Abstract

In [5], an axiomatic model for contradiction measures on Atanassov Intuitionistic fuzzy sets was presented; there, different kinds of those measures, depending on the continuity conditions required, were established. But in previous papers (see [4]), not only the contradiction in general, but also the contradiction with respect to a given strong intuitionistic fuzzy negation were studied. This is due to the fact that in some applications, in order to fix a suitable model, not any negation is valid, but it is necessary to use a particular one. Thus, the problem of the axiomatization of the different types of contradiction measures regarding a given strong intuitionistic fuzzy negation was studied. This is due to the fact that in some applications, in order to fix a suitable model, not any negation is valid, but it is necessary to use a particular one. Thus, the problem of the axiomatization of the different types of contradiction measures regarding a given strong intuitionistic fuzzy negation remained open. This is the main aim of the present work.

Keywords: Atanassov Intuitionistic fuzzy sets, N-contradiction measures, continuity from below and from above.

1 Preliminaries

1.1 An Atanassov intuitionistic fuzzy set (AIFS) is a set \( A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \), where \( \mu_A : X \rightarrow [0, 1] \), \( \nu_A : X \rightarrow [0, 1] \) are called the membership and non-membership functions, respectively, and such that, for all \( x \in X \), \( \mu_A(x) + \nu_A(x) \leq 1 \) (see [1]). Let us denote the set of all intuitionistic fuzzy sets on \( X \) as \( \mathcal{IF}(X) \).

An AIFS could also be considered as an \( L \)-fuzzy set as defined by Goguen in [10], where the lattice \( L \) is the set
\[
L = \left\{ (\alpha_1, \alpha_2) \in [0, 1]^2 : \alpha_1 + \alpha_2 \leq 1 \right\},
\]
with the partial order \( \leq \) defined as follows: given \( \alpha = (\alpha_1, \alpha_2) \in L \), \( \beta = (\beta_1, \beta_2) \in L \),
\[
\alpha \leq \beta \iff \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \geq \beta_2.
\]

\( (L, \leq) \) is a complete lattice with smallest element \( 0_L = (0, 1) \), and greatest element \( 1_L = (1, 0) \).

So, an AIFS \( A \) is an \( L \)-fuzzy set whose \( L \)-membership function \( \chi_A \in L^X = \{ \chi : X \rightarrow L \} \) is defined for each \( x \in X \) as \( \chi_A(x) = (\mu_A(x), \nu_A(x)) \). The order \( \leq_L \) induces, in a natural way, a partial order in \( L^X \), that we denote in the same way. In this way \( (L^X, \leq_L) \) is a bounded and complete lattice.

Furthermore, let us recall that a decreasing function \( N : L \rightarrow L \) is an intuitionistic fuzzy negation (IFN) if \( N(0_L) = 1_L \) and \( N(1_L) = 0_L \) hold. Moreover, \( N \) is a strong IFN if the equality \( N(N(\alpha)) = \alpha \) holds for all \( \alpha \in L \).

Bustince et al. introduced in [3] the intuitionistic fuzzy generators, which can be used to construct intuitionistic fuzzy negations, and Deschrijver et al. focused on this problem in [8] and [9], and proved that any strong IFN \( N \) is characterized by a strong negation \( N : [0, 1] \rightarrow [0, 1] \) by means of the formula \( N(\alpha_1, \alpha_2) = (N(1 - \alpha_2), 1 - N(\alpha_1)) \), for all \( (\alpha_1, \alpha_2) \in L \). It will be said that \( N \) is the...
negation associated to \( \mathcal{N} \).

1.2 The study of contradiction in the framework of intuitionistic fuzzy sets was initiated in \([6]\). Similarly to the fuzzy case, an AIFS \( A \), or alternatively \( \chi^A \), is said to be contradictory with respect to some strong IFN \( \mathcal{N} \), or, to be short, \( \mathcal{N} \)-contradictory, if \( \chi^A(x) \leq_L (\mathcal{N} \circ \chi^A)(x) \) for all \( x \in X \). Also \( A \), or \( \chi^A \), is said to be contradictory (without depending on any specific negation) if there exists a strong negation \( \mathcal{N} \), such that \( A \) is \( \mathcal{N} \)-contradictory.

