcal{A}^* \mathcal{M}_{\Omega} &= \mathcal{M}_S \mathcal{A}^+, \\ \mathcal{M}_S^{-1} \mathcal{A}^* \mathcal{M}_{\Omega} &= \mathcal{A}^+, \end{aligned} \quad (2.44)$$

an expression which relates the conjugate transpose of the Jacobian to its adjoint. Eq. (2.40) and (2.41) applied to the LHS and RHS respectively in Eq. (2.43).

2.4.4.3 Lagrangian Method

For our optimisation problem we define a Lagrangian functional

$$L(\bar{\mathbf{q}}, \sigma, \hat{\mathbf{q}}, \mathbf{w}, \bar{\mathbf{q}}^+, \hat{\mathbf{q}}^+) = \sigma + \langle \hat{\mathbf{q}}^+, \mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}} - \sigma\hat{\mathbf{q}} \rangle_{\Omega} - \langle \bar{\mathbf{q}}^+, \mathcal{F}(\bar{\mathbf{q}}, \mathbf{w}) \rangle_{\Omega}. \quad (2.45)$$

The base flow $\bar{\mathbf{q}}$ and $\mathbf{w}$ can be considered independent variables for this analysis [1].

Differentiating with respect to the state variables, $(\bar{\mathbf{q}}, \sigma, \hat{\mathbf{q}})$:

$$\frac{\partial L}{\partial \hat{\mathbf{q}}} \delta \hat{\mathbf{q}} = \langle \hat{\mathbf{q}}^+, (\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w}) - \sigma \mathcal{I}) \delta \hat{\mathbf{q}} \rangle_{\Omega} = 0, \quad (2.46)$$

$$= \langle (\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w}) - \sigma \mathcal{I})^+ \hat{\mathbf{q}}^+, \delta \hat{\mathbf{q}} \rangle_{\Omega} = 0, \quad (2.47)$$

$$\mathcal{J}^+(\bar{\mathbf{q}}, \mathbf{w}) \hat{\mathbf{q}}^+ = \sigma \hat{\mathbf{q}}^+, \quad (2.48)$$

$$\frac{\partial L}{\partial \sigma} \delta \sigma = \delta \sigma + \langle \hat{\mathbf{q}}^+, -\delta \sigma \mathcal{I} \hat{\mathbf{q}} \rangle_{\Omega} = 0, \quad (2.49)$$

$$= \delta \sigma - \delta \sigma \langle \hat{\mathbf{q}}^+, \mathcal{I} \hat{\mathbf{q}} \rangle_{\Omega} = 0, \quad (2.50)$$

$$\langle \hat{\mathbf{q}}^+, \hat{\mathbf{q}} \rangle_{\Omega} = 1, \quad (2.51)$$

$$\frac{\partial L}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}} = \langle \hat{\mathbf{q}}^+, \frac{\partial(\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}})}{\partial \bar{\mathbf{q}}} \delta \bar{\mathbf{q}} \rangle_{\Omega} - \langle \bar{\mathbf{q}}^+, \mathcal{J}(\bar{\mathbf{q}}, \mathbf{w}) \delta \bar{\mathbf{q}} \rangle_{\Omega} = 0, \quad (2.52)$$

$$\langle \mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}}, \mathbf{w}) \hat{\mathbf{q}}^+, \delta \bar{\mathbf{q}} \rangle_{\Omega} - \langle \mathcal{J}^+(\bar{\mathbf{q}}, \mathbf{w}) \bar{\mathbf{q}}^+, \delta \bar{\mathbf{q}} \rangle_{\Omega} = 0, \quad (2.53)$$

$$\mathcal{J}^+(\bar{\mathbf{q}}, \mathbf{w}) \bar{\mathbf{q}}^+ = \mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}}, \mathbf{w}) \hat{\mathbf{q}}^+, \quad (2.54)$$

Differentiating w.r.t. $\mathbf{w}$ gives

$$\frac{\partial L}{\partial \mathbf{w}} \delta \mathbf{w} = \langle \hat{\mathbf{q}}^+, \frac{\partial(\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}})}{\partial \mathbf{w}} \delta \mathbf{w} \rangle_{\Omega} - \langle \bar{\mathbf{q}}^+, \frac{\partial(\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w}))}{\partial \mathbf{w}} \delta \mathbf{w} \rangle_{\Omega}, \quad (2.55)$$

$$= \{\mathcal{B}_{\mathbf{w}}^+(\bar{\mathbf{q}}, \mathbf{w}, \hat{\mathbf{q}}) \hat{\mathbf{q}}^+, \delta \mathbf{w}\}_S - \{\mathcal{J}_{\mathbf{w}}^+(\bar{\mathbf{q}}, \mathbf{w}) \bar{\mathbf{q}}^+, \delta \mathbf{w}\}_S, \quad (2.56)$$

$$\nabla_{\mathbf{w}} \sigma = \mathcal{B}_{\mathbf{w}}^+(\bar{\mathbf{q}}, \mathbf{w}, \hat{\mathbf{q}}) \hat{\mathbf{q}}^+ - \mathcal{J}_{\mathbf{w}}^+(\bar{\mathbf{q}}, \mathbf{w}) \bar{\mathbf{q}}^+. \quad (2.57)$$

2.4.4.4 Alternative Method

An alternative method that does not require the adjoint information is also presented. As with the Lagrangian approach (Equation (2.45)), we consider the effect of the governing equations $\mathcal{F}(\bar{\mathbf{q}}, \mathbf{w})$ and the eigenvalue problem $\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}} - \sigma\hat{\mathbf{q}}$. Using chain rule and implicit differentiation, differentiating with respect to $\mathbf{w}$ we get

$$\frac{d\mathcal{F}(\bar{\mathbf{q}}, \mathbf{w})}{d\mathbf{w}} = \frac{\partial \mathcal{F}(\bar{\mathbf{q}}, \mathbf{w})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{w}} + \frac{\partial \mathcal{F}(\bar{\mathbf{q}}, \mathbf{w})}{\partial \bar{\mathbf{q}}} \frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}}, \quad (2.58)$$

$$\frac{d(\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}} - \sigma\hat{\mathbf{q}})}{d\mathbf{w}} = \frac{\partial(\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}})}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{w}} + \frac{\partial(\mathcal{J}(\bar{\mathbf{q}}, \mathbf{w})\hat{\mathbf{q}})}{\partial \bar{\mathbf{q}}} \frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}}. \quad (2.59)$$

Both left hand side of Equation (2.58) and (2.59) are zero because of the constraints of the analysis. By rearranging

$$\mathcal{J}_{\bar{\mathbf{q}}}(\bar{\mathbf{q}}, \mathbf{w}) \frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}} = -\mathcal{J}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w}), \quad (2.60)$$

$$\mathcal{B}_{\bar{\mathbf{q}}}(\bar{\mathbf{q}}, \mathbf{w}) \frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}} = -\mathcal{B}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w}), \quad (2.61)$$

where the second term in both equations on the RHS is the sensitivity of the flow to modifications to the surface geometry. The systems are

$$\left(\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}} \right)_{\mathcal{J}} = -(\mathcal{J}_{\bar{\mathbf{q}}}(\bar{\mathbf{q}}, \mathbf{w}))^{-1} \mathcal{J}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w}), \quad (2.62)$$

$$\left(\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}} \right)_{\mathcal{B}} = -(\mathcal{B}_{\bar{\mathbf{q}}}(\bar{\mathbf{q}}, \mathbf{w}))^{-1} \mathcal{B}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w}). \quad (2.63)$$

Due to the sizes of the arrays in the equations above more than one linear solve of the system will be required. The systems have the following dimensions

$$\begin{aligned} & \left(\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}} \right)_{\mathcal{J}} \quad \left(\mathcal{J}_{\bar{\mathbf{q}}}(\bar{\mathbf{q}}, \mathbf{w}) \right)^{-1} \quad \mathcal{J}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w}) \\ & \begin{bmatrix} \bullet_{1,1} & \cdots & \bullet_{1,m} \\ \vdots & \ddots & \vdots \\ \bullet_{n-1,1} & \cdots & \bullet_{n-1,m} \\ \bullet_{n,1} & \cdots & \bullet_{n,n} \end{bmatrix} = - \begin{bmatrix} \bullet_{1,1} & \cdots & \cdots & \bullet_{1,m} \\ \vdots & \ddots & \cdots & \vdots \\ \bullet_{n-1,1} & \cdots & \cdots & \bullet_{n-1,m} \\ \bullet_{n,1} & \cdots & \cdots & \bullet_{n,n} \end{bmatrix} \begin{bmatrix} \bullet_{1,1} & \cdots & \bullet_{1,m} \\ \vdots & \ddots & \vdots \\ \bullet_{n-1,1} & \cdots & \bullet_{n-1,m} \\ \bullet_{n,1} & \cdots & \bullet_{n,n} \end{bmatrix}, \\ & n \times m \quad \quad \quad n \times n \quad \quad \quad n \times m \end{aligned} \quad (2.64)$$

which will therefore require m linear solves of the system to obtain $\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}}$ in each case.

By summing the individual components of the sensitivity of the base flow to geometry modifications $\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}}$ and with information we already have we can employ the chain rule derive the following equation

$$\left(\frac{\partial\sigma}{\partial\mathbf{w}}\right)^T = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^T \frac{\partial\bar{\mathbf{q}}}{\partial\mathbf{w}}, \quad (2.65)$$

$$\left(\frac{\partial\sigma}{\partial\mathbf{w}}\right)^T = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^T \left(\left(\frac{\partial\bar{\mathbf{q}}}{\partial\mathbf{w}}\right)_{\mathcal{J}} + \left(\frac{\partial\bar{\mathbf{q}}}{\partial\mathbf{w}}\right)_{\mathcal{B}} \right), \quad (2.66)$$

which gives the sensitivity of the eigenvalue to the changes of the surface geometry. For simplicity and clarity, the parentheses defining the dependent variables is dropped for the rest of the section. We now have two equations for computing the sensitivity of the eigenvalue to modifications of the surface geometry: one which uses information from the adjoint problem and one which uses only information from the direct problem:

$$\nabla_{\mathbf{w}}\sigma = \mathcal{B}_{\mathbf{w}}^+\hat{\mathbf{q}}^+ - \mathcal{J}_{\mathbf{w}}^+\bar{\mathbf{q}}^+, \quad (2.67)$$

$$\left(\frac{\partial\sigma}{\partial\mathbf{w}}\right)^T = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^T \frac{\partial\bar{\mathbf{q}}}{\partial\mathbf{w}}. \quad (2.68)$$

The Lagrangian method exploits the use of the adjoint which in terms of computational cost is the optimal of the two methods - the alternative approach requires $2m$ linear solves (Equation (2.64) for both Jacobian and sensitivity matrix systems). However, the Lagrangian approach requires the definition of a mass matrix (rectangular) for the surface geometry to define the adjoint matrix (Equation (2.44)). This definition will dependent on the spatial discretisation method applied and may not always be immediately obvious/simple to define. This problem is negated by the alternative approach which does not require this definition.

Following alternative method we can also find the sensitivities to other input parameters, such as

$$\left(\frac{\partial\sigma}{\partial Re}\right)^T = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^T \frac{\partial\bar{\mathbf{q}}}{\partial Re}, \quad (2.69)$$

$$\left(\frac{\partial\sigma}{\partial\alpha}\right)^T = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^T \frac{\partial\bar{\mathbf{q}}}{\partial\alpha}, \quad (2.70)$$

$$\left(\frac{\partial\sigma}{\partial Ma}\right)^T = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^T \frac{\partial\bar{\mathbf{q}}}{\partial Ma}. \quad (2.71)$$

2.5 Summary

The procedure for performing sensitivity analysis (sensitivity to base flow modifications and to a steady force), when using a discrete approach, is summarised here and Figure 2.2:

1. Calculate the base flow by solving a steady solution of system of partial differential equations, Eq. (2.1).
2. Compute the Jacobian matrix,
3. Compute the adjoint matrix,
4. Solve the direct and adjoint eigenvalue problems, Eq. (2.7) and (2.11).
5. Normalize adjoint modes, Eq. (2.24).
6. Compute the sensitivity matrix, multiply with leading adjoint mode to get sensitivity to base flow modifications, Eq. (2.37).
7. Solve system, Eq. (2.32), to obtain the sensitivity to a steady force.

Figure 2.2: General outline for the discrete sensitivity analysis methodology presented

Chapter 3

Matrix Computation

3.1 Introduction

The core of this chapter will lay out the numerical way in which the matrices, namely the Jacobian and sensitivity matrix, are evaluated. The current conventional standard forward finite difference approximation for computing the numerical Jacobian is compared with the complex-step derivative approximation, which is presented here as a more accurate alternative. Furthermore, it is also shown how sensitivity matrix and related sensitivity fields are numerically computed.

