Generalized sampling: From shift-invariant to $U$-invariant spaces

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The aim of this article is to derive a sampling theory in $U$-invariant subspaces of a separable Hilbert space $\mathcal{H}$ where $U$ denotes a unitary operator defined on $\mathcal{H}$. To this end, we use some special dual frames for $L^2(0,1)$, and the fact that any $U$-invariant subspace with stable generator is the image of $L^2(0,1)$ by means of a bounded invertible operator. The used mathematical technique mimics some previous sampling work for shift-invariant subspaces of $L^2(\mathbb{R})$. Thus, sampling frame expansions in $U$-invariant spaces are obtained. In order to generalize convolution systems and deal with the time-jitter error in this new setting we consider a continuous group of unitary operators which includes the operator $U$.

Keywords: Stationary sequences; $U$-invariant subspaces; frames; dual frames; time-jitter error; group of unitary operators; pseudo-dual frames.

1. By Way of Motivation

The aim in this paper is to derive a generalized sampling theory for $U$-invariant subspaces of a separable Hilbert space $\mathcal{H}$, where $U : \mathcal{H} \to \mathcal{H}$ denotes a unitary
operator. The motivation for our work can be found in the generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$; there $\mathcal{H} := L^2(\mathbb{R})$ and the unitary operator is the shift $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$. In that setting, the functions (signals) belong to some (principal) shift-invariant subspace $V^2_\varphi := \text{span}_{L^2(\mathbb{R})}\{\varphi(u-n), \ n \in \mathbb{Z}\}$, where the generator function $\varphi$ belongs to $L^2(\mathbb{R})$ and the sequence $\{\varphi(u-n)\}_{n \in \mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$. Thus, the shift-invariant space $V^2_\varphi$ can be described as

$$V^2_\varphi = \left\{ \sum_{n \in \mathbb{Z}} \alpha_n \varphi(u-n) : \{\alpha_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}.$$ 

On the other hand, in many common situations the available data are samples of some filtered versions $f * h_j$ of the signal $f$ itself, where the average function $h_j$ reflects the characteristics of the acquisition device.

For $s$ convolution systems (linear time-invariant systems or filters in engineering jargon) $L_j f := f * h_j$, $j = 1, 2, \ldots, s$, defined on $V^2_\varphi$, and assuming also that the sequence of samples

$$\{ (L_j f)(rm) \}_{m \in \mathbb{Z}; j=1,2,\ldots,s},$$

where $r \in \mathbb{N}$, is available for any $f$ in $V^2_\varphi$, the generalized sampling problem mathematically consists of the stable recovery of any $f \in V^2_\varphi$ from the above sequence of samples. In other words, it deals with the construction of sampling formulas in $V^2_\varphi$ having the form

$$f(u) = \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} (L_j f)(rm)S_j(u-rm), \quad u \in \mathbb{R},$$

where the sequence of reconstruction functions $\{S_j(\cdot-rm)\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}$ is a frame for the shift-invariant space $V^2_\varphi$.

Sampling in shift-invariant spaces of $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^d)$), with one or multiple generators, has been profusely treated in the mathematical literature. A few selected references are: [4, 5, 9–14, 18, 23, 27–31].

In this work we provide a generalization of the above problem in the following sense. Let $U$ be a unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by $\mathcal{A}_a := \text{span}\{U^n a, n \in \mathbb{Z}\}$. In case that the sequence $\{U^n a\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \left\{ \sum_{n \in \mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}.$$ 

In order to generalize convolution systems and mainly to obtain some perturbation results in this new setting, we assume that the operator $U$ is included in a continuous group of unitary operators $\{U^t\}_{t \in \mathbb{R}}$ in $\mathcal{H}$ as $U := U^1$. Recall that $\{U^t\}_{t \in \mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying (see [2, Vol. 2; p. 29]):

1. $U^t U^{t'} = U^{t+t'}$, 

(2) $U^0 = I_H$.
(3) $\langle U^t x, y \rangle_H$ is a continuous function of $t$ for any $x, y \in H$.

Note that $(U^t)^{-1} = U^{-t}$, and since $(U^t)^* = (U^t)^{-1}$, we have $(U^t)^* = U^{-t}$.

Thus, for $b \in H$ we consider the linear operator $H \ni x \mapsto L_b x \in C(\mathbb{R})$ such that $(L_b x)(t) := \langle x, U^t b \rangle_H$ for every $t \in \mathbb{R}$. These operators $L_b$, which will be called $U$-systems, can be seen as a generalization of the convolution systems in $L^2(\mathbb{R})$. Indeed, for the shift operator $T : f(u) \mapsto f(u - 1)$ in $L^2(\mathbb{R})$ we have

$$
\langle f, T^t b \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(u) b(u-t) du = (f * h)(t), \quad t \in \mathbb{R},
$$

where $h(u) := \overline{b(-u)}$.

Given $U$-systems $L_j$, $j = 1, 2, \ldots, s$, corresponding to $s$ elements $b_j \in H$, i.e. $L_j \equiv L_{b_j}$ for each $j = 1, 2, \ldots, s$, the generalized regular sampling problem in $A_a$ consists of the stable recovery of any $x \in A_a$ from the sequence of the samples

$$
\{L_j x(rm)\}_{m \in \mathbb{Z}; j = 1, 2, \ldots, s} \quad \text{where } r \in \mathbb{N}, \ r \geq 1.
$$

This $U$-sampling problem has been treated, for the first time, in some recent papers [22, 24]. Sampling in shift-invariant subspaces or in modulation-invariant subspaces of $L^2(\mathbb{R})$ becomes a particular case of $U$-sampling associated with the shift operator $T : f(u) \mapsto f(u - 1)$ or with the modulation operator $M : f(u) \mapsto e^{2\pi i u} f(u)$ in $L^2(\mathbb{R})$ respectively.

In this paper, we propose a completely different approach which allows to analyze in depth the $U$-sampling problem. In Sec. 3, we prove the existence of frames in $A_a$, having the form $\{U^m c_j\}_{m \in \mathbb{Z}; j = 1, 2, \ldots, s}$, where $c_j \in A_a$ for $j = 1, 2, \ldots, s$, such that for each $x \in A_a$ the sampling expansion

$$
x = \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} L_j x(rm) U^m c_j \quad \text{in } H \tag{1.1}
$$

holds. To this end, as in the shift-invariant case (see, for instance, [13, 14]), we use that the above sampling formula is intimately related with some special dual frames in $L^2(0,1)$ (see Sec. 2) via the isomorphism $T_{U,a} : L^2(0,1) \rightarrow A_a$ which maps the orthonormal basis $\{e^{2\pi i n u}\}_{n \in \mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n \in \mathbb{Z}}$ for $A_a$.

In [24], regular sampling expansions like (1.1) are obtained by using a completely different technique; basically, they use the cross-covariance function $R_{a,b_j}(n) := \langle U^n a, b_j \rangle_H$ between the sequences $\{U^n a\}_{n \in \mathbb{Z}}$ and $\{U^n b_j\}_{n \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$.

Strictly speaking, we do not need the formalism of the continuous group of unitary operators to derive the sampling results in Sec. 3 since we only use the discrete group $\{U^n\}_{n \in \mathbb{Z}}$ completely determined by $U$. However, for the study, in Sec. 4, of the time-jitter error in sampling formulas as in (1.1), the continuous group of unitary operators $\{U^t\}_{t \in \mathbb{R}}$ becomes essential. In this case, we dispose of a perturbed sequence of samples $\{(L_j x)(rm + \epsilon_{mj})\}_{m \in \mathbb{Z}; j = 1, 2, \ldots, s}$, with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in A_a$. We prove that, for small enough errors $\epsilon_{mj}$,
the stable recovery of any \( x \in A_a \) is still possible. Finally, in Sec. 5 we deal with the case of multiple stable generators. We only sketch the procedure since it is essentially identical to the one-generator case.

2. On Sampling in \( U \)-Invariant Subspaces

For a fixed \( a \in \mathcal{H} \), assume that the sequence \( \{U^n a\}_{n \in \mathbb{Z}} \) is a Riesz sequence in \( \mathcal{H} \). Recall that a Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis \( \{x_n\}_{n \in \mathbb{Z}} \) has a unique biorthogonal (dual) Riesz basis \( \{y_n\}_{n \in \mathbb{Z}} \), i.e. \( \langle x_n, y_m \rangle_{\mathcal{H}} = \delta_{n,m} \), such that the expansions

\[
x = \sum_{n \in \mathbb{Z}} \langle x, y_n \rangle y_n = \sum_{n \in \mathbb{Z}} \langle x, x_n \rangle y_n,
\]

hold for every \( x \in \mathcal{H} \). We state the definition by considering the integers set \( \mathbb{Z} \) as the index set since throughout the paper most of sequences are indexed in \( \mathbb{Z} \). A Riesz sequence in \( \mathcal{H} \) is a Riesz basis for its closed span (see, for instance, [8]). Thus, the \( U \)-invariant subspace \( A_a = \text{span}\{U^n a, n \in \mathbb{Z}\} \) can be expressed as

\[
A_a = \left\{ \sum_{n \in \mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}.
\]

For simplicity and ease of notation we are considering the one-generator setting; as we have already said, the same sampling results for the general case can be obtained by analogy, and it will be drawn in Sec. 5. The sequence \( \{U^n a\}_{n \in \mathbb{Z}} \) is a stationary sequence since the inner product \( \langle U^n a, U^m a \rangle_{\mathcal{H}} \) depends only on the difference \( n - m \in \mathbb{Z} \). Moreover, the auto-covariance \( R_a \) of the sequence \( \{U^n a\}_{n \in \mathbb{Z}} \) admits the integral representation

\[
R_a(k) := \langle U^k a, a \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\mu_a(\theta), \quad k \in \mathbb{Z},
\]

in terms of a positive Borel measure \( \mu_a \) on \((-\pi, \pi)\) called the spectral measure of the sequence (see [19]). This is obtained from the integral representation of the unitary operator \( U \) on \( \mathcal{H} \) (see, for instance, [2, 33]). The spectral measure \( \mu_a \) can be decomposed into an absolute continuous and a singular part as \( d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta) \). A necessary and sufficient condition in order for the sequence \( \{U^n a\}_{n \in \mathbb{Z}} \) to be a Riesz sequence for \( \mathcal{H} \) is given in next theorem in terms of the decomposition of the spectral measure \( \mu_a \).