Nevertheless, it is interesting to know not only if a set is contradictory, but also the extent to which this property holds; that is, it is necessary to measure somehow the degree of contradiction of any AIFS. In order to do this, in \([4]\) some functions were proposed to measure both the degree of \( \mathcal{N} \)-contradiction with respect to a strong negation \( \mathcal{N} \), and the degree of contradiction of an AIFS. And in \([5]\), an axiomatic model to measure contradiction is given. In a similar way, this paper focuses on establishing an axiomatic model to measure \( \mathcal{N} \)-contradiction.

1.3. In the previous paper \([4]\), Castiñeira et al. analyzed the regions of \( \mathbb{L} \) in which contradictory sets with respect to a given negation are located, with the purpose of suggesting the way to measure how contradictory an AIFS is. In \([6]\) it was proved that, given \( \chi^A = (\mu_A, \nu_A) \in \mathbb{L}^X \), and \( \mathcal{N} \) a strong IFN associated with the strong negation \( \mathcal{N} \), \( \chi^A \) is \( \mathcal{N} \)-contradictory if and only if \( \mathcal{N}(\mu_A(x)) + \nu_A(x) \geq 1 \), for all \( x \in X \). Thus a region free of contradiction is determined in \( \mathbb{L} \), as well as other region where contradictory sets remain. Being more specific, if \( \chi^A(X) = \{\chi^A(x) : x \in X\} \) is the range of \( \chi^A \), the set \( A \) is \( \mathcal{N} \)-contradictory if and only if

\[
\chi^A(X) \subset \{(\alpha_1, \alpha_2) \in \mathbb{L} : \mathcal{N}(\alpha_1) + \alpha_2 \geq 1\}
\]

Moreover, let \( \mathbb{L}_N = \{(\alpha_1, \alpha_2) \in \mathbb{L} : \mathcal{N}(\alpha_1) + \alpha_2 \leq 1\} \), and the boundary curve \( \mathcal{N}(\alpha_1) + \alpha_2 = 1 \) satisfies the following properties:

1) It determines an increasing function of \( \alpha_1 \).
2) It contains the point \((0,0)\).
3) Its intersection with the line \( \alpha_1 + \alpha_2 = 1 \)

is the point \((\alpha_N, 1 - \alpha_N)\), being \( \alpha_N \) the equilibrium point of the negation \( \mathcal{N} \).

Figure 1: Regions of \( \mathcal{N} \)-contradiction and non-\( \mathcal{N} \)-contradiction

2 Measures of \( \mathcal{N} \)-Contradiction

In \([4]\), in order to measure the \( \mathcal{N} \)-contradiction of AIFS, the following functions \( C_i^N : L^X \rightarrow [0,1] \), \( i = 1, 2, 3 \), were proposed. If \( \chi = (\mu, \nu) \in L^X \), then:

\[
\begin{align*}
C_1^N(\chi) &= \max(0, \inf_{x \in X}(\mathcal{N}(\mu(x)) + \nu(x) - 1)) \\
C_2^N(\chi) &= \max(0, 1 - \sup_{x \in X}(g(\mu(x)) + g(1 - \nu(x))))
\end{align*}
\]

where \( g : [0,1] \rightarrow [0,1] \) is an order automorphism satisfying \( \mathcal{N}(x) = g^{-1}(1 - g(x)) \) for all \( x \in [0,1] \).

\[
C_3^N(\chi) = \frac{d(\chi(X), \mathbb{L}_N)}{d(0_\mathbb{L}, \mathbb{L}_N)}, \text{ where } d \text{ is the Euclidean distance.}
\]

But it is necessary to determine what is understood as a measure of \( \mathcal{N} \)-contradiction. That is, which are the properties demanded to a function to accept it measures adequately the \( \mathcal{N} \)-contradiction.

Before introducing the \( \mathcal{N} \)-contradiction measures, we need a previous definition.

**Definition 2.1.** Let \( \chi \in L^X \); we say that \( \chi \) is \( L_N \)-normal if \( \chi(X) \cap \mathbb{L}_N \neq \emptyset \), where \( \chi(X) \) is the closure of \( \chi(X) \) in the usual topology in \( \mathbb{R}^2 \).