3.2 Jacobian Matrix

In mathematics the Jacobian is, by considering all partial variables, the first order derivative of a function. The Jacobian will be one of the cornerstones of this work both for stability/sensitivity analysis and the numerics (Newton iteration). Alternatives for computing the Jacobian are provided in Chapter 1 but we proceed here by considering the numerical evaluation of the Jacobian. The numerical evaluation of the Jacobian is well established and is commonly used within the linear stability computational fluid dynamics

community [38, 48, 69] to name a few. Firstly, a function $\mathcal{F}(\bar{\mathbf{q}})$ is expanded with a Taylor series around the base state $\bar{\mathbf{q}}$ and a small increment ε

$$\mathcal{F}(\bar{\mathbf{q}} + \varepsilon \mathbf{k}) = \mathcal{F}(\bar{\mathbf{q}}) + \varepsilon \frac{\partial \mathcal{F}(\bar{\mathbf{q}})}{\partial \bar{\mathbf{q}}} \mathbf{k} + \mathcal{O}(\varepsilon^2), \quad (3.1)$$

Where $\mathbf{k}$ is a vector which is iterated on to loop through each degree of freedom. Rearranging gives

$$\mathcal{J}(\bar{\mathbf{q}}) \mathbf{k} = \frac{\mathcal{F}(\bar{\mathbf{q}} + \varepsilon \mathbf{k}) - \mathcal{F}(\bar{\mathbf{q}})}{\varepsilon} + \mathcal{O}(\varepsilon). \quad (3.2)$$

As $\varepsilon \ll 1$, we can neglect the higher order ε terms. This gives the standard first order forward finite difference approximation to evaluate the Jacobian matrix. To increase the level of accuracy a 1st order centred difference approximation can be employed. We expand the function $\mathcal{F}(\bar{\mathbf{q}})$ again but with a negative increment ε to give

$$\mathcal{F}(\bar{\mathbf{q}} - \varepsilon \mathbf{k}) = \mathcal{F}(\bar{\mathbf{q}}) - \varepsilon \frac{\partial \mathcal{F}(\bar{\mathbf{q}})}{\partial \bar{\mathbf{q}}} \mathbf{k} + \mathcal{O}(\varepsilon^2). \quad (3.3)$$

Equation (3.3) is then subtracted from Equation (3.1) to give

$$\mathcal{J}(\bar{\mathbf{q}}) \mathbf{k} = \frac{\mathcal{F}(\bar{\mathbf{q}} + \varepsilon \mathbf{k}) - \mathcal{F}(\bar{\mathbf{q}} - \varepsilon \mathbf{k})}{2\varepsilon} + \mathcal{O}(\varepsilon^2). \quad (3.4)$$

The centred approximation (Eq. (3.4)) reduces the truncation error $\mathcal{O}(\varepsilon) \rightarrow \mathcal{O}(\varepsilon^2)$ but requires that the $\mathcal{F}$ operator be evaluated twice, $\mathcal{F}(\bar{\mathbf{q}} + \varepsilon)$ and $\mathcal{F}(\bar{\mathbf{q}} - \varepsilon)$, therefore doubling the computational cost of evaluating the Jacobian matrix. In addition, higher order

difference schemes for computing the Jacobian can improve the accuracy of the numerical Jacobian: these schemes have small numerical errors, can be easily implemented but at the expense of additional computation of fluxes $\mathcal{F}$ [47].

Finite difference approximations require an appropriate choice of the perturbation or increment term ε . ε must be chosen so that it is small enough such that the truncation error is not significant while remaining not too small to keep round-off errors controlled when subtracting numbers which are almost equal. This is illustrated in Figure 3.1.

Figure 3.1: Comparison of round-off error and truncation error

An alternative to improve the accuracy of the Jacobian without sacrificing the computational cost, is to use complex variables. The complex-step derivative approximation can be used to determine first derivatives providing near analytic accuracy. Early work on obtaining derivatives via a complex-step approximation in order to improve overall accuracy can be found in [63, 62]

The advantages of the complex-step derivative approach over a standard finite difference include: 1) the Jacobian approximation is not subject to subtractive cancellations inherent

to round-off errors, 2) it can be used on discontinuous functions, and 3) it is easy to implement in a black-box manner, thereby making it applicable to general non-linear functions.

Additional information can be found in [55, 54], which shows that not only a wider range of step sizes can be selected for the complex approach, but a higher level of precision can be achieved (near to machine precision). On the downside, the use of this approach means that it is necessary to implement a complex version of the NS fluxes. However, modern computer languages (e.g. Fortran, C++, etc) enable the ease of implementation of these changes through function overloading or template techniques.

The derivation of the complex-step derivative approximation is accomplished by approximating the non-linear function with a complex variable using a Taylors series expansion

$$\mathcal{F}(\bar{\mathbf{q}} + i\varepsilon\mathbf{k}) = \mathcal{F}(\bar{\mathbf{q}}) + \frac{\partial\mathcal{F}(\bar{\mathbf{q}})}{\partial\bar{\mathbf{q}}}i\varepsilon\mathbf{k} - \frac{\partial^2\mathcal{F}(\bar{\mathbf{q}})}{\partial\bar{\mathbf{q}}^2}\frac{\varepsilon^2}{2}\mathbf{k} - \mathcal{O}(i\varepsilon^3 + \varepsilon^4). \quad (3.5)$$

Taking the imaginary part of each term and rearranging, we obtain

$$\frac{\partial\mathcal{F}(\bar{\mathbf{q}})}{\partial\bar{\mathbf{q}}}\mathbf{k} = \frac{\Im[\mathcal{F}(\bar{\mathbf{q}} + i\varepsilon\mathbf{k})]}{\varepsilon} + \mathcal{O}(\varepsilon^2), \quad (3.6)$$

where $\Im$ denotes the imaginary part.

From a practical implementation point of view, to calculate the j^{th} column of the Jacobian, the j^{th} row of the $\mathbf{k}$ vector is set to one while the other rows are set to zero. This process is successively repeated for as many number of degrees of freedom of the matrix. The first order formula, Eq. (3.6), requires n computations of numerical fluxes, with n being the dimension of $\bar{\mathbf{q}}$.

$$\begin{pmatrix} J_{11} & J_{12} & \cdots & J_{1N} \\ J_{21} & J_{2,2} & \cdots & J_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ J_{N1} & J_{N2} & \cdots & J_{NN} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{pmatrix} = \frac{\Im \left[\mathcal{F} \left(\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_N \end{pmatrix} + i\varepsilon \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{pmatrix} \right) \right]}{\varepsilon} \quad (3.7)$$

$$\begin{pmatrix} J_{11} & J_{12} & \cdots & J_{1N} \\ J_{21} & J_{2,2} & \cdots & J_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ J_{N1} & J_{N2} & \cdots & J_{NN} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \frac{\Im \left[\mathcal{F} \left(\begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_N \end{pmatrix} + i\varepsilon \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right) \right]}{\varepsilon} \quad (3.8)$$

$$= \begin{pmatrix} J_{12} \\ J_{22} \\ \vdots \\ J_{N2} \end{pmatrix}$$

In compact schemes, the perturbation introduced in a element will only affect a limited number of elements (perturbation stencil). In order to reduce the computational cost, only the fluxes in the elements that are part of the stencil need be evaluated. An example of the stencil is given in Figure 3.2.

Figure 3.2: Figure showing an example of the compact stencil used in Jacobian computation. The green square indicates where the perturbation is applied, and the red squares indicate the elements to that the information travels.

3.2.1 Computational Cost

Later sections of this Chapter will look at the accuracy but here the computational cost of the standard forward finite difference, centred finite difference and complex-step derivative approximation is considered. The validation test case of flow past circular cylinder presented in Section 5.2 is used for comparing the methods. Table 3.1 presents the computational times t_{method} for evaluating the Jacobian matrix, of the aforementioned methods, normalised with the computational time of the forward finite difference $t_{\text{forward}^{\mathbb{R}}}$ method. It can be seen that while $\text{forward}^{\mathbb{C}}$ takes approximately double the amount of computational time of $t_{\text{forward}^{\mathbb{R}}}$, the $\text{centred}^{\mathbb{R}}$ method takes 2.27 times the computational time. As explained in the previous section, the increase in computational time for the centred method is explained by the need to re-evaluate the fluxes twice. However, the $\text{forward}^{\mathbb{C}}$ method, although needing the fluxes to be evaluated once, has double the computational time of $\text{forward}^{\mathbb{R}}$ due to the complex arithmetic operations $(a + ib)$ needed for evaluating the Jacobian. The point to highlight is that complex-step derivative offers reduced computational cost over the centred while also not having any round-off errors associated

with subtraction.

Method	$\frac{t_{\text{method}}}{t_{\text{forward}^{\mathbb{R}}}}$
forward $^{\mathbb{R}}$: $\mathcal{J}(\bar{\mathbf{q}})\mathbf{k} = \frac{\mathcal{F}(\bar{\mathbf{q}}+\varepsilon\mathbf{k})-\mathcal{F}(\bar{\mathbf{q}})}{\varepsilon}$	1
centred $^{\mathbb{R}}$: $\mathcal{J}(\bar{\mathbf{q}})\mathbf{k} = \frac{\mathcal{F}(\bar{\mathbf{q}}+\varepsilon\mathbf{k})-\mathcal{F}(\bar{\mathbf{q}}-\varepsilon\mathbf{k})}{2\varepsilon}$	~ 2.27
forward $^{\mathbb{C}}$: $\mathcal{J}(\bar{\mathbf{q}})\mathbf{k} = \frac{\Im[\mathcal{F}(\bar{\mathbf{q}}+i\varepsilon\mathbf{k})]}{\varepsilon}$	~ 2.01

Table 3.1: The computational cost for evaluating the Jacobian matrix of the standard forward finite difference, centred finite difference and complex-step derivative approximation. The computational time of the forward finite difference $t_{1\text{st}^{\mathbb{R}}}$ is taken as the *datum*.

3.2.2 Jacobian With Respect to $\mathbf{w}$

Calculating the Jacobian, $\mathcal{J}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w})$, can be done in a similar way as before

$$\frac{\partial \mathcal{F}(\bar{\mathbf{q}}, \mathbf{w})}{\partial \mathbf{w}} = \mathcal{J}_{\mathbf{w}}(\bar{\mathbf{q}}, \mathbf{w})\mathbf{k} = \frac{\mathcal{F}(\mathbf{w} + \delta(x, y)\mathbf{k}) - \mathcal{F}(\mathbf{w})}{\delta(x, y)}. \quad (3.9)$$

We perturb the old grid nodes by a small amount, $\delta(x, y)$, and then recompute the fluxes.

3.3 Sensitivity Matrix

The computation of the sensitivity matrix, $\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}})$, is performed in the following way, since, the only resource available is the NS fluxes $\mathcal{F}$. Moreover, we have to keep in mind that an explicit version of the Jacobian is not known, only its numerical values. From its definition, it follows that $\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}})$ is a linear function of $\hat{\mathbf{q}} = \hat{\mathbf{q}}_{\mathbf{R}} + i\hat{\mathbf{q}}_{\mathbf{I}}$ and a non-linear function of $\bar{\mathbf{q}}$ through the Jacobian of $\mathcal{F}$. Thus

$$\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \frac{\partial(\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}})}{\partial\bar{\mathbf{q}}}, \quad (3.10)$$

can be decomposed into its real and imaginary part

$$\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \mathcal{B}_R(\bar{\mathbf{q}}, \hat{\mathbf{q}}) + i\mathcal{B}_I(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \frac{\partial(\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}}_R)}{\partial\bar{\mathbf{q}}} + i\frac{\partial(\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}}_I)}{\partial\bar{\mathbf{q}}}. \quad (3.11)$$

Additionally

$$\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}} = \left. \frac{\partial\mathcal{F}(\bar{\mathbf{q}})}{\partial\mathbf{q}} \right|_{\bar{\mathbf{q}}} \hat{\mathbf{q}}, \quad (3.12)$$

As the eigenmode $\hat{\mathbf{q}}$ is assumed independent of the base state $\bar{\mathbf{q}}$, Equation (3.10) can be written as

$$\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \frac{\partial^2\mathcal{F}(\bar{\mathbf{q}})}{\partial\bar{\mathbf{q}}^2}\hat{\mathbf{q}}, \quad (3.13)$$

where the first term on the left hand side is the Hessian of the vector operator $\mathcal{F}(\bar{\mathbf{q}})$. If the real terms of the complex variable Taylor series expansion in Equation (3.5) are taken

$$\Re[\mathcal{F}(\bar{\mathbf{q}} + i\varepsilon\mathbf{k})] = \mathcal{F}(\bar{\mathbf{q}}) - \Re\left[\frac{\partial^2\mathcal{F}(\bar{\mathbf{q}})}{\partial\bar{\mathbf{q}}^2}\frac{\varepsilon^2}{2}\mathbf{k}\right] - \mathcal{O}(\varepsilon^4). \quad (3.14)$$

Rearranging, a definition for the Hessian matrix can be found

$$\frac{\partial^2 \mathcal{F}(\bar{\mathbf{q}})}{\partial \bar{\mathbf{q}}^2} \mathbf{k} = \frac{2}{\varepsilon^2} (\mathcal{F}(\bar{\mathbf{q}}) - \Re [\mathcal{F}(\bar{\mathbf{q}} + i\varepsilon \mathbf{k})]) - \mathcal{O}(\varepsilon^2) \quad (3.15)$$

The corresponding standard 2nd order forward derivative finite difference approximation is

$$\frac{\partial^2 \mathcal{F}(\bar{\mathbf{q}})}{\partial \bar{\mathbf{q}}^2} \mathbf{k} = \frac{\mathcal{F}(\bar{\mathbf{q}} + 2\varepsilon \mathbf{k}) - 2\mathcal{F}(\bar{\mathbf{q}} + \varepsilon \mathbf{k}) + \mathcal{F}(\bar{\mathbf{q}})}{\varepsilon^2} + \mathcal{O}(\varepsilon^2). \quad (3.16)$$

It can be seen that the standard 2nd order forward derivative finite difference approximation (Eq. 3.16) requires an extra evaluation of the fluxes when compared to the complex form (Eq. 3.15). However, the complex form now introduces a subtraction which means that it is no longer round-off error free as opposed to the first order evaluation (Eq. 3.6).