**Theorem 2.1.** Let \( \{U^n a\}_{n \in \mathbb{Z}} \) be a sequence obtained from a unitary operator in a separable Hilbert space \( \mathcal{H} \) with spectral measure \( d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta) \), and let \( A_a \) be the closed subspace spanned by \( \{U^n a\}_{n \in \mathbb{Z}} \). Then the sequence \( \{U^n a\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( A_a \) if and only if the singular part \( \mu_a^s \equiv 0 \) and

\[
0 < \inf_{\theta \in (-\pi, \pi)} \phi_a(\theta) \leq \sup_{\theta \in (-\pi, \pi)} \phi_a(\theta) < \infty.
\]
Theorem 2.1 is just the one-generator case \((L = 1)\) of Theorem 5.1 proved below. It is worth to mention that a straightforward computation shows that the dual Riesz basis of \(\{U^n a\}_{n \in \mathbb{Z}}\) in \(\mathcal{A}_a\) is given by \(\{U^n b\}_{n \in \mathbb{Z}}\) with \(b = \sum_{k \in \mathbb{Z}} b_k U^k a \in \mathcal{A}_a\), where the terms of the sequence \(\{b_k\}_{k \in \mathbb{Z}} \subseteq \ell^2(\mathbb{Z})\) are the Fourier coefficients of the function \(1/\phi_a(\theta) \in L^2(-\pi, \pi)\). Indeed, for \(b = \sum_{k \in \mathbb{Z}} b_k U^k a \in \mathcal{A}_a\), the biorthogonality between the sequences \(\{U^n a\}_{n \in \mathbb{Z}}\) and \(\{U^n b\}_{n \in \mathbb{Z}}\) means

\[
\delta_{m,0} = \langle U^m a, b \rangle_{\mathcal{H}} = \sum_{k \in \mathbb{Z}} \overline{b_k} U^k a \quad \Rightarrow \quad \sum_{k \in \mathbb{Z}} b_k \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-k)\theta} \phi_a(\theta) d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k \in \mathbb{Z}} b_k e^{-ik\theta} \right) \phi_a(\theta) e^{im\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} B(\theta) \phi_a(\theta) e^{-im\theta} d\theta,
\]

where \(B(\theta) := \sum_{k \in \mathbb{Z}} b_k e^{ik\theta}\); in other words, we have \(B(\theta) \phi_a(\theta) \equiv 1\) in \(L^2(-\pi, \pi)\). Moreover, it is easy to deduce that \(\phi_b(\theta) = 1/\phi_a(\theta), \theta \in (-\pi, \pi)\); that is, for \(k \in \mathbb{Z}\) we obtain \(\langle U^k b, b \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta} \phi_a(\theta) d\theta\).

Finally, for the shift operator \(T : f(u) \mapsto f(u - 1)\) in \(L^2(\mathbb{R})\), Theorem 2.1 allows to recover the classical necessary and sufficient condition for the sequence \(\{\varphi(t - n)\}_{n \in \mathbb{Z}}\), where \(\varphi \in L^2(\mathbb{R})\), to be a Riesz basis for the corresponding shift-invariant subspace \(\mathcal{A}_\varphi\) in \(L^2(\mathbb{R})\). Indeed, consider the Fourier transform as \(\hat{\varphi}(\theta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{-it\theta} d\theta\) in \(L^1(\mathbb{R}) \cap L^2(\mathbb{R})\); using the Parseval’s equality one easily gets

\[
\langle T^k \varphi, \varphi \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \varphi(u - k) \overline{\varphi}(u) du = \int_{-\infty}^{\infty} \overline{\varphi(u - k)}(\theta) \hat{\varphi}(\theta) d\theta
\]

\[
= \int_{-\infty}^{\infty} |\hat{\varphi}(\theta)|^2 e^{-ik\theta} d\theta = \int_{-\infty}^{\infty} \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\theta + 2\pi n)|^2 e^{-ik\theta} d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} 2\pi \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\theta + 2\pi n)|^2 d\theta,
\]

that is, \(\phi_\varphi(\theta) = 2\pi \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\theta + 2\pi n)|^2, \theta \in (-\pi, \pi)\). Thus, Theorem 2.1 yields the classical condition (see, for instance, [8]):

\[
0 < \text{ess inf}_{\theta \in (-\pi, \pi)} \left| \hat{\varphi}(\theta + 2\pi n) \right|^2 \leq \text{ess sup}_{\theta \in (-\pi, \pi)} \sum_{n \in \mathbb{Z}} \left| \hat{\varphi}(\theta + 2\pi n) \right|^2 < \infty.
\]

The following isomorphism between \(L^2(0, 1)\) and \(\mathcal{A}_a\) will be crucial along this paper.

The isomorphism \(T_{U,a}\)

We define the isomorphism \(T_{U,a}\) which maps the orthonormal basis \(\{e^{2\pi inw}\}_{n \in \mathbb{Z}}\) for \(L^2(0, 1)\) onto the Riesz basis \(\{U^n a\}_{n \in \mathbb{Z}}\) for \(\mathcal{A}_a\), that is,

\[
T_{U,a} : L^2(0, 1) \rightarrow \mathcal{A}_a,
\]

\[
F = \sum_{n \in \mathbb{Z}} \alpha_n e^{2\pi inw} \mapsto x = \sum_{n \in \mathbb{Z}} \alpha_n U^n a.
\]
The following $U$-shift property holds: For any $F \in L^2(0,1)$ and $N \in \mathbb{Z}$, we have
\[ T_{U,a}(Fe^{2\pi i Nw}) = U_N(T_{U,a}F). \] (2.1)

**The $U$-systems**

For any fixed $b \in \mathcal{H}$ we define the $U$-system $\mathcal{L}_b$ as the linear operator between $\mathcal{H}$ and the set $C(\mathbb{R})$ of the continuous functions on $\mathbb{R}$ given by
\[ \mathcal{H} \ni x \mapsto \mathcal{L}_b x \in C(\mathbb{R}) \quad \text{such that} \quad \mathcal{L}_b x(t) := \langle x, U^t b \rangle_{\mathcal{H}}, \quad t \in \mathbb{R}. \]

For any $x \in \mathcal{A}_a$ and $t \in \mathbb{R}$, by using the Plancherel equality for the orthonormal basis $\{e^{2\pi inw}\}_{n \in \mathbb{Z}}$ in $L^2(0,1)$, we have
\[ \mathcal{L}_b x(t) = \langle x, U^t b \rangle_{\mathcal{H}} = \left\langle \sum_{n \in \mathbb{Z}} \alpha_n U^n a, U^t b \right\rangle_{\mathcal{H}} = \sum_{n \in \mathbb{Z}} \alpha_n \langle U^t b, U^n a \rangle_{\mathcal{H}} = \langle F, K_t \rangle_{L^2(0,1)}, \] (2.2)

where $T_{U,a}F = x$, and the function
\[ K_t(w) := \sum_{n \in \mathbb{Z}} \langle U^t b, U^n a \rangle_{\mathcal{H}} e^{2\pi i nw} = \sum_{n \in \mathbb{Z}} \mathcal{L}_b a(t - n) e^{2\pi i nw} \]

belongs to $L^2(0,1)$ since the sequence $\{(U^t b, U^n a)_{\mathcal{H}}\}_{n \in \mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ for each $t \in \mathbb{R}$.

**An expression for the generalized samples**

Suppose that $s$ vectors $b_j \in \mathcal{H}$, $j = 1, 2, \ldots, s$, are given and consider their associated $U$-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$, $j = 1, 2, \ldots, s$. Our aim is the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of samples $\{\mathcal{L}_j x(rm)\}_{m \in \mathbb{Z}, j = 1, 2, \ldots, s}$ where $r \geq 1$. To this end, first we obtain a suitable expression for the samples. For $x \in \mathcal{A}_a$ let $F \in L^2(0,1)$ such that $T_{U,a}F = x$; by using (2.2), for $j = 1, 2, \ldots, s$ and $m \in \mathbb{Z}$ we have
\[ \mathcal{L}_j x(rm) = \left\langle F, \sum_{n \in \mathbb{Z}} \langle U^{rm} b_j, U^n a \rangle_{\mathcal{H}} e^{2\pi i nw} \right\rangle_{L^2(0,1)} = \left\langle F, \sum_{k \in \mathbb{Z}} \langle U^k b_j, a \rangle_{\mathcal{H}} e^{2\pi i (rm - k)w} \right\rangle_{L^2(0,1)} = \left\langle F, \left[ \sum_{k \in \mathbb{Z}} \langle a, U^k b_j \rangle_{\mathcal{H}} e^{-2\pi ikw} \right] e^{2\pi irmw} \right\rangle_{L^2(0,1)}, \]

where the change in the summation’s index $k := rm - n$ has been done. Hence,
\[ \mathcal{L}_j x(rm) = \langle F, \overline{g_j(w)} e^{2\pi irmw} \rangle_{L^2(0,1)} \quad \text{for} \quad m \in \mathbb{Z} \quad \text{and} \quad j = 1, 2, \ldots, s, \] (2.3)
where the function
\[ g_j(w) := \sum_{k \in \mathbb{Z}} \mathcal{L}_j a(k) e^{2\pi i k w} \] (2.4)
belongs to \( L^2(0, 1) \) for each \( j = 1, 2, \ldots, s \).

As a consequence of (2.3), the stable recovery of any \( x \in \mathcal{A} \) depends on whether the sequence \( \{g_j(w) e^{2\pi i r m w} \}_{m \in \mathbb{Z}, j=1,2,\ldots,s} \) forms a frame for \( L^2(0,1) \). Recall that a sequence \( \{x_n\}_{n \in \mathbb{Z}} \) is a frame for a separable Hilbert space \( \mathcal{H} \) if there exist two constants \( A, B > 0 \) (frame bounds) such that
\[ A\|x\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle x, x_n \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}. \]

A sequence \( \{x_n\}_{n \in \mathbb{Z}} \) in \( \mathcal{H} \) satisfying only the right-hand side inequality above is said to be a Bessel sequence for \( \mathcal{H} \). Given a frame \( \{x_n\}_{n \in \mathbb{Z}} \) for \( \mathcal{H} \) the representation property of any vector \( x \in \mathcal{H} \) as a series \( x = \sum_{n \in \mathbb{Z}} c_n x_n \) is retained, but, unlike the case of Riesz bases (exact frames), the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients \( c_n \) which depend continuously and linearly on \( x \) are obtained by using the dual frames \( \{y_n\}_{n \in \mathbb{Z}} \) of \( \{x_n\}_{n \in \mathbb{Z}} \), i.e. \( \{y_n\}_{n \in \mathbb{Z}} \) is another frame for \( \mathcal{H} \) such that \( x = \sum_{n \in \mathbb{Z}} \langle x, y_n \rangle x_n = \sum_{n \in \mathbb{Z}} \langle x, x_n \rangle y_n \) for each \( x \in \mathcal{H} \). For more details on frame theory see [8].

A deep study of sequences having the form of \( \{g_j(w) e^{2\pi i r m w} \}_{m \in \mathbb{Z}, j=1,2,\ldots,s} \) was done in [13, 14]. Namely, consider the \( s \times r \) matrix of functions in \( L^2(0,1) \)

\[
G(w) := \begin{bmatrix}
g_1(w) & g_1\left(w + \frac{1}{r}\right) & \cdots & g_1\left(w + \frac{r-1}{r}\right) \\
g_2(w) & g_2\left(w + \frac{1}{r}\right) & \cdots & g_2\left(w + \frac{r-1}{r}\right) \\
\vdots & \vdots & \ddots & \vdots \\
g_s(w) & g_s\left(w + \frac{1}{r}\right) & \cdots & g_s\left(w + \frac{r-1}{r}\right)
\end{bmatrix}
\]

and its related constants
\[ \alpha_G := \text{ess inf}_{w \in (0,1/r)} \lambda_{\min}[G^*(w)G(w)], \quad \beta_G := \text{ess sup}_{w \in (0,1/r)} \lambda_{\max}[G^*(w)G(w)], \]

where \( \lambda_{\min} \) (respectively, \( \lambda_{\max} \)) denotes the smallest (respectively, the largest) eigenvalue of the positive semidefinite matrix \( G^*(w)G(w) \). Observe that \( 0 \leq \alpha_G \leq \beta_G \leq \infty \). Notice that in the definition of the matrix \( G(w) \) we are considering 1-periodic extensions of the involved functions \( g_j, j = 1, 2, \ldots, s \).
A complete characterization of the sequence \( \{g_j(w)e^{2\pi irmw}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \) is given in the next lemma (see [14, Lemma 3] or [13, Lemma 2] for the proof).