Furthermore, \( \chi \) is said to be \( L \)-normal if \( \chi(X) \cap \{(\alpha_1, \alpha_2) \in \mathbb{L} : \alpha_2 = 0\} \neq \emptyset \).

The set of all \( L_N \)-normal AIFS will be denoted by \( L^X_N \). And the set of all \( L \)-normal AIFS, \( L^X_0 \).
Let us observe that $\chi \in \mathbb{L}^X$ is $L$-normal if and only if it is $\mathbb{L}_N$-normal for all strong IFN $\mathcal{N}$. That is, $\bigcap_{\mathcal{N}} \mathbb{L}_N^X = \mathbb{L}_0^X$.

Now a first proposal is given.

**Definition 2.2.** Let $X \neq \emptyset$ be a universe of discourse and $\mathcal{N}$ a strong IFN; a function $\mathcal{C}_N : \mathbb{L}^X \rightarrow [0, 1]$ is a *measure of $\mathcal{N}$-contradiction* on $\mathcal{I} \mathcal{F}(X)$, or equivalently on $\mathbb{L}^X$, if the following is satisfied:

(c.i) $\mathcal{C}_N(\chi^0_{\mathcal{N}}) = 1$, where $\chi^0_{\mathcal{N}}(x) = 0_L$ for all $x \in X$.

(c.ii) If $\chi \in \mathbb{L}_N^X$, then $\mathcal{C}_N(\chi) = 0$.

(c.iii) Anti-monotonicity: If $\chi^A, \chi^B \in \mathbb{L}^X$ verify $\chi^A(x) \leq \chi^B(x)$ for all $x \in X$, then $\mathcal{C}_N(\chi^A) \geq \mathcal{C}_N(\chi^B)$.

**Remark.** If in the axiom (c.ii) we replace $\mathbb{L}_N^X$ with $\mathbb{L}_0^X$, the definition is just that of contradiction measure given in [5].

The set of all measures of $\mathcal{N}$-contradiction on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}(\mathbb{L}^X)$. Recall that the set of all contradiction measures is denoted by $\mathcal{CM}(\mathbb{L}^X)$.

**Remark.** Obviously, $\mathcal{NCM}(\mathbb{L}^X) \subset \mathcal{CM}(\mathbb{L}^X)$.

In [4] it was proved that the functions $\mathcal{C}_1^N$, $\mathcal{C}_2^N$, $\mathcal{C}_3^N$ defined above satisfy the axioms (c.i) and (c.iii), moreover it is not difficult to show that they also satisfy axiom (c.ii); hence $\mathcal{C}_1^N$, $\mathcal{C}_2^N$, $\mathcal{C}_3^N$ are measures of $\mathcal{N}$-contradiction.

Furthermore, those $\mathcal{N}$-contradiction measures seem to vary their values in a gradual way; nevertheless the previous definition does not guarantee any kind of continuity in the measures, as the following example shows: The function $\mathcal{C}_N : \mathbb{L}^X \rightarrow [0, 1]$, given by

$$\mathcal{C}_N(\chi) = \begin{cases} 1, & \text{if } \chi = \chi^0_{\mathcal{N}} \\ 0, & \text{otherwise} \end{cases}$$

is a measure of $\mathcal{N}$-contradiction, that changes sharply in $\chi^0_{\mathcal{N}}$.

So, if we want to modelize the continuity in the $\mathcal{N}$-contradiction measures, we need to impose some additional conditions. The following two sections are devoted to this subject.

### 3 Completely Semi-continuous $\mathcal{N}$-Contradiction measures

In order to demand a measure changes smoothly, we propose a new definition.

**Definition 3.1.** Let $X \neq \emptyset$ and $\mathcal{N}$ a strong IFN; an $\mathcal{N}$-contradiction measure $\mathcal{C}_N : \mathbb{L}^X \rightarrow [0, 1]$ is to be said *completely semi-continuous from below* on $\mathbb{L}^X$ if the following axiom is satisfied:

(c.iv) For all $\{\chi^i\}_{i \in \mathcal{I}} \subset \mathbb{L}^X$, where $\mathcal{I}$ is an arbitrary set of indexes,

$$\inf_{i \in \mathcal{I}} \mathcal{C}_N(\chi^i) = \mathcal{C}_N\left(\sup_{i \in \mathcal{I}} \chi^i\right)$$

holds, where $\sup_{i \in \mathcal{I}} \chi^i \in \mathbb{L}^X$ is defined as

$$\left(\sup_{i \in \mathcal{I}} \chi^i\right)(x) = \sup_{i \in \mathcal{I}} \chi^i(x), \text{ for all } x \in X.$$

It is easy to prove that (c.iv) implies (c.iii).