From a practical implementation, let us say that our function $\mathcal{F}(\bar{\mathbf{q}})$ has a size of n by 1. As an example, to evaluate the Jacobian with the real first order finite difference method, n evaluations are needed to compute the Jacobian matrix which has a size of n by n . Subsequently to compute the Hessian matrix, n^2 evaluations are needed and the size of the Hessian will be n^2 by n (a 3rd order tensor). For systems where n is small there will be no issue but for larger and larger systems there will be two issues which will become problematic very quickly: firstly, the memory needed to store the Hessian matrix and secondly, the number of evaluations needed to compute the Hessian i.e. the amount of computational time. By again writing the definition as it was originally posed

$$\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \frac{\partial(\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}})}{\partial \bar{\mathbf{q}}}, \quad (3.17)$$

and by first multiplying the Jacobian $\mathcal{J}(\bar{\mathbf{q}})$ and eigenmode $\hat{\mathbf{q}}$ then differentiating we have

reduced term to be differentiated from an n by n to an n by 1. By taking advantage of this the sensitivity matrix is the same size as the Jacobian and there is no problem as we do not have to compute the Hessian directly. Given that the Jacobian and sensitivity matrix are computed by perturbing the fluxes $\mathcal{F}(\bar{\mathbf{q}})$ it is convenient to write the Jacobian $\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}}$ and eigenmode $\hat{\mathbf{q}}$ (for either the real or imaginary part of $\hat{\mathbf{q}}$) as simply the directional derivative of $\mathcal{F}(\bar{\mathbf{q}})$ along $\hat{\mathbf{q}}$, thus

$$\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}} = \frac{\mathcal{F}(\bar{\mathbf{q}} + \varepsilon_1 \hat{\mathbf{q}}) - \mathcal{F}(\bar{\mathbf{q}})}{\varepsilon_1} + \mathcal{O}(\varepsilon_1), \quad (3.18)$$

Therefore, neglecting order ε_1 terms

$$\frac{\partial(\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}})}{\partial \bar{\mathbf{q}}} = \frac{1}{\varepsilon_1} \left(\left. \frac{\partial \mathcal{F}(\bar{\mathbf{q}})}{\partial \mathbf{q}} \right|_{\bar{\mathbf{q}} + \varepsilon_1 \hat{\mathbf{q}}} - \left. \frac{\partial \mathcal{F}(\bar{\mathbf{q}})}{\partial \mathbf{q}} \right|_{\bar{\mathbf{q}}} \right) = \frac{1}{\varepsilon_1} \left(\left. \frac{\partial \mathcal{F}(\bar{\mathbf{q}})}{\partial \mathbf{q}} \right|_{\bar{\mathbf{q}} + \varepsilon_1 \hat{\mathbf{q}}} - \mathcal{J}(\bar{\mathbf{q}}) \right), \quad (3.19)$$

where the definition of the Jacobian, Eq. (2.7), has been used. The right hand side expression can be evaluated using complex variables, thus giving the following expression for the sensitivity matrix

$$\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \frac{1}{\varepsilon_1} \left[\left(\frac{\Im(\mathcal{F}(\bar{\mathbf{q}} + \varepsilon_1 \hat{\mathbf{q}}_{\mathbf{R}} + i\varepsilon \mathbf{k}))}{\varepsilon} - \mathcal{J}(\bar{\mathbf{q}}) \right) + i \left(\frac{\Im(\mathcal{F}(\bar{\mathbf{q}} + \varepsilon_1 \hat{\mathbf{q}}_{\mathbf{I}} + i\varepsilon \mathbf{k}))}{\varepsilon} - \mathcal{J}(\bar{\mathbf{q}}) \right) \right], \quad (3.20)$$

In a similar fashion to the computation of the Jacobian in Eq. (3.8), Eq. (3.20) is evaluated by successively computing columns of $\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}})$ by setting one row of vector $\mathbf{k}$ to one and the rest to zero. In the previous expression only one column of the Jacobian $\mathcal{J}(\bar{\mathbf{q}})$ is considered at each time although one can consider computing, in one iteration, all the non-interacting columns. Additionally, the sensitivity matrix, $\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}})$, or its conjugate

transpose, need never be formed explicitly for the calculation of Eq. (2.37) and Eq. (2.32). From expression (2.10), it can be seen that Eq. (2.37) can be written explicitly as;

$$\nabla_{\bar{\mathbf{q}}}\sigma = \mathcal{M}^{-1} \mathcal{B}^*(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \mathcal{M} \hat{\mathbf{q}}^+, \quad (3.21)$$

where $\mathcal{B}^*(\bar{\mathbf{q}}, \hat{\mathbf{q}})$ is the conjugate transpose of the sensitivity matrix. It is a trivial task to multiply the mass matrix, $\mathcal{M}$, with the adjoint mode $\hat{\mathbf{q}}^+$ to give a new vector $\mathbf{d}$;

$$\nabla_{\bar{\mathbf{q}}}\sigma = \mathcal{M}^{-1} \mathcal{B}^*(\bar{\mathbf{q}}, \hat{\mathbf{q}}) \mathbf{d}. \quad (3.22)$$

Thus, by setting the first row of the $\mathbf{k}$ vector equal to one and the rest zero, we obtain the first column of the sensitivity matrix, $\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}})$. If we take the complex conjugate of this vector we will get the first row of the transpose of the sensitivity matrix, $\mathcal{B}^*(\bar{\mathbf{q}}, \hat{\mathbf{q}})$. By multiplying this row vector now with the vector $\mathbf{d}$ we obtain a scalar value. We proceed to multiply this with the first row of the inverse of the mass matrix, $\mathcal{M}^{-1}$, which returns the first row of the sensitivity to base flow modifications vector, $\nabla_{\bar{\mathbf{q}}}\sigma$.

The previous algorithm obtains first order accuracy in ε_1 and second order in ε and requires three computation of the fluxes per grid node. Second order precision in ε can be easily obtained by using a central difference scheme in Eq. (3.18), but at the additional cost of a new evaluation of the fluxes.

3.4 Analytical Test Case

Let us define the flux vector as

$$\mathcal{F}(\bar{\mathbf{q}}) = \begin{bmatrix} \bar{u}\bar{v} \\ \bar{u}\bar{w}^3 \\ \bar{v}\bar{w} \end{bmatrix}, \quad (3.23)$$

where

$$\bar{\mathbf{q}} = (\bar{u}, \bar{v}, \bar{w})^T. \quad (3.24)$$

It may be easily checked that the analytical Jacobian and sensitivity matrices are given by

$$\mathcal{J}(\bar{\mathbf{q}}) = \begin{bmatrix} \bar{v} & \bar{u} & 0 \\ \bar{w}^3 & 0 & 3\bar{u}\bar{w}^2 \\ 0 & \bar{w} & \bar{v} \end{bmatrix}, \quad \mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}}) = \begin{bmatrix} \hat{q}_2 & \hat{q}_1 & 0 \\ 3\bar{w}^2\hat{q}_3 & 0 & 3\bar{w}^2\hat{q}_1 + 6\bar{u}\bar{w}\hat{q}_3 \\ 0 & \hat{q}_3 & \hat{q}_2 \end{bmatrix}, \quad (3.25)$$

where

$$\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3)^T. \quad (3.26)$$

The Jacobian and sensitivity matrices are then calculated using the algorithms, detailed in this Section, subject to epsilon, ε and ε_1 , values ranging from 10^5 to 10^{-15} . Results using real and complex variables are compared. The absolute maximum matrix component value of the relative error between the numerical and analytical matrices is computed

$$\max \left(\left| \frac{\mathcal{J}_{\text{numerical}} - \mathcal{J}_{\text{analytical}}}{\mathcal{J}_{\text{analytical}}} \right| \right) ; \max \left(\left| \frac{\mathcal{B}_{\text{numerical}} - \mathcal{B}_{\text{analytical}}}{\mathcal{B}_{\text{analytical}}} \right| \right) \quad (3.27)$$

The results are plotted in Figure 3.3 and 3.4.

Figure 3.3: Comparison of numerical and analytical Jacobian (relative error, Equation (3.27)), a) and b), for test case Eq. (3.25) as a function of ε , Eq. (3.8)

Figure 3.4: Comparison of numerical and analytical sensitivity matrix (relative error, Equation (3.27)), a) and b), for test case Eq. (3.25) as a function of ϵ_1 , Eq. (3.20)

If we arbitrarily take 10^{-5} as the maximum allowable difference, which is acceptable and examine Figure 3.3 and 3.4, the following observations conclusions can be drawn

1. Figure 3.3 a) Calculating the Jacobian with real variables using a first order approach restricts the choice of ϵ to a narrow range 10^{-5} to 10^{-11} . Using a 2nd order central difference approach for the numerical evaluation of the Jacobian enables more flexibility in the choice of epsilon, but a 2nd order approach is more computationally expensive because it requires the calculation the fluxes twice.
2. Figure 3.3 b) Calculating the Jacobian with the complex step approximation gives

an acceptable result for any value of epsilon smaller than $10^{-2.5}$. Furthermore, the error levels out at machine precision ($\sim 10^{-15}$) can be obtained for $\varepsilon = 10^{-7}$.

3. Figure 3.4 c) Using a first order method, with real variables and the epsilon value, ε , held at its optimal value from Figure 3.3a) (10^{-8}), for the calculation of the sensitivity matrix $\mathcal{B}(\bar{\mathbf{q}}, \hat{\mathbf{q}})$ gives acceptable result only when a value of epsilon, ε_1 , equal to 10^{-2} is selected. In addition, it can be seen that little improvement on relative error and the range of acceptable epsilon, ε_1 , values is achieved by moving to a second order method ($\varepsilon = 10^{-8}$).
4. Figure 3.4 d) A first order method with the complex step approximation for the sensitivity matrix enables a greater range of epsilon, ε_1 , values (10^{-2} to 10^{-10}) with improved accuracy.

3.4.1 Lid Driven Davity: ε and ε_1

The effect of the choice of epsilon values for the high order compressible NS solver will now be explored. It was shown in Section 3.4 and in [65] that for $\varepsilon > \approx 10^{-7}$ the relative error reduces to machine precision when using the complex-step derivative approximation. The lid driven cavity is run with the following conditions: $Re = 200$, $Ma = 0.2$, 4 quadrilateral elements each with $P=8$. A base flow is obtained and then the Jacobian is computed using a first order, Eq. (3.2), a second order (see [69]), and a complex-step approximation, Eq. (3.6). Figure 3.5 shows relative error of the two most unstable eigenvalues for the three aforementioned methods with varying epsilon, ε . It is assumed that the true values for the eigenvalues are calculated with Eq. (3.6) using $\varepsilon = 10^{-15}$ and these are then used as the reference values when calculating the relative error.

It can be from Figure 3.5 that the first order complex-step approximation method con-

Figure 3.5: Comparison of forward, centred difference and the complex-step approximation methods for the numerical differentiation of the fluxes to calculate the Jacobian. The relative error for the 2 most unstable eigenvalues are shown for the lid driven cavity case.

verges at $\varepsilon = 10^{-6}$ for a relative error of 10^{-16} . Furthermore, both the first order and second order methods initially decrease (truncation error is decreasing) to a relative error of $\sim 10^{-8}$ for epsilon, ε , ranging from 10^{-4} to 10^{-7} . The relative error begins to increase (round-off error becomes significant) for $\varepsilon > 10^{-7}$. Close inspection of ε values ranging from 10^{-1} to 10^{-3} reveals that the first and second order methods converge with slopes of one and two respectively. The complex-step approximation method converges with a gradient of two and significantly, but unlike the second order method, it only requires one evaluation of the fluxes. Furthermore, in our implementation the cost for one solve is comparable between the real and complex formulations. Indeed, the complex approximation provides greater accuracy whilst having a lower computational cost.

The choice of ε_1 is more challenging. Mettot *et al.* [69] present a value of $\varepsilon_1 = (1 \times 10^{-8}) \|\bar{\mathbf{q}}\| / \|\hat{\mathbf{q}}\|$ based on the work of ([48, 5]). This value corresponds well with the optimal value for ε_1 given for the analytical case in Figure 3.4 d). There is only a necessity to tune ε_1 to reduce the error in Eq. (3.20) because the Jacobian, $\mathcal{J}(\bar{q})$, is computed to near analytical accuracy. For the rest of this thesis we take $\varepsilon = 10^{-10}$ and $\varepsilon_1 =$

$$(1 \times 10^{-8})\|\bar{\mathbf{q}}\|/\|\hat{\mathbf{q}}\|.$$

3.5 Summary

This chapter has established the approach for numerically computing the Jacobian and sensitivity matrix. It has been shown that the complex-step derivative approximation is advantageous when resolving higher order derivatives in sensitivity analysis. Furthermore, down to machine precision accuracy can be achieved when evaluating the Jacobian matrix. An analytical and a small fluid flow problem have shown the gains of the aforementioned method. The relative size of the solver will have to be considered as to whether transforming all the algorithms to accept complex numbers is feasible.

Chapter 4

Numerical Methods

4.1 Introduction

The formulation of the methodology in previous section has been kept general. We now detail the governing equations which will be used to firstly validate the methodology and its implementation within our code and secondly to then apply this to a more advanced case. We also present the numerical methods that we employ to solve the governing equations by integrating in time and in space.

4.2 Compressible Navier-Stokes Equations

Fluid flow is described by the Navier-Stokes equations. These equations are based on three laws of conservation: the conservation of mass, the conservation of momentum, and the conservation of mass. The 2-D compressible Navier-Stokes equations are written in vector form

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{\partial \mathcal{F}_1^a}{\partial x} - \frac{\partial \mathcal{F}_2^a}{\partial y} + \frac{1}{\text{Re}} \left(\frac{\partial \mathcal{F}_1^v}{\partial x} + \frac{\partial \mathcal{F}_2^v}{\partial y} \right), \quad (4.1)$$

where superscript a and v denote the advective and viscous fluxes respectively. The state vector, $\mathbf{q}$, and advective fluxes are given as

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{bmatrix}, \quad \mathcal{F}_1^a = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ u(\rho e + p) \end{bmatrix}, \quad \mathcal{F}_2^a = \begin{bmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ v(\rho e + p) \end{bmatrix}, \quad (4.2)$$

where ρ is the density of the fluid, u and v are the x and y components of the velocity, p is the pressure and e is the internal energy. The pressure, p , is computed from state variables using the ideal gas relation given as

$$p = (\gamma - 1) \left(\rho e - \frac{\rho(u^2 + v^2)}{2} \right) \quad (4.3)$$

The viscous fluxes are given as

$$\mathcal{F}_1^v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ u\tau_{xx} + v\tau_{yx} + \frac{\gamma\kappa}{(\gamma-1)Pr}T_x \end{bmatrix}, \quad \mathcal{F}_2^v = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + \frac{\gamma\kappa}{(\gamma-1)Pr}T_y \end{bmatrix}, \quad (4.4)$$

where

$$\begin{aligned} \tau_{xx} &= 2\mu(u_x - (u_x + v_y)/3) \\ \tau_{yy} &= 2\mu(v_y - (u_x + v_y)/3) \\ \tau_{xy} &= \tau_{yx} = \mu(v_x + u_y) \end{aligned} \quad (4.5)$$

Equations (4.1)(4.5), have been scaled with respect to reference quantities in the same way as [51].

