The Zero-Removing Property in Hilbert Spaces of Entire Functions Arising in Sampling Theory

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Abstract. In the topic of sampling in reproducing kernel Hilbert spaces, sampling in Paley–Wiener spaces is the paradigmatic example. A natural generalization of Paley–Wiener spaces is obtained by substituting the Fourier kernel with an analytic Hilbert-space-valued kernel K. Thus we obtain a reproducing kernel Hilbert space $\mathcal{H}_K$ of entire functions in which the Kramer property allows to prove a sampling theorem. A necessary and sufficient condition ensuring that this sampling formula can be written as a Lagrange-type interpolation series concerns the stability under removal of a finite number of zeros of the functions belonging to the space $\mathcal{H}_K$; this is the so-called zero-removing property. This work is devoted to the study of the zero-removing property in $\mathcal{H}_K$ spaces, regardless of the Kramer property, revealing its connections with other mathematical fields.

Mathematics Subject Classification. 46E22, 42C15, 94A20.

Keywords. Analytic Kramer kernel, Lagrange-type interpolation series, zero-removing property.

1. Introduction

Sampling in reproducing kernel Hilbert spaces is nowadays an interesting mathematical topic (see, for instance, Refs. [8, 19, 21]). Besides, it has opened new research lines: sampling in unitarily translation invariant reproducing kernel Hilbert spaces or sampling in reproducing Banach spaces (see, for instance, Refs. [13, 18, 20, 21]). The present work is intimately related with this subject, and an easy motivation can be found in the Lagrange-type interpolatory character of the Shannon sampling theorem which holds for Paley–Wiener spaces. Namely, the Paley–Wiener space PW_π of bandlimited functions to $[-\pi, \pi]$, i.e.,

$$PW_\pi := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \mid \text{supp}\, \widehat{f} \subseteq [-\pi, \pi]\},$$

where $\widehat{f}$ stands for the Fourier transform of f, coincides, via the classical Paley–Wiener theorem [28, p. 85], with the space of entire functions f such that $|f(z)| \leq A\, \mathrm{e}^{\pi|z|}$ on $\mathbb{C}$ for some positive constant A, and $f|_{\mathbb{R}} \in L^2(\mathbb{R})$. In PW_π the classical Shannon sampling theorem holds: Any $f \in PW_\pi$ can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad z \in \mathbb{C}. \tag{1}$$

The series converges absolutely and uniformly on horizontal strips of the complex plane. Moreover, the sampling expansion (1) can be written as the Lagrange-type interpolation series

$$f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{P(z)}{(z-n)P'(n)}, \quad z \in \mathbb{C},$$

where P stands for the entire function $P(z) = (\sin \pi z)/\pi$, which has only simple zeros at $\mathbb{Z}$.

Since any function $f \in PW_\pi$ can be written as

$$f(z) = \left\langle \frac{\mathrm{e}^{iz\cdot}}{\sqrt{2\pi}}, F \right\rangle_{L^2[-\pi,\pi]}, \quad z \in \mathbb{C},$$

for some function $F \in L^2[-\pi, \pi]$, Shannon sampling theory admits a straightforward generalization by substituting the Fourier kernel

$$\mathbb{C} \ni z \longmapsto K(z) \in L^2[-\pi, \pi] \text{ such that } K(z)(w) := \mathrm{e}^{izw}/\sqrt{2\pi}, \quad w \in [-\pi, \pi],$$

by another abstract kernel K valued in a Hilbert space $\mathcal{H}$. The analytic Kramer sampling theory accomplishes this generalization. Indeed, let $\mathcal{H}$ be a complex, separable Hilbert space with inner product $\langle \cdot, - \rangle_{\mathcal{H}}$ and suppose K is an $\mathcal{H}$-valued analytic function defined on $\mathbb{C}$. For each $x \in \mathcal{H}$, define the function $f_x(z) = \langle K(z), x \rangle_{\mathcal{H}}$ on $\mathbb{C}$, and let $\mathcal{H}_K$ denote the collection of all such functions f_x. Furthermore, each element in $\mathcal{H}_K$ is an entire function since K is analytic on $\mathbb{C}$. In this setting, an abstract version of the analytic Kramer theorem [16] is obtained by assuming the Kramer property, that is, the existence of two sequences, $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{0\}$, and a Riesz basis for $\mathcal{H}$ $\{x_n\}_{n=1}^{\infty}$ such that $K(z_n) = a_n x_n$ for each $n \in \mathbb{N}$. Namely, for any $f_x \in \mathcal{H}_K$ we have

$$f_x(z) = \sum_{n=1}^{\infty} f_x(z_n) \frac{S_n(z)}{a_n}, \quad z \in \mathbb{C}, \tag{2}$$

where, for each $n \in \mathbb{N}, S_n(z) = \langle K(z), y_n \rangle, z \in \mathbb{C}$, and $\{y_n\}_{n=1}^{\infty}$ stands for the dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$ (see Sect. 2 below for the details).

A challenging problem is to give a necessary and sufficient condition to ensure that the above sampling formula can be written as a Lagrange-type interpolation series, that is

$$f_x(z) = \sum_{n=1}^{\infty} f_x(z_n) \frac{P(z)}{(z - z_n)P(z_n)}, \quad z \in \mathbb{C}, \tag{3}$$

where P denotes an entire function having only simple zeros at all points of the sequence $\{z_n\}_{n=1}^{\infty}$. The necessary and sufficient condition ensuring when a Kramer sampling expansion (2) can be written as a Lagrange-type interpolation series (3) was proved in [8] for orthogonal sampling formulas, and in [9] for non-orthogonal Riesz basis sampling formulas. Roughly speaking, the aforesaid necessary and sufficient condition concerns the stability of the functions belonging to the space $\mathcal{H}_K$ under removal of a finite number of their zeros; in other words,

$$f \in \mathcal{H}_K \text{ and } f(a) = 0 \text{ implies that } \frac{f(z)}{z - a} \in \mathcal{H}_K.$$

This is an ubiquitous algebraic property in the mathematical literature (see Sect. 2 below) and it will be called the zero-removing property (ZR property in short) throughout the paper. The main aim in this paper is to thoroughly study the ZR property in $\mathcal{H}_K$ spaces, regardless of the Kramer property, revealing its relationships with other mathematical fields. For instance, Paley–Wiener spaces are particular cases of de Branges spaces [4] where the ZR property holds, and de Branges spaces are particular cases of $\mathcal{H}_K$ spaces as well [10].

Next, we outline the organization of the paper, highlighting its significant contributions. In Sect. 2 we introduce the needed preliminaries on spaces $\mathcal{H}_K$: these spaces are reproducing kernel Hilbert spaces (RKHS in short) of entire functions; we briefly recall the sampling result in $\mathcal{H}_K$. In Sect. 3 we study some properties of $\mathcal{H}_K$ obtained from the Taylor coefficients of the kernel K at a fixed complex point. In particular, the relationship between $\mathcal{H}_K$ and the set $\mathcal{P}(\mathbb{C})$ of complex polynomials. In Section 4 we study the zero-removing property at a fixed point; this property can be reduced to a general moment problem. Thus, the zero-removing property at a fixed point depends on the continuity of a certain associated operator which looks like the classical shift operator. Moreover, we give a sufficient condition for the continuity of this operator. The section is closed by studying the local zero-removing property: If the zero-removing property holds for a fixed point, say 0, it also holds for any $a \in \mathbb{C}$ with $|a|$ small enough. This study is carried out by using the well-known Fredholm operator theory. Finally, in Sect. 5 we close the paper with an study of the differentiation operator in an $\mathcal{H}_K$ space.

2. Preliminaries on $\mathcal{H}_K$ Spaces

Suppose we are given a separable complex Hilbert space and an abstract kernel K which is nothing but an $\mathcal{H}$-valued function on $\mathbb{C}$. For each $z \in \mathbb{C}$, set $f_x(z) := \langle K(z), x\rangle_{\mathcal{H}}$ for $z \in \mathbb{C}$, and denote by $\mathcal{H}_K$ the collection of all such functions f_x, $x \in \mathcal{H}$, and let $\mathcal{T}_K$ be the mapping

$$\mathcal{H} \ni x \xmapsto{\mathcal{T}_K} f_x \in \mathcal{H}_K \tag{4}$$

If we define the norm $\|f\|_{\mathcal{H}_K} := \inf\{\|x\|_{\mathcal{H}} \,:\, f = \mathcal{T}_K x\}$ in $\mathcal{H}_K$ (in the sequel we omit the subscript x in f_x), we obtain a reproducing kernel Hilbert space whose reproducing kernel is given by

$$k(z, w) = \langle K(z), K(w)\rangle_{\mathcal{H}}, \quad z, w \in \mathbb{C}.$$

(see [24] for the details). Notice that the mapping $\mathcal{T}_K$ is an antilinear mapping from $\mathcal{H}$ onto $\mathcal{H}_K$. It is injective if and only if the set $\{K(z)\}_{z\in\mathbb{C}}$ is complete in $\mathcal{H}$. In particular, if there exist sequences $\{z_n\}_{n=1}^\infty \subset \mathbb{C}$, $\{a_n\}_{n=1}^\infty \in \mathbb{C}\setminus\{0\}$ and a Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$ such that $K(z_n) = a_n\, x_n$ for any $n \in \mathbb{N}$, then the mapping $\mathcal{T}_K$ is an anti-linear isometry from $\mathcal{H}$ onto $\mathcal{H}_K$. Recall that a Riesz basis in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a boundedly invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^\infty \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^\infty \langle x, x_n\rangle_{\mathcal{H}}\, y_n$$

hold for every $x \in \mathcal{H}$ (see [6, 28] for more details and proofs).

The convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which is uniform on those subsets of $\mathbb{C}$ where the function $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded; in particular, in compact subsets of $\mathbb{C}$ whenever K is a continuous kernel.

Like in the classical case the following result holds: The space $\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel K is analytic in $\mathbb{C}$ ([26, p. 266]). Another characterization of the analyticity of the functions in $\mathcal{H}_K$ is given in terms of Riesz bases. Suppose that a Riesz basis $\{x_n\}_{n=1}^\infty$ for $\mathcal{H}$ is given and let $\{y_n\}_{n=1}^\infty$ be its dual Riesz basis; expanding $K(z)$, where $z \in \mathbb{C}$ is fixed, with respect to the basis $\{x_n\}_{n=1}^\infty$ we obtain

$$K(z) = \sum_{n=1}^\infty \langle K(z), y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^\infty S_n(z)\, x_n \quad \text{in } \mathcal{H},$$

where the coefficients

$$S_n(z) := \langle K(z), y_n\rangle_{\mathcal{H}}, \quad z \in \mathbb{C}, \tag{5}$$

as functions in $z \in \mathbb{C}$, are in $\mathcal{H}_K$. The following result holds [10]: Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be a pair of dual Riesz bases for $\mathcal{H}$. Then, $\mathcal{H}_K$ is RKHS of entire functions if and only if all the functions S_n, $n \in \mathbb{N}$, are entire and the function $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded on compact sets of $\mathbb{C}$.

2.1. Sampling and the Zero-Removing Property in $\mathcal{H}_K$ Spaces

Consider the data

$$\{z_n\}_{n=1}^{\infty} \in \mathbb{C} \quad \text{and} \quad \{a_n\}_{n=1}^{\infty} \in \mathbb{C} \setminus \{0\}. \tag{6}$$

Definition 1. An analytic kernel $K : \mathbb{C} \longrightarrow \mathcal{H}$ is said to be an analytic Kramer kernel (with respect to the data (6)) if it satisfies $K(z_n) = a_n\, x_n, n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^{\infty}$ of $\mathcal{H}$. A sequence $\{S_n\}_{n=1}^{\infty}$ of functions in $\mathcal{H}_K$ is said to have the interpolation property (with respect to the data (6)) if

$$S_n(z_m) = a_n\, \delta_{n,m}\, . \tag{7}$$

An analytic kernel K is an analytic Kramer one if and only if the sequence of functions $\{S_n\}_{n=1}^{\infty}$ in $\mathcal{H}_K$ given by (5), where $\{y_n\}_{n=1}^{\infty}$ is the dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$, has the interpolation property with respect to the same data (6).

Under the notation introduced so far an abstract version of the classical Kramer sampling theorem [16] holds: First notice that $\lim_{m\to\infty} |z_m| = +\infty$; otherwise we obtain that any entire function S_n is identically zero in $\mathbb{C}$. The anti-linear mapping $\mathcal{T}_K$ is a bijective isometry between $\mathcal{H}$ and $\mathcal{H}_K$. As a consequence, the functions $\{S_n = \mathcal{T}_K(y_n)\}_{n=1}^{\infty}$ form a Riesz basis for $\mathcal{H}_K$; the sequence $\{T_n := \mathcal{T}_K(x_n)\}_{n=1}^{\infty}$ is its dual Riesz basis. Expanding any $f \in \mathcal{H}_K$ with respect the basis $\{S_n\}_{n=1}^{\infty}$ we obtain

$$f(z) = \sum_{n=1}^{\infty} \langle f, T_n\rangle_{\mathcal{H}_K} S_n(z) \quad \text{in } \mathcal{H}_K.$$

Besides,

$$\langle f, T_n\rangle_{\mathcal{H}_K} = \overline{\langle x, x_n\rangle_{\mathcal{H}}} = \left\langle \frac{K(z_n)}{a_n}, x \right\rangle_{\mathcal{H}} = \frac{f(z_n)}{a_n}.$$

Since a Riesz basis is an unconditional basis, the sampling series will be pointwise unconditionally convergent and hence, absolutely convergent. The uniform convergence is a standard result in the setting of the RKHS theory since $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded on compact subsets of $\mathbb{C}$. Thus we have proved an abstract version of the *classical Kramer sampling theorem* [16]:

Theorem 1. *Let $K : \mathbb{C} \longrightarrow \mathcal{H}$ be an analytic Kramer kernel, and assume that the interpolation property (7) holds for some sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C} \setminus \{0\}$. Let $\mathcal{H}_K$ be the corresponding RKHS of entire functions. Then any $f \in \mathcal{H}_K$ can be recovered from the sequence of its samples $\{f(z_n)\}_{n=1}^{\infty}$ by means of the sampling series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\, \frac{S_n(z)}{a_n}, \quad z \in \mathbb{C}. \tag{8}$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Equivalently, the Kramer property in Definition 1 can be seen as a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ such that the sequence of reproducing kernels $\{k(\cdot, z_n)\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}_K$. An interesting problem is to characterize the sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ having this property in some structured RKHS spaces of entire functions like Hardy or Bergman spaces (see, for instance, Refs. [2,3,25] and references therein).

Concerning the sampling formula (8) in $\mathcal{H}_K$, a challenging problem is to give a necessary and sufficient condition to ensure that it can be written as a Lagrange-type interpolation series (see, Eq. (9) below). As it was pointed out in the introduction, it concerns the stability of the functions belonging to the space $\mathcal{H}_K$ on removing a finite number of their zeros; it will be called the *zero-removing property*:

Definition 2. A set $\mathcal{A}$ of entire functions has the zero-removing property (ZR property in short) if for any $g \in \mathcal{A}$ and any zero w of g the function $g(z)/(z-w)$ belongs to $\mathcal{A}$.

A set $\mathcal{A}$ of entire functions has the zero-removing property at a point $a \in \mathbb{C}$ (ZR_a property in short) if for any $g \in \mathcal{A}$ with $g(a) = 0$ the function $g(z)/(z-a)$ belongs to $\mathcal{A}$.

In fact, the following result holds (see [8,9] for the proof):

Theorem 2. *Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the data $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $\{a_n\}_{n=1}^{\infty} \in \mathbb{C}\setminus\{0\}$, i.e., $K(z_n) = a_n\, x_n$, $n \in \mathbb{N}$, for some Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathcal{H}$. Then, the sampling formula (8) for $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series*

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{(z-z_n)P'(z_n)}, \quad z \in \mathbb{C}, \tag{9}$$

where P denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$ if and only if the space $\mathcal{H}_K$ satisfies the ZR property.

The ZR property (also called the *division property*; see [11]) is ubiquitous in mathematics; for instance, the set $\mathcal{P}_N(\mathbb{C})$ of polynomials with complex coefficients of degree less or equal than N has the ZR property. Another more involved examples sharing this property are:

(a) The entire functions in the Pólya class have the ZR property [4, p. 15]. Recall that an entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if $|E(x-iy)| \le |E(x+iy)|$ for $y > 0$, and if $|E(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed x.

(b) The Paley–Wiener space PW_π satisfies the ZR property; it follows immediately from its characterization as the space of entire functions f such that $|f(z)| \le A\,\mathrm{e}^{\pi|z|}$ on $\mathbb{C}$ for some positive constant A, and $f|_{\mathbb{R}} \in L^2(\mathbb{R})$, i.e., the classical Paley–Wiener theorem [28, p. 85]. For a direct proof, consider $f \in PW_\pi$ such that $f(a) = 0$, i.e.,

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \mathrm{e}^{iwz}\,\widehat{f}(w)\,dw,\ z\in\mathbb{C},\quad \text{such that} \int_{-\pi}^{\pi} \mathrm{e}^{iwa}\,\widehat{f}(w)\,dw=0,$$

where $\widehat{f}$ stands for the Fourier transform of f. Consider the function $g(w)=\int_{-\pi}^{w} \mathrm{e}^{iax}\widehat{f}(x)\,dx$ which satisfies $g(-\pi)=g(\pi)=0$. Integrating by parts one obtains

$$\frac{f(z)}{z-a}=\frac{1}{\sqrt{2\pi}}\frac{1}{z-a}\int_{-\pi}^{\pi} \mathrm{e}^{i(z-a)w}\,\mathrm{e}^{iwa}\,\widehat{f}(w)\,dw=\frac{-1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \mathrm{e}^{iwz}(i\,\mathrm{e}^{-iaw}\,g(w))\,dw.$$

In other words, since the function $-i\,\mathrm{e}^{-iaw}g(w)$ belongs to $L^2[-\pi,\pi]$, the function $f(z)/(z-a)$ belongs to PW_π.
(c) In general, a de Branges space $\mathcal{H}(E)$ with strict de Branges (structure) function E has the ZR property [4, p. 52]. Let E be an entire function verifying $|E(x-iy)|<|E(x+iy)|$ for all $y>0$. The de Branges space $\mathcal{H}(E)$ is the set of all entire functions f such that

$$\|f\|_E^2:=\int_{-\infty}^{\infty}\left|\frac{f(t)}{E(t)}\right|^2 dt<\infty,$$

and such that both ratios f/E and f^*/E, where $f^*(z):=\overline{f(\overline{z})}, z\in\mathbb{C}$, are of bounded type and of nonpositive mean type in the upper half-plane. The structure function or de Branges function E has no zeros in the upper half plane. A de Branges function E is said to be strict if it has no zeros on the real axis. We require f/E and f^*/E to be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function is of bounded type if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$ and it is of nonpositive mean type if it grows no faster than $\mathrm{e}^{\varepsilon y}$ for each $\varepsilon>0$ as $y\to\infty$ on the positive imaginary axis $\{iy\,:\,y>0\}$. Note that the Paley–Wiener space PW_π is a de Branges space with strict structure function $E_\pi(z)=\exp(-i\pi z)$.