The set of all completely semi-continuous from below $\mathcal{N}$-contradiction measures on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}_{csc}(\mathbb{L}^X)$.

**Remark.** $\mathcal{NCM}_{csc}(\mathbb{L}^X) \subset \mathcal{CM}_{csc}(\mathbb{L}^X)$, where $\mathcal{CM}_{csc}(\mathbb{L}^X)$ is the set of contradiction measures satisfying axiom (c.iv).

**Proposition 3.2.** Let $\mathcal{N}$ be a strong IFN, $\mathcal{N}$ the strong fuzzy negation associated with $\mathcal{N}$ and $\alpha_{\mathcal{N}}$ the equilibrium point of $\mathcal{N}$. For each $p \in (0, \alpha_{\mathcal{N}})$, let $\mathcal{C}_{\mathcal{N}, p} : \mathbb{L}^X \rightarrow [0, 1]$ be the function defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ by:

$$\mathcal{C}_{\mathcal{N}, p}(\chi) = \begin{cases} 0, & \text{if } \sup_{x \in X} \mu(x) > p \\
\max\left(0, \frac{\inf_{x \in X} \nu(x) - 1 + N(p)}{N(p)}\right), & \text{else} \end{cases}$$

Then $\mathcal{C}_{\mathcal{N}, p} \in \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

**Proof.** Before confirming the axioms, let us notice that the function has a simple geometrical interpretation (see figure 2) since it can be written as

$$\mathcal{C}_{\mathcal{N}, p}(\chi) = \begin{cases} 0, & \text{if } \sup_{x \in X} \mu(x) > p \text{ or } \inf_{x \in X} \nu(x) \leq 1 - N(p) \\
\frac{\inf_{x \in X} \nu(x) - 1 + N(p)}{N(p)}, & \text{otherwise} \end{cases}$$
Now, let us prove the conditions.

(c.i) \( C_{\mathcal{N}, p}(\chi^{0_L}) = \frac{\inf_{x \in X} \nu(x) - 1 + N(p)}{N(p)} = 1 \)

(c.ii) Let \( \chi = (\mu, \nu) \in \mathbb{L}^2 \), then if there exists \( x \in X \) such that \( \mu(x) > p \) or \( \nu(x) < 1 - N(p) \) then \( C_{\mathcal{N}, p}(\chi) = 0 \) by the definition; if on the contrary, there is not such an \( x \), then there exists \( (x_n)_{n \in \mathbb{N}} \subset X \) such that \( \lim_{n \to \infty} \chi(x_n) = (p, 1 - N(p)) \), thus \( C_{\mathcal{N}, p}(\chi) = \lim_{n \to \infty} \nu(x_n) - 1 + N(p) = 0. \)

(c.iv) Let \( \{\chi^i\}_{i \in I} \) be a family of AIFS.

a) If \( \sup_{i \in I} \chi^i = (\sup_{i \in I} \mu_i, \inf_{i \in I} \nu_i) \) is such that \( \sup_{x \in X} \sup_{i \in I} \mu_i(x) > p \), by definition \( C_{\mathcal{N}, p}(\sup \chi^i) = 0 \) is satisfied, and furthermore, there exist \( x \in X \) and \( j \in I \) satisfying \( \mu_j(x) > p \). Then \( C_{\mathcal{N}, p}(\chi^i) = 0 \) and \( \inf_{i \in I} \mathcal{C}_{\mathcal{N}, p}(\chi^i) = 0 = C_{\mathcal{N}, p}(\sup \chi^i). \)

b) If \( \sup_{x \in X} \sup_{i \in I} \mu_i(x) \leq p \), then \( C_{\mathcal{N}, p}(\sup \chi^i) = \max_{i \in I} \left( 0, \frac{\inf_{x \in X} \nu_i(x) - 1 + N(p)}{N(p)} \right) \)

Furthermore, for all \( x \in X \) and \( i \in I \), \( \mu_i(x) \leq p \), and so,

\[
\inf_{i \in I} \mathcal{C}_{\mathcal{N}, p}(\chi^i) = \inf_{i \in I} \max_{i \in I} \left( 0, \frac{\inf_{x \in X} \nu_i(x) - 1 + N(p)}{N(p)} \right) = \max_{i \in I} \left( 0, \frac{\inf_{x \in X} \nu_i(x) - 1 + N(p)}{N(p)} \right)
\]

From now on, many proofs will be omitted due to limits of space.

