UNIVERSIDAD POLITÉCNICA DE MADRID
ESCUELA TÉCNICA SUPERIOR DE INGENIEROS AERONÁUTICOS

**Linear Instability Mechanisms on Airfoils at Low Reynolds
Number: Massive Separation, Wingtip Vortex Formation
and the Trailing Vortex System**

Doctoral Thesis

By
Wei He
MEng Aeronautical Engineering

MADRID, JULY 2016

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Ph.D. in Aeronautical Engineering

MADRID, JULY 2016

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Piano, piano, si va molto lontano!
– from an Italian proverb

Abstract

The global linear instability analysis of separated flow on homogeneous infinite and finite wings are numerically investigated, including analysis of the wingtip vortex.

Two- and three-dimensional modal and non-modal instability mechanisms of steady spanwise homogeneous laminar separated flows over airfoil profiles, placed at large angles of attack against the oncoming flow, have been investigated using linear global theory. Three NACA profiles of distinct thickness and camber were considered, in order to assess geometry effects on the laminar-turbulent transition paths discussed. At the conditions investigated large-scale steady separation occurs, such that Tollmien-Schlichting and cross-flow mechanisms were not considered. It is found that the leading modal instability on all three airfoils is that associated with the Kelvin-Helmholtz (KH) mechanism, taking the form of the eigenmodes known from analysis of generic bluff bodies. The three-dimensional stationary eigenmode of the two-dimensional laminar separation bubble, associated in earlier analyses with the formation on the airfoil surface of large-scale separation patterns akin to stall-cell, is shown to be stronger damped than the KH mode. Non-modal instability analysis reveals the potential of the flows to sustain transient growth which becomes stronger with increasing angle of attack and Reynolds number. Optimal initial conditions were computed and were found to be analogous to those on a cascade of Low Pressure Turbine blades. By changing the time-horizon of the analysis these linear optimal initial conditions are found to evolve into the KH mode. Steady Navier-Stokes equations for a high angle of attack and Reynolds number flow are achieved through selective damping frequency method, and its modal analysis is performed. The most unstable mode is oscillating after the airfoil and dominates about $O(10)$ chord length. The sub-leading mode is a KH type as appeared in the low Reynolds number steady flow. The stationary mode starts immediately behind the airfoil and then decays into the wake.

The time-periodic base flows ensuing linear amplification of the KH mode are analyzed here via temporal Floquet theory. Two amplified modes are discovered, having characteristic spanwise wavelengths of approximately 0.6 and 2 chord lengths, respectively. Unlike secondary instabilities on the circular cylinder, three-dimensional short-wavelength perturbations are the first to become linearly unstable on all airfoils. Long-wavelength perturbations are quasi-periodic, standing or traveling-wave perturbation that also become unstable as the Reynolds number is increased further. The dominant short-wavelength instability gives rise to spanwise periodic wall-shear patterns, akin to the separation cells encountered on airfoils at low angles of attack and the stall cells found in flight at conditions close to stall. Thickness and camber have quantitative but not qualitative effect on the secondary instability analysis results obtained.

The previous analysis assumes an idealistic wing flow which has a homogeneous boundary conditions in the spanwise direction. A generalized wingtip developed downstream

should be taken into account. To this purpose, a finite wing laminar flow has been considered. Instability analysis of flow in the wake of a low aspect ratio three-dimensional wing of elliptic platform, constructed with appropriately scaled Eppler E387 airfoils, has been performed. The flow field over the airfoil and in its wake was computed by full three-dimensional direct numerical simulation at a chord Reynolds number of $Re_c = \frac{uc}{\nu} = 1750$ and two angles of attack, $AoA = 0^\circ$ and 5° . The spatial eigenvalue problem governing linear global small-amplitude perturbations superposed upon this base flow has been solved and results were used to initialize a linear PSE-3D marching procedure without any simplifying assumptions regarding the form of the trailing vortex system, other than weak dependence of all flow quantities on the axial spatial direction. Two classes of linearly unstable perturbations were identified, namely stronger-amplified symmetric modes and weaker-amplified antisymmetric disturbances, both peaking at the vortex sheet which connects the trailing vortices. The amplitude functions of both classes of modes were documented and N-factor predictions for potential laminar breakdown have been computed.