As a consequence of Theorem 2, any sampling formula like Eq. 8 in a de Branges space can be written as a Lagrange-type interpolation series.
(d) Whenever the space $\mathcal{H}_K$ is associated with a polynomial kernel $K(z):=\sum_{n=0}^{N} c_n z^n$, where $c_n\in\mathcal{H}$ and $c_N\neq 0$, it is easy to give a characterization for the ZR property in $\mathcal{H}_K$. Namely, the ZR property holds in $\mathcal{H}_K$ if and only if the set $\{c_0,c_1,\dots,c_N\}$ is linearly independent in $\mathcal{H}$ (see [9] for a proof). A more involved problem is to deal with a general entire $\mathcal{H}$-valued kernel $K(z)=\sum_{n=0}^{\infty} c_n z^n$, $z\in\mathbb{C}$; the aim of this paper is to obtain some results in this direction.
(e) In a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n\}_{n=0}^{\infty}$ consider the kernel

$$\begin{aligned} K_\gamma:\mathbb{C}&\longrightarrow\mathcal{H}\\ z&\longmapsto K_\gamma(z):=\sum_{n=0}^{\infty}\frac{e_n}{\gamma_n}z^n\,, \end{aligned}$$

where $\gamma := \{\gamma_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers such that the sequence of quotients $\{\gamma_n/\gamma_{n+1}\}_{n\in\mathbb{N}_0}$ decreases to zero as n increases to infinity. The corresponding spaces $\mathcal{H}_{K_\gamma}$ constructed from this family of analytic kernels were introduced by Chan and Shapiro in [5]. Obviously, an entire function $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ belongs to the space $\mathcal{H}_{K_\gamma}$ if and only if the sequence $\{\gamma_n\alpha_n\}_{n\in\mathbb{N}_0}$ belongs to $\ell^2(\mathbb{N}_0)$, where, as usual, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Therefore, it is straightforward to show that if $f\in\mathcal{H}_{K_\gamma}$, with $f(0)=0$, then $f(z)/z$ belongs to $\mathcal{H}_{K_\gamma}$, i.e., the space $\mathcal{H}_{K_\gamma}$ satisfies the ZR_0 property. If the sequence $\{\gamma_{n+1}/\gamma_n\}_{n\in\mathbb{N}_0}$ is $O(1/n)$ as $n\to\infty$, then, for any $a\in\mathbb{C}$, the translation operator given by $T_af(z):=f(z-a)$, $z\in\mathbb{C}$, is a well-defined bounded operator $T_a:\mathcal{H}_{K_\gamma}\longrightarrow\mathcal{H}_{K_\gamma}$ (see [5] for the details). As a consequence of this fact, the space $\mathcal{H}_{K_\gamma}$ satisfies the ZR property (for the details, see Eq. (20) below).

Next we include some examples of spaces $\mathcal{H}_K$ where the ZR property fails:

(f) Let $K:\mathbb{C}\to\mathcal{H}$ be an analytic kernel and assume that there exist two distinct points z_1 and $z_2\in\mathbb{C}$ such that $K(z_1)=K(z_2)$. Then the space $\mathcal{H}_K$ does not hold the ZR property. Indeed, for $x\neq 0$ in $\mathcal{H}$, orthogonal to $K(z_1)$, consider the function $f(z)=\langle K(z),x\rangle$, $z\in\mathbb{C}$. Assume that r is the order of the zero z_1 of f. If the property ZR holds in $\mathcal{H}_K$, the function

$$g(z)=\frac{f(z)}{(z-z_1)^r},\quad z\in\mathbb{C}$$

belongs to $\mathcal{H}_K$, and $g(z_1)\neq 0$. Let $y\in\mathcal{H}$ be such that $g(z)=\langle K(z),y\rangle$, $z\in\mathbb{C}$. Since $g(z_2)=0$ we have that y is orthogonal to $K(z_2)$; but $g(z_1)\neq 0$ implies that y is not orthogonal to $K(z_1)$, that is, a contradiction.

(g) Finally, we exhibit a nontrivial example taken from [9] of a RKHS $\mathcal{H}_K$, built from the Sobolev Hilbert space $\mathcal{H}:=H^1(-\pi,\pi)$, where the ZR property fails. Namely: consider the Sobolev Hilbert space $H^1(-\pi,\pi)$ with its usual inner product

$$\langle f,g\rangle_1=\int_{-\pi}^{\pi}f(x)\,\overline{g(x)}\,dx+\int_{-\pi}^{\pi}f'(x)\,\overline{g'(x)}\,dx\,,\quad f,g\in H^1(-\pi,\pi).$$

The sequence $\{e^{inx}\}_{n\in\mathbb{Z}}\cup\{\sinh x\}$ forms an orthogonal basis for H^1 $(-\pi,\pi)$: It is straightforward to prove that the orthogonal complement of $\{e^{inx}\}_{n\in\mathbb{Z}}$ in $H^1(-\pi,\pi)$ is a one-dimensional space for which $\sinh x$ is a basis. For a fixed $a\in\mathbb{C}\setminus\mathbb{Z}$ we define a kernel

$$\begin{aligned} K_a:\mathbb{C}&\longrightarrow H^1(-\pi,\pi)\\ z&\longmapsto\ K_a(z)\,, \end{aligned}$$

by setting

$$[K_a(z)](x)=(z-a)\,e^{izx}+\sin\pi z\,\sinh x\quad\text{for } x\in(-\pi,\pi)\,.$$

Clearly, K_a defines an analytic Kramer kernel. Expanding $K_a(z) \in H^1(-\pi, \pi)$ in the former orthogonal basis we obtain

$$K_a(z) = [1 - i(z-a)] \sin \pi z \sinh x + (z-a) \sum_{n=-\infty}^{\infty} \frac{1+zn}{1+n^2} \operatorname{sinc}(z-n) e^{inx} \text{in } H^1(-\pi, \pi)\,.$$

where sinc denotes the cardinal sine function $\operatorname{sinc}(z) = \sin \pi z / \pi z$, if $z \neq 0$, and $\operatorname{sinc}(0) = 1$. As a consequence, Theorem 1 gives the following sampling result in $\mathcal{H}_{K_a}$: Any function $f \in \mathcal{H}_{K_a}$ can be recovered from its samples $\{f(a)\} \cup \{f(n)\}_{n \in \mathbb{Z}}$ by means of the sampling formula

$$f(z) = [1 - i(z-a)] \frac{\sin \pi z}{\sin \pi a} f(a) + \sum_{n=-\infty}^{\infty} f(n) \frac{z-a}{n-a} \frac{1+zn}{1+n^2} \operatorname{sinc}(z-n)\,, \quad z \in \mathbb{C}\,.$$

The function $(z-a) \operatorname{sinc} z$ belongs to $\mathcal{H}_{K_a}$ since $(z-a) \operatorname{sinc} z = \langle K_a(z), 1/2\pi \rangle_1$ for each $z \in \mathbb{C}$. However, by using the above sampling formula for $\mathcal{H}_{K_a}$ it is straightforward to check that the function $\operatorname{sinc} z$ does not belong to $\mathcal{H}_{K_a}$. Analogously, one can prove that the zero-removing property also fails for any $n \in \mathbb{Z}$ by considering the function $f(z) = \sin \pi z$ which belongs to $\mathcal{H}_{K_a}$.

3. Some Properties on $\mathcal{H}_K$ Related to the Kernel K

In this section we obtain some properties of the Hilbert space $\mathcal{H}_K$ derived from the sequence of Taylor coefficients of the entire kernel K at a point $a \in \mathbb{C}$. Indeed, for each $a \in \mathbb{C}$ we have the Taylor expansion

$$K(z) = \sum_{n=0}^{\infty} c_n(a)(z-a)^n, \quad z \in \mathbb{C},$$

where the coefficient $c_n(a) \in \mathcal{H}$ for each $n \in \mathbb{N}_0$. By using Cauchy's integral formula for derivatives (see [26, p. 268] we have

$$c_n(a) = \frac{1}{n!} K^{(n)}(a) = \frac{1}{2\pi i} \int_{|z-a|=R} \frac{K(z)}{(z-a)^{n+1}}\, dz, \quad n = 0, 1, \dots,$$

from which

$$\|c_n(a)\|_{\mathcal{H}} \leq \frac{1}{R^{n+1}} \sup_{|z-a|=R} \|K(z)\|_{\mathcal{H}} = \frac{M_R(a)}{R^{n+1}}\,, \tag{10}$$

where $M_R(a) := \sup_{|z-a|=R} \|K(z)\|_{\mathcal{H}}$. Taking $R > 1$, the above inequality shows that the sequence $\{\|c_n(a)\|\}_{n \in \mathbb{N}_0}$ belongs to $\ell^1(\mathbb{N}_0) \subset \ell^2(\mathbb{N}_0)$.