**Remark.** Would we change in the definition of \( C_{\mathcal{N}, p} \) the condition \( \sup \mu(x) > p \) by \( \sup_{x \in X} \mu(x) \geq p? \)

If we want to preserve the continuity of the measure, the answer is not. In fact, if we would have

\[
\mathcal{C}(\chi) = \begin{cases} 
0, & \text{if } \sup_{x \in X} \mu(x) \geq p \\
\max \left( 0, \frac{\inf_{x \in X} \nu(x) - 1 + N(p)}{N(p)} \right), & \text{else}
\end{cases}
\]

taking \( m \), with \( 1 - N(p) < m < 1 \), and the family of constant AIFS \( \{\chi^n\}_{n \in \mathbb{N}} \), defined by (see figure 3)

\[\chi^n(x) = \left( p - \frac{p}{n}, m \right) \text{ for all } x \in X, \]

it holds \( \sup_{n \in \mathbb{N}} \chi^n(x) = (p, m) \) and \( \mathcal{C}(\sup \chi^n) = 0. \)

Nevertheless, for all \( n \in \mathbb{N} \), \( \mathcal{C}(\chi^n) = \frac{m - 1 + N(p)}{N(p)} > 0 \), and thus

\[
\inf_{n \in \mathbb{N}} \mathcal{C}(\chi^n) = \frac{m - 1 + N(p)}{N(p)} \neq \mathcal{C}(\sup \chi^n)
\]

**Remark.** In the extremal case \( p = \alpha_N \), the measure will be given as (see figure 4)

\[\mathcal{C}_{\mathcal{N}}(\chi) = \max \left( 0, \frac{\inf_{x \in X} \nu(x) - 1 + \alpha_N}{\alpha_N} \right), \]

Figure 2: Measure \( C_{\mathcal{N}, p} \in NCM_{\text{loc}}(\mathbb{L}^2 X) \).

Figure 3: Countereexample.
**Proposition 3.3.** Let \( f : [0, 1] \to [0, 1] \) be a continuous and strictly decreasing function such that \( f(1) = 0 \) and \( \alpha + f(\alpha) < 1 \) for all \( \alpha \in (0, 1) \). Let \((p, f(p)) \in L\) satisfying \( f(p) + N(p) = 1 \). For all \( \beta \in (f(p), f(0)) \) let us consider the region

\[
L_\beta = \{(\alpha_1, \beta) \mid \alpha_1 \in [0, f^{-1}(\beta)]\} \\
\bigcup \{(f^{-1}(\beta), \alpha_2) \mid \alpha_2 \in [\beta, 1 - f^{-1}(\beta)]\}
\]

and \( L_{f(0)} = \{(0,0_2) \mid \alpha_2 \in [f(0), 1]\} \). Then the function \( C^i_N : L^X \to [0, 1] \) defined for each \( \chi = (\mu, \nu) \in L^X \) as (see figure 5)

\[
C^i_N(\chi) = \begin{cases} 
1, & \text{if } \sup x \in L_{f(0)} \\
\beta - f(p) \quad 1 - f(p), & \text{if } \sup x \in L_\beta \text{ for some } \beta \\
0, & \text{otherwise}
\end{cases}
\]

satisfies that \( C^i_N \in NCM_{\text{csc}}(L^X) \).