Resumen

En esta tesis se investiga numericamente el análisis global de estabilidad lineal en flujo desprendido sobre alas homogéneamente infinitas y finitas, incluyendo el análisis de los vórtices de punta de ala.

Los mecanismos de inestabilidad modal y no modal, bidimensional y tridimensional, de flujos separados estacionarios laminares sobre superficies sustentadoras homogéneas a lo largo de la envergadura han sido investigados a altos ángulos de ataque, con respecto al flujo de entrada, mediante el uso de la teoría lineal global. Para ello se han considerado tres perfiles NACA de distinto grosor y curvatura con el fin de evaluar los efectos de la geometría en la ruta seguida por el flujo en la transición de flujo laminar a flujo turbulento. En las condiciones investigadas, no se considera la separación estacionaria a gran escala generada por mecanismos del tipo Tollmien-Schlichting o inestabilidad de flujo cruzado (crossflow).

En estas condiciones se encontró que la inestabilidad modal dominante en las tres superficies sustentadoras están asociadas con el mecanismo de tipo Kelvin-Helmholtz (KH), siendo su forma similar a los modos propios obtenidos en la literatura del análisis de inestabilidad lineal alrededor de cuerpos romos. En este estudio se muestra que el modo propio estacionario tridimensional obtenido en la burbuja de separación laminar bidimensional (asociado en estudios previos con la formación de los patrones de separación de gran-escala parecidos a los “Stall-cells”) es más fuertemente amortiguado que los modos KH.

El análisis de estabilidad no-modal revela el potencial que tienen estos flujos para permitir mecanismos de crecimiento transitorio los cuales son más fuertes cuanto mayor es el ángulo de ataque y el número de Reynolds. Se computaron las condiciones iniciales óptimas, las cuales se encontraron que son análogas a las que surgen en la cascada de álabes en turbinas de baja presión. Mediante el cambio del horizonte de tiempo en el análisis de estas condiciones iniciales óptimas se encontró que éstas evolucionan a modos de tipo KH.

Por otro lado, las soluciones de Navier-Stokes estacionarias para ángulos de ataque y números de Reynolds considerados se calcularon mediante el uso de un método de amortiguación selectivo de frecuencias (“selective damping frequency method”), sobre las cuales se realizaron los análisis modales y no-modales. El modo más inestable es oscilante detrás de la superficie sustentadora y domina el comportamiento fluido sobre longitudes $O(10)$ la longitud de la cuerda. El siguiente modo dominante es el modo de tipo KH que aparece en el flujo estacionario a números de Reynolds bajos. Finalmente, el modo estacionario comienza inmediatamente después de la superficie sustentadora y decae en la estela. El flujo base periódico en el tiempo resultante de la subsiguiente amplificación lineal del modo KH ha sido analizado aquí mediante la teoría temporal Floquet. Se

descubrieron dos modos con longitudes típicas a lo largo de la envergadura del orden de 0.6 y 2 longitudes de cuerda, respectivamente. A diferencia de las inestabilidades secundarias sobre el cilindro, las inestabilidades tridimensionales de longitud de onda corta son las dominantes en todas las superficies sustentadoras. Las perturbaciones de longitud de onda larga son cuasi-periódicas, se vuelven también inestables cuando el número de Reynolds se incrementa. Las inestabilidades dominantes de corta longitud de onda dan lugar a patrones periódicos a lo largo de la envergadura de las líneas de corriente similares a las celdas de separación encontradas sobre las superficies sustentadoras a bajos ángulos de ataque, y las encontradas en condiciones de vuelo próximas a la entrada en pérdida (“Stall cells”). El espesor y la curvatura de las superficies tienen un efecto cuantitativo pero no cualitativo en el análisis de las inestabilidades secundarias realizado.

El análisis anterior asume un flujo ideal sobre un ala homogénea en la dirección a lo largo de la envergadura. Un caso más general debería tener en cuenta el efecto de la punta del ala en el flujo aguas abajo. Para este objetivo se considero el flujo laminar entorno a un ala finita. Se realizó