**Lemma 2.2.** For the functions \( g_j \in L^2(0,1) \), \( j = 1, 2, \ldots, s \), consider the associated matrix \( G(w) \) given in (2.5). Then, the following results hold:

(a) The sequence \( \{g_j(w)e^{2\pi irmw}\}_{n \in \mathbb{Z}; j=1,2,\ldots,s} \) is a complete system for \( L^2(0,1) \) if and only if the rank of the matrix \( G(w) \) is \( r \) a.e. in \((0,1/r)\).

(b) The sequence \( \{g_j(w)e^{2\pi irmw}\}_{n \in \mathbb{Z}; j=1,2,\ldots,s} \) is a Bessel sequence for \( L^2(0,1) \) if and only if \( g_j \in L^\infty(0,1) \) (or equivalently \( \beta_G < \infty \)). In this case, the optimal Bessel bound is \( \beta_G/r \).

(c) The sequence \( \{g_j(w)e^{2\pi irmw}\}_{n \in \mathbb{Z}; j=1,2,\ldots,s} \) is a frame for \( L^2(0,1) \) if and only if \( 0 < \alpha_G \leq \beta_G < \infty \). In this case, the optimal frame bounds are \( \alpha_G/r \) and \( \beta_G/r \).

(d) The sequence \( \{g_j(w)e^{2\pi irmw}\}_{n \in \mathbb{Z}; j=1,2,\ldots,s} \) is a Riesz basis for \( L^2(0,1) \) if and only if it is a frame and \( s = r \).

A comment about Lemma 2.2 in terms of the average sampling terminology introduced by Aldroubi et al. in [6] is in order. According to [6], we say that:

1. The set \( \{L_1, L_2, \ldots, L_s\} \) is an r-determining U-sampler for \( A_a \) if the only vector \( x \in A_a \), satisfying \( L_jx(rm) = 0 \) for all \( j = 1, 2, \ldots, s \) and \( m \in \mathbb{Z} \), is \( x = 0 \).

2. The set \( \{L_1, L_2, \ldots, L_s\} \) is an r-stable U-sampler for \( A_a \) if there exist positive constants \( A \) and \( B \) such that

\[
A\|x\|^2 \leq \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} |L_jx(rm)|^2 \leq B\|x\|^2 \quad \text{for all } x \in A_a.
\]

Hence, parts (a) and (c) of Lemma 2.2 can be read, by using (2.3), as follows:

(i) The set \( \{L_1, L_2, \ldots, L_s\} \) is an r-determining U-sampler for \( A_a \) if and only if \( \text{rank} \ G(w) = r \) a.e. in \((0,1)\) (and hence, necessarily, \( s \geq r \)).

(ii) The set \( \{L_1, L_2, \ldots, L_s\} \) is an r-stable U-sampler for \( A_a \) if and only if \( 0 < \alpha_G \leq \beta_G < \infty \).

An r-determining U-sampler for \( A_a \) can distinguish between two distinct elements in \( A_a \), but the recovery, if any, is not necessarily stable. If the system \( \{L_1, L_2, \ldots, L_s\} \) is an r-stable U-sampler for \( A_a \), then any \( x \in A_a \) can be recovered, in a stable way, from the sequence of generalized samples \( \{L_jx(rm)\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \), where necessarily \( s \geq r \). Roughly speaking, the operator which maps

\[
A_a \ni x \mapsto \{L_jx(rm)\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \in \ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z}) \quad (s \text{ times})
\]

has a bounded inverse.

Having in mind (2.3), from the sequence of samples \( \{L_jx(rm)\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \) we recover \( F \in L^2(0,1) \), and by means of the isomorphism \( T_{U,a} \), the vector \( x = T_{U,a}F \) in \( A_a \). This will be the main goal in the next section.
3. Generalized Regular Sampling in $\mathcal{A}_a$

Along with the characterization of the sequence $\{g_j(w)e^{2\pi i nw}\}_{n \in \mathbb{Z}, j=1,2,\ldots,s}$ as a frame in $L^2(0,1)$, in [14] a family of dual frames are also given: Choose functions $h_j$ in $L^\infty(0,1)$, $j = 1, 2, \ldots, s$, such that

$$[h_1(w), h_2(w), \ldots, h_s(w)]G(w) = [1, 0, \ldots, 0] \text{ a.e. in } (0,1). \quad (3.1)$$

It was proven in [14] that the sequence $\{rh_j(w)e^{2\pi i nw}\}_{n \in \mathbb{Z}, j=1,2,\ldots,s}$ is a dual frame of the sequence $\{g_j(w)e^{2\pi i nw}\}_{n \in \mathbb{Z}, j=1,2,\ldots,s}$ in $L^2(0,1)$. In other words, taking into account (2.3), we have for any $F \in L^2(0,1)$ the expansion

$$F = \sum_{j=1}^s \sum_{m \in \mathbb{Z}} \mathcal{L}_j x(rm)rh_j(w)e^{2\pi irmw} \quad \text{in } L^2(0,1). \quad (3.2)$$

Concerning to the existence of the functions $h_j$, $j = 1, 2, \ldots, s$, consider the first row of the $r \times s$ Moore–Penrose pseudo-inverse $G^\dagger(w)$ of $G(w)$ given by

$$G^\dagger(w) := [G^*(w)G(w)]^{-1}G^*(w).$$

Its entries are essentially bounded in $(0,1)$ since the functions $g_j$, $j = 1, 2, \ldots, s$, and $\det^{-1}[G^*(w)G(w)]$ are essentially bounded in $(0,1)$, and (3.1) trivially holds. All the possible solutions of (3.1) are given by the first row of the $r \times s$ matrices given by

$$H_K(w) := G^\dagger(w) + K(w)[I_s - G(w)G^\dagger(w)], \quad (3.3)$$

where $K(w)$ denotes any $r \times s$ matrix with entries in $L^\infty(0,1)$, and $I_s$ is the identity matrix of order $s$.

Applying the isomorphism $\mathcal{T}_{U,a}$ in (3.2), for $x = \mathcal{T}_{U,a}F \in \mathcal{A}_a$ we obtain the sampling expansion:

$$x = \sum_{j=1}^s \sum_{m \in \mathbb{Z}} \mathcal{L}_j x(rm)\mathcal{T}_{U,a}[rh_j(\cdot)e^{2\pi irm}] = \sum_{j=1}^s \sum_{m \in \mathbb{Z}} \mathcal{L}_j x(rm)U^{rm}[\mathcal{T}_{U,a}(rh_j)] \quad (3.4)$$

$$= \sum_{j=1}^s \sum_{m \in \mathbb{Z}} \mathcal{L}_j x(rm)U^{rm}c_{j,h} \quad \text{in } \mathcal{H},$$

where $c_{j,h} := \mathcal{T}_{U,a}(rh_j) \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, and we have used the $U$-shift property (2.1). Besides, the sequence $\{U^{rm}c_{j,h}\}_{m \in \mathbb{Z}, j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds.

**Theorem 3.1.** Let $b_j \in \mathcal{H}$ and let $\mathcal{L}_j$ be its associated $U$-system for $j = 1, 2, \ldots, s$. Assume that the function $g_j$, $j = 1, 2, \ldots, s$, given in (2.4) belongs to $L^\infty(0,1)$; or equivalently, that $\beta_G < \infty$ for the associated $s \times r$ matrix $G(w)$. The following statements are equivalent:

(a) $\alpha_G > 0$. 

(b) There exists a vector \([h_1(w), h_2(w), \ldots, h_s(w)]\) with entries in \(L^\infty(0, 1)\) satisfying
\[
[h_1(w), h_2(w), \ldots, h_s(w)]G(w) = [1, 0, \ldots, 0] \quad \text{a.e. in } (0, 1).
\]

(c) There exist elements \(c_j \in \mathcal{A}_a, j = 1, 2, \ldots, s,\) such that the sequence \(\{U^{rk}c_j\}_{k \in \mathbb{Z}; j=1,2,\ldots,s}\) is a frame for \(\mathcal{A}_a,\) and for any \(x \in \mathcal{A}_a\) the expansion
\[
x = \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} L_jx(rk)U^{rk}c_j \quad \text{in } \mathcal{H}, \quad (3.5)
\]
holds.

(d) There exists a frame \(\{C_{j,k}\}_{k \in \mathbb{Z}; j=1,2,\ldots,s}\) for \(\mathcal{A}_a\) such that, for each \(x \in \mathcal{A}_a\) the expansion
\[
x = \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} L_jx(rk)C_{j,k} \quad \text{in } \mathcal{H},
\]
holds.

**Proof.** We have already proved that (a) implies (b) and that (b) implies (c). Obviously, (c) implies (d). As a consequence, we only need to prove that (d) implies (a). Applying the isomorphism \(T_{U,a}^{-1}\) to the expansion in (d), and taking into account (2.3) we obtain
\[
F = T_{U,a}^{-1}x = \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} L_jx(rk)T_{U,a}^{-1}(C_{j,k})
= \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} (F, g_j(w)e^{2\pi irkw})_{L^2(0,1)} T_{U,a}^{-1}(C_{j,k}) \quad \text{in } L^2(0, 1),
\]
where the sequence \(\{T_{U,a}^{-1}(C_{j,k})\}_{k \in \mathbb{Z}; j=1,2,\ldots,s}\) is a frame for \(L^2(0, 1).\) The sequence \(\{g_j(w)e^{2\pi irkw}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}\) is a Bessel sequence in \(L^2(0,1)\) since \(\beta_\mathcal{S} < \infty,\) and satisfying the above expansion in \(L^2(0, 1).\) According to [8, Lemma 5.6.2], the sequences \(\{T_{U,a}^{-1}(C_{j,k})\}_{k \in \mathbb{Z}; j=1,2,\ldots,s}\) and \(\{g_j(w)e^{2\pi irkw}\}_{k \in \mathbb{Z}; j=1,2,\ldots,s}\) form a pair of dual frames in \(L^2(0,1);\) in particular, by using Lemma 2.2 we obtain that \(\alpha_\mathcal{S} > 0\) which concludes the proof. \(\square\)

In case the functions \(g_j, j = 1, 2, \ldots, s\) are continuous on \(\mathbb{R},\) condition (a) in Theorem 3.1 can be expressed in terms of the rank of the matrix \(G(w);\) notice that this occurs, for example, whenever the sequences \(\{L_ja(k)\}_{k \in \mathbb{Z}, j=1,2,\ldots,s}\) belong to \(\ell^1(\mathbb{Z}).\)

**Corollary 3.2.** Assume that the 1-periodic extension of the functions \(g_j, j = 1, 2, \ldots, s,\) given in (2.4) are continuous on \(\mathbb{R}.\) Then, the following conditions are
(i) \( \text{rank} G(w) = r \) for all \( w \in \mathbb{R} \).

(ii) There exist \( c_j \in A_a \), \( j = 1, 2, \ldots, s \), such that the sequence \( \{U^{rk} c_j \}_{k \in \mathbb{Z} : j=1,2,\ldots,s} \) is a frame for \( A_a \), and the sampling formula (3.5) holds for each \( x \in A_a \).