**Definition 3.4.** Let \( X \neq \emptyset \) and \( N \) a strong IFN; an \( N \)-contradiction measure \( C^i_N : L^X \to [0, 1] \) is to be said completely semi-continuous from above on \( L^X \) if the following axiom is satisfied:

(c,v) For all \( \{\chi^i\}_{i \in I} \subset L^X \setminus L^X_N \),

\[
\sup_{i \in I} C^i_N(\chi^i) = C_N \left( \inf_{i \in I} \chi^i \right)
\]

holds, where \( \inf_{i \in I} \chi^i \in L^X \) is defined as \( \left( \inf_{i \in I} \chi^i \right)(x) = \inf_{i \in I} \chi^i(x) \) for all \( x \in X \).

**Remark.** Notice that it is necessary to consider the AIFS are not \( L^X_N \)-normal in the previous axiom. Indeed, let \( X = \{x_1, x_2\} \) and the AIFS defined as follows:

\[
\chi^1(x_1) = \begin{cases} 
0, & \text{if } i = 1 \\
(\alpha_N, 1 - \alpha_N), & \text{if } i = 2
\end{cases}
\]

\[
\chi^2(x_2) = \begin{cases} 
(\alpha_N, 1 - \alpha_N), & \text{if } i = 1 \\
0, & \text{if } i = 2
\end{cases}
\]

Then \( \inf_{i \in I} \chi^1, \chi^2 \}(x_i) = 0 \), for \( i = 1, 2 \), and thus \( C_N(\inf_{i \in I} \chi^1, \chi^2) \) = 1, nevertheless \( C_N(\chi^1) = C_N(\chi^2) = 0 \) as \( \chi^1, \chi^2 \in L^X_N \).

Once again, axiom (c,v) implies axiom (c,iii). The set of all completely semi-continuous \( N \)-contradiction measures from above on \( L^X \) will be denoted by \( NCM_{\text{csc}}(L^X) \).

**Remark.** \( NCM_{\text{csc}}(L^X) \subset CM_{\text{csc}}(L^X) \), where \( CM_{\text{csc}}(L^X) \) is the set of contradiction measures satisfying axiom (c,iv).

**Example 3.5.** Let \( C^i_N : L^X \to [0, 1] \) be a function defined for each \( \chi = (\mu, \nu) \in L^X \) by (see figure 6):

\[
C^i_N(\chi) = \begin{cases} 
0, & \text{if } \chi \in L^X_N \\
\sup \nu(x), & \text{otherwise}
\end{cases}
\]

Then \( C^i_N \in NCM_{\text{csc}}(L^X) \). Furthermore, \( C^i_N \notin NCM_{\text{csc}}(L^X) \).

**Remark.** The measure \( C^i_N \) is not a completely semi-continuous \( N \)-contradiction measure from above.

Indeed, let \( X \) be a universe of discourse with \( \text{Card}(X) \geq 2 \), and \( x_1, x_2 \in X \) such that \( x_1 \neq x_2 \). Let us take for \( i = 1, 2 \) the AIFS

\[
\chi^i(x) = \begin{cases} 
(0, f(p)), & \text{if } x = x_i \\
0, & \text{otherwise}
\end{cases}
\]
Then \( \left( \inf_{i=1,2} \chi^i \right)(x) = 0 \) for all \( x \in X \). So,
\[
C_N^i(\inf_{i=1,2} \chi^i) = C_N^i(\chi^0) = 1.
\]

But, \( C_N^i(\chi^1) = C_N^i(\chi^2) = \sup_i C_N^i(\chi^i) = 0 \).

**Proposition 3.6.** Let \( f : [0,1] \to [0,1] \) be a continuous and strictly decreasing function such that \( f(1) = 0 \) and \( \alpha + f(\alpha) < 1 \) for all \( \alpha \in (0,1) \). Let \( (p, f(p)) \in \mathbb{I} \) satisfying \( f(p) + N(p) = 1 \). For all \( \beta \in [f(p), f(0)] \) let us consider the region

\[
M_\beta = \{ (f^{-1}(\beta), \alpha_2) \mid \alpha_2 \in [0, \beta] \}
\]

and \( M_\beta = \{ (\alpha_1, \beta) \mid \alpha_1 \in [0, 1 - \beta] \} \) if \( \beta \in (f(0), 1] \). The function \( C_N^i : \mathbb{I} \to [0,1] \) defined for each \( \chi = (\mu, \nu) \in \mathbb{I} \) as (see fig. 7):

\[
C_N^i(\chi) = \begin{cases} 
0, & \text{if } \chi \in \mathbb{I}_N^X \\
\frac{x-f(p)}{1-f(p)}, & \text{if } \chi \notin \mathbb{I}_N^X & \text{and } \inf_{x \in X} \chi(x) \in M_\beta
\end{cases}
\]

satisfies \( C_N^i \in \mathcal{NCM}^{\text{csc}}(\mathbb{I}^X) \). Furthermore, \( C_N^i \notin \mathcal{NCM}^{\text{csc}}(\mathbb{I}^X) \).