4.3 Spatial Discretisation

4.3.1 Discontinuous Galerkin Spectral Element Method

Discontinuous Galerkin (DG) methods were developed by the pioneering work of Reed and Hill [76] in 1973 in the framework of hyperbolic partial differential equations for the neutron transport equation, and it was only in the late 1990's that the method was generalized to elliptic and convection-diffusion problems; e.g. Bassi and Rebay in 1997 [15], Baumann in 1997 [16] or Cockburn and Shu in 1998 [28]. Since, DG methods have proven useful in solving the compressible, e.g. [15, 57, 72, 75, 22], and the incompressible NS equations, e.g. [14, 79, 27, 88, 36, 37, 35].

High order methods (e.g. Spectral methods or discontinuous Galerkin) offer an alternative to low order methods and the classical h-refinement strategy. Such methods enable the use of high degree polynomials inside each computational element to approximate the numerical solution, which decreases the error exponentially when smooth flows are considered [101, 46, 33].

To apply discontinuous Galerkin method, we first tessellate the physical domain to obtain its discrete version. This variational technique requires multiplying the conservation law at an elemental level by a test function (typically polynomials) and integrate. By doing so, it is ensured that the residual of the equation is orthogonal (in the L_2 norm) to the space spanned by the test functions. Summing over all elemental contributions in the mesh domain and integrating by parts leads to the weak formulation of the problem and includes volume and surface contributions. Since in the discontinuous framework, discontinuities are allowed at edge interfaces, dual valued functions co-exist at the edge elements. To resolve the discontinuity associated to the convective fluxes, a Roe approximate Riemann

solver is used (see Toro’s monograph [99] for details). In addition, discontinuities associated to the viscous terms are resolved using a Bassi-Rebay formulation [15, 6]. Finally, boundary conditions are weakly imposed at domain boundaries. The resulting volume and surface integrals are numerically approximated using Gauss quadrature rules and lead to a discrete set of equations for each nodal point. A nodal variant of discontinuous Galerkin technique that uses a quad/hexa mesh topology and tensor product expansions for the polynomial spaces is known as Discontinuous Galerkin Spectral Element Method (DGSEM), as detailed in Kopriva [53]. The compressible Navier-Stokes equations can be discretised using the DGSEM approach [50, 53, 49]. The temporal terms are discretised using a third order Runge-Kutta method and are converged until steady state, which is reached once the infinite norm of the conservation law residual (for all flow equations) falls below a tolerance not larger than 10^{-10} . In our simulations, the Mach number is low ($M=0.2$) such that compressibility effects are negligible.

The computationally demanding nature of the NS equations, in the stability analysis context, leads to the selection of high-order numerical schemes for the numerical discretisation of system Eq. (4.6). High order methods (Spectral type methods) have been extensively used in computational fluid dynamics due to their accuracy and efficiency in the simulation of fluid flows. In particular, these methods are suitable for problems where high accuracy is required and, hence, are well suited to track the evolution of small perturbations as in stability analysis.

The numerical scheme used in this work discretises the compressible NS equations following a Discontinuous Galerkin Spectral Element Method (DGSEM [52]) for the spatial operators, coupled with a time integration scheme of order three. The DGSEM is a nodal collocation form of the discontinuous Galerkin method where the solution is approximated by a tensor product of Lagrange polynomials, and where inner products are evaluated using Gauss-Legendre quadrature [23, 52]. A summary of the mathematical formulation of the DGSEM is now presented. The general 3D formulation is detailed hereafter but the test cases considered, which employ the DGSEM method, in this thesis are strictly 2D.

The NS equations constitute a system of partial differential equations which can be shortly written in vector form as;

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{q}) = \mathbf{0}, \quad (4.6)$$

where $\mathbf{q}$ represents the vector of conservative variables and $\mathcal{F}(\mathbf{q})$ represent the 3D fluxes; including convective, diffusive in the three coordinate directions.

To solve our set of equations in general two or three dimensional geometries, the original domain is divided into non-overlapping hexahedral sub-domains, E_{sd} , such that $\Omega = \sum_{sd} E_{sd}$. Inside each sub-domain, a polynomial of degree, P , is used to approximate the unknowns and the fluxes, $\mathbf{q}, \mathcal{F}$ thus;

$$\mathbf{q}^P = \sum_{i,j,k=0}^P \mathbf{q}_{i,j,k} \Phi_{i,j,k}, \quad \mathcal{F}^P = \sum_{i,j,k=0}^P \mathcal{F}(\mathbf{q}_{i,j,k}) \Phi_{i,j,k}, \quad (4.7)$$

where

$$\Phi_{i,j,k} = L_i(x) L_j(y) L_k(z),$$

is the tensor product of the Lagrange interpolant in the nodes i, j, k and $\mathbf{q}_{i,j,k}$ is the value of the unknown in each computational node. In this work, the nodes in each direction follow a Gauss-Legendre distribution, and the basis functions $L_{(i,j,k)}$ are taken as the Lagrange interpolant at these nodes.

Reconsidering Eq. (4.6), we obtain, at an element level, the following discretised equation;

$$\frac{\partial \mathbf{q}^P}{\partial t} + \nabla \cdot \mathcal{F}^P = \mathbf{0}, \quad (4.8)$$

DGSEM makes use of the Galerkin weak form of the equations and a discontinuous treatment of the interfaces and boundaries. Thus, Eq. (4.8) is multiplied by a test function (the same function as the basis for the Galerkin method) and integrated in the computational space, then the error is forced to be orthogonal at each test function $\Phi_{i,j,k}$

in a mesh element E_{sd} , yielding;

$$\left(\frac{\partial \mathbf{q}^P}{\partial t}, \Phi_{i,j,k} \right)_{E_{sd}} + (\nabla \cdot \mathcal{F}^P, \Phi_{i,j,k})_{E_{sd}} = \mathbf{0}, \quad (i, j, k) = 0, \dots, P.$$

with $(a, b)_{E_{sd}} = \int_{E_{sd}} a b d\Omega$, defining an inner product (typically the L_2 inner product).

After integrating by parts we obtain;

$$\left(\frac{\partial \mathbf{q}^P}{\partial t}, \Phi_{i,j,k} \right)_{E_{sd}} - (\nabla \Phi_{i,j,k}, \cdot \mathcal{F}^P)_{E_k} + \int_{\partial E_{sd}} \Phi_{i,j,k} \mathcal{F}^P \cdot \mathbf{n} d\vec{S} = 0, \quad (4.9)$$

where the third term (the surface integral) extends over the boundary ∂E_{sd} of the computational element E_{sd} , with external pointing normal $\mathbf{n}$. This boundary may lie at the interface between two elements or at a physical boundary conditions, and in both cases, the treatment is similar. Note that all integrals in Eq. (4.9) can be numerically evaluated using Gauss quadrature.

To obtain a solution over the complete discretised computational domain ($\Omega = \sum_{sd} E_{sd}$) it is necessary to sum all the element contributions;

$$\left(\frac{\partial \mathbf{q}^P}{\partial t}, \Phi_{i,j,k} \right)_{\Omega} - (\nabla \Phi_{i,j,k}, \cdot \mathcal{F}^P)_{\Omega} + \sum_{\gamma \in \Gamma} \int_{\gamma} \Phi_{i,j,k} \mathcal{F}^*(\mathbf{n}, \mathbf{q}^L, \mathbf{q}^R) d\vec{S} = 0, \quad (4.10)$$

where Γ denotes the set of internal edges in the mesh Ω . In addition, note that we have replaced $\mathcal{F}^P$ by $\mathcal{F}^*(\mathbf{n}, \mathbf{q}^L, \mathbf{q}^R)$ in the surface integral. $\mathcal{F}^*(\mathbf{n}, \mathbf{q}^L, \mathbf{q}^R)$ represents the numerical flux between two consecutive elements in the mesh (**Left** and **Right**). These numerical fluxes arise from the discontinuous Galerkin setting where we consider that each element is disconnected from the next and hence contains a complete set of degrees of freedom to represent a polynomial of order P .

Taking into account the decomposition of the unknown in Eq. (4.7) and the orthogonality of the Lagrange basis in the Gauss nodes, the following expression is finally obtained for the integrals of Eq. (4.10);

$$\frac{\partial \mathbf{q}_{i,j,k}}{\partial t} + \mathcal{D}_x \mathcal{F}_{i,j,k}^1 + \mathcal{D}_y \mathcal{F}_{i,j,k}^2 + \mathcal{D}_z \mathcal{F}_{i,j,k}^3 = 0, \quad (i, j, k) = 0, \dots, P. \quad (4.11)$$

The discrete divergence (second term of the previous equation) is obtained after the numerical integration of the second and third terms of Eq. (4.10). Gauss quadrature is used to evaluate these integrals, giving;

$$\begin{aligned} \mathcal{D}_x \mathcal{F}_{i,j,k}^1 &= \mathcal{F}^{1*}(x, y_j, z_k) \frac{L_i(x)}{w_i} \Big|_{x=0}^{x=1} - \sum_{m=0}^P \mathcal{F}_{m,j,k}^1 d_{i,m} \\ \mathcal{D}_y \mathcal{F}_{i,j,k}^2 &= \mathcal{F}^{2*}(x_i, y, z_k) \frac{L_j(y)}{w_j} \Big|_{y=0}^{y=1} - \sum_{m=0}^P \mathcal{F}_{i,m,k}^2 d_{j,m} \\ \mathcal{D}_z \mathcal{F}_{i,j,k}^3 &= \mathcal{F}^{3*}(x_i, y_j, z) \frac{L_k(z)}{w_k} \Big|_{z=0}^{z=1} - \sum_{m=0}^P \mathcal{F}_{i,j,m}^3 d_{k,m}, \text{ with } d_{m,n} = \frac{L'_m(s_n)w_n}{w_m}, \end{aligned} \quad (4.12)$$

in the previous expression, w_n are the Gauss integration weights in direction $n = x, y$, or z , $L'_m(s_n)$ is the derivative of the Lagrange interpolant evaluate in the node s_n , and $\mathcal{F}^*$ are the interface fluxes. These fluxes can be differentiated into viscous or inviscid. Computation of inviscid fluxes requires taking into account the left and right values of the unknowns at each interface. Let us note that by taking the average value of the unknowns $\mathbf{q}$ at the interface (equivalent to a central scheme) provides a numerically unstable scheme when the convective terms dominates and is only recommended at very low Reynolds numbers. For larger Re , an up-winding scheme should be used instead. The most common way to introduce up-winding in the scheme is by solving the equivalent Riemann problem at the interface. The particular case of Euler equations (or inviscid NS) different Riemann solver has been already developed. In this work, a Roe Riemann solver has been used in the computations [99].

The viscous fluxes require discretisations for elliptic type equations. A simple approach consists of averaging the right and left viscous fluxes at the interface, but this solution has been proved numerically unstable for implicit schemes. A more general framework for derivation and analysis of discontinuous Galerkin methods for elliptic equations (e.g. Interior Penalty, Local Discontinuous Galerkin, Bassi-Rebay) was derived in [6]. Addi-

tional information of this methodology can be found in [52]. In our explicit time marching method we use the BR1 (Bassi-Rebay) approach to calculate the diffusive fluxes.

4.3.2 Mass Matrix Computation

As well as the DGSEM discretisation, a mass matrix which defines the volume and a mass matrix which defines the geometry surface are required to compute the adjoint matrix (Equations (2.10) and (2.43)). In the context of the DGSEM framework, the following relations for volume and surface integrals map a function from physical space, (x, y) to the corresponding computational space, (ξ, η) ;

$$\int_{el} f(x, y) dx dy = \int_{I_{el}} f(\xi, \eta) |J_\Psi| d\xi d\eta = \sum_{i=1}^{Pol_i} \sum_{j=1}^{Pol_j} f(\xi_i, \eta_j) |J_\Psi| \omega_i \omega_j = f(\xi, \eta) \mathcal{M}_{el} \quad (4.13)$$

$$\int_{\Gamma} f(S) dS = \int_{I_\Gamma} f(\xi) |J_S| d\xi = \sum_{i=1}^{Pol_i} f(\xi_i) |J_S| \omega_i = f(\xi) \mathcal{M}_\Gamma \quad (4.14)$$

where ω_i, ω_j are the Gauss Legendre weights and $|J_\Psi|, |J_S|$ are the volume and surface (this example is the upper surface of the element) Jacobian's given as

$$|J_\Psi| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \quad (4.15)$$

$$|J_S| = \frac{|J_\Psi|}{J_\Psi} \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} \quad (4.16)$$

where

$$\mathcal{M}_\Omega = \sum_{n=1}^{N_{el}} \mathcal{M}_{el_n} \quad (4.17)$$

$$\mathcal{M}_S = \sum_{n=1}^{N_\Gamma} \mathcal{M}_{\Gamma_n} \quad (4.18)$$

4.4 Temporal Discretisation

There are two temporal integration methods implemented within the code: 1) an explicit 3rd order Runge-Kutta method and 2) an implicit Newton iteration. The explicit method is perhaps the simpler of the two methods as it does not require the formation of the Jacobian matrix, does not require any external linear algebra libraries and the only parameter we need assign is the time step which is defined implicitly through the CFL number. However, the nature of the explicit solver requires a very small time step and many iterations are needed to obtain a fully converged steady solution. The implicit Newton iteration allows us to define greater time steps but the bottleneck is the formation of the Jacobian and subsequent solve of a linear system.