**Proof.** Whenever the functions \( g_j, j = 1, 2, \ldots, s \), are continuous on \( \mathbb{R} \), the condition \( \alpha_G > 0 \) is equivalent to \( \det [G^*(w)G(w)] \neq 0 \) for all \( w \in \mathbb{R} \). Indeed, if \( \det G^*(w) \times G(w) > 0 \) then the first row of the matrix \( G^*(w) : = [G^*(w)G(w)]^{-1}G^*(w) \), gives a vector \( [h_1, h_2, \ldots, h_s] \) satisfying the statement (b) in Theorem 3.1 and, as a consequence, \( \alpha_G > 0 \). The converse follows from the fact that \( \det [G^*(w)G(w)] \geq \alpha_G^r \) for all \( w \in \mathbb{R} \). Since, \( \det [G^*(w)G(w)] \neq 0 \) is equivalent to \( \text{rank} G(w) = r \) for all \( w \in \mathbb{R} \), the result is a consequence of Theorem 3.1. \( \square \)

Whenever the sampling period \( r \) equals the number of \( U \)-systems \( s \) we are in the presence of Riesz bases, and there exists a unique sampling expansion in Theorem 3.1.

**Corollary 3.3.** Let \( b_j \in H \) for \( j = 1, 2, \ldots, r \), i.e. \( r = s \) in Theorem 3.1. Let \( L_j \) be its associated \( U \)-system for \( j = 1, 2, \ldots, r \). Assume that the function \( g_j, j = 1, 2, \ldots, r \), given in (2.4) belongs to \( L^\infty(0,1) \); or equivalently, \( \beta_G < \infty \) for the associated \( r \times r \) matrix \( G(w) \). The following statements are equivalent:

(a) \( \alpha_G > 0 \).

(b) There exists a Riesz basis \( \{C_{j,k}\}_{k \in \mathbb{Z} : j=1,2,\ldots,r} \) such that for any \( x \in A_a \) the expansion

\[
x = \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}} L_j x(rk) C_{j,k} \quad \text{in } H \tag{3.6}
\]

holds.

In case the equivalent conditions are satisfied, necessarily there exist \( c_j \in A_a \), \( j = 1, 2, \ldots, r \), such that \( C_{j,k} = U^{rk} c_j \) for \( k \in \mathbb{Z} \) and \( j = 1, 2, \ldots, r \). Moreover, the interpolation property \( L_j c_j(rk) = \delta_j \delta_{k,0} \), where \( k \in \mathbb{Z} \) and \( j, j' = 1, 2, \ldots, r \), holds.

**Proof.** Assume that \( \alpha_G > 0 \); since \( G(w) \) is a square matrix, this implies that

\[\text{ess inf}_{w \in \mathbb{R}} |\det G(w)| > 0.\]

Thus, the unique solution \( [h_1(w), h_2(w), \ldots, h_r(w)] \) of (3.1) with \( h_j \in L^\infty(0,1) \) for \( j = 1, 2, \ldots, r \) is given by the first row of the matrix \( G^{-1}(w) \). According to Theorem 3.1, the sequence \( \{C_{j,k}\}_{k \in \mathbb{Z} : j=1,2,\ldots,r} : = \{U^{rk} c_j\}_{k \in \mathbb{Z} : j=1,2,\ldots,r} \), where \( c_j = T_{U,a}(rh_j) \), satisfies the sampling formula (3.6). Moreover, the sequence \( \{rh_j(w)e^{2\pi i k w}\}_{k \in \mathbb{Z} : j=1,2,\ldots,r} = \{T_{U,a}^{-1}(U^{rk} c_j)\}_{k \in \mathbb{Z} : j=1,2,\ldots,r} \) is a frame for \( L^2(0,1) \).
Since $r = s$, according to Lemma 2.2, it is a Riesz basis. Hence, \( \{ U^r c_j \}_{k \in \mathbb{Z}; j=1,2,...,r} \) is a Riesz basis for \( A_a \) and (b) is proved.

Conversely, assume now that \( \{ C_{j,k} \}_{k \in \mathbb{Z} j=1,2,...,r} \) is a Riesz basis for \( A_a \) satisfying (3.6). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition \( (L_j C_{j,k})(rk') = \delta_{j,j'} \delta_{k,k'} \) holds for \( j, j' = 1, 2, \ldots, r \) and \( k, k' \in \mathbb{Z} \). Since \( T_{U,a}^{-1} \) is an isomorphism, the sequence \( \{ T_{U,a}^{-1}(C_{j,k}) \}_{k \in \mathbb{Z}; j=1,2,...,r} \) is a Riesz basis for \( L^2(0,1) \). Expanding the function \( g_{j'}(w)e^{-2\pi irk'w} \) with respect to the dual basis of \( \{ T_{U,a}^{-1}(C_{j,k}) \}_{j \in \mathbb{Z}; j=1,2,...,r} \), denoted by \( \{ D_{j,k} \}_{k \in \mathbb{Z}; j=1,2,...,r} \), and having in mind (2.3) we obtain

\[
\sum_{j=1}^{r} \sum_{k \in \mathbb{Z}} \langle g_{j'}(\cdot)e^{2\pi irk' \cdot \cdot}, T_{U,a}^{-1}(C_{j,k}) \rangle_{L^2(0,1)} D_{j,k}(w) = D_{j',k'}(w).
\]

Therefore, the sequence \( \{ g_{j'}(w)e^{2\pi irk'w} \}_{k \in \mathbb{Z}; j=1,2,...,r} \) is the dual basis of the Riesz basis \( \{ T_{U,a}^{-1}(C_{j,k}) \}_{k \in \mathbb{Z}; j=1,2,...,r} \). In particular, it is a Riesz basis for \( L^2(0,1) \), which implies, according to Lemma 2.2, that \( \alpha_{C} > 0 \), i.e. condition (a). Moreover, the sequence \( \{ T_{U,a}^{-1}(C_{j,k}) \}_{k \in \mathbb{Z}; j=1,2,...,r} \) is necessarily the unique dual basis of the Riesz basis \( \{ g_{j'}(w)e^{2\pi irk'w} \}_{k \in \mathbb{Z}; j=1,2,...,r} \). Therefore, this proves the uniqueness of the Riesz basis \( \{ C_{j,k} \}_{k \in \mathbb{Z}; j=1,2,...,r} \) for \( A_a \) satisfying (3.6).

Some comments on the sequence \( \{ U^r b_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \)

Concerning Theorem 3.1, more can be said about the sequence \( \{ U^r b_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \), where the vectors \( b_j \in H \) define the \( U \)-systems \( L_j \equiv L_{b_j}, j = 1, 2, \ldots, s \). Having in mind (2.3) and the isomorphism \( T_{U,a} \), we obtain that

\[
\frac{\alpha_{C}}{r} \| T_{U,a}^{-1} \|^2 \| x \|^2 \leq \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} | \langle x, U^r b_j \rangle |^2 \leq \frac{\beta_{C}}{r} \| T_{U,a} \|^2 \| x \|^2 \text{ for all } x \in A_a.
\]  

(3.7)

- In case that \( b_j \in A_a \) for each \( j = 1, 2, \ldots, s \), we derive that \( \{ U^r b_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) is a frame for \( A_a \), and it is dual to the frame \( \{ U^r c_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) in \( A_a \). Thus, the sampling expansion (3.5) is nothing but a frame expansion in \( A_a \).
- In case that some \( b_j \notin A_a \), the sequence \( \{ U^r b_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) is not contained in \( A_a \). However, inequalities (3.7) hold. Therefore, the sequence \( \{ U^r b_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) is a pseudo-dual frame for the frame \( \{ U^r c_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) in \( A_a \) (see [20, 21]). Denoting by \( P_{A_a} \) the orthogonal projection onto \( A_a \), we derive from (3.7) that the sequence \( \{ P_{A_a}(U^r b_j) \}_{k \in \mathbb{Z}; j=1,2,...,s} \) is a dual frame of \( \{ U^r c_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) in \( A_a \).
- Whenever \( r = s \), according to the above cases, the sequence \( \{ U^r b_j \}_{k \in \mathbb{Z}; j=1,2,...,s} \) is a Riesz basis or a pseudo-Riesz basis for \( A_a \).
Sampling formulas with prescribed properties

The sampling formula (3.5) can be thought as a filter-bank. Indeed, assume that for $j = 1, 2, \ldots, s$ we have

$$c_{j,h} = T_{U,a}(rh_j) = r \sum_{n \in \mathbb{Z}} \hat{h}_j(n) U^m a$$

where $\hat{h}_j(n) = \int_0^1 h_j(w)e^{-2\pi inw} dw, \ n \in \mathbb{Z}$.

Substituting in (3.5), after the change of summation index $m := rk + n$ we obtain

$$x = \sum_{m \in \mathbb{Z}} \left\{ \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} r \mathcal{L}_j x(rk) \hat{h}_j(m - rk) \right\} U^m a,$$

that is, the relevant data is the output of a filter-bank:

$$\alpha_m := \sum_{j=1}^{s} \sum_{k \in \mathbb{Z}} r \mathcal{L}_j x(rk) \hat{h}_j(m - rk), \ m \in \mathbb{Z}$$

where the input is the given samples and the impulse responses depend on the sampling vectors $c_{j,h}, \ j = 1, 2, \ldots, s$. In the oversampling setting, i.e. $s > r$, according to (3.3) there exist infinitely many sampling vectors $c_{j,h}, \ j = 1, 2, \ldots, s$, for which the sampling formula (3.5) holds. A natural question is whether we can choose the sampling vectors $c_{j,h}, \ j = 1, 2, \ldots, s$, with prescribed properties.

For instance, a challenging problem is to ask under what conditions we are in the presence of a finite impulse response filter-bank; i.e. $c_{j,h} = r \sum_{\text{finite}} \hat{h}_j(n) U^m a, \ j = 1, 2, \ldots, s$, or equivalently, when the functions $h_j, \ j = 1, \ldots, s$, are $2\pi$-periodic trigonometric polynomials. Instead, we deal with Laurent polynomials by using the variable $z = e^{2\pi i w}$, that is, $g_j(z) := \sum_{k \in \mathbb{Z}} \mathcal{L}_j a(k) z^k, \ j = 1, 2, \ldots, s$. We introduce the $s \times r$ matrix

$$G(z) := \begin{bmatrix} g_1(z) & g_1(zW) & \cdots & g_1(zW^{r-1}) \\ g_2(z) & g_2(zW) & \cdots & g_2(zW^{r-1}) \\ \vdots & \vdots & \ddots & \vdots \\ g_s(z) & g_s(zW) & \cdots & g_s(zW^{r-1}) \end{bmatrix} = \begin{bmatrix} g_j(zW^k) \end{bmatrix}_{j=1,2,\ldots,s \atop k=0,1,\ldots,r-1},$$

where $W := e^{2\pi i / r}$. In case the functions $g_j(z), \ j = 1, 2, \ldots, s$, are Laurent polynomials, the matrix $G(z)$ has Laurent polynomials entries. Besides, the relationship $G(w) = G(e^{2\pi i w}), \ w \in (0, 1)$, holds.