Proposition 1. *Let $\{c_n(a)\}_{n \in \mathbb{N}_0}$ be the sequence of Taylor coefficients of K at any $a \in \mathbb{C}$.*

1. *The sequence* $\{c_n(a)\}_{n\in\mathbb{N}_0}$ *is a Bessel sequence for* $\mathcal{H}$.
2. *Assume that the mapping* $\mathcal{T}_K$ *in* (4) *is injective. Then the sequence* $\{c_n(a)\}_{n\in\mathbb{N}_0}$ *is a complete sequence in* $\mathcal{H}$.

Proof. For any $x \in \mathcal{H}$ we have $|\langle c_n(a), x\rangle|^2 \le \|c_n(a)\|^2_{\mathcal{H}}\|x\|^2_{\mathcal{H}}$ for each $n \in \mathbb{N}_0$. Thus, having in mind (10) we obtain

$$\sum_{n=0}^{\infty} |\langle c_n(a), x\rangle|^2 \le \left(\sum_{n=0}^{\infty} \|c_n(a)\|^2_{\mathcal{H}}\right) \|x\|^2_{\mathcal{H}} \le B\|x\|^2_{\mathcal{H}},$$

where $B := \frac{M_R^2(a)}{R^2-1}$ and $R > 1$.

Assume now that $\langle c_n(a), x\rangle = 0$ for all $n \in \mathbb{N}_0$. For the function $f(z) := \langle K(z), x\rangle$, $z \in \mathbb{C}$, we have the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \langle c_n(a), x\rangle (z-a)^n = 0 \quad \text{for all } z \in \mathbb{C}.$$

Since the anti-linear mapping $\mathcal{T}_K$ is injective we deduce that $x = 0$. □

The Bessel property in Proposition 1 implies that the space $\mathcal{H}_K$ is a subspace of the Hardy space $H^2(\mathbb{D})$ with continuous inclusion (see [22]). It will be a closed subspace if and only if the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is a frame for $\mathcal{H}$ (see, for instance, [6,23]). In this paper we often assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is also minimal (see Definition 3 below); as a consequence, $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is a Riesz basis where necessarily $0 < m \le \|c_n(0)\| \le M < \infty$ for all $n \in \mathbb{N}_0$ (see [6, p. 124]). This is not the case in our setting since the sequence of Taylor coefficients $c_n(0) \to 0$ in $\mathcal{H}$ as $n \to \infty$. In other words, the space $\mathcal{H}_K$ is not, in general, a closed subspace of the Hardy space $H^2(\mathbb{D})$.

As it was mentioned in Sect. 2, whenever K is a polynomial kernel with coefficients in $\mathcal{H}$, a necessary and sufficient condition for $\mathcal{H}_K$ satisfying the ZR property is the linear independence in $\mathcal{H}$ of the coefficients of K. In the general case, the linear independence of the Taylor coefficients $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of K at 0 is only a necessary condition for the ZR_0 property (clearly it is not a sufficient condition; see, for instance, example (g) in Sect. 2):

Proposition 2. *Assume that the space* $\mathcal{H}_K$ *satisfies the* ZR_0 *property and consider the Taylor expansion* $K(z) = \sum_{n=0}^{\infty} c_n(0)z^n$ *of* K *around* 0*. Then, the sequence* $\{c_n(0)\}_{n\in\mathbb{N}_0}$ *is linearly independent in* $\mathcal{H}$.

Proof. Assume that there exists an index N such that the coefficient $c_N(0)$ depends linearly on $\{c_0(0), c_1(0), \ldots, c_{N-1}(0)\}$, and consider a non-zero $x \in \{c_0(0), c_1(0), \ldots, c_{N-1}(0)\}^{\perp}$. Then, the function $\langle K(z), x\rangle$ satisfies

$$\langle K(z), x\rangle = z^m(\langle c_m(0), x\rangle + \langle c_{m+1}(0), x\rangle z + \langle c_{m+2}(0), x\rangle z^2 + \cdots)$$

with $m \ge N+1$ and $\langle c_m(0), x\rangle \ne 0$. If $\mathcal{H}_K$ satisfies the ZR_0 property, then the entire function

$$g(z) = z^N(\langle c_m(0), x\rangle + \langle c_{m+1}(0), x\rangle z + \langle c_{m+2}(0), x\rangle z^2 + \cdots)$$

belongs to $\mathcal{H}_K$, that is, there exists $y \in \mathcal{H}$ such that

$$\langle c_0(0), y\rangle = \langle c_1(0), y\rangle = \cdots = \langle c_{N-1}(0), y\rangle = 0$$

and

$$\langle c_{m+k}(0), x\rangle = \langle c_{N+k}(0), y\rangle \quad \text{for all } k \geq 0\,.$$

Since $c_N(0)$ depends linearly on $\{c_0(0), c_1(0), \ldots, c_{N-1}(0)\}$ and $\langle c_N(0), y\rangle \neq 0$ we get a contradiction. □

As a consequence of the above result, if the space $\mathcal{H}_K$ satisfies the ZR property then, for each $a \in \mathbb{C}$, the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent in $\mathcal{H}$. In other words, if there exists $a \in \mathbb{C}$ such that $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly dependent in $\mathcal{H}$, then the ZR property does not hold in $\mathcal{H}_K$.

A classical problem in a de Branges space $\mathcal{H}(E)$ is to determine when the set of polynomials $\mathcal{P}(\mathbb{C})$ is included in $\mathcal{H}(E)$ (see [1] and references therein). Next, we study the relationship between the set of polynomials $\mathcal{P}(\mathbb{C})$ and our spaces $\mathcal{H}_K$ via the Taylor coefficients $\{c_n(a)\}_{n\in\mathbb{N}_0}$ of the kernel K at a point $a \in \mathbb{C}$.

Definition 3. A sequence $\{c_n\}_{n\in\mathbb{N}_0}$ is said to be minimal in $\mathcal{H}$ if $c_m \notin \overline{\text{span}}\,\{c_n\}_{n\neq m}$ for each $m \in \mathbb{N}_0$. A sequence $\{c_n\}_{n\in\mathbb{N}_0}$ is said to be supercomplete in $\mathcal{H}$ if the sequence $\{c_n\}_{n\geq m}$ is complete in $\mathcal{H}$ for each $m \in \mathbb{N}_0$.

Obviously, each minimal sequence $\{c_n\}_{n=0}^{\infty}$ is linearly independent in $\mathcal{H}$. In this section we will assume that the mapping $\mathcal{T}_K$ in (4) is injective; consequently, the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at any $a \in \mathbb{C}$ is a complete sequence in $\mathcal{H}$ (see Proposition 1).

Proposition 3. *The set of polynomials $\mathcal{P}(\mathbb{C})$ is contained in $\mathcal{H}_K$ if and only if the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is minimal in $\mathcal{H}$. Moreover, the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is minimal in $\mathcal{H}$ if and only if the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is minimal in $\mathcal{H}$ for each $a \in \mathbb{C}$.*

Proof. For each $n \in \mathbb{N}_0$ the monomial z^n belongs to $\mathcal{H}_K$ if and only if there exists $x_n \in \mathcal{H}$ such that $\langle c_m(0), x_n\rangle = \delta_{m,n}$, where $\delta_{m,n}$ denotes the Kronecker delta. Equivalently, $\{z^n\}_{n=0}^{\infty} \subset \mathcal{H}_K$ if and only if there exists a biorthogonal sequence $\{x_n\}_{n=0}^{\infty} \subset \mathcal{H}$ of $\{c_n(0)\}_{n\in\mathbb{N}_0}$. This is known to be equivalent to the minimality of $\{c_n(0)\}_{n\in\mathbb{N}_0}$ (see [28]).

Now, suppose that for some $a \in \mathbb{C}$ the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ fails to be minimal. Then there exists $N \in \mathbb{N}_0$ such that

$$c_N(a) \in \overline{\text{span}}\{c_0(a), \ldots, c_{N-1}(a), c_{N+1}(a), \ldots\}$$

Having in mind the completeness of the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$ we deduce that the sequence $\{c_0(a), \ldots, c_{N-1}(a), c_{N+1}(a), \ldots\}$ is complete in $\mathcal{H}$. Therefore, if $x \in \{c_m\}_{m\neq N}^{\perp}$ then $x = 0$ and, consequently, the polynomial $(z-a)^N$ does not belong to $\mathcal{H}_K$. □

Proposition 4. *The sequence* $\{c_n(0)\}_{n\in\mathbb{N}_0}$ *of Taylor coefficients of* K *at* 0 *is supercomplete in* $\mathcal{H}$ *if and only if the space* $\mathcal{H}_K$ *does not contain any non-zero polynomial. Moreover, the sequence* $\{c_n(0)\}_{n\in\mathbb{N}_0}$ *is supercomplete in* $\mathcal{H}$ *if and only if the sequence* $\{c_n(a)\}_{n\in\mathbb{N}_0}$ *is supercomplete in* $\mathcal{H}$ *for each* $a\in\mathbb{C}$.

Proof. A non-zero polynomial $a_N z^N + a_{N-1}z^{N-1}+\cdots+a_0$ belongs to $\mathcal{H}_K$ if and only if there exists $x\in\mathcal{H}$, $x\neq 0$, such that

$$\begin{aligned}\langle c_0(0),x\rangle &= a_0, \langle c_1(0),x\rangle = a_1,\dots,\langle c_N(0),x\rangle\\ &= a_N \quad \text{and} \quad \langle c_m(0),x\rangle = 0 \quad \text{ for } m>N.\end{aligned}$$

Hence, a non-zero polynomial is in $\mathcal{H}_K$ if and only if the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is not supercomplete in $\mathcal{H}$.