4.4.1 Runge-Kutta 3rd Order

For explicit time integration we employ the 3rd Order Runge-Kutta method to solve the system of differential equations (Compressible Navier-Stokes equations Eq. (4.1))

$$\frac{d\mathbf{q}}{dt} = \mathcal{F}(t, \mathbf{q}(t)), \quad \mathbf{q}(t = t_0) = \mathbf{q}_0 \quad (4.19)$$

where the solution is approximated with

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h \sum_{i=1}^3 b_i k_i,$$

$$k_i = \mathcal{F} \left(t_n + c_i h, \mathbf{q}_n + \sum_{j=1}^s a_{ij} k_j \right). \quad (4.20)$$

The coefficients are obtained from the Butcher tableau

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \end{array} \longrightarrow \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

4.4.2 Newton's Iteration

Equation (4.1) is discretised with a implicit time backward Euler formulation

$$\frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} = \mathcal{F}(\mathbf{q}^{n+1}), \quad (4.21)$$

where Δt is the time step. A new operator $\mathcal{G}$ and variable $\check{\mathbf{q}}$ are defined

$$\mathcal{G}(\check{\mathbf{q}}) = \frac{\check{\mathbf{q}} - \mathbf{q}^n}{\Delta t} - \mathcal{F}(\check{\mathbf{q}}), \quad (4.22)$$

for solving the Newton iteration at each step. When the RHS converges to zero (i.e. $\mathcal{G}(\check{\mathbf{q}}) = 0$) then $\check{\mathbf{q}} = \mathbf{q}^{n+1}$ which indicates the Newton iteration has converged at that step. Next we take a Taylor expansion of $\mathcal{G}$

$$\mathcal{G}(\check{\mathbf{q}}^{k+1}) = \mathcal{G}(\check{\mathbf{q}}) + \mathcal{G}'(\check{\mathbf{q}}^k)\Delta\check{\mathbf{q}}, \quad (4.23)$$

where the higher order terms have been ignored. Setting the RHS equal to zero, subbing in terms from Equation (4.22) and rearranging, we obtain an equation for the classical Newton iteration

$$\underbrace{\left(\frac{\mathcal{I}}{\Delta t} - \mathcal{J}(\mathbf{q}^n)\right)}_{\check{\mathcal{J}}} \Delta\check{\mathbf{q}} = - \underbrace{\left(\frac{\check{\mathbf{q}} - \mathbf{q}^n}{\Delta t} - \mathcal{F}(\check{\mathbf{q}})\right)}_{\check{\mathcal{F}}} \quad (4.24)$$

where the solution is updated after each iteration with $\check{\mathbf{q}}^{k+1} = \check{\mathbf{q}}^k + \Delta\check{\mathbf{q}}$. The linear system solved is $\check{\mathcal{J}}\Delta\check{\mathbf{q}} = -\check{\mathcal{F}}$.

4.5 Eigenvalue/Eigenvector Computation

The numerical method for computing the eigenvectors and associated eigenvalues of the given matrices $(\mathcal{J}, \mathcal{J}^+)$ is described here. Firstly, the computational size in terms of memory must be considered. Based on the number of degrees of freedom and the spatial discretisation (stencil size) the number of non-zeroes in the matrix can be estimated. We use the example of the flow past a circular cylinder case presented in Chapter 5. The of degrees of freedom equal $n = 64000$ and to form the matrix explicitly in double precision would require 31GB of memory. However, it was shown in Figure 3.2 that when to perturbation is applied, to compute the Jacobian, the information only travels to neighbouring cells. Therefore when computing a column of the Jacobian matrix (Equation 3.2 or 3.6) the non-zeroes will only exist for DoF within the elements shown in Figure 3.2 and zero everywhere else. Again using the flow past a circular cylinder example, based on: 640 elements, global polynomial order of 4, stencil size of 13, there

will be no more than $(13*(4+1)^2)*64000 = 83.2$ million non-zeros which, again in double precision, equates to 0.66GB.

Following each computation of the j^{th} column of the Jacobian matrix, only the non-zero values which are part of the stencil are stored in the aptly named compressed sparse column (CSC) format. With this format, the matrix is now comprised of 3 vectors; a vector containing the cumulative number of non-zeros per column which has a dimension equal to that of $n + 1$, a vector containing the row index for each of the non-zero values which intuitively has a dimension equal to the number of non-zeros, a vector containing the non-zero values of the Jacobian which again has the dimension equal to the number of non-zeros of the problem.

It is evident that the number of non-zero's per row (note: the above vector contains the number of non-zero's per column) for the Jacobian matrix is equal to the number of non-zeros per column for the adjoint matrix. Thus, a simple algorithm is used to transpose the Jacobian to form the adjoint matrix without explicitly forming the adjoint.

The standard eigenvalue is as follows

$$\mathcal{J}(\bar{\mathbf{q}})\hat{\mathbf{q}} = \sigma\hat{\mathbf{q}}. \quad (4.25)$$

Given that our interest is in eigenvalues which are most unstable or close to becoming unstable we can use the shift and invert Arnoldi method.

$$(\mathcal{J} - \mu\mathcal{I})^{-1}\hat{\mathbf{q}} = \frac{1}{\sigma - \mu}\hat{\mathbf{q}} \quad (4.26)$$

The Arnoldi method iterates and builds an orthogonal basis of k -dimensional Krylov subspace

$$\begin{aligned}\mathcal{K}(\mathcal{J}^{-1}, \mathring{\mathbf{q}}_1) &= \text{span}\{\mathring{\mathbf{q}}_1, \mathcal{J}^{-1}\mathring{\mathbf{q}}_1, \mathcal{J}^{-1^2}\mathring{\mathbf{q}}_1, \dots, \mathcal{J}^{-1^{k-1}}\mathring{\mathbf{q}}_1\} \\ &= \text{span}\{\mathring{\mathbf{q}}_1, \mathring{\mathbf{q}}_2, \mathring{\mathbf{q}}_3, \dots, \mathring{\mathbf{q}}_k\}\end{aligned}\tag{4.27}$$

where $\mathring{\mathbf{q}}_1$ is an initial guess or random vector and at each subsequent iteration we solve a linear system

$$\mathcal{J}^{-1}\mathring{\mathbf{q}}_j = \mathring{\mathbf{q}}_{j+1},\tag{4.28}$$

to obtain the Arnoldi vectors $\mathring{\mathbf{q}}_2, \dots, \mathring{\mathbf{q}}_k$. The Arnoldi vectors are orthogonal and combined together to form the matrix $\mathcal{Q}$. The Arnoldi iteration (Saad [83, 84]) computes the entries $\mathring{h}_{i,j}$ of the Hessenberg matrix $\mathcal{H}$ and we can write the following relations

$$\mathcal{J}^{-1}\mathcal{Q}_k = \mathcal{Q}_{k+1}\tilde{\mathcal{H}}_k\tag{4.29}$$

$$\mathcal{J}^{-1}\mathcal{Q}_k = \mathcal{Q}_k\mathcal{H}_k + \mathring{\mathbf{q}}_{k+1}\mathring{h}_{k+1,k}\mathbf{e}_k^T,\tag{4.30}$$

$$\mathcal{Q}^T\mathcal{Q} = \mathcal{I},\tag{4.31}$$

where $\mathring{\mathbf{q}}_{k+1}\mathring{h}_{k+1,k}\mathbf{e}_k^T$ is the last row of $\tilde{\mathcal{H}}$. The eigenvalues and Ritz vectors of the Hessenberg matrix $\mathcal{H}$, the ones close to the origin for this formulation, will be good approximations for the respective eigenvalue and eigenvectors of $\mathcal{J}^{-1}$ (Saad [83]).

4.5.1 Implementation

The Arnoldi iteration algorithm (Saad [83, 84]) is written in the Fortran programming language. The matrices $(\mathcal{J}, \mathcal{J}^+)$ are passed to the linear algebra solver MUMPS [4] in

CSC (Compressed Sparse Column) format to solve Equation 4.28. The library MUMPS performs an LU factorisation instead of directly inverting the matrix. The computation of the eigenvalues and Ritz vectors of the Hessenberg matrix $\mathcal{H}$ is handled by the subroutine *dgeev* of LAPACK [43].

Chapter 5

Validation Cases

5.1 Introduction

This chapter will now present results for an application case using the methodology presented in Chapter 2 coupled with the numerical recipes described in Chapter 4. The governing equations are the 2D compressible Navier-Stokes equations presented in Chapter 4. The application case will be flow past a circular cylinder at Reynolds number = 45. Flow past a circular is a classical fluid dynamics problem and there have been hundreds of papers which have looked at different aspects of the flow (see Williamson 1996 [103] for a comprehensive review). The evolution of the wake with increasing Reynolds number can be seen in Figure 5.1. $Re = 20$ is depicted as being stable while at $Re = 100$ is unstable. At the critical Reynolds number ($Re_{crit} \approx 47$), the flow transitions from a stable flow to an unstable flow where a global instability (Kármán vortex street) develops. Efforts to prevent the Kármán vortex street from developing is desirable because unwanted physical phenomena can result: periodic and fluctuating forces can undermine the structural integrity of the body, acoustic noise can be generated due to periodic nature of the unsteady flow, oscillation of the body which can lead to reduced operating life, aerodynamic effects such as stall and flutter etc.

With linear stability analysis we will identify the most unstable eigenvalue i.e. the mode

Figure 5.1: Evolution of the flow past a circular cylinder with increasing Reynolds number.

responsible for the global instability. Then with sensitivity analysis we will identify the regions of flow to perturb in order to defer the transition to instability to a higher Reynolds number. To validate the method, the results will be compared with the experimental work [93] and the computational work (continuous approach) of [64].

In an attempt to also show the robustness of the method we also present a convectively unstable case where eigenspectrum presents pseudo-continuous branches. For a system to be convectively unstable, the instabilities must grow in space but not in time. The open cavity is chosen to demonstrate the methodology in the context of a convectively unstable flow. The open cavity flow shows a global instability inside the cavity while a convectively instability develops upstream of the cavity. Flow past an open cavity at $Re = 1400$ is investigated.

5.2 Flow Around a Circular Cylinder at $Re = 45$

5.2.1 Base flow Computations

The flow aft a circular cylinder is studied for $Re = 45$ (below critical Reynolds number). Contrary to other authors, [64, 40, 34], the compressible NS are used in this study, therefore, in order to reproduce previous results and for validation purposes, the Mach number Ma is set to 0.2, being a compromise between numerical accuracy and compressible effects.

After a preliminary study for the convergence of the solution (not shown), the final mesh is composed of 640 quadrilateral elements with a polynomials of order $P = 4$. With a compressible Navier-Stokes equations state vector of $\mathbf{q} = (\rho, \rho u, \rho v, \rho E)^T$, the number of degrees of freedom (N) for the problem is 64,000 (No of DoF = $640 \times (P+1)^2 \times$ No of Equations). The Reynolds number, $Re = U_\infty D / \nu$, is based on the uniform inflow velocity, U_∞ , the kinematic viscosity, ν , and the diameter of the cylinder D . The computational domain and outline of the boundary conditions are given in Figure 5.2. The standard inflow ∂_{in} and outflow ∂_{out} are employed while the no slip condition is enforced on the cylinder wall ∂_{wall} . An example of the polynomial order is also provided.

Figure 5.3 a) and b) show the steady fields for flow past a circular cylinder at $Re = 45$ with the former being the $\bar{u}$ and the latter the vorticity magnitude $\bar{\omega}$. The base flow is a steady solution ($\mathcal{F}(\bar{\mathbf{q}}) = 0$) and is computed by time marching the unsteady solver (third order Runge-Kutta method) until convergence. A symmetric and stable recirculation bubble can be seen aft of the cylinder (5.3a). As the Reynolds number is increased, the recirculation bubble increases in size until the critical Reynolds number is reached at which point instabilities develop. Unstable configurations, $Re = 50$ and 60 , are shown in Figure 5.3 c):f). In this unstable regime the wake will oscillate periodically with the frequency of the oscillation being a function of the Reynolds number. These unsteady flow fields show that there is no longer a stable recirculation bubble but an unstable Karmen vortex wake. The efforts of later sections of this chapter will be to suppress, or more precisely defer the onset, of this instability.

Figure 5.2: Overview of the dimensions and boundary conditions for cylinder flow case

Figure 5.3: Cylinder flow fields: steady fields at $Re = 45$ a) $\bar{u}$ and b) $\bar{\omega}$, unsteady fields at $Re = 50$ c) u and d) ω and 60 e) u and f) ω

5.2.2 Linear Stability Analysis

This problem, studied by several authors (see Table 5.1), shows a Hopf bifurcation at the at $Re_{crit} \approx 47$. Above this, a global instability (Kármán vortex street) develops. We have chosen $Re = 45$ to compare the accuracy of our methodology with other authors work in the literature [64, 69].

	I/C	σ_i	frequency ($f_\sigma = \frac{\sigma_i}{2\pi}$)
Marquet <i>et al.</i> [64] $Re = 46.8$	I	-	~ 0.116
Giannetti and Luchini [40] $Re = 45$	I	-	~ 0.1177
Ferrer <i>et al.</i> [34] $Re = 45$	I	-	~ 0.118
Current Study $Re = 45$	C	0.742	~ 0.118

Table 5.1: Numerical comparison of the non-dimensional frequency of the least stable eigenvalue for the circular test case (**I** - incompressible Navier-Stokes, **C** - compressible Navier-Stokes).

Figure 5.4: The eigenvalue spectrum for flow past a circular cylinder at $Re = 45$. The circled eigenvalue corresponds global mode associated to the (Kármán vortex instability).

Once the steady base flow is obtained, the Jacobian and its eigenvalues are computed following the procedures described in previous sections. For this configuration we are only concerned with the least stable eigenmode (the one which defines the onset of unsteadiness). Figure 5.4 partially shows the eigenspectrum with the red circle denoting the

eigenvalue (complex pair) which corresponds the least stable eigenmode and the mode responsible for the onset of the (Kármán vortex instability). Finally, the frequency associated to this eigenvalue is compared in Table 5.1 against solutions provided by other authors, validating our methodology and analysis.

Figure 5.5 shows the real part of the direct eigenmodes associated to the least stable eigenvalue for the $\widehat{\rho u}$ (left) and $\widehat{\rho v}$ (right) components of velocity. The $\widehat{\rho u}$ eigenmode is antisymmetric about the centreline ($y=0$) whereas the $\widehat{\rho v}$ eigenmode is symmetric about the same line. Both show similar structures to that of the Kármán vortex street, which is present at Reynolds numbers close to the critical value.

Figure 5.5: The real part of the cylinder direct eigenmode fields; a) $\widehat{\rho u}$ and b) $\widehat{\rho v}$

Similarly, Figure 5.6 shows the real part of the adjoint velocities, $\widehat{\rho u}^+$ (left) and $\widehat{\rho v}^+$

(right). As expected, but contrariwise to the direct eigenmode, the structures of interest for the adjoint are located upstream of the cylinder [59]. Figure 5.5 and 5.6 both show excellent agreement with other authors [64, 69, 34]. We can therefore be confident in the complex variable implementation for the numerical computation of the Jacobian and the formulation/implementation of the adjoint matrix with corresponding mass matrix.