So that, we are interested in finding Laurent polynomials $h_j(z), \ j = 1, 2, \ldots, s$, satisfying

$$[h_1(z), h_2(z), \ldots, h_s(z)]G(z) = [1, 0, \ldots, 0].$$

Thus, the trigonometric polynomials $h_j(w) := h_j(e^{2\pi i w}), \ j = 1, 2, \ldots, s$, satisfy (3.1), and the corresponding reconstruction vectors $c_{j,h} = T_{U,a}(rh_j), \ j = 1, 2, \ldots, s$,
can be expanded in $A_a$ with just a finite number of terms. Namely,
\[ c_{j,h} = r \sum_{\text{finite}} \hat{h}_j(n)U^n a, \quad \text{where} \quad h_j(z) = \sum_{\text{finite}} \hat{h}_j(n)z^n, \quad j = 1, 2, \ldots, s. \]

The following result holds:

**Theorem 3.4.** Assume that the sequences $\{L_ja(k)\}_{k \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$, contain only a finite number of nonzero terms. Then, there exists a vector $h(z) := [h_1(z), h_2(z), \ldots, h_s(z)]$ whose entries are Laurent polynomials, and satisfying $h(z)G(z) = [1, 0, \ldots, 0]$ if and only if
\[ \text{rank } G(z) = r \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \]

**Proof.** This result is a consequence of the next lemma which proof can be found in [34, Theorems 5.1 and 5.6]. \( \square \)

**Lemma 3.5.** Let $G(z)$ be an $s \times r$ matrix whose entries are Laurent polynomials. Then, there exists an $r \times s$ matrix $H(z)$ whose entries are also Laurent polynomials satisfying $H(z)G(z) = I_r$ if and only if $\text{rank } G(z) = r$ for all $z \in \mathbb{C} \setminus \{0\}$.

Analogously we can consider the case where the coefficients of the reconstruction vectors $c_{j,h} = r \sum_{n \in \mathbb{Z}} \hat{h}_j(n)U^n a$, $j = 1, 2, \ldots, s$, have exponential decay, i.e. there exist $C > 0$ and $q \in (0,1)$ such that $|\hat{h}_j(n)| \leq Cq^n$, $n \in \mathbb{Z}$, $j = 1, 2, \ldots, s$. Assuming that the sequences $\{L_ja(k)\}_{k \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay then, we can find reconstruction vectors $c_{j,h}$ such that the sequences $\{\hat{h}_j(n)\}_{n \in \mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay if and only if $\text{rank } G(z) = r$ for all $z \in \mathbb{C}$ such that $|z| = 1$. For the details, see [16] and references therein.

4. **Time-Jitter Error: Irregular Sampling in $A_a$**

A close look to Sec. 3 shows that all the regular sampling results have been proved without the formalism of a continuous group of unitary operators $\{U^t\}_{t \in \mathbb{R}}$ in $H$: we have only used the integer powers $\{U^n\}_{n \in \mathbb{Z}}$ which are completely determined from the unitary operator $U$. However, if we are concerned with the jitter error in a sampling formula as (3.5), the group of unitary operators becomes essential. Here, we dispose of a perturbed sequence of samples $\{(L_jx)(rm + \epsilon_m)\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}$, with errors $\epsilon_m \in \mathbb{R}$, for the recovery of $x \in A_a$. By using (2.3) and (2.2) we obtain:
\[ L_jx(rm) = \langle F, g_j(w)e^{2\pi irmw} \rangle_{L^2(0,1)} \quad \text{and} \]
\[ L_jx(rm + \epsilon_m) = \langle F, g_{m,j}(w)e^{2\pi irmw} \rangle_{L^2(0,1)}, \]
where the functions
\[ g_j(w) := \sum_{k \in \mathbb{Z}} L_ja(k)e^{2\pi ikw} \quad \text{and} \quad g_{m,j}(w) := \sum_{k \in \mathbb{Z}} L_ja(k + \epsilon_m)e^{2\pi ikw}, \]
belong to $L^2(0,1)$. Let $G(w)$ be the $s \times r$ matrix given in (2.5), associated with the functions $g_{j}$, $j = 1, 2, \ldots, s$. In the case that $0 < \alpha_G \leq \beta_G < \infty$, the sequence
\{g_j(w)e^{2\pi irm_w}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \text{ is a frame for } L^2(0,1) \text{ with optimal frame bounds } \alpha_G/r \text{ and } \beta_G/r. \text{ Thus, as in [15], we can see the sequence } \\
\{g_{m,j}(w)e^{2\pi irm_w}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \text{ in } L^2(0,1) \text{ as a perturbation of the frame } \\
\{g_j(w)e^{2\pi irm_w}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \text{ in } L^2(0,1). \text{ The following result on frame perturbation, which proof can be found in [8, p. 354] will be used later.}

**Lemma 4.1.** Let \(\{x_n\}_{n=1}^\infty\) be a frame for the Hilbert space \(\mathcal{H}\) with frame bounds \(A, B\), and let \(\{y_n\}_{n=1}^\infty\) be a sequence in \(\mathcal{H}\). If there exists a constant \(R < A\) such that \\
\[\sum_{n=1}^\infty |\langle x_n - y_n, x \rangle|^2 \leq R\|x\|^2 \quad \text{for each } x \in \mathcal{H},\]
then the sequence \(\{y_n\}_{n=1}^\infty\) is also a frame for \(\mathcal{H}\) with bounds \(A(1 - \sqrt{R/A})^2\) and \(B(1 + \sqrt{R/B})^2\). If the sequence \(\{x_n\}_{n=1}^\infty\) is a Riesz basis, then the sequence \(\{y_n\}_{n=1}^\infty\) is also a Riesz basis.

**The time-jitter error sampling expansion**

Given an error sequence \(e := \{e_{mj}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}\), assume that the operator \\
\[D_e : \ell^2(\mathbb{Z}) \to \ell_s^2(\mathbb{Z}),\]
\[c = \{c_i\}_{i \in \mathbb{Z}} \mapsto D_e c := (D_{e,1}c, \ldots, D_{e,s}c),\]
is well-defined, where, for \(j = 1, 2, \ldots, s,\)
\[D_{e,j}c := \left\{\sum_{k \in \mathbb{Z}} [L_ja(rm - k + e_{mj}) - L_ja(rm - k)]c_k\right\}_{m \in \mathbb{Z}}. \tag{4.1}\]

The operator norm (it could be infinity) is defined as usual \\
\[\|D_e\| := \sup_{c \in \ell^2(\mathbb{Z})\setminus\{0\}} \frac{\|D_e c\|_{\ell^2_s(\mathbb{Z})}}{\|c\|_{\ell^2(\mathbb{Z})}},\]
where \(\|D_e c\|_{\ell^2_s(\mathbb{Z})} := \sum_{j=1}^s \|D_{e,j}c\|_{\ell^2_s(\mathbb{Z})}^2\) for each \(c \in \ell^2(\mathbb{Z}).\)

**Theorem 4.2.** Assume that for the functions \(g_j, j = 1, 2, \ldots, s\), given in (2.4) we have \(0 < \alpha_G \leq \beta_G < \infty\). Let \(e := \{e_{mj}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}\) be an error sequence satisfying the inequality \(\|D_e\|^2 < \alpha_G/r\). Then, there exists a frame \(\{C^{e}_{m,j}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}\) for \(A_e\) such that, for any \(x \in A_a\), the sampling expansion \\
\[x = \sum_{j=1}^s \sum_{m \in \mathbb{Z}} L_j x(rm + e_{mj}) C^{e}_{j,m} \text{ in } \mathcal{H}, \tag{4.2}\]
holds. Moreover, when \(r = s\) the sequence \(\{C^{e}_{j,m}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}\) is a Riesz basis for \(A_a\), and the interpolation property \((L_i C^{e}_{j,n})(rm + e_{mj}) = \delta_{j,i}\delta_{n,m}\) holds.
Proof. The sequence \( \{ g_j(w) e^{2\pi i rmw} \}_{m\in\mathbb{Z}; j=1,2,...,s} \) is a frame (a Riesz basis if \( r = s \)) for \( L^2(0,1) \) with optimal frame (Riesz) bounds \( \alpha r / r \) and \( \beta r / r \). For any \( F(w) = \sum_{l \in \mathbb{Z}} a_l e^{2\pi ilw} \) in \( L^2(0,1) \) we have

\[
\sum_{m \in \mathbb{Z}} \sum_{j=1}^{s} |\langle g_{m,j}(\cdot) e^{2\pi i rm\cdot} - g_j(\cdot) e^{2\pi i rm\cdot}, F(\cdot) \rangle_{L^2(0,1)}|^2 \leq \sum_{m \in \mathbb{Z}} \sum_{j=1}^{s} \left| \sum_{k \in \mathbb{Z}} (\mathcal{L}_j a(r m - k + \epsilon_{m_j}) - \overline{\mathcal{L}_j a(r m - k)}) e^{2\pi i k \cdot}, F(\cdot) \rangle_{L^2(0,1)} \right|^2 \leq 2 \sum_{m \in \mathbb{Z}} \sum_{j=1}^{s} \left| \sum_{k \in \mathbb{Z}} (\mathcal{L}_j a(r m - k + \epsilon_{m_j}) - \overline{\mathcal{L}_j a(r m - k)}) \overline{a_k} \right|^2 \leq 2 \sum_{j=1}^{s} \| D_{\epsilon,j} \{ a_l \}_{l \in \mathbb{Z}} \|^2_{\ell^2(\mathbb{Z})} \leq 2 \| D_{\epsilon} \|^2 \| \{ a_l \}_{l \in \mathbb{Z}} \|^2_{\ell^2(\mathbb{Z})} \leq 2 \| D_{\epsilon} \|^2 \| F \|^2_{L^2(0,1)}. \tag{4.3}
\]

By using Lemma 4.1 we obtain that the sequence \( \{ g_{m,j}(w) e^{2\pi i rmw} \}_{m\in\mathbb{Z}; j=1,2,...,s} \) is a frame for \( L^2(0,1) \) (a Riesz basis if \( r = s \)). Let \( \{ h^\epsilon_{j,m} \}_{m\in\mathbb{Z}; j=1,2,...,s} \) be its canonical dual frame. Hence, for any \( F \in L^2(0,1) \)

\[
F = \sum_{m \in \mathbb{Z}} \sum_{j=1}^{s} \langle F(\cdot), g_{m,j}(\cdot) e^{2\pi i rm\cdot} \rangle_{L^2(0,1)} h^\epsilon_{j,m} = \sum_{m \in \mathbb{Z}} \sum_{j=1}^{s} \mathcal{L}_j x(r m + \epsilon_{m_j}) h^\epsilon_{j,m} \quad \text{in} \quad L^2(0,1).
\]

Applying the isomorphism \( T_{U,a} \), one gets (4.2), where \( C_{j,m}^\epsilon := T_{U,a}(h^\epsilon_{j,m}) \) for \( m \in \mathbb{Z} \) and \( j = 1,2,\ldots,s \). Since \( T_{U,a} \) is an isomorphism between \( L^2(0,1) \) and \( A_a \), the sequence \( \{ C_{j,m}^\epsilon \}_{m\in\mathbb{Z}; j=1,2,...,s} \) is a frame for \( A_a \) (a Riesz basis if \( r = s \)). The interpolatory property in the case \( r = s \) follows from the uniqueness of the coefficients with respect to a Riesz basis. \( \square \)

Sampling formula (4.2) is useless from a practical point of view: it is impossible to determine the involved frame \( \{ C_{j,m}^\epsilon \}_{m\in\mathbb{Z}; j=1,2,...,s} \). As a consequence, in order to recover \( x \in A_a \) from the sequence of samples \( \{ (\mathcal{L}_j x)(r m + \epsilon_{m_j}) \}_{m\in\mathbb{Z}; j=1,2,...,s} \) we should implement a frame algorithm in \( \ell^2(\mathbb{Z}) \) (see [15]); another possibility is given in [1].
In order to prove the existence of sequences $\epsilon := \{\epsilon_{mj}\}_{m \in \mathbb{Z}, j = 1, \ldots, s}$ such that $\|D_\epsilon\|^2 < \alpha_G/r$ we need some results from the group of unitary operators theory.