Now, suppose that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is supercomplete in $\mathcal{H}$ and that, for some $b\in\mathbb{C}$, the sequence $\{c_n(b)\}_{n\in\mathbb{N}_0}$ is not supercomplete in $\mathcal{H}$. Then, there exists $N\in\mathbb{N}_0$ such that sequence $\{c_{N+1}(b),c_{N+2}(b),\dots\}$ is not complete in $\mathcal{H}$. Therefore, there exists a non-zero $x\in\mathcal{H}$ such that $\langle c_m(b),x\rangle = 0$ for all $m>N$. As a consequence, the non-zero polynomial

$$\langle c_0(b),x\rangle + \langle c_1(b),x\rangle(z-b)+\cdots+\langle c_N(b),x\rangle(z-b)^N$$

belongs to $\mathcal{H}_K$, that is, a contradiction. $\square$

In the Paley–Wiener case, the Fourier kernel $K(z)(w)=\frac{1}{\sqrt{2\pi}}e^{izw}$, $w\in[-\pi,\pi]$, can be expanded, around $a\in\mathbb{C}$, as

$$K(z)(w)=\frac{1}{\sqrt{2\pi}}e^{izw}=\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}e^{iaw}\frac{(iw)^n}{n!}(z-a)^n\,,\quad z\in\mathbb{C}\,.$$

Hence, for $n\in\mathbb{N}_0$, we get that $c_n(0)(w)=\frac{1}{\sqrt{2\pi}}\frac{(iw)^n}{n!}$, $w\in[-\pi,\pi]$.

As a by-product, since the Paley–Wiener space PW_π does not contain any non-zero polynomial, from Proposition 4 we deduce that the sequence of monomials $\{1,w,w^2,\dots\}$ is supercomplete (and hence, it is not minimal) in $L^2[-\pi,\pi]$.

Concerning the ZR property in $\mathcal{H}_K$ and the relationship between the set $\mathcal{P}(\mathbb{C})$ of polynomials and $\mathcal{H}_K$ we have the following result:

Proposition 5. *Suppose that the space* $\mathcal{H}_K$ *satisfies the ZR property. Then, only one of the following three cases hold:*

(a) *For any* $a\in\mathbb{C}$ *the sequence* $\{c_n(a)\}_{n\in\mathbb{N}_0}$ *is minimal in* $\mathcal{H}$. *In this case the space* $\mathcal{H}_K$ *contains any polynomial.*
(b) *For any* $a\in\mathbb{C}$ *the sequence* $\{c_n(a)\}_{n\in\mathbb{N}_0}$ *is supercomplete in* $\mathcal{H}$. *In this case the space* $\mathcal{H}_K$ *does not contain non-zero polynomials.*
(c) *There exists* $N\in\mathbb{N}_0$ *such that the polynomials belonging to* $\mathcal{H}_K$ *are precisely the set of polynomials of degree less or equal than* N. *In this case, for each* $a\in\mathbb{C}$ *the sequence* $\{c_n(a)\}_{n\geq N+1}$ *is supercomplete in its closed span and* $c_r(a)\notin\overline{\operatorname{span}}\{c_n(a)\}_{n\neq r}$ *for* $r=0,1,\dots,N$.

Proof. We denote by ∂p the degree of a polynomial p. Assume that there exists a polynomial p belonging to the space $\mathcal{H}_K$. If the space $\mathcal{H}_K$ satisfies the ZR property then the set of polynomials whose degree is less or equal than ∂p is included in $\mathcal{H}_K$.

If the case a. does not hold, consider $N := \max_{r\in\mathbb{N}_0}\{q \in \mathcal{H}_K \,|\, q$ polynomial and $\partial q = r\}$ which is finite. Since the ZR property holds, the set of polynomials of degree less or equal than N is included in $\mathcal{H}_K$. □

4. The Zero-Removing Property at a Fixed Point

In this section we study conditions under which, for a fixed $a \in \mathbb{C}$, the ZR_a property holds in $\mathcal{H}_K$. Reducing the ZR_a property to a moment problem, a sufficient condition assuring that the ZR_a property holds involves the continuity of a shift-type operator.

4.1. A Sufficient Condition for the ZR_a Property

Consider a function $f \in \mathcal{H}_K$, i.e., $f(z) = \langle K(z), x\rangle_{\mathcal{H}}$ on $\mathbb{C}$ for some $x \in \mathcal{H}$, such that $f(a) = 0$. Then $\langle c_0(a), x\rangle = 0$ and

$$\frac{f(z)}{z-a} = \sum_{n=0}^{\infty} \langle c_{n+1}(a), x\rangle (z-a)^n\,, \quad z \in \mathbb{C}\,.$$

As a consequence, the space $\mathcal{H}_K$ satisfies the property ZR_a if and only if for each $x \in \{c_0(a)\}^{\perp}$ there exists $y \in \mathcal{H}$ such that

$$\langle c_n(a), y\rangle = \langle c_{n+1}(a), x\rangle\,, \quad n \in \mathbb{N}_0\,.$$

For the sake of completeness we include the following result on general moment problems whose proof can be found in [28, p. 126]:

Theorem 3. *Let $\{f_1, f_2, f_3, \dots\}$ be a sequence of vectors belonging to a Hilbert space $\mathcal{H}$ and $\{d_1, d_2, d_3, \dots\}$ a sequence of scalars. In order that the equations*

$$\langle f, f_n\rangle = d_n\,, \quad n \in \mathbb{N}$$

shall admit at least one solution $f \in \mathcal{H}$ for which $\|f\| \le M$, it is necessary and sufficient that

$$\left|\sum_n a_n \overline{d}_n\right| \le M \left\|\sum_n a_n f_n\right\|$$

for every finite sequence of scalars $\{a_n\}$. If the sequence $\{f_1, f_2, f_3, \dots\}$ is complete in $\mathcal{H}$, then the solution is unique.

As a consequence of the above result we obtain:

Proposition 6. *The space $\mathcal{H}_K$ satisfies the ZR$_a$ property if and only if for each $x \in \{c_0(a)\}^\perp$ the linear functional $\mu_{a,x}$ defined on $Y_a := \operatorname{span}\{c_n(a)\}_{n\in\mathbb{N}_0}$ as*

$$\mu_{a,x}\left(\sum_n a_n c_n(a)\right) = \sum_n a_n \langle c_{n+1}(a), x\rangle$$

for every finite sequence of scalars $\{a_n\}$, is bounded.

Assume that the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent. The linear functional $\mu_{a,x} : Y_a \to \mathbb{C}$ can be decomposed as $T_{a,x} \circ S_a$ where $T_{a,x} : Y_a \to \mathbb{C}$ is the linear operator given by

$$T_{a,x}\left(\sum_n a_n c_n(a)\right) = \sum_n a_n \langle c_n(a), x\rangle$$

and $S_a : Y_a \to Y_a$, is the linear operator given by

$$S_a\left(\sum a_n c_n(a)\right) = \sum a_n c_{n+1}(a) \tag{11}$$

for every finite sequence of scalars $\{a_n\}$. Observe that S_a is a well-defined linear operator since the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent. From now on we will assume that the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent (see Proposition 2 above). Note that operator S_a is nothing but a generalization of the classical shift operator defined by means of an orthonormal basis [22]; here it is defined by means of a linear independent Bessel sequence in $\mathcal{H}$.

The operator $T_{a,x}$ is obviously bounded since

$$T_{a,x}\left(\sum_n a_n c_n(a)\right) = \sum_n a_n \langle c_n(a), x\rangle = \left\langle \sum_n a_n c_n(a), x \right\rangle .$$

Thus, we have obtained the following result:

Theorem 4. *Assume that, for each $a \in \mathbb{C}$, the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is linearly independent and the corresponding operator S_a given by* (11) *is bounded. Then the space $\mathcal{H}_K$ satisfies the ZR property.*

Notice that, for the Paley–Wiener space PW_π, the corresponding operator S_a is bounded for any $a \in \mathbb{C}$. Indeed, for $a = 0$, we have that $\frac{d}{dw} c_{n+1}(0)(w) = i c_n(0)(w)$, $w \in (-\pi, \pi)$, from which we deduce that

$$S_0 f(w) := i \int_0^w f(s)\, ds \qquad \text{for any } f \in L^2[-\pi, \pi]\,.$$

Since $\|S_0 f\|_2 \le 2\pi \|f\|_2$ for any $f \in L^2[-\pi,\pi]$, S_0 is bounded on $\overline{\operatorname{span}}\,\{c_n(0)\}_{n\in\mathbb{N}_0} = L^2[-\pi,\pi]$. For a non-zero $a \in \mathbb{C}$ we have that $S_a f(w) = \mathrm{e}^{iaw} S_0 f(w)$ and, as a consequence, the operator S_a is bounded for each $a \in \mathbb{C}$.

The reciprocal of Theorem 4 remains true under the hypothesis that the function $1 \in \mathcal{H}_K$.