Figure 5.6: The real part of the cylinder adjoint eigenmode fields; a) $\widehat{\rho u}^+$ and b) $\widehat{\rho v}^+$

5.2.3 Sensitivity Analysis

The structural sensitivity identifies the most sensitive regions to localised feedback and origin of the instability. This was derived in [40] and the structural sensitivity of the flow can be calculated using the dot product of the direct and adjoint eigenmodes $\|\hat{\mathbf{q}}\| \cdot \|\hat{\mathbf{q}}^+\|$ and is shown in Figure 5.7. According to these results the maximum sensitivity region (also called the wavemaker) occurs at the centre of the two distorted ovals at ($x = 2.5$ and $y = \pm 0.5$), namely the region of flow where small modifications produce the larger variations in the eigenvalues. Despite the use of the compressible NS equations, a quantitative and qualitative agreement is obtained with the results of other authors [64, 40, 34]. The boundary conditions for the adjoint matrix are automatically applied in the discrete framework, when solving the system, Eq. (2.11), as stated by [40]. Furthermore, it is also clear from Figure 5.7 that the wavemaker is far enough from the inlet and farfield boundaries and therefore the influence of boundary conditions is minimised for this case.

Figure 5.7: Structural sensitivity for the circular cylinder at $Re = 45$

Figure 5.8 shows the real (5.8 a)) and imaginary (5.8 b)) part of the sensitivity to a steady force, $\nabla_{\mathbf{q}}\sigma$, with Figure 5.9 showing $\nabla_{\mathbf{q}}\sigma_r$ with streamlines representing the momentum components of the sensitivity terms. It can be seen that the magnitude of the sensitivity decays to zero far from the cylinder and approaching the boundaries of the domain as the influence of the direct and adjoint modes is extremely small. The highest magnitude

regions for $\nabla_{\mathbf{q}}\sigma_r$ are located downstream of the cylinder. By following the streamlines in Figure 5.9, and applying a base flow modification in the direction of the streamlines, would result in a increase in a positive increase in the growth rate, producing a destabilising effect. Therefore, applying a base flow modification aft the cylinder, in the high magnitude regions, orientated downstream would have a stabilising effect. Good agreement with [69] is obtained for the sensitivity to base flow modifications.

Finally, Figure 5.10 and 5.11 shows the real (5.10 a)) and imaginary (5.10 b)) part of the sensitivity to a steady force, $\nabla_{\mathbf{q}_f}\sigma$. The results are in excellent agreement with the sensitivity maps published by Marquet *et al.* [64]. Three main regions are identified where the flow is most sensitive: 1) 1 diameter to 2 diameters downstream of the cylinder along $y = 0$, 2) very close (downstream) the cylinder at plus 45 degrees from $y = 0$ and 3) very close (downstream) the cylinder at minus 45 degrees from $y = 0$.

Moreover, Figure 5.10 gives us the regions of the flow where inputting a steady force would have the greatest effect or to be more specific; the greatest shift in eigenvalue and hence, stabilise or destabilise the flow. By examining Figure 5.11a), it can be delineated that by inputting a steady force at $(x=1, y=0)$ and orientated downstream would decrease the growth rate, σ_r , and subsequently stabilise the flow. In Figure 5.10b) the same force would result in an increase in the frequency, σ_i . Furthermore, the streamlines in Figure 5.11 shows similar regions to the regions for stabilising the flow found experimentally by Strykowski and Sreenivasan [93].

Figure 5.8: Sensitivity to base flow modifications; a) real part, $\nabla_{\bar{q}}\sigma_r$, and b) imaginary part, $\nabla_{\bar{q}}\sigma_i$, for the circular cylinder.

Figure 5.9: Sensitivity to base flow modifications real part, $\nabla_{\bar{q}}\sigma_r$, for the circular cylinder.

Figure 5.10: Sensitivity to a steady force; a) real part, $\nabla_{\mathbf{qf}} \sigma_r$, and b) imaginary part, $\nabla_{\mathbf{qf}} \sigma_i$, for the circular cylinder.

Figure 5.11: Sensitivity to a steady force real part, $\nabla_{\mathbf{qf}} \sigma_r$, for the circular cylinder.

5.3 Sensitivity Derivatives

It was seen in the previous section that the sensitivity to steady forcing, Figure 5.10, agreed very well with the work of Marquet *et al.* [64] (continuous approach, incompressible Navier-Stokes eqn's) and the work of Mettot *et al.* [69] (discrete approach, compressible Navier-Stokes eqn's) whereas for the sensitivity to base flow modifications, Figure 5.8, the same level of agreement to Marquet *et al.* was not achieved. However, good agreement with Mettot *et al.* was found. Mettot *et al.* point out that there can be differences when comparing the incompressible and compressible form of the equations and the fact that there is some difference is not in itself surprising. Marquet *et al.* results are the velocity components (u, v) while this work and Mettot *et al.* present the momentum terms $(\rho u, \rho v)$. Further analysis into derivation of the sensitivity equations alludes to more reasons for the discrepancies.

In the continuous framework, for the incompressible Navier-Stokes eqn's, the following is derived for computing the sensitivity to base flow modifications [64]

$$\nabla_{\mathbf{u}}\sigma = -(\nabla\hat{\mathbf{u}})^H \cdot \hat{\mathbf{u}}^+ + \nabla\hat{\mathbf{u}}^+ \cdot \hat{\mathbf{u}}^*, \quad (5.1)$$

where H is the Hermitian transpose. It can be seen that the computation of $\nabla_{\mathbf{u}}\sigma$ is done by taking first order derivatives. We now look at the discrete framework. If the equations for the sensitivity fields

$$\nabla_{\bar{\mathbf{q}}}\sigma = -\mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}})\hat{\mathbf{q}}^+ \quad (5.2)$$

$$\nabla_{\mathbf{q}_f}\sigma = (\mathcal{J}^+(\bar{\mathbf{q}}))^{-1} \mathcal{B}^+(\bar{\mathbf{q}}, \hat{\mathbf{q}})\hat{\mathbf{q}}^+, \quad (5.3)$$

are rewritten in differential form then we obtain

$$\nabla_{\bar{\mathbf{q}}}\sigma = \frac{\partial}{\partial \bar{\mathbf{q}}^+} \left(\frac{\partial \mathcal{F}^+(\bar{\mathbf{q}}^+)}{\partial \bar{\mathbf{q}}^+} \hat{\mathbf{q}} \right) \hat{\mathbf{q}}^+, \quad (5.4)$$

$$\nabla_{\mathbf{q}_f}\sigma = \left(\frac{\partial \mathcal{F}^+(\bar{\mathbf{q}}^+)}{\partial \bar{\mathbf{q}}^+} \right)^{-1} \frac{\partial}{\partial \bar{\mathbf{q}}^+} \left(\frac{\partial \mathcal{F}^+(\bar{\mathbf{q}}^+)}{\partial \bar{\mathbf{q}}^+} \hat{\mathbf{q}} \right) \hat{\mathbf{q}}^+, \quad (5.5)$$

where $\mathcal{F}^+$ is the adjoint operator. It can be seen that Equation 5.4 is of second order due to the differentiation of the adjoint operator and the direct mode twice.

Therefore, the discrete approach requires a higher order derivative to be taken in order to compute $\nabla_{\bar{\mathbf{q}}}\sigma$. Furthermore, the spatial discretisation method employed for this analysis (Discontinuous Galerkin Spectral Element Methods) does not enforce that the solution $\bar{\mathbf{q}}$ at the boundaries of neighbouring elements be equal (C^0 continuous). Given this and coupled with $\nabla_{\bar{\mathbf{q}}}\sigma$ needing a second order derivative, the discontinuities between different elements in Figure 5.8 can be accounted for. Finally, given that $\nabla_{\bar{\mathbf{q}}}\sigma$ is subsequently used in the computation of $\nabla_{\mathbf{q}_f}\sigma$ ($\nabla_{\mathbf{q}_f}\sigma = (\mathcal{J}^+(\bar{\mathbf{q}}))^{-1} \nabla_{\bar{\mathbf{q}}}\sigma$) and good agreement is found for $\nabla_{\mathbf{q}_f}\sigma$, we can be convinced in the results.

5.4 Open Cavity Flow at $Re = 1400$

5.4.1 Base Flow Computations

To further validate the method, a convectively unstable case is presented. For a system to be convectively unstable, the instabilities must grow in space but not in time. The open cavity is chosen to demonstrate the methodology in the context of a convectively unstable flow. The open cavity flow shows a global instability inside the cavity while a convectively instability develops upstream of the cavity.

Figure 5.12: Overview of the dimensions and boundary conditions for the open cavity

The discretised computational domain can be seen in Figure 5.12 with the corresponding boundary conditions. The inflow boundary condition is computed by solving the Blasius boundary layer equations based on the Reynolds number, Re , and a desired boundary layer thickness, Θ_{inlet} . The Reynolds number, $Re = \frac{U_{\infty} D}{\nu} = 1400$, based on the cavity depth, D , free-stream velocity, U_{∞} and kinematic viscosity, ν , and dimensionless boundary layer thickness, $\Theta_{inlet} = \frac{\Theta}{D} = 0.0337$, are both given as a function of the cavity depth, D . The computational mesh comprises of 394 quadrilateral elements, each with $P=4$. The problem has been extensively studied by de Vicente [100] and Meseguer *et al.* [68] who show that convective stability of the system is strongly dependent on the outflow boundary

condition. Their work is used for validating the base flow at $Re = 1400$. The outflow boundary condition does not affect the frequency, σ_i , but can affect the growth rate, σ_r , of the system because the outflow boundary condition can modify the spatial growth of the disturbance for this convectively unstable case.

The base flow fields, $\overline{\rho u}$, $\overline{\rho v}$ and $\overline{\omega}$, for the open cavity can be seen in Figures 5.13 a), b) and c) respectively. These agree well with other studies in the literature [100, 68].

Figure 5.13: Open cavity base flow fields at $Re = 1400$; a) $\overline{\rho u}$, b) $\overline{\rho v}$ and c) $\overline{\omega}$

5.4.2 Linear Stability Analysis

Figure 5.14 shows the eigenvalue spectrum computed for the open cavity. The eigenvalue which corresponds to the first Rossiter mode is selected. This corresponds to the eigenvalue which is circled in Figure 5.14. In this case, this is not the most unstable eigenvalue but it is known from [68, 82, 13], that as the Reynolds number increases the growth rate, σ_r , for this eigenvalue will eventually become positive. We achieve a frequency of $f_\sigma = 0.236$ while Table 5.2 compares this result with other studies. It can be seen that relatively good agreement has been achieved. The real part of the eigenmodes corresponding to this eigenvalue are shown in Figure 5.15. In both cases the significant part of the mode extends downstream of the trailing edge of the cavity and a structure similar to a Tollmien–Schlichting wave, present in boundary layer flows, can be seen for both $\widehat{\rho u}$ and $\widehat{\rho v}$. The adjoint modes can be seen in Figure 5.16. These modes give information about the most receptive regions and, as would be expected, these are found just upstream of the cavity.

	I/C	σ_i	frequency ($f_\sigma = \frac{\sigma_i}{2\pi}$)
Meseguer <i>et al.</i> [68]	I	-	~ 0.25
Rowley <i>et al.</i> [82]	I	-	~ 0.26
Current Study	C	1.505	~ 0.24

Table 5.2: Numerical comparison of the non-dimensional frequency of the least stable eigenvalue for the flow past an open cavity

Figure 5.14: The eigenvalue spectrum for flow past an open cavity at $\text{Re} = 1400$. The circled eigenvalue corresponds to the first Rossiter mode.

Figure 5.15: The real part of the open cavity direct eigenmode fields; a) $\widehat{\rho u}$ and b) $\widehat{\rho v}$

Figure 5.16: The real part of the open cavity adjoint eigenmode fields; a) $\widehat{(\rho u)^+}$ and b) $\widehat{(\rho v)^+}$

5.4.3 Sensitivity Analysis

Figure 5.17 shows the structural sensitivity for flow past an open cavity. It can be seen that the “wavemaker” region is located just upstream of the leading edge of the cavity. This is the region just before the boundary layer detaches, so it is intuitive that this would be the wavemaker.

Figure 5.17: Structural sensitivity for the open cavity at $Re = 1400$

Figure 5.18 shows the sensitivity of the open cavity to base flow modifications, $\nabla_{\bar{\mathbf{q}}} \sigma$. It can be seen that for both the growth rate, $\nabla_{\bar{\mathbf{q}}} \sigma_r$, and the frequency, $\nabla_{\bar{\mathbf{q}}} \sigma_i$, the region of the flow which is most sensitive is located just upstream of the cavity. This indicates here that a base flow perturbation could have a stabilising effect on the wave which can be seen in Figure 5.15 for $\widehat{\rho u}$ and $\widehat{\rho v}$. The most sensitive region in both Figure 5.18 a) and b) coincides with the most sensitive region identified by the structural sensitivity in Figure 5.17.

Figure 5.19 shows the sensitivity to a steady force, $\nabla_{\mathbf{q}_f} \sigma$. Contrariwise to Figure 5.18, Figure 5.19 indicates that putting a steady force inside the cavity will the greatest stabilising effect. There is also a region downstream of the leading edge of the cavity where a force could also be put to have the same effect. This could be due to the global instability, which, as mentioned, originates from inside the cavity.

Figure 5.18: Sensitivity to base flow modifications, a) $\nabla_{\bar{q}} \sigma_r$ and b) $\nabla_{\bar{q}} \sigma_i$, for the open cavity at $Re = 1400$

Figure 5.19: Sensitivity to a steady force, a) real part, $\nabla_{\mathbf{qf}} \sigma_r$, and b) imaginary part, $\nabla_{\mathbf{qf}} \sigma_i$, for the open cavity at $\text{Re} = 1400$

Chapter 6

Towards Industrial Cases

6.1 Actuator Disc

The linear momentum actuator disc theory (LMADT) often referred to as momentum theory or simply actuator disc theory was first introduced by [39] to model ship propellers and subsequently extended by [56] and [18], who derived limits on the efficiency for this simplified model. More recently, various authors have particularised the theory to model helicopter rotors [58], centrifugal pumps, turbochargers and wind turbines [94]. During the last decade, the rise of new engineering devices for energy extraction such as tidal turbines has required the modification of the classic theories to accommodate for blockage effects due to confinement (between sea surface and ground) [102, 73].