A brief excursion on groups of unitary operators

Let $\{U_t\}_{t \in \mathbb{R}}$ denote a continuous group of unitary operators in $\mathcal{H}$. Classical Stone’s theorem [26] assures us the existence of a self-adjoint operator $T$ (may be unbounded) such that $U_t \equiv e^{itT}$. This self-adjoint operator $T$, defined on the dense domain of $\mathcal{H}$

$$D_T := \left\{ x \in \mathcal{H} \text{ such that } \int_{-\infty}^{\infty} w^2 d\|E_w x\|^2 < \infty \right\},$$

admits the spectral representation $T = \int_{-\infty}^{\infty}wdE_w$ which means:

$$\langle Tx, y \rangle = \int_{-\infty}^{\infty} wd\langle E_w x, y \rangle \text{ for any } x \in D_T \text{ and } y \in \mathcal{H},$$

where $\{E_w\}_{w \in \mathbb{R}}$ is the corresponding resolution of the identity, i.e. a one-parameter family of projection operators $E_w$ in $\mathcal{H}$ such that

(i) $E_{-\infty} := \lim_{w \to -\infty} E_w = 0_{\mathcal{H}}, E_{\infty} := \lim_{w \to \infty} E_w = I_{\mathcal{H}},$

(ii) $E_w^{-} = E_w$ for any $-\infty < w < \infty$,

(iii) $E_u E_v = E_w$ where $w = \min\{u, v\}$.

Recall that $\|E_w x\|^2$ and $\langle E_w x, y \rangle$, as functions of $w$, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Furthermore, for any $x \in D_T$ we have that $\lim_{t \to 0} \frac{U^t x - x}{t} = iTx$ and the operator $iT$ is said to be the infinitesimal generator of the group $\{U_t\}_{t \in \mathbb{R}}$. For each $x \in D_T$, $U^t x$ is a continuous differentiable function of $t$. Notice that, whenever the self-adjoint operator $T$ is bounded, $D_T = \mathcal{H}$ and $e^{itT}$ can be defined as the usual exponential series; in any case, $U^t \equiv e^{itT}$ means that

$$\langle U^t x, y \rangle = \int_{-\infty}^{\infty} e^{iwt} d\langle E_w x, y \rangle, \quad t \in \mathbb{R},$$

where $x \in D_T$ and $y \in \mathcal{H}$.

Finally, a comment on the continuity of a group of unitary operators: The group is said to be strongly continuous if, for each $x \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, $U^t x \to U^{t_0} x$ as $t \to t_0$. If $\mathcal{H}$ is a separable Hilbert space, strong continuity can be deduced from continuity and even from weak measurability, i.e. $\langle U^t x, y \rangle_{\mathcal{H}}$ is a Lebesgue measurable function of $t$ for any $x, y \in \mathcal{H}$. See, for instance, [2, 7, 32, 33].

On the existence of sequences $\epsilon$ such that $\|D_\epsilon\|^2 < \alpha_G/r$

Assuming that $b_j \in D_T$, $j = 1, 2, \ldots, s$, the functions $L_j a(t)$, $j = 1, 2, \ldots, s$, are continuously differentiable on $\mathbb{R}$. If, for instance, we demand in addition that, for
each \( j = 1, 2, \ldots, s \), there exists \( \eta_j > 0 \) such that
\[
(L_j a)'(t) = O(|t|^{-(1+\eta_j)}) \quad \text{whenever } |t| \to \infty, \quad (4.4)
\]
then we can find out a finite bound for the norm \( \|D_\varepsilon\|_2^2 \). Indeed, for \( j = 1, 2, \ldots, s \) and \( n, m \in \mathbb{Z} \) denote
\[
d^{(j)}_{m,k} := L_j a(rm - k + \varepsilon_{m,j}) - L_j a(rm - k).
\]
Taking into account (4.1), for any sequence \( c = \{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \) we have
\[
\|D_\varepsilon c\|_2^2(\mathbb{Z}) = \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} d^{(j)}_{m,k} c_k \right|^2
\]
\[
\leq \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} \left| \sum_{l, k \in \mathbb{Z}} |d^{(j)}_{m,l} c_l d^{(j)}_{m,k} c_k| \right|
\]
\[
= \sum_{j=1}^{s} \sum_{l, k \in \mathbb{Z}} \left| c_l \right| \left| c_k \right| \sum_{m \in \mathbb{Z}} \left| d^{(j)}_{m,l} d^{(j)}_{m,k} \right|
\]
\[
\leq \sum_{j=1}^{s} \sum_{l, k \in \mathbb{Z}} \left| c_l \right|^2 + \left| c_k \right|^2 \sum_{m \in \mathbb{Z}} \left| d^{(j)}_{m,l} d^{(j)}_{m,k} \right|
\]
\[
= \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} \left| c_l \right|^2 \sum_{k, m \in \mathbb{Z}} \left| d^{(j)}_{m,l} d^{(j)}_{m,k} \right|. \quad (4.5)
\]
Under the decay conditions (4.4), for \( |\gamma| \leq 1/2 \) we define the continuous functions,
\[
M(L_j a)'(\gamma) := \sum_{m \in \mathbb{Z}} \max_{t \in [k - \gamma, k + \gamma]} |(L_j a)'(t)|,
\]
and
\[
N(L_j a)'(\gamma) := \max_{k=0,1,\ldots,r-1} \sum_{m \in \mathbb{Z}} \max_{t \in [rm + k - \gamma, rm + k + \gamma]} |(L_j a)'(t)|.
\]
Notice that \( N(L_j a)'(\gamma) \leq M(L_j a)'(\gamma) \) and for \( r = 1 \) the equality holds.

**Theorem 4.3.** Given an error sequence \( \varepsilon := \{\varepsilon_{m,j}\}_{m \in \mathbb{Z}, j=1,\ldots,s} \), define the constant \( \gamma_j := \sup_{m \in \mathbb{Z}} |\varepsilon_{m,j}| \) for each \( j = 1, 2, \ldots, s \). Then, the inequality
\[
\|D_\varepsilon\|_2^2 \leq \sum_{j=1}^{s} \gamma_j^2 N(L_j a)'(\gamma_j) M(L_j a)'(\gamma_j)
\]
holds and, as a consequence, condition
\[
\sum_{j=1}^{s} \gamma_j^2 N(L_j a)'(\gamma_j) M(L_j a)'(\gamma_j) < \frac{\alpha_\varepsilon}{r}
\]
ensures the hypothesis \( \|D_\varepsilon\|_2^2 < \alpha_\varepsilon/r \) in Theorem 4.2.
Proof. For each \( j = 1, 2, \ldots, s \), the mean value theorem gives
\[
\sup_{d \in [-\gamma_j, \gamma_j]} \sum_{n \in \mathbb{Z}} |L_j a(n + d) - L_j a(n)| \leq \gamma_j M(L_j a)'(\gamma_j),
\]
(4.6)
and
\[
\sup_{d \in [-\gamma_j, \gamma_j]} \sum_{n \in \mathbb{Z}} |L_j a(n + d) - L_j a(n)| \leq \gamma_j N(L_j a)'(\gamma_j).
\]
(4.7)
Thus, using (4.6) and (4.7), inequality (4.5) becomes
\[
\|D c\|_{\ell^2(\mathbb{Z})}^2 \leq \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} |c_l|^2 \sum_{k, m \in \mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}|
\leq \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} |c_l|^2 \sum_{m \in \mathbb{Z}} |d_{m,l}^{(j)}| \gamma_j M(L_j a)'(\gamma_j)
\leq \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} |c_l|^2 (\gamma_j)^2 M(L_j a)'(\gamma_j) N(L_j a)'(\gamma_j)
= \|c\|_{\ell^2(\mathbb{Z})}^2 \sum_{j=1}^{s} \gamma_j^2 N(L_j a)'(\gamma_j) M(L_j a)'(\gamma_j),
\]
which concludes the proof. \( \square \)

5. The Case of Multiple Generators

The case of \( L \) generators can be analogously derived. Indeed, consider the \( U \)-invariant subspace generated by \( a := \{a_1, a_2, \ldots, a_L\} \subset \mathcal{H} \), i.e.
\[
\mathcal{A}_a := \overline{\text{span}}\{U^n a_l, n \in \mathbb{Z}; l = 1, 2, \ldots, L\}.
\]
Assuming that the sequence \( \{U^n a_l\}_{n \in \mathbb{Z}; l = 1, 2, \ldots, L} \) is a Riesz sequence in \( \mathcal{H} \), the \( U \)-invariant subspace \( \mathcal{A}_a \) can be expressed as
\[
\mathcal{A}_a = \left\{ \sum_{l=1}^{L} \alpha_l U^n a_l : \{\alpha_l^n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}); l = 1, 2, \ldots, L \right\}.
\]
The sequence \( \{U^n a_l\}_{n \in \mathbb{Z}; l = 1, 2, \ldots, L} \) can be thought as an \( L \)-dimensional stationary sequence. Its covariance matrix \( R_a(k) \) is the \( L \times L \) matrix
\[
R_a(k) := \langle U^k a_m, a_n \rangle_{\mathcal{H}}, n, m = 1, 2, \ldots, L, \quad k \in \mathbb{Z}.
\]
It admits the spectral representation [19]:
\[
R_a(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\mu_a(\theta), \quad k \in \mathbb{Z}.
\]
The spectral measure \( \mu_a \) is an \( L \times L \) matrix; its entries are the spectral measures associated with the cross-correlation functions \( R_{m,n}(k) := \langle U^k a_m, a_n \rangle_{\mathcal{H}} \). It can be
decomposed into an absolute continuous part and its singular part. Thus we can write
\[ d\mu_a(\theta) = \Phi_a(\theta)d\theta + d\mu^s_a(\theta). \]
In case that the singular part \( \mu^s_a \equiv 0 \), the hermitian \( L \times L \) matrix \( \Phi_a(\theta) \) is called the spectral density of the sequence \( \{U^n a_i\}_{n \in \mathbb{Z}, l=1,2,...,L} \). The following theorem holds.

**Theorem 5.1.** Let \( \{U^n a_i\}_{n \in \mathbb{Z}, l=1,2,...,L} \) be a sequence obtained from a unitary operator in a separable Hilbert space \( \mathcal{H} \) with spectral measure \( d\mu_a(\theta) = \Phi_a(\theta)d\theta + d\mu^s_a(\theta) \), and let \( \mathcal{A}_a \) be the closed subspace spanned by \( \{U^n a_i\}_{n \in \mathbb{Z}, l=1,2,...,L} \). Then the sequence \( \{U^n a_i\}_{n \in \mathbb{Z}, l=1,2,...,L} \) is a Riesz basis for \( \mathcal{A}_a \) if and only if the singular part \( \mu^s_a \equiv 0 \) and

\[ 0 < \text{ess inf}_{\theta \in (-\pi, \pi)} \lambda_{\min} [\Phi_a(\theta)] \leq \text{ess sup}_{\theta \in (-\pi, \pi)} \lambda_{\max} [\Phi_a(\theta)] < \infty. \] (5.1)

**Proof.** For a fixed \( \ell^2 \)-sequence \( c := \{c_n\}_{n \in \mathbb{Z}, l=1,2,...,L} \) we have

\[
\left\| \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} c_k U^k a_l \right\|_2^2 = \sum_{i,j=1}^{L} \sum_{m,n \in \mathbb{Z}} c^i_k c^j_k \langle U^m a_i, U^n a_j \rangle \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{im\theta} e^{-in\theta} d\mu_{a_i,a_j}(\theta) \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n \in \mathbb{Z}} (c^i_m e^{im\theta})^* (\Phi_a(\theta)c^n_n e^{-in\theta}) d\mu_a(\theta),
\] (5.2)

where \( c_k = (c^1_k, c^2_k, \ldots, c^L_k)^T \) for every \( k \in \mathbb{Z} \).