Theorem 5. *Assume that the mapping T_K in (4) is injective, the sequence $\{c_n(a)\}_{n=1}^\infty$ is linearly independent for any $a \in \mathbb{C}$, and $1 \in \mathcal{H}_K$. Then, the space $\mathcal{H}_K$ satisfies the ZR property if and only if the operator S_a is bounded for each $a \in \mathbb{C}$.*

Proof. From Theorem 4 it is enough to show that if $1 \in \mathcal{H}_K$ and the ZR_a property holds, then the operator S_a is bounded. Let $\sum_{n=0}^{L} a_n c_n(a)$ be a vector in Y_a. We have,

$$\begin{aligned}\left|\mu_{a,x}\left(\sum_{n=0}^{L} a_n c_n(a)\right)\right| &= \left|\left\langle \sum_{n=0}^{L} a_n c_{n+1}(a), x\right\rangle\right| \\ &= \left\|S_a\left(\sum_{n=0}^{L} a_n c_n(a)\right)\right\| \|x\| \left|\cos\left(\sum_{n=0}^{L} a_n c_{n+1}(a), x\right)\right|\end{aligned}$$

First we prove that the function $1 \in \mathcal{H}_K$ if and only if the condition $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$ holds for each $a \in \mathbb{C}$. Indeed, $1 = \langle K(z), x\rangle$ for some $x \in \mathcal{H}$ (necessarily $x \neq 0$) implies that, for each $a \in \mathbb{C}$, $\langle c_n(a), x\rangle = 0$, $n \geq 1$. From the completeness of $\{c_n(a)\}_{n\in\mathbb{N}_0}$ (see Proposition 1) we deduce that $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$. For the sufficient condition, let $b \in \mathbb{C}$ such that $c_0(b) \notin \overline{\operatorname{span}}\{c_n(b)\}_{n=1}^\infty$; there exists $x \neq 0$ in $(\{c_n(b)\}_{n=1}^\infty)^\perp$ and, as a consequence, the non-zero constant $\langle K(z), x\rangle$ belongs to $\mathcal{H}_K$. Note that the condition $c_0(b) \notin \overline{\operatorname{span}}\{c_n(b)\}_{n=1}^\infty$ for some $b \in \mathbb{C}$ is equivalent to the condition $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$ for every $a \in \mathbb{C}$. Therefore, the hypothesis $1 \in \mathcal{H}_K$ implies the existence of a positive number α such that $|\cos(\widehat{v,x})| = \frac{|\langle v,x\rangle|}{\|v\|\,\|x\|} > \alpha > 0$ for any nonzero $v \in \operatorname{span}\{c_n(a)\}_{n=1}^\infty$ and any $x \in c_0(a)^\perp \setminus \{0\}$. Hence,

$$\left\|S_a\left(\sum_{n=0}^{L} a_n c_n(a)\right)\right\| \leq \frac{1}{\alpha\|x\|}\left|\mu_{a,x}\left(\sum_{n=0}^{L} a_n c_n(a)\right)\right|. \tag{12}$$

Since the ZR_a property holds, the linear functional $\mu_{a,x}$ is bounded in Y_a. Thus, the inequality (12) implies the boundedness of the operator S_a on Y_a. □

In the context of de Branges spaces, Baranov [1], improving a previous work in [14,15], solved the problem of finding the structure functions E of zero exponential type for which $1 \in \mathcal{H}(E)$. Since de Branges spaces are particular cases of $\mathcal{H}_K$ spaces [10], the condition $1 \in \mathcal{H}(E)$ could be replaced by the equivalent geometric condition $c_0(a) \notin \overline{\operatorname{span}}\{c_n(a)\}_{n=1}^\infty$ for some $a \in \mathbb{C}$.

4.2. A Sufficient Condition for the Global ZR Property in a $\mathcal{H}_K$ Space

In this section we give a sufficient condition on the continuity of the operator, say S_0, under the assumption of the minimality of the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$, i.e., any polynomial belongs to $\mathcal{H}_K$ (see Proposition 3). Following Ref.

[12, p. 27], the minimality of the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$ implies that the numbers δ_k given by

$$\delta_k := \inf_{\theta\in\mathbb{R}} \rho\left(e^{i\theta}\frac{c_k(0)}{\|c_k(0)\|}, \overline{\operatorname{span}}\{c_n(0)\}_{n\neq k}\right), \quad k \in \mathbb{N}_0\,, \tag{13}$$

are strictly positive for every $k \in \mathbb{N}_0$. Note that the number δ_k denotes the inclination in $\mathcal{H}$ of the straight line spanned by $c_k(0)$ to the closed subspace $\overline{\operatorname{span}}\{c_n(0)\}_{n\neq k}$, being ρ the distance with respect to the metric given by the norm in $\mathcal{H}$.

Besides (see [12, pp. 27–28]), for any $x = \sum_k a_k c_k(0)$ (finite or convergent sum) the inequalities

$$|a_k| \le \frac{\|x\|}{\delta_k \|c_k(0)\|} \quad \text{hold for each } k \in \mathbb{N}_0\,. \tag{14}$$

Lemma 6. *Assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is complete and minimal in $\mathcal{H}$. The convergence of the series*

$$\sum_{n=0}^{\infty} \frac{1}{\delta_n}\frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}, \tag{15}$$

where the numbers $\delta_n > 0$, $n \in \mathbb{N}_0$, are given in (13)*, implies that the operator S_0 is bounded.*

Proof. For any finite sum $x = \sum_n a_n c_n(0)$, using inequalities (14), we have

$$\|S_0\, x\| \le \sum_n |a_n| \|c_{n+1}(0)\| \le \left(\sum_n \frac{1}{\delta_n}\frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}\right)\|x\| \le M\|x\|\,,$$

where M denotes the sum of the series in (15). Therefore, the operator S_0 is bounded on $\operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$; the completeness of $\{c_n(0)\}_{n\in\mathbb{N}_0}$ proves that S_0 is bounded on $\mathcal{H}$. □

In fact, the following result holds:

Theorem 7. *Assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is complete and minimal in $\mathcal{H}$. Suppose also that the series in* (15) *converges and the sequence of quotients $\{\|c_{n+1}(0)\|/\|c_n(0)\|\}_{n\in\mathbb{N}_0}$ is monotonically decreasing. Then, the ZR property in $\mathcal{H}_K$ holds.*

Proof. By Lemma 6 the ZR_0 property holds. Let a be a nonzero complex number and let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in \mathcal{H}_K$ be such that $f(a) = 0$. Then,

$$g(z) = \frac{f(z)}{z-a} = -\frac{1}{a}\sum_{n=0}^{\infty} c_n z^n \quad \text{where} \quad c_n = \frac{1}{a^n}\sum_{k=0}^{n} \alpha_k a^k\,.$$

Since $f(a)=\sum_{k=0}^{\infty}\alpha_k a^k=0$, we have that $\sum_{k=0}^{n}\alpha_k a^k=-\sum_{k=n+1}^{\infty}\alpha_k a^k$. Hence,

$$g(z)=\sum_{n=0}^{\infty}\left(\frac{1}{a^{n+1}}\sum_{k=n+1}^{\infty}\alpha_k a^k\right)z^n$$

The entire function g belongs to $\mathcal{H}_K$ if and only if the linear functional defined on $Y_0=\operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$ as

$$\nu_f\left(\sum_n b_n c_n(0)\right)=\sum_n b_n\left(\frac{1}{a^{n+1}}\sum_{k=n+1}^{\infty}\alpha_k a^k\right),$$

for every finite sequence of scalars $\{b_n\}$, is bounded. For any $y=\sum_n b_n c_n(0)$, using inequalities (14) we have

$$|\nu_f(y)|\le\|y\|\|x\|\sum_n\frac{\|c_{n+1}(0)\|}{\delta_n\|c_n(0)\|}\sum_{m=0}^{\infty}\frac{\|c_{m+n+1}(0)\|}{\|c_{n+1}(0)\|}|a|^m,$$

where $f=\mathcal{T}_K x$. Applying the ratio test it is straightforward to prove that the series $\sum_{m=0}^{\infty}\frac{\|c_{m+n+1}(0)\|}{\|c_{n+1}(0)\|}z^m$ defines an entire function G_n fon any $n\in\mathbb{N}_0$. Moreover, since the sequence $\{\|c_{l+1}(0)\|/\|c_l(0)\|\}_{l\in\mathbb{N}_0}$ is monotonically decreasing, we have that $G_n(|a|)\ge G_{n+1}(|a|)$ for any $a\in\mathbb{C}$. As a consequence, for any $y\in Y_0=\operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$ we obtain that

$$|\nu_f(y)|\le\left(\|x\|G_0(|a|)\sum_{n=0}^{\infty}\frac{\|c_{n+1}(0)\|}{\delta_n\|c_n(0)\|}\right)\|y\|=M_{f,a}\|y\|,$$

i.e., the boundedness of ν_f. □

4.3. On the Local Zero-Removing Property

In this section we will assume that S_0 is well-defined bounded operator on $\operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0}$. As a consequence, the ZR_0 property holds in $\mathcal{H}_K$. This means that for each function $f\in\mathcal{H}_K$ with $f(0)=0$, the function $f(z)/z$ belongs to $\mathcal{H}_K$. Our goal here is to prove that the ZR_a property also holds for $a\in\mathbb{C}$ with $|a|$ small enough.

We will also assume that the operator $\mathcal{T}_K:\mathcal{H}\longrightarrow\mathcal{H}_K$ given in (4) is injective. Therefore, for any $a\in\mathbb{C}$ the sequence $\{c_n(a)\}_{n\in\mathbb{N}_0}$ is complete in $\mathcal{H}$. The completeness of the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ implies that S_0 can be extended to $\mathcal{H}$ as a bounded operator. Let S_0^* be its adjoint bounded operator, i.e., for each x, y in $\mathcal{H}$ we have $\langle x,S_0y\rangle=\langle S_0^*x,y\rangle$.