On the other hand, measures \( C_N^1, C_N^2 \) and \( C_N^3 \) defined in [4] do not satisfy the conditions demanded in this section, as we are going to show.

**Proposition 3.7.** If \( X \neq \emptyset \) and \( N \) is a strong negation, \( N \)-contradiction measures on \( \mathbb{I}^X \) \( C_N^1, C_N^2 \) and \( C_N^3 \), defined at the beginning of section 2, are neither completely semicontinuous from below nor from above.

**Proof.** First, let us see that, for \( i = 1, 2, 3 \), \( C_i^N \notin \mathcal{NCM}^{\text{csc}}(\mathbb{I}^X) \). Let us fix \( \beta \) such that

\[0 < \beta < 1 - \alpha_N, \text{ and let } \alpha \text{ such that } N^{-1}(1 - \beta) < \alpha < \alpha_N. \]

We consider the AIFS

\[
\begin{align*}
\chi^1(x) &= (0, \beta) \\
\chi^2(x) &= (\alpha, 1 - \alpha)
\end{align*}
\]

for all \( x \in X \).

Then \( \left( \sup_{j=1,2} \chi^j \right)(x) = (\alpha, \beta) \) for all \( x \in X \), and it is easy to prove that for \( i = 1, 2, 3 \),

\[0 < \inf_{j=1,2} \chi^j(x) \neq \chi_i(\inf_{j=1,2} \chi^j(x)). \]

Second, let us see that, for \( i = 1, 2, 3 \), \( C_i^N \notin \mathcal{NCM}^{\text{csc}}(\mathbb{I}^X) \). Let us fix \( \alpha \) such that \( 1 - \alpha_N < \alpha < 1 \), and \( \beta \) with \( \alpha < 1 - \beta \). Now, we consider the AIFS

\[
\begin{align*}
\chi^1(x) &= (0, \alpha) \\
\chi^2(x) &= (\beta, 1 - \beta)
\end{align*}
\]

for all \( x \in X \).

Then \( \left( \inf_{j=1,2} \chi^j \right)(x) = (0, 1 - \beta) \) for all \( x \), and it can be proved that for \( i = 1, 2, 3 \),

\[
\sup \chi_i(\inf_{j=1,2} \chi^j) \neq \chi_i(\inf_{i=1,2} \chi^j). \]

\( \Box \)
4 Semi-continuous \( \mathcal{N} \)-Contradiction measures

Let us remember that a set \( S \subseteq \mathbb{L}^X \) is a semilattice from below if for all \( \chi^A, \chi^B \in S \), \( \text{Sup}\{\chi^A, \chi^B\} \in S \) holds; and similarly, a set \( S \subseteq \mathbb{L}^X \) is a semilattice from above if for all \( \chi^A, \chi^B \in S \), \( \text{Inf}\{\chi^A, \chi^B\} \in S \) holds (see, for example, [2]).

**Definition 4.1.** Let \( X \neq \emptyset \) and \( \mathcal{N} \) a strong IFN, an \( \mathcal{N} \)-contradiction measure \( C_N : \mathbb{L}^X \to [0, 1] \) is to be said semicontinuous from below if the following axiom is satisfied:

\( \text{(c.vi)} \) For all semi-lattice from below \( \{\chi^i\}_{i \in I} \subseteq \mathbb{L}^X \), where \( I \) is an arbitrary set, the following is satisfied

\[
\text{Inf}_{i \in I} C_N(\chi^i) = C_N\left(\text{Sup}_{i \in I} \chi^i\right)
\]

Notice that axiom \( \text{(c.vi)} \) implies axiom \( \text{(c.iii)} \).

The set of all semi-continuous from below \( \mathcal{N} \)-contradiction measures on \( \mathbb{L}^X \) will be denoted by \( \mathcal{N}\mathcal{CM}_{sc}(\mathbb{L}^X) \).