First, we show that if the measure \( \mu_a \) is not absolutely continuous with respect to Lebesgue measure \( \lambda \) then \( \{U^n a_i\}_{n \in \mathbb{Z}, l=1,2,...,L} \) is not a Riesz basis for \( \mathcal{A}_a \). Indeed, if the spectral measure \( \mu_a \) is not absolutely continuous with respect to Lebesgue measure then there exists \( i \in \{1, 2, \ldots, L\} \) such that the positive spectral measure \( \mu_{a_i,a_i} \) is not absolutely continuous with respect to Lebesgue measure; this comes from the fact that, if any spectral measure in the diagonal \( \mu_{a_j,a_j} \) is absolutely continuous with respect to Lebesgue measure, the same occurs for each measure \( \mu_{a_j,a_k} \) with \( k \neq j \) (see [7, p. 137]). Then, \( \mu_{a_i,a_i}(B) > 0 \) for a (Lebesgue) measurable set \( B \subset (-\pi, \pi) \) of Lebesgue measure zero. Bearing in mind that every measurable set is included in a Borel set, actually an intersection of a countable collection of open sets, having the same Lebesgue measure (see [25, p. 63]), we take \( B \) to be a Borel set. Moreover, since every finite Borel measure on \( (-\pi, \pi) \) is inner regular (see [25, p. 340]) we may also assume that \( B \) is a compact set. For any \( \varepsilon > 0 \) there exists a sequence of disjoint open intervals \( I_j \subset (-\pi, \pi) \) such that

\[ B \subset \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(I_j) \leq \lambda(B) + \varepsilon = \varepsilon, \]
(see [25, pp. 58 and 42]). Since B is compact we may take the sequence to be finite. Hence, for every N ∈ ℕ there exist open disjoint intervals $I^N_1, I^N_2, \ldots, I^N_j$ in $(-\pi, \pi)$ such that

$$B \subset \bigcup_{j=1}^{j_N} I^N_j \quad \text{and} \quad \sum_{j=1}^{j_N} \lambda(I^N_j) \leq \frac{1}{3N}.$$ 

Besides, $\sum_{j=1}^{j_N} \mu_{a_i, a_i}(I^N_j) \geq \mu_{a_i, a_i}(B)$. Consider the function $g_N : (-\pi, \pi) \to \mathbb{R}$, where $g_N = 2^{N/2} \chi_{\bigcup_{j=1}^{j_N} I^N_j}$, that satisfies

$$\|g_N\|^2 = 2^N \sum_{j=1}^{j_N} \lambda(I^N_j) \leq \frac{2^N}{3N} < 1.$$ 

We modify and extend each $g_N$ to obtain a 2π-periodic function $f_N : \mathbb{R} \to \mathbb{R}$ such that $f_N$ and its derivative are continuous on $\mathbb{R}$, $\|f_N\|^2 \leq 1$ and $f_N(\theta) = g_N(\theta)$ for every $\theta \in \bigcup_{j=1}^{j_N} I^N_j$. Let $\sum_k c^N_k e^{ik\theta}$ be the Fourier series of $f_N$. First, by using Parseval’s identity we have

$$\|c^N_k\|^2 = \frac{1}{2\pi} \|f_N\|^2 \leq \frac{1}{2\pi} \quad \text{for every } N \in \mathbb{N},$$

so that $\{c^N\}_{N=1}^\infty$ is a bounded sequence in $\ell^2(\mathbb{Z})$. Besides, the regularity of each $f_N$ ensures that each Fourier series converges uniformly to $f_N$. Therefore, each series $\sum_k c^N_k e^{ik\theta}$ converges to $f_N$ in $L^2_{\mu_{a_i, a_i}}(-\pi, \pi)$ and consequently,

$$\left\| \sum_k c^N_k e^{ik\theta} \right\|^2_{L^2_{\mu_{a_i, a_i}}(-\pi, \pi)} = \int_{-\pi}^{\pi} |f_N|^2 d\mu_{a_i, a_i} \geq \int_{-\pi}^{\pi} |g_N|^2 d\mu_{a_i, a_i}$$

$$= 2^N \sum_{j=1}^{j_N} \mu_{a_i, a_i}(I^N_j) \geq 2^N \mu_{a_i, a_i}(B).$$

For every $c^N \in \ell^2(\mathbb{Z})$ we consider the $\ell^2$-sequence $\{c^N_n\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$ given by $c^N_n = c^N_l$ and $c^N_n = 0$ if $l \neq i$. Substituting each $\{c^N_n\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$ in (5.2) we have that

$$\left\| \sum_{l=1}^L c^N_l U^k a_l \right\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k \in \mathbb{Z}} c^N_k e^{ik\theta} \right|^2 d\mu_{a_i, a_i}(\theta)$$

tends to infinity with $N$, so $\{U^l a_l\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$ cannot be a Bessel sequence, therefore, not a Riesz basis.

For the remainder of the proof we assume that the singular part $\mu^*_\alpha \equiv 0$ and that $d\mu^*_\alpha(\theta) = \Phi_\alpha(\theta) d\theta$. Then (5.2) yields that

$$\left\| \sum_{l=1}^L \sum_{k \in \mathbb{Z}} c^l_k U^k a_l \right\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{m \in \mathbb{Z}} c_m e^{im\theta} \right)^\top \Phi_\alpha(\theta) \sum_{n \in \mathbb{Z}} c_n e^{in\theta} d\theta. \quad (5.3)$$
We have to show that \( \{U^n a_l\}_{n \in \mathbb{Z}, l=1,2,\ldots,L} \) is a Riesz basis for \( A_a \) if and only if (5.1) holds. Rayleigh–Ritz theorem (see [17, p. 176]) provides the inequalities

\[
\lambda_{\min}[\Phi_a(\theta)] \left| \sum_{k \in \mathbb{Z}} c_k e^{i k \theta} \right|^2 \leq \lambda_{\max}[\Phi_a(\theta)] \left| \sum_{n \in \mathbb{Z}} c_n e^{i n \theta} \right|^2,
\]

and taking into account (5.3) we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\min}[\Phi_a(\theta)] \left| \sum_{k \in \mathbb{Z}} c_k e^{i k \theta} \right|^2 d\theta \leq \left\| \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} c_k^l U^k a_l \right\|^2
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\max}[\Phi_a(\theta)] \left| \sum_{k \in \mathbb{Z}} c_k e^{i k \theta} \right|^2 d\theta,
\]

so that

\[
\text{ess inf}_{\theta \in (-\pi, \pi)} \lambda_{\min}[\Phi_a(\theta)] \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |c_k^l|^2 \leq \left\| \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} c_k^l U^k a_l \right\|^2
\]

\[
\leq \text{ess sup}_{\theta \in (-\pi, \pi)} \lambda_{\max}[\Phi_a(\theta)] \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |c_k^l|^2.
\]

Therefore, (5.1) implies that \( \{U^n a_l\}_{n \in \mathbb{Z}, l=1,2,\ldots,L} \) is a Riesz basis for \( A_a \).

Conversely, if \( \{U^n a_l\}_{n \in \mathbb{Z}, l=1,2,\ldots,L} \) is a Riesz basis for \( A_a \) then there exist constants \( 0 < A \leq B < \infty \) such that

\[
A \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |c_k^l|^2 \leq \left\| \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} c_k^l U^k a_l \right\|^2 \leq B \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |c_k^l|^2 \quad (5.4)
\]

for every \( \ell_2^L \)-sequence \( c := \{c_n^l\}_{n \in \mathbb{Z}, l=1,2,\ldots,L} \). Let us prove that

\[
A \leq \text{ess inf}_{\theta \in (-\pi, \pi)} \lambda_{\min}[\Phi_a(\theta)] \leq \text{ess sup}_{\theta \in (-\pi, \pi)} \lambda_{\max}[\Phi_a(\theta)] \leq B. \quad (5.5)
\]

Proceeding by contradiction, if (5.5) would not hold, then

\[
A \leq \lambda_{\min}[\Phi_a(\theta)] \leq \lambda_{\max}[\Phi_a(\theta)] \leq B
\]

does not hold on a subset of \((\pi, \pi)\) with positive Lebesgue measure. In case the set \( \Gamma_B := \{\theta \in (-\pi, \pi) : \lambda_{\max}[\Phi_a(\theta)] > B\} \) has positive Lebesgue measure we introduce the Fourier expansion of the function \( F \in L_2^L(-\pi, \pi) \) \( L_2^2(-\pi, \pi) \) denotes the usual product Hilbert space \( L_2^2(-\pi, \pi) \times \cdots \times L_2^2(-\pi, \pi)(L \text{ times}) \) in (5.3),
where \( F(\theta) = X(\theta)\chi_{\Gamma_B}(\theta) \) and \( X(\theta) \) is an eigenvector of norm 1 associated with the biggest eigenvalue of \( \Phi_a(\theta) \). We get

\[
\left\| \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} c_{l,k} U^k a_l \right\|^2 = \frac{1}{2\pi} \int_{\Gamma_B} \lambda_{\max}[\Phi_a(\theta)] d\theta > \frac{1}{2\pi} \int_{\Gamma_B} Bd\theta
\]

which contradicts the right inequality in (5.4) for such a Fourier expansion. Whenever Lebesgue measure of the set \( \Gamma_B \) is zero then we proceed in a similar way with the set of positive Lebesgue measure \( \Gamma_A := \{ \theta \in (-\pi, \pi) : \lambda_{\min}[\Phi_a(\theta)] < A \} \). \( \square \)

The above proof is similar to that of [24, Lemma 2], except we do not exclude the case in which the singular measure is atomless. Another characterization for being \( \{U^n a_l\}_{n \in \mathbb{Z}; l=1,2,...,L} \) a Riesz basis for \( \mathcal{A}_a \) can be found in [3].

The resulting regular sampling formulas

As in the one-generator case, the space \( \mathcal{A}_a \) is the image of the usual product Hilbert space \( L^2_\mathcal{B}(0,1) \) by means of the isomorphism \( T_{U,a} : L^2_\mathcal{B}(0,1) \rightarrow \mathcal{A}_a \), which maps the orthonormal basis \( \{e^{-2\pi i nw} e_l\}_{n \in \mathbb{Z}; l=1,2,...,L} \) for \( L^2_\mathcal{B}(0,1) \) (here, \( \{e_l\}_{l=1}^L \) denotes the canonical basis for \( \mathbb{C}^L \)) onto the Riesz basis \( \{U^n a_l\}_{n \in \mathbb{Z}; l=1,2,...,L} \) for \( \mathcal{A}_a \), i.e.

\[
T_{U,a} \mathbf{F} := \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}} \langle F_l, e^{2\pi i n \cdot} \rangle_{L^2(0,1)} U^n a_l = \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}} \alpha_n U^n a_l,
\]

where \( \mathbf{F} = (F_1, F_2, \ldots, F_L)^T \in L^2_\mathcal{B}(0,1) \).