By using the bijective anti-linear isometry $\mathcal{T}_K$, we define two bounded operators on $\mathcal{H}_K$ as $\mathfrak{S}_0=\mathcal{T}_K\,S_0\,\mathcal{T}_K^{-1}$ and $\mathfrak{S}_0^*=\mathcal{T}_K\,S_0^*\,\mathcal{T}_K^{-1}$.

For each $x\in\mathcal{H}$, having in mind that $K(z)=\sum_{n=0}^{\infty}c_n(0)z^n$, we have

$$\langle K(z),S_0^*x\rangle=\langle S_0K(z),x\rangle=\sum_{n=0}^{\infty}\langle c_{n+1}(0),x\rangle z^n=\frac{f(z)-f(0)}{z},$$

being $f(z) = \langle K(z), x \rangle = \sum_{n=0}^{\infty} \langle c_n(0), x \rangle z^n$. Since $\mathcal{T}_K S_0^* x = \mathfrak{S}_0^* \mathcal{T}_K x = \mathfrak{S}_0^*(f)$, we deduce that

$$\mathfrak{S}_0^* f(z) = \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{C}.$$

In general, assuming that S_a is bounded, the ZR_a property holds and the bounded operator $\mathfrak{S}_a^* := \mathcal{T}_K S_a^* \mathcal{T}_K^{-1}$ from $\mathcal{H}_K \to \mathcal{H}_K$ satisfies that

$$\mathfrak{S}_a^* f(z) = \frac{f(z) - f(a)}{z - a}, \quad z \in \mathbb{C}. \tag{16}$$

Notice that in the de Branges spaces theory, a natural question is whether the space is closed under forming difference quotients as in (16), which means that the function 1 is an associated function (see, for instance, [4,27]).

For each $a \in \mathbb{C}$ we denote by $\mathcal{H}_a$ the set

$$\mathcal{H}_a := \{f \in \mathcal{H}_K \text{ such that } f(a) = 0\}.$$

It is straightforward to prove that $\mathcal{H}_a$ is a closed subspace of $\mathcal{H}_K$ for each $a \in \mathbb{C}$. Indeed, Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_a$ be a sequence converging to g in the $\mathcal{H}_K$ norm. Since $\mathcal{H}_K$ is a RKHS, for each $z \in \mathbb{C}$ we have that $f_n(z) \to g(z)$; in particular $g(a) = \lim_{n\to\infty} f_n(a) = 0$.

The following lemma relates de ZR_a property with the subspace $\mathcal{H}_a$ via the restriction of the operator $\mathfrak{S}_0^*$ to the subspace $\mathcal{H}_0$:

Lemma 8. *Assume that the operator S_0 is bounded. Let $\widetilde{\mathfrak{S}_0^*}$ be the restriction of the operator $\mathfrak{S}_0^*$ to the closed subspace $\mathcal{H}_0$ of $\mathcal{H}_K$. Given $a \in \mathbb{C}$, the ZR_a property holds in $\mathcal{H}_K$ if and only if the range of the operator $I - a\widetilde{\mathfrak{S}_0^*}$ is $\mathcal{H}_a$.*

Proof. Assume that the operator S_0 is bounded and, therefore, the ZR_0 property holds. The range of the operator $I_0 - a\widetilde{\mathfrak{S}_0^*}$ is a subspace included in $\mathcal{H}_a$, where $I_0 := I_{|\mathcal{H}_0}$. If the ZR_a property holds in $\mathcal{H}_K$ then any entire function g in $\mathcal{H}_a$ can be written as $g(z) = zh(z) - ah(z)$ where $h \in \mathcal{H}_K$. The entire function $zh(z)$ belongs to $\mathcal{H}_0$ and $g = (I - a\widetilde{\mathfrak{S}_0^*})(zh)$, i.e., g belongs to the range of $I - a\widetilde{\mathfrak{S}_0^*}$.

Now, suppose that the range of $I - a\widetilde{\mathfrak{S}_0^*}$ is $\mathcal{H}_a$. For any $g \in \mathcal{H}_a$ there exists $f \in \mathcal{H}_0$ such that $g(z) = f(z) - af(z)/z = (z-a)f(z)/z$. Hence, since the ZR_0 property holds, the entire function $g(z)/(z-a) = f(z)/z$ belongs to $\mathcal{H}_K$. □

In the sequel we follow the Fredholm operator theory as it appears in [7].

Theorem 9. *Assume that the operator S_0 is bounded. Then, there exists $\delta > 0$ such that the ZR_a property holds in $\mathcal{H}_K$ for $|a| < \delta$.*

Proof. The identity operator I restricted to $\mathcal{H}_0$, i.e., I_0, is a Fredholm operator. Indeed, I_0 is bounded; its range is $R(I_0) = \mathcal{H}_0$, hence, closed; the kernel of I_0, $N(I_0) = \{0\}$ is finite dimensional and the codimension of the range is finite and equal to 1 (recall that $\mathcal{H}_0^\perp$ is the subspace of $\mathcal{H}_K$ generated by $f_0(z) = \langle K(z), c_n(0)\rangle,\ z \in \mathbb{C}$). The index of I_0 is $\dim N(I_0) - \operatorname{codim} R(I_0) = -1$.

For any $a \in \mathbb{C}$ the operator $I_0 - a\widetilde{\mathfrak{S}}_0^*$ is injective. Indeed, let $f \in \mathcal{H}_0$ such that $(I_0 - a\widetilde{\mathfrak{S}}_0^*)f = 0$ or, equivalently, such that $\frac{z-a}{z} f(z) = 0$, for any $z \in \mathbb{C}$. This implies that f is the zero function since f is an entire function. Following see [7, p. 34], there exists $\delta > 0$ such that if $|a| < \delta$ the operator $I_0 - a\widetilde{\mathfrak{S}}_0^*$ is Fredholm and its index verifies $\operatorname{ind}(I_0 - a\widetilde{\mathfrak{S}}_0^*) = \operatorname{ind} I_0 = -1$. Since $I_0 - a\widetilde{\mathfrak{S}}_0^*$ is an injective Fredholm operator we have that $R(I_0 - a\widetilde{\mathfrak{S}}_0^*) = \mathcal{H}_a$. Hence, by Lemma 8 the ZR$_a$ property holds in $\mathcal{H}_K$ □

An estimation of the constant δ is given in next proposition:

Proposition 7. *Assume that the operator S_0 is bounded. Then, the ZR$_a$ holds for each $a \in \mathbb{C}$ such that $|a| < \|\widetilde{\mathfrak{S}}_0^*\|^{-1}$.*

Proof. The *numerical range* of the operator $\widetilde{\mathfrak{S}}_0^*$ is defined by:

$$\Theta(\widetilde{\mathfrak{S}}_0^*) = \{\langle \widetilde{\mathfrak{S}}_0^* f, f\rangle \mid f \in \mathcal{H}_0 \text{ and } \|f\| = 1\}.$$

Since $\widetilde{\mathfrak{S}}_0^*$ is bounded we have that $\Theta(\widetilde{\mathfrak{S}}_0^*)$ is bounded in $\mathbb{C}$; indeed, $|\langle \widetilde{\mathfrak{S}}_0^* F, F\rangle| \leq \|\widetilde{\mathfrak{S}}_0^*\|$.

It is known that if $|\lambda| > \|\widetilde{\mathfrak{S}}_0^*\|$ then $\lambda I_0 - \widetilde{\mathfrak{S}}_0^*$ is an injective semi-Fredholm operator, whose range, $R(\lambda I_0 - \widetilde{\mathfrak{S}}_0^*)$, is closed and $\operatorname{codim} R(\lambda I_0 - \widetilde{\mathfrak{S}}_0^*)$ is constant in the set $\{\mu \in \mathbb{C}$ such that $|\mu| > \|\widetilde{\mathfrak{S}}_0^*\|\}$ (see [7, p. 100]).

Let $\lambda = a^{-1}$, taking into account that $a^{-1} I_0 - \widetilde{\mathfrak{S}}_0^* : \mathcal{H}_0 \to \mathcal{H}_a$ we obtain that if $|a| < \|\widetilde{\mathfrak{S}}_0^*\|^{-1}$ then $R(a^{-1} I_0 - \widetilde{\mathfrak{S}}_0^*)$ is a closed subspace in $\mathcal{H}_a$, therefore, $\operatorname{codim} R(a^{-1} I_0 - \widetilde{\mathfrak{S}}_0^*) = C$ with $C \geq 1$ for each $a \neq 0$ satisfying $|a| < \|\widetilde{\mathfrak{S}}_0^*\|^{-1}$. From Theorem 9 we have that if $b \neq 0$ is close to 0 then $\operatorname{codim} R(I_0 - b\widetilde{\mathfrak{S}}_0^*) = \operatorname{codim} R(b^{-1} I_0 - \widetilde{\mathfrak{S}}_0^*) = 1$. Hence, $C = 1$ and the ZR$_a$ property holds in $\mathcal{H}_K$ whenever $|a| < \|\widetilde{\mathfrak{S}}_0^*\|^{-1}$. □

Corollary 10. *Assume that the mapping T_K in (4) is injective, the sequence $\{c_n(a)\}_{n=1}^\infty$ is linearly independent for any $a \in \mathbb{C}$ and $1 \in \mathcal{H}_K$. Then the set*

$$\{b \in \mathbb{C} \mid \textit{property } ZR_b \textit{ holds}\}$$

is an open set in $\mathbb{C}$

Proof. It is a straightforward consequence of Theorems 5 and 9. □

Remark As far as Theorem 9 is concerned, one can construct kernels K such that the ZR$_a$ property at a fixed point $a \in \mathbb{C}$ implies that the zero-removing property holds in $\mathcal{H}_K$ for every $b \in \mathbb{C}$. It remains the open question whether this is true for every space $\mathcal{H}_K$.