The actuator disc model and the LMADT theory (see section 6.2.1 for details) provide a simplified means to model rotors that provide thrust (e.g. propellers or helicopters) or that extract energy from the flow (e.g. wind, tidal turbines). These simplified rotors provide a useful framework for the analysis of wake structures with a lower computational cost compared with simulating rotating blades. Indeed, the actuators disc may be used to represent the rotor disc, whilst the far field is computed using a numerical approach. The wake resulting from the disc can be simulated using a Navier-Stokes solver and thus can evolve freely.

The wake unsteadiness aft of simple bodies (e.g. cylinders) is governed by flow parameters, such as the Reynolds number, the free stream turbulence, and geometric shapes and constraints (e.g. geometry of the body, flow confinement) [103, 25].

Predicting the wake and its steadiness is of particular importance for wind and tidal turbines for clean energy generation. These devices are typically distributed in clusters or farms. Ideally, the spacing between devices should be small to allocate a maximum number of turbines in a minimum space, which should maximise energy extraction. However, locating devices close to each other may not only limit the energy extracted by shadowing effects, but can also lead to problems related to the interactions of one wakes with neighbouring devices. Unsteady wakes and associated fluctuating forces are important since they are correlated to undesired phenomena, such as unpredictable unsteady effects (e.g. dynamic stall on the blades), increased forces, reduced fatigue life or noise generation. Some of these effects can undermine the structural integrity and reduce the operating life of the turbine and hence its prediction and control is paramount.

Two recent studies [91, 90] have presented two and three dimensional instability analysis of an Optimum Glauert rotor [94] for various Reynolds numbers. It was shown that over the range of Reynolds numbers two dimensional and three dimensional computations provided similar unsteady perturbation shapes (i.e. direct modes). In addition, they show that two dimensional computations were more unstable than three dimensional computations for the flow conditions (i.e. the same Reynolds number). Other works [74, 92, 85] have performed stability analysis on turbine wakes using an actuator line method combined with a Large Eddy simulation far field to simulate the wake behind a wind turbine. These papers focus on capturing the physics of the wake but not controlling it.

Figure 6.1 shows how a Kelvin-Helmholtz instability develops in the case of a fluid moving at different velocities. As the velocity gradient increases, a shear layer develops at the interface and this leads to overturning in fluid, resulting in a localised spinning motion. The K-H instability presents itself in many areas in engineering such as fluids, geo-physical and ocean/marine, which has resulted in it has becoming a well investigated problem. Two

Figure 6.1: Kelvin-Helmholtz instability

early works by Thorpe [97, 98] showed experimentally the onset of the K-H instability as a consequence of the growth of waves at the shear layer interface and found good agreement with theoretical results. In the context of turbines, Hodson and Howell [42] present a review of low pressure turbine wakes, detailing the breakdown to turbulence, one mechanism being due to the K-H instability, and how to reduce the losses caused by such mechanisms.

6.2 Problem Formulation

6.2.1 Actuator Disc Modelling

In this section we briefly summarise the linear momentum actuator disc theory (LMADT) to provide context to our study. A complete overview of this theory can be found elsewhere [58, 94, 17]. The main results of the theory include the determination of the Power (C_p) and Thrust (C_t) coefficients for an actuator disc. In the idealised inviscid and steady case, these are functions of the axial induction factor a , that relates to the deceleration of the flow through the actuator disc. Namely, a is

$$U_t = U_\infty(1 - a), \quad (6.1)$$

Figure 6.2: Power and Thrust coefficients based on LMADT

where U_t is the velocity of the flow through the actuator and U_∞ is the velocity in the unperturbed upstream far field. Using mass, momentum-force conservation and Bernoulli's principle, one can derive the following expressions for the Power and Thrust coefficients:

$$C_p = 4a(1 - a)^2 \quad ; \quad C_t = 4a(1 - a). \quad (6.2)$$

In 1920, Betz [18] derived using the LMADT theory the maximum power that can be provided or extracted from the flow using an idealised rotor as modelled by an actuator disc. To do so it suffices to derive and equate to zero the previous expression for the power coefficient to obtain the optimal induction $a^{opt} = 1/3$ and corresponding maximum power coefficient $C_p^{opt} = 16/27$. The power and thrust coefficient are illustrated in Figure 6.2.

In our study, we select an actuator forcing through a deficit of momentum (a body force in the Navier-Stokes equations). Therefore the condensed form of the 2D compressible Navier-Stokes equations becomes

$$\frac{d\mathbf{q}}{dt} = \mathcal{F}(\mathbf{q}) + \mathbf{f}, \quad (6.3)$$

where $\mathbf{f} = (0, f, 0, 0)$. The pressure drop (Δp) is computed across the disc. This pressure difference can be related to the coefficient of thrust using $C_t = 2\Delta p / \varrho U_\infty^2$, where ϱ denotes the fluid density. Finally, our study comprises cases for $0.2 < C_t < 0.45$ corresponding to induction factors $0.05 < a < 0.13$. In all cases the force applied by the actuator to the flow is uniform and constant. Note that other more sophisticated approaches that account for radial variations in rotor geometry can be used (e.g. Glauert rotor in [94, 90, 91]) but are not retained in our work.

6.3 Results

After a preliminary mesh convergence and domain size study, the final computational domain, depicted in Figure 6.3, has 742 quadrilateral elements with a global polynomial order of 4. The left Dirichlet boundary is a uniform inflow for velocity whilst the upper, right and lower boundaries are treated with a Neumann outflow boundaries. The zoomed region, in Figure 6.3, details the location where the forcing term, $\mathbf{f} = (0, f, 0, 0)^T$, is applied (in red) and shows the effective degrees of freedoms for the high order technique.

6.3.1 The Onset of Wake Unsteadiness

We consider a range of Reynolds numbers (Re), based on the rotor diameter D and the free stream velocity, from $\text{Re} = 1000$ to 3500 and actuator disc forcing (f) from $f = 0.5$ to 1.2 . Note that $f = 0.5$ correspond to an induction factor of $a = 0.05$, whilst $f = 1.0$ corresponds to $a = 0.13$ (see section 6.2.1). The simulated cases together with their wake behaviour (i.e. steady or unsteady) are summarised in Table 6.1. The table reports that for a fixed forcing $f = 1.0$ a stable wake becomes unstable if the Reynolds number

Actuator forcing f	Re=1000	Re=1500	Re=2000	Re=2500	Re=3000	Re=3500
0.50	S	S	S	S	S	S
0.75	S	S	S	S	S	S
1.00	S	S	S	S	S	U
1.20	S	S	U	U	U	U

Table 6.1: Stability of the wake of an actuator disc when varying the Reynolds number (Re) and the actuator forcing (f). The table details the wake behaviour: steady (S) or unsteady (**U**).

is increased from $\text{Re} = 3000$ to $\text{Re} = 3500$. Alternatively, the table shows that for a fixed Reynolds $\text{Re} = 3000$, an increase in the actuator forcing from $f = 1.0$ to $f = 1.2$ destabilises the wake. We illustrate the destabilising effect of increasing the Reynolds number, in Figures 6.4 a):f) . Figures 6.4 a), b) and c) depict the streamwise, transverse velocity fields and the vorticity magnitude for a stable case at $\text{Re} = 3000$ and $f = 1.0$, whilst Figures 6.4 d), e) and f) show an unstable wake for $\text{Re} = 3500$ and $f = 1.0$.

Figure 6.3: Domain and mesh for the actuator disc case.

Figures 6.5 a) to c) and Figures 6.5 d) to f) present streamwise and transverse velocities in sections parallel to the actuator at various streamwise locations for a steady ($\text{Re} = 1000$) and an unsteady ($\text{Re} = 3500$) case. Figures 6.4 f) and 6.5 suggest a “Kelvin-Helmholtz” (KH) type instability. The KH instability can be theoretically analysed in inviscid fluids [31].

Since viscosity smooths the steep velocity gradients, responsible of KH instabilities, lower

Figure 6.4: Contours of streamwise velocity a) and d), transverse velocity b) and e), and vorticity magnitude c) and f). Steady flow shown for $Re = 3000$, in a), b) and c), and unsteady flow for $Re = 3500$ d), e) and f). Both flows are computed with the same actuator forcing $f = 1.0$.

Figure 6.5: Streamwise velocity for $Re = 1000$ (stable) and 3500 (unstable), at a) $x/D = 5$, b) $x/D = 15$ and c) $x/D = 20$. Transverse velocity for $Re = 1000$ (stable) and 3500 (unstable), at d) $x/D = 5$, e) $x/D = 15$ and f) $x/D = 20$.

Figure 6.6: a) Vorticity magnitude along the streamwise line $y/D = 0.57$ for $Re = 1000$ (stable), 3000 (stable), and 3500 (unstable) are shown. b) zoom of 3500 (unstable)

Reynolds numbers provide more stable wakes. Similarly, a reduced actuator forcing results in a smoother velocity gradient, which produces a more stable wake (not shown). Figures 6.5 a) to 6.5 f) show that the velocity fields present a step gradient near the edge of the

actuator that becomes smoother for sections further downstream. These gradients are smoother for decreased Reynolds numbers, which illustrates the stabilising effect of the viscosity. Figure 6.6 a) presents the evolution of the vorticity magnitude along the line $y/D = 0.57$, which lies at the edge of the actuator disc and in the streamwise direction along the high vorticity region depicted in figures 6.4 c) and 6.4 f). For both $Re = 1000$ and $Re = 3000$, the vorticity decays downstream of the actuator whereas for $Re = 3500$ (unsteady case) the vorticity oscillates aft 10 diameters downstream of the actuator disc. This illustrates that the onset of the instability is not at the edge of the actuator but 10 to 15 diameters downstream. Finally, the spatial location for the instability onset moves upstream for higher Reynolds number or higher actuator forcing (not shown). The aim of this work is to defer the onset of wake unsteadiness such that the wake remains steady for significantly higher Reynolds numbers or actuator forcing.

6.3.2 Linear Stability Analysis

The complex eigenvalues σ are obtained from the direct eigenvalue problem as detailed in previous sections. As mentioned, when $\sigma_r > 0$ the system is said to be linearly unstable in time (i.e. absolute instability).

Firstly, Figure 6.7 shows the convergence of the a) growth rate (σ_r) and b) frequency (σ_i) with respect to polynomial order. The evolution of the most unstable eigenvalue is presented in Figures 6.8 a) to 6.8 d), when varying the Reynolds number (Re) and the actuator forcing (f). These Figures confirm and quantify the destabilising effect of increasing the Reynolds number or the actuator forcing.

It can be seen that increasing the actuator forcing (e.g. extracting more energy using a wind turbine), has a destabilising effect on the wake, which is comparable to the destabilising effect of an increase in the Reynolds number. Note that for wind/tidal turbines there is little or no control over the Reynolds number as this is dictated by the environment. However, the resistance of the turbine can be modified by adjusting the speed at which its blades rotate. In addition, small variations were observed in the frequencies (imaginary

Figure 6.7: Convergence of the a) growth rate (σ_r) and b) frequency (σ_i) with respect to polynomial order.

part of the eigenvalues) when varying the Reynolds number or the actuator forcing but are not shown.

Second order polynomials are used to model the evolution of σ_r . We find the polynomials for Figure 6.8 a) ($f = 1.0$) and Figure 6.8 c) ($Re = 3000$) respectively

$$\begin{aligned}\sigma_r(Re) &= 2.4 \times 10^{-10} Re^2 - 3.8 \times 10^{-7} Re - 1.9 \times 10^{-3}, \\ \sigma_r(f) &= 7.6 \times 10^{-3} f^2 - 9.9 \times 10^{-3} f + 1.5 \times 10^{-3}.\end{aligned}$$

Computing the roots of the polynomial provides an approximation of the critical values for Reynolds and forcing for which the system becomes unstable. We find $Re_{crit} \approx 3643$ and $f_{crit} = 1.12$. These estimates are validated by brute force computations to find that the wake becomes unsteady in the range $3250 < Re_{crit} < 3300$ (for $f = 1.0$) and for $f_{crit} > 1.01$ (for $Re = 3000$). Furthermore, taking derivatives of the models, we can obtain

Figure 6.8: a) and c) Growth rates for the most unstable eigenvalue at various Reynolds (Re) and actuator forcing f . b) and d) frequencies for the most unstable eigenvalue at various Reynolds (Re) and actuator forcing f .

the sensitivity of the growth rates with respect to the Reynolds and actuator forcing

$$\left. \frac{\partial \sigma_r}{\partial Re} \right|_{Re=3000} = 1.1 \times 10^{-6},$$

$$\left. \frac{\partial \sigma_r}{\partial f} \right|_{f=1.0} = 5.0 \times 10^{-3}.$$

These sensitivities quantify that a variation of the actuator forcing $\delta f \approx 0.1$ has a similar effect to a variation of Reynolds $\delta Re \approx 450$.

For completeness, Figure 6.9 a) and 6.9 b) depict the direct and adjoint streamwise-

Figure 6.9: a) Direct mode (real part streamwise momentum) and b) Adjoint mode (real part streamwise momentum), for $Re = 3000$ and $f = 1$ (steady case).

momentum components (real part), respectively, corresponding to the most unstable eigenvalue for $Re = 3000$ and $f = 1.0$. It can be seen for the direct mode that the perturbations begins aft on the actuator and its amplitude grows when moving downstream. The adjoint mode propagates upstream, which is explained by the upstreaming advective nature of the adjoint equations [59].

6.3.3 Wake Control Using Sensitivity Analysis

6.3.3.1 Sensitivity Analysis

Our interest is in stabilising the most unstable eigenvalue identified with linear instability analysis. In this section we compare three sensitivities: the structural sensitivity, the sensitivities to base flow modifications and localised forcing. The structural sensitivity or “wavemaker” region [40] is shown in Figure 6.10. This identifies the most sensitive regions to localized feedback and results from computing the dot product of direct and adjoint modes.

Figure 6.10 shows that the wavemaker region becomes important in the region downstream of the actuator ($x/D > 20$) and that a much smaller region appears near the edges of the actuator disc.

Figure 6.10: structural sensitivity (wavemaker region).