Here, for \( \mathbf{F} \in L^2_\mathcal{B}(0,1) \) and \( N \in \mathbb{Z} \) the \( U \)-shift property reads:

\[
T_{U,a}(F e^{2\pi i N w}) = U^N(T_{U,a} \mathbf{F}).
\]

Concerning the representation of an \( U \)-system \( \mathcal{L}_b \), for \( x \in \mathcal{A}_a \) we have

\[
\mathcal{L}_b x(t) = \langle x, U^t b \rangle_{\mathcal{H}} = \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}} \alpha_n U^n a_l \mathcal{L}_b x(t) = \sum_{l=1}^{L} \left( \sum_{n \in \mathbb{Z}} \langle U^t b, U^n a_l \rangle_{\mathcal{H}} e^{2\pi i n w} \right)_{L^2(0,1)} = \langle \mathbf{F}, \mathbf{K}_t \rangle_{L^2_\mathcal{B}(0,1)},
\]

where \( T_{U,a} \mathbf{F} = x \), \( \mathbf{F} = (F_1, F_2, \ldots, F_L)^T \in L^2_\mathcal{B}(0,1) \), and the function

\[
\mathbf{K}_t(w) := \left( \sum_{n \in \mathbb{Z}} a_1(t-n)e^{2\pi i nw}, \sum_{n \in \mathbb{Z}} a_2(t-n)e^{2\pi i nw}, \ldots, \sum_{n \in \mathbb{Z}} a_L(t-n)e^{2\pi i nw} \right)^T.
\]
belongs to $L^2_L(0,1)$. In particular, given $s$ $U$-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$ associated with $b_j$ in $\mathcal{H}, j = 1, 2, \ldots, s$, we get the expression for the samples $\{\mathcal{L}_j x(rm)\}_{m \in \mathbb{Z}, j = 1, 2, \ldots, s}$:

$$\mathcal{L}_j x(rm) = \langle F, g_j(w) e^{2\pi irmw} \rangle_{L^2_L(0,1)} \text{ for } m \in \mathbb{Z} \text{ and } j = 1, 2, \ldots, s,$$

(5.8)

where $T_u a F = x$ and for $j = 1, 2, \ldots, s$

$$g_j(w) := \left( \sum_{k \in \mathbb{Z}} \mathcal{L}_j a_1(k) e^{2\pi ikw}, \sum_{k \in \mathbb{Z}} \mathcal{L}_j a_2(k) e^{2\pi ikw}, \ldots, \sum_{k \in \mathbb{Z}} \mathcal{L}_j a_L(k) e^{2\pi ikw} \right)^\top$$

belongs to $L^2_L(0,1)$. As in the one-generator case we must study the sequence $\{g_j(w) e^{2\pi irmw}\}_{m \in \mathbb{Z}, j = 1, 2, \ldots, s}$ in $L^2_L(0,1)$. Consider the $s \times rL$ matrix of functions in $L^2(0,1)$

$$G(w) := \begin{bmatrix}
\mathbf{g}_1^\top(w) & \mathbf{g}_1^\top\left(w + \frac{1}{r}\right) & \cdots & \mathbf{g}_1^\top\left(w + \frac{r-1}{r}\right) \\
\mathbf{g}_2^\top(w) & \mathbf{g}_2^\top\left(w + \frac{1}{r}\right) & \cdots & \mathbf{g}_2^\top\left(w + \frac{r-1}{r}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{g}_s^\top(w) & \mathbf{g}_s^\top\left(w + \frac{1}{r}\right) & \cdots & \mathbf{g}_s^\top\left(w + \frac{r-1}{r}\right)
\end{bmatrix}_{j=1,2,\ldots,s \atop k=1,2,\ldots,r}$$

(5.9)

and its related constants

$$\alpha_G := \operatorname{ess \inf}_{w \in (0,1/r)} \lambda_{\min}[G^*(w)G(w)], \quad \beta_G := \operatorname{ess \sup}_{w \in (0,1/r)} \lambda_{\max}[G^*(w)G(w)].$$

In [13, Lemma 2], one can find the proof of the following lemma.

**Lemma 5.2.** Let $g_j$ be in $L^2_L(0,1)$ for $j = 1, 2, \ldots, s$ and let $G(w)$ be its associated matrix given in (5.9). Then, the following results hold:

(a) The sequence $\{g_j(w) e^{2\pi irmw}\}_{n \in \mathbb{Z}, j = 1, 2, \ldots, s}$ is a complete system for $L^2_L(0,1)$ if and only if the rank of the matrix $G(w)$ is $rL$ a.e. in $(0,1/r)$.

(b) The sequence $\{g_j(w) e^{2\pi irmw}\}_{n \in \mathbb{Z}, j = 1, 2, \ldots, s}$ is a Bessel sequence for $L^2_L(0,1)$ if and only if $g_j \in L^\infty_L(0,1)$ (or equivalently $\beta_G < \infty$). In this case, the optimal Bessel bound is $\beta_G/r$.

(c) The sequence $\{g_j(w) e^{2\pi irmw}\}_{n \in \mathbb{Z}, j = 1, 2, \ldots, s}$ is a frame for $L^2_L(0,1)$ if and only if $0 < \alpha_G \leq \beta_G < \infty$. In this case, the optimal frame bounds are $\alpha_G/r$ and $\beta_G/r$.

(d) The sequence $\{g_j(w) e^{2\pi irmw}\}_{n \in \mathbb{Z}, j = 1, 2, \ldots, s}$ is a Riesz basis for $L^2_L(0,1)$ if and only if $G$ is a frame and $s = rL$.

In case that the sequence $\{g_j(w) e^{2\pi irmw}\}_{n \in \mathbb{Z}, j = 1, 2, \ldots, s}$ is a frame for $L^2_L(0,1)$ (here, necessarily $s \geq rL$), a dual frame is given by $\{r h_j(w) e^{2\pi irmw}\}_{n \in \mathbb{Z}, j = 1, 2, \ldots, s}$. 
where the functions $h_j$, $j = 1, 2, \ldots, s$, form an $L \times s$ matrix $h(w) := [h_1(w), h_2(w), \ldots, h_s(w)]$ with entries in $L^\infty(0, 1)$, and satisfying

$$[h_1(w), h_2(w), \ldots, h_s(w)]G(w) = [I, 0, 0, \ldots] \text{ a.e. in } (0, 1)$$

(see [13] for the details). That is, the matrix $h(w)$ is formed with the first $L$ rows of a left-inverse of the matrix $G(w)$ having essentially bounded entries in $(0, 1)$. In other words, all the dual frames of $\{g_j(e^{2\pi i \nu_n})\}$ $\nu \in \mathbb{Z}; j = 1, 2, \ldots, s$ with the above property are obtained by taking the first $L$ rows of the $rL \times s$ matrices given by

$$H_k(w) := G^k(w) + K(w)[I_s - G(w)G^k(w)],$$

where $K(w)$ denotes any $rL \times s$ matrix with entries in $L^\infty(0, 1)$.

Thus, any $F \in L^2_L(0, 1)$ can be expanded as

$$F = \sum_{j=1}^s \sum_{n \in \mathbb{Z}} \langle F, g_j(e^{2\pi i \nu_n}) \rangle_L^* h_j(w) e^{2\pi i \nu_n} \text{ in } L^2_L(0, 1).$$

Applying the isomorphism $T_{U,a}$ and taken into account (5.8), for each $x = T_{U,a}F \in A_a$ we get the sampling expansion

$$x = \sum_{j=1}^s \sum_{n \in \mathbb{Z}} L_j x(rn) U^{rn} [T_{U,a}(r h_j)] = \sum_{j=1}^s \sum_{n \in \mathbb{Z}} L_j x(rn) U^{rn} c_{j,h} \text{ in } H,$$

where the sampling elements $c_{j,h} = T_{U,a}(r h_j) \in A_a$, $j = 1, 2, \ldots, s$, and the sequence $\{U^{rn} c_{j,h}\}_{n \in \mathbb{Z}; j = 1, 2, \ldots, s}$ is a frame for $A_a$. Proceeding as in Sec. 3, it is straightforward to state and prove the corresponding results.

The time-jitter error sampling formulas

Under appropriate slight changes, the time-jitter error results in Sec. 4 still remain valid for the case of multiple generators. Namely, given an error sequence $\epsilon := \{\epsilon_m\}_{m \in \mathbb{Z}; j = 1, 2, \ldots, s}$, assume that the operator

$$D_\epsilon : \ell^2_L(\mathbb{Z}) \to L^2_L(\mathbb{Z}),$$

$$c \mapsto D_\epsilon c := (D_{\epsilon,1} c, \ldots, D_{\epsilon,s} c),$$

is well-defined, where $c := \{\{c_1^1\}_{k \in \mathbb{Z}}, \{c_2^1\}_{k \in \mathbb{Z}}, \ldots, \{c_s^1\}_{k \in \mathbb{Z}}\} \in \ell^2_L(\mathbb{Z})$ and, for $j = 1, 2, \ldots, s$,

$$D_{\epsilon,j} c := \left\{ \sum_{l=1}^L \sum_{k \in \mathbb{Z}} [L_j a_l(rm - k + \epsilon_m) - L_j a_l(rm - k)] c_k^j \right\}_{m \in \mathbb{Z}}.$$

The operator norm (it could be infinity) is defined as usual

$$\|D_\epsilon\| := \sup_{c \in \ell^2_L(\mathbb{Z}) \setminus \{0\}} \frac{\|D_\epsilon c\|_{\ell^2_{2,L}(\mathbb{Z})}}{\|c\|_{\ell^2_{2,L}(\mathbb{Z})}},$$

where $\|D_\epsilon c\|_{\ell^2_{2,L}(\mathbb{Z})} := \sum_{j=1}^s \|D_{\epsilon,j} c\|_{\ell^2_{2,L}(\mathbb{Z})}$ and $\|c\|_{\ell^2_{2,L}(\mathbb{Z})} = \sum_{j=1}^L \sum_{k \in \mathbb{Z}} |c_k^j|^2$ for each $c \in \ell^2_L(\mathbb{Z})$. Assume that the matrix $G$ in (5.9) satisfies $0 < \alpha_G \leq \beta_G < \infty$, and
let \( \varepsilon := \{\varepsilon_{m,j}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \) be an error sequence satisfying the inequality \( \|D\varepsilon\|_2^2 < \alpha_\varepsilon/r \). Then, proceeding as in Sec. 4, there exists a frame \( \{C_{f,m}^\varepsilon\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \) for \( \mathcal{A}_\varepsilon \) such that, for any \( x \in \mathcal{A}_\varepsilon \), a sampling formula as in (4.2) holds.

Now assume that \( b_j \in \mathcal{D}_T, j = 1,2,\ldots,s \); thus the functions \( \mathcal{L}_{b_j} a_l(t) \equiv \mathcal{L}_{b_j} a_l(t), j = 1,2,\ldots,s \) and \( l = 1,2,\ldots,L \), are continuously differentiable on \( \mathbb{R} \). Again, as in Sec. 4, under the decay condition (4.4) for each \( (\mathcal{L}_j a_l)'(t), j = 1,2,\ldots,s \) and \( l = 1,2,\ldots,L \), one can easily prove that there exists \( \delta > 0 \) such that \( \gamma_j := \sup_{m \in \mathbb{Z}} |\varepsilon_{m,j}| < \delta \) for each \( j = 1,2,\ldots,s \), implies that \( \|D\varepsilon\|_2^2 < \alpha_\varepsilon/r \) for the error sequence \( \varepsilon := \{\varepsilon_{m,j}\}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \).

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References