5. The Differentiation Operator in $\mathcal{H}_K$

In general, the differentiation operator $\mathcal{D} : \mathcal{H}_K \longrightarrow \mathcal{H}_K$ given by $\mathcal{D}(f) = f'$, $f \in \mathcal{H}_K$, is not well-defined as the following example shows. In example (e) in Sect. 2 with $\gamma := \{\sqrt{n!}\}_{n\in\mathbb{N}_0}$, an entire function $f(z) = \sum_{n=0}^\infty \alpha_n z^n$ belongs to $\mathcal{H}_{K_\gamma}$ if and only if the sequence $\{\sqrt{n!}\,\alpha_n\}_{n=0}^\infty$ belongs to $\ell^2(\mathbb{N}_0)$. In particular, for the sequence $\alpha_n = 1/(n\sqrt{n!})$, $n \in \mathbb{N}_0$, the corresponding function f belongs to $\mathcal{H}_{K_\gamma}$; however its derivative f' does not belong to $\mathcal{H}_{K_\gamma}$. A sufficient condition is given in the next result:

Theorem 11. *Suppose that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ of Taylor coefficients of K at 0 is complete and minimal in $\mathcal{H}$. Consider the numbers $\delta_n > 0$, $n \in \mathbb{N}_0$, given in* (13)*. If the series*

$$\sum_{n=0}^{\infty} \frac{(n+1)}{\delta_n} \frac{\|c_{n+1}(0)\|}{\|c_n(0)\|} \tag{17}$$

converges, then the differentiation operator $\mathcal{D}$ is a well-defined bounded operator on $\mathcal{H}_K$.

Proof. Let f be in $\mathcal{H}_K$; there exists $x \in \mathcal{H}$ such that $f(z) = \langle K(z), x\rangle$, for any $z \in \mathbb{C}$, and $f(z) = \sum_{n=0}^\infty \langle c_n(0), x\rangle z^n$. Therefore,

$$f'(z) = \sum_{n=1}^{\infty} \langle c_n(0), x\rangle n z^{n-1}, \quad z \in \mathbb{C}.$$

The derivative f' of the entire function f belongs to $\mathcal{H}_K$ if and only if there exists $y \in \mathcal{H}$ such that

$$\langle c_n(0), y\rangle = (n+1)\langle c_{n+1}(0), x\rangle \quad \text{for any } n \in \mathbb{N}_0. \tag{18}$$

Proceeding as in the proof of Theorem 4, the set of equations (18) has a solution $y \in \mathcal{H}$ if and only if the operator

$$\begin{array}{rccc} \mathfrak{D} : & \operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0} & \longrightarrow & \operatorname{span}\{c_n(0)\}_{n\in\mathbb{N}_0} \\ & c_n(0) & \longmapsto & (n+1)c_{n+1}(0)\,, \end{array}$$

is bounded. Let $u = \sum_n a_n\, c_n(0)$ be a finite sum in $\mathcal{H}$ with $\|u\|_{\mathcal{H}} = 1$. By using inequalities in (14), we obtain

$$\begin{aligned} \|\mathfrak{D}u\| = \left\| \sum_n a_n (n+1) c_{n+1}(0) \right\| &\le \sum_n (n+1)|a_n| \|c_{n+1}(0)\| \\ &\le \sum_n \frac{(n+1)}{\delta_n} \frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}. \end{aligned}$$

Hence, the convergence of the series in (17) implies the continuity of the operator $\mathfrak{D}$. Moreover, the boundedness of the operator $\mathfrak{D}$ implies the boundedness of the differentiation operator $\mathcal{D}$. Indeed, if $\mathfrak{D}$ is bounded on $\operatorname{span}\{c_n(0)\}_{n=0}^\infty$ then it can be extended by continuity to the whole space $\mathcal{H}$.

In this case, the adjoint operator of $\mathfrak{D}$, $\mathfrak{D}^* : \mathcal{H} \to \mathcal{H}$ is bounded and it is straightforward to prove that $\mathcal{D} = \mathcal{T}_K \mathfrak{D}^* \mathcal{T}_K^{-1}$ where $\mathcal{T}_K : \mathcal{H} \to \mathcal{H}_K$ is the anti-linear isometry defined in (4). $\square$

Moreover, whenever the differentiation operator $\mathcal{D}$ is a well-defined bounded operator on $\mathcal{H}_K$, the translation operator given by $T_a f(z) := f(z-a)$, $z \in \mathbb{C}$, is also a well-defined bounded operator $T_a : \mathcal{H}_K \longrightarrow \mathcal{H}_K$ for each $a \in \mathbb{C}$. Indeed, adapting a result from [5] we obtain:

Proposition 8. *Suppose that the differentiation operator $\mathcal{D}$ defined as $\mathcal{D}(f) = f'$ is a well-defined bounded operator $\mathcal{D} : \mathcal{H}_K \to \mathcal{H}_K$. Then, for each $a \in \mathbb{C}$, the translation operator $T_a : \mathcal{H}_K \to \mathcal{H}_K$ is a well-defined, bounded operator. Moreover, we have the following expansion for T_a converging in the operator norm*

$$T_a = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \mathcal{D}^n \,. \tag{19}$$

Proof. It is a well-known result that (19) holds in $\mathcal{E}$, the space of entire functions endowed with the topology of the uniform convergence on compact sets (see, for instance, [5]). Since the differentiation operator $\mathcal{D}$ is bounded on the Hilbert space $\mathcal{H}_K$, the series on the right side of (19) converges absolutely, and hence in the operator norm to a bounded operator on $\mathcal{H}_K$. As the convergence in $\mathcal{H}_K$ implies convergence in the space $\mathcal{E}$, this operator must be T_a. $\square$

Note that, under the hypotheses of Theorem 11, in the corresponding $\mathcal{H}_K$ space the ZR property holds. Indeed, by using Lemma (6)) the ZR$_0$ property holds. For $a \in \mathbb{C} \setminus \{0\}$, let g be an entire function in $\mathcal{H}_K$ such that $g(a) = 0$. The entire function $f = T_{-a}g$ belongs to $\mathcal{H}_K$ and $f(0) = g(a) = 0$. Since the ZR$_0$ property holds we have

$$h(z) = \frac{f(z)}{z} = \frac{g(z+a)}{z} \in \mathcal{H}_K \,, \tag{20}$$

and hence $g(z)/(z-a) = (T_a h)(z) \in \mathcal{H}_K$.

Closing the paper, it is worth to mention that the convergence of the series in (17) imposes a condition on the rate of decay of the sequence $\{\|c_n(0)\|\}_{n \in \mathbb{N}_0}$ and therefore, on the growth of the functions in $\mathcal{H}_K$. Indeed, let F be the entire function defined by $F(z) = \sum_{n=0}^{\infty} \|c_n(0)\| z^n$. Then, for any $f \in \mathcal{H}_K$, we have

$$|f(z)| \le \|f\| F(|z|) = \|f\| \sum_{n=0}^{\infty} \|c_n(0)\| |z|^n \quad \text{for all } z \in \mathbb{C} \,. \tag{21}$$

In order to illustrate the relationship between the decaying of the sequence $\{\|c_n(0)\|\}_{n \in \mathbb{N}_0}$, the growth of functions in $\mathcal{H}_K$ and the ZR property, suppose that

$$\lim_{n \to \infty} n^r \frac{\|c_n(0)\|}{\|c_{n-1}(0)\|} = \alpha \quad \text{for some } r > 2 \,. \tag{22}$$

Assume that the sequence $\{c_n(0)\}_{n\in\mathbb{N}_0}$ is uniformly minimal, i.e., there exists δ such that $\delta_n > \delta > 0$ for any $n \in \mathbb{N}_0$ (see [12, p. 27]). Notice that the existence of the limit in (22) implies the convergence of the series $\sum_{n=0}^{\infty}(n+1)\frac{\|c_{n+1}(0)\|}{\|c_n(0)\|}$ and, the boundedness of the differentiation operator on $\mathcal{H}_K$; as a consequence, the ZR property holds.

Let $\gamma > \alpha$; following [5], condition (22) implies the existence of a positive constant C, depending only on γ, such that

$$\|c_n(0)\| \leq C\left(\frac{\mathrm{e}\,\gamma^{1/r}}{n}\right)^{nr}, \quad n \in \mathbb{N}_0. \tag{23}$$

Now according to [17, p. 7], the entire function g defined by

$$g(z) = \sum_{n=1}^{\infty}\left(\frac{\mathrm{e}Mr^{-1}}{n}\right)^{nr} z^n\,,$$

where $M = r\gamma^{1/r}$, has order r^{-1}. Having in mind (23) we obtain that $F(|z|) \leq Cg(|z|)$ for any $z \in \mathbb{C}$. Hence, inequality (21) implies that any function f in $\mathcal{H}_K$ has order less or equal than $r^{-1} < 1/2$.

Acknowledgements

The authors are indebted to the anonymous referees for their valuable suggestions. This work has been supported by the grant MTM2009-08345 from the Spanish *Ministerio de Ciencia e Innovación* (MICINN).

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Received: September 18, 2013.
Accepted: September 4, 2014.