Figure 6.11 a) reports the sensitivity to a base flow modifications (real part of $\nabla_{\bar{q}}\sigma$ in equation (2.37)) and confirms the two regions detected by the structural sensitivity. Figure 6.11 b) presents the sensitivity to a steady force (real part of $\nabla_{\mathbf{q}_f}\sigma$ in equation (2.37)). The latter sensitivity detects a flow region closer to the actuator ($-5 < x/D < 5$) and within the shear layer arising from the edges of the actuator disc.

Figure 6.11: a) base flow modifications ($\nabla_{\bar{q}}\sigma$) and b) a steady force ($\nabla_{\mathbf{q}_f}\sigma$).

In summary, two regions are localised by the sensitivity analysis. The first one is located far downstream of the actuator as indicated by the structural sensitivity and the sensitivity to base flow modifications. A second zone appears clearly when the sensitivity to a steady forcing is computed. Note that for engineering applications (e.g. wind turbines), having regions relatively close to the actuator is desirable as it eases the implementation of a passive control technique to stabilise the flow.

6.3.3.2 Wake Stabilisation

To validate the methodology and provide guidelines for flow control of actuator discs, infinitesimal forces are placed in the regions identified by the various sensitivity contour maps. Two sets of forces (applied in the opposite direction to the freestream velocity) are added: $f1$ is located near the actuator disc $(x/D, y/D) = (1.6, \pm 0.75)$, whilst $f2$ is situated far downstream $(x/D, y/D) = (27.5, \pm 0.75)$, both within the shear layer region. The locations for $f1$ and $f2$ are shown in Figure 6.12.

Figure 6.12: Positions of infinitesimal forces to enhance stability: near disc control $f1$ (red squares) and downstream control $f2$ (blue squares).

Growth rates for uncontrolled (original) and controlled cases $f1$ and $f2$ (applied separately) for a range of Reynolds numbers $2000 < \text{Re} < 5000$ (with fixed actuator forcing $f = 1.0$) are shown in Figure 6.13 a), whilst growth rates for a variety of actuator forcing $0.5 < f < 1.2$ (at fixed $\text{Re} = 3000$) are plotted in Figure 6.13 b). The Figures confirm that both controls $f1$ and $f2$ stabilise considerably the wake.

The effect of the near disc control $f1$ and the downstream control $f2$ increases when increasing the Reynolds numbers or the actuator forcing, but $f1$ shows a more effective stabilising effect for the range of Reynolds and forcing considered. As for the uncontrolled case, we fit a second order polynomial to the data for the $f2$ control

$$\begin{aligned}\sigma_r(\text{Re}) &= 2.3 \times 10^{-10} \text{Re}^2 - 7.2 \times 10^{-7} \text{Re} - 1.3 \times 10^{-3}, \\ \sigma_r(f) &= 3.3 \times 10^{-3} f^2 - 4.6 \times 10^{-3} f - 1.1 \times 10^{-4},\end{aligned}$$

which predicts a $\text{Re}_{crit} = 4396$, quantifying the enhanced stabilisation by 21% (the uncontrolled case showed $\text{Re}_{crit} \approx 3643$). For completeness, we compute the sensitivities of

Figure 6.13: Uncontrolled wake (original) compared to near disc control ($f1$) and downstream control ($f2$): Growth rates for the most unstable eigenvalue for a) various Reynolds numbers (Re) (fixed actuator forcing $f = 1.0$) and b) various actuator forcing f (and fixed $Re = 3000$).

Figure 6.14: Uncontrolled wake (original) compared to near disc control ($f1$) and downstream control ($f2$): Vorticity magnitude along the horizontal line $y/D = 0.57$ for $Re = 3500$ and $f = 1.0$.

the growth rate (σ_r) to a change in Re and actuator forcing (f) by taking derivatives of the models.

$$\left. \frac{\partial \sigma_r}{\partial Re} \right|_{Re=3000} = 6.65 \times 10^{-7},$$

$$\left. \frac{\partial \sigma_r}{\partial f} \right|_{f=1.0} = 2.0 \times 10^{-3}.$$

The percentage reduction in sensitivity to the Reynolds is 39% and to the actuator forcing

Figure 6.15: Streamwise velocity at sections $x/D = 5$, $x/D = 10$, $x/D = 15$, $x/D = 20$, $x/D = 25$ and $x/D = 30$ for $Re = 3500$ and $f = 1.0$.

Figure 6.16: Direct mode (real part streamwise momentum) for $Re = 3000$ and $f = 1$ for a) no forcing, b) $f1$ and c) $f2$.

60%, when compared to the uncontrolled cases. Again, we quantify these sensitivities by noting that a variation of the actuator forcing $\delta f \approx 0.1$ is now comparable to a variation of Reynolds $\delta Re \approx 300$. In addition, let us note that the sensitivities for the control $f1$ are close to zero, which follows from the curves being almost horizontal near $Re = 3000$ and $f = 1.0$. For the $f1$ control the wake becomes unstable at $Re_{crit} \approx 7500$ and a different eigenmode is responsible of the instability.

In addition, Figure 6.14 reports the vorticity evolution for the modified actuator with the control forcing $f1$ and $f2$ to confirm the stabilising effect at the shear layer.

Figure 6.16 shows the direct mode (real part streamwise momentum) for $Re = 3000$ and $f = 1$ for no forcing, $f1$ and $f2$. Figure 6.13 showed that the *absolute stability* of the system was increased by the forcing but Figure 6.16 shows that the *convective stability* is also increased. Particularly for Figure 6.16 b) it can be seen the amplitude of the mode structure is reduced far downstream of the actuator in comparison to Figure 6.16 a).

We finally report the streamwise velocity at downstream sections parallel to the disc in Figure 6.15 d) for the original unsteady wake, and the controlled $f1$ and $f2$ at $Re = 3500$ and $f = 1.0$. It can be seen for the $f1$ case that the forcing initialises a smoother velocity gradient which has a stabilising effect on the wake (Figure 6.15 a)). A similar effect is seen for $f2$ (Figure 6.13 f)) but the latter, acting further downstream, results in a less efficient stabilisation mechanism. We conclude that the $f1$ control located close to the actuator is best suited to enhance the stability of actuator wakes.

6.4 Summary

This chapter shows how to stabilise the wake arising from an actuator disc. Using linear stability and sensitivity analysis based on adjoints, we find the regions of the flow that when modified provide a more stable wake. We test this procedure for a range of Reynolds numbers and actuator forcing to show its usability at stabilising or delaying unsteadiness in the wake of an actuator disc. It is shown that locating point forces close behind the

disc modifies the shear layer profile stabilising the wake. This mechanism is more efficient than stabilising the wake far downstream, where instabilities first arise. It is expected that these results are applicable to a wide range of engineering devices, where the actuator disc theory (LMADT) holds such as propellers, helicopter rotors or wind and tidal turbines.

Chapter 7

Conclusions

7.1 Summary of Thesis Achievements

This thesis has presented a general numerical methodology for conducting sensitivity and stability analysis. Although the method could be applied to any partial differential equation containing a spatial operator and a time derivative, in this thesis, it has been investigated for the compressible NS equations. It was found that it is insufficient to use a standard finite difference approximation to obtain an accurate sensitivity matrix when analysing a small problem with an analytical solution. To overcome this problem the method of complex step approximation has been used and has shown to provide a far superior prediction of the sensitivity matrix at a reduced computational cost. It was also shown for the lid driven cavity case that near analytic accuracy can be achieved for Jacobian calculation when using complex variables. The method has been validated using the flow past a circular cylinder at a Reynolds number below critical Reynolds number. Good quantitative and qualitative agreement has been found with published data for the same test case, where the continuous approach was used together with the incompressible NS equations. The sensitivity maps correctly show the regions of the flow which are most sensitive when subjected to a steady force. Flow past an open cavity was also presented to give an example of a convectively unstable case. The proposed method also agrees

well in the prediction of the eigenvalues with other authors for this case. The discrete approach offers advantages over the continuous:

1. The adjoint base flow equations do not need to be computed. This is the case for the continuous approach but this is achieved for the discrete approach with Eq. (2.10),
2. If a continuous approach was opted for, then to compute the sensitivity to a steady force, the NS equations (condensed form Eq. (2.1)) would need to be recomputed but with an extra source term, $\nabla_{\mathbf{q}}\sigma$, inserted.
3. Near to analytic accuracy is achieved for the Jacobian matrix when employing complex variables.

The application of this methodology to more complex models, including turbulence models and highly compressible flows, is straightforward and may give valuable information of the regions of the flow where the applications of a small perturbation can produce the maximum output.

We also present results that shows how to stabilise the wake arising from an actuator disc. Using linear stability and sensitivity analysis based on adjoints, we find the regions of the flow that when modified provide a more stable wake. We test this procedure for a range of Reynolds numbers and actuator forcing to show its usability at stabilising or delaying unsteadiness in the wake of an actuator disc. It is shown that locating point forces close behind the disc modifies the shear layer profile stabilising the wake. This mechanism is more efficient than stabilising the wake far downstream, where instabilities first arise. It is expected that these results are applicable to a wide range of engineering devices, where the actuator disc theory (LMADT) holds such as propellers, helicopter rotors or wind and tidal turbines.

7.2 Future Work

The sensitivity analysis for the sensitivity to base flow modifications and to a steady force has been investigated for well studied 2D fluid dynamics problems and for the actuator disc in this thesis. This analysis can be taken forward by now applying this to more complex problems such as turbulent cases or 3D cases. These will be more computationally demanding but will provide relevant results for industrial type problems. Furthermore, this analysis can also be tested for cases where the flow is unstable - to stabilise flow that is already unstable.

Section 2.4.4 presents the mathematical framework for computing the sensitivity to geometric modifications. This methodology can be extended and applied to test cases to investigate its usefulness.

Appendix A

Equivalence of Lagrangian and Alternative Method

From section 2.4.4 we have

$$\nabla_{\mathbf{w}}\sigma = \mathcal{B}_{\mathbf{w}}^+\hat{\mathbf{q}}^+ + \mathcal{J}_{\mathbf{w}}^+\bar{\mathbf{q}}^+, \quad (\text{A.1})$$

$$\left(\frac{\partial\sigma}{\partial\mathbf{w}}\right)^* = \left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}\right)^* \frac{\partial\bar{\mathbf{q}}}{\partial\mathbf{w}}. \quad (\text{A.2})$$

By writing Equation (A.2) in terms of matrices and vectors (noting the transpose from above and remembering $\left(\frac{\partial\sigma}{\partial\bar{\mathbf{q}}} = \mathcal{B}_{\bar{\mathbf{q}}}^+\hat{\mathbf{q}}^+\right)$ we get

$$\nabla_{\mathbf{w}}\sigma = - \underbrace{(\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\bar{\mathbf{q}}}^+)^*}_{\frac{\partial\sigma}{\partial\bar{\mathbf{q}}}} \underbrace{(\mathcal{J}_{\bar{\mathbf{q}}}^{-1}\mathcal{J}_{\mathbf{w}} + \mathcal{B}_{\bar{\mathbf{q}}}^{-1}\mathcal{B}_{\mathbf{w}})}_{\frac{\partial\bar{\mathbf{q}}}{\partial\mathbf{w}}}. \quad (\text{A.3})$$

By subbing the RHS of Equation (A.1) into the parentheses on the LHS of (A.2) (i.e. equating the two equations) we arrive at

$$\underbrace{(\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\mathbf{w}}^+)^* + (\bar{\mathbf{q}}^+)^*(\mathcal{J}_{\mathbf{w}}^+)^*}_{\frac{\partial \sigma}{\partial \mathbf{w}}} = \underbrace{(\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\bar{\mathbf{q}}}^+)^*}_{\frac{\partial \sigma}{\partial \bar{\mathbf{q}}}} \underbrace{(\mathcal{J}_{\bar{\mathbf{q}}}^{-1} \mathcal{J}_{\mathbf{w}} + \mathcal{B}_{\bar{\mathbf{q}}}^{-1} \mathcal{B}_{\mathbf{w}})}_{\frac{\partial \bar{\mathbf{q}}}{\partial \mathbf{w}}}. \quad (\text{A.4})$$

expanding brackets

$$(\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\mathbf{w}}^+)^* + (\bar{\mathbf{q}}^+)^*(\mathcal{J}_{\mathbf{w}}^+)^* = (\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\bar{\mathbf{q}}}^+)^* \mathcal{J}_{\bar{\mathbf{q}}}^{-1} \mathcal{J}_{\mathbf{w}} + (\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\bar{\mathbf{q}}}^+)^* \mathcal{B}_{\bar{\mathbf{q}}}^{-1} \mathcal{B}_{\mathbf{w}}. \quad (\text{A.5})$$

With the generalisation⁵ that the conjugate transpose $\bullet^*$ corresponds to the adjoint operator ($\mathcal{A}^* = \mathcal{A}^+$), the equation can be manipulated in the following way

$$(\hat{\mathbf{q}}^+)^* \mathcal{B}_{\mathbf{w}} + (\bar{\mathbf{q}}^+)^* \mathcal{J}_{\mathbf{w}} = \underbrace{(\hat{\mathbf{q}}^+)^*(\mathcal{B}_{\bar{\mathbf{q}}}^+)^* \mathcal{J}_{\bar{\mathbf{q}}}^{-1}}_{\left(\frac{\partial \sigma}{\partial \mathbf{q}_{\mathbf{f}}}\right)^* = (\bar{\mathbf{q}}^+)^*} \mathcal{J}_{\mathbf{w}} + (\hat{\mathbf{q}}^+)^* \underbrace{(\mathcal{B}_{\bar{\mathbf{q}}}^+)^* \mathcal{B}_{\bar{\mathbf{q}}}^{-1}}_{\mathcal{I}} \mathcal{B}_{\mathbf{w}}. \quad (\text{A.6})$$

Which leaves

$$(\hat{\mathbf{q}}^+)^* \mathcal{B}_{\mathbf{w}} + (\bar{\mathbf{q}}^+)^* \mathcal{J}_{\mathbf{w}} = (\bar{\mathbf{q}}^+)^* \mathcal{J}_{\mathbf{w}} + (\hat{\mathbf{q}}^+)^* \mathcal{B}_{\mathbf{w}}. \quad (\text{A.7})$$

showing the methods are equivalent and converge to the same result.

⁵We employ this generalisation but for equidistant mesh points this is unequivocally valid ($\mathcal{M} \propto \mathcal{I}$)

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