**Remark.** Obviously, \( \mathcal{N}\mathcal{CM}_{csc}(\mathbb{L}^X) \subset \mathcal{N}\mathcal{CM}_{sc}(\mathbb{L}^X) \).

**Proposition 4.2.** Let \( X \neq \emptyset \) and \( \mathcal{N} \) and strong IFN. Given a fixed \( p \in (0, +\infty) \), for all \( \beta \in [0, 1] \) let us consider the following region

\[
L_\beta = \left\{(\alpha_1, \alpha_2) \in \mathbb{L} : \alpha_1 \in [0, \beta], \quad \alpha_2 = \frac{\alpha_1 + p(1-\beta)}{\beta + p}\right\},
\]

that is, \( L_\beta \) is a segment on the line joining the points \((-p, 0)\) and \((\beta, 1-\beta)\).

Given the function \( C^L_N : \mathbb{L}^X \to [0, 1] \) defined for each \( \chi = (\mu, \nu) \in \mathbb{L}^X \) by (see figure 10):

\[
C^L_N(\chi) = \begin{cases} 0, & \text{if } \chi \notin \mathbb{L}^X_N \land \text{Sup}_{x \in X} \chi(x) \in L_\beta \setminus \mathbb{L}^X_N, \\ 1 - \beta, & \text{if } \chi \notin \mathbb{L}^X_N \land \text{Sup}_{x \in X} \chi(x) \in L_\beta \end{cases}
\]

we have \( C^L_N \in \mathcal{N}\mathcal{CM}_{sc}(\mathbb{L}^X) \setminus \mathcal{N}\mathcal{CM}_{csc}(\mathbb{L}^X) \).

**Proposition 4.4.** Consider for any \( \beta \in [0, 1] \), the segment \( L_\beta \) defined in Proposition 4.2.
Let \( C^U_N : \mathbb{L}^X \rightarrow [0, 1] \) be the function defined for each \( \chi = (\mu, \nu) \in \mathbb{L}^X \) by (see figure 11):

\[
C^U_N(\chi) = \begin{cases} 
0, & \text{if } \chi \in \mathbb{L}^X_N, \\
1 - \beta, & \text{if } \chi \notin \mathbb{L}^X_N \& \inf_{x \in X} \chi(x) \in L_\beta 
\end{cases}
\]

Then \( C^U_N \in \mathcal{NCM}^{sc}_{\mathbb{L}^X}(\mathbb{L}^X) \setminus \mathcal{NCM}^{sc}_{\mathbb{L}^X}(\mathbb{L}^X) \).

Now, we have the following result.

**Proposition 4.5.** For \( i = 1, 2, 3 \), each measure \( C^N_i \) defined at the beginning of section 2 satisfies that \( C^N_i \in \mathcal{NCM}^{sc}_{\mathbb{L}^X}(\mathbb{L}^X) \), but, in general, \( C^N_i \in \mathcal{NCM}^{sc}_{\mathbb{L}^X}(\mathbb{L}^X) \) do not hold.

Finally, the functions presented through this paper show the following result.

**Proposition 4.6.** For any strong IFN \( \mathcal{N} \), the following inequalities hold:

\[
\mathcal{NCM}^{sc}_{\mathbb{L}^X} \subseteq \mathcal{NCM}^{sc}_{\mathbb{L}^X} \subseteq \mathcal{NCM} \quad \mathcal{NCM}^{sc}_{\mathbb{L}^X} \subseteq \mathcal{NCM}^{sc}_{\mathbb{L}^X} \subseteq \mathcal{NCM}_{\mathbb{L}^X} 
\]

**Conclusions**

Contradictory sets can result inconvenient in certain applications, for instance, in the processes of fuzzy inference. Until now, a mathematic model had been defined to measure in which degree an AIFS is contradictory. However, demanding that an object have a small contradictory degree can be very restrictive and it may result more interesting to measure that degree regarding a given negation, if that negation is the one used in a specific application. That is why, in this work, we have presented a mathematic model to measure the \( \mathcal{N} \)-contradiction of an AIFS. Moreover, we have obtained families of measures that satisfy different kinds of continuity.

**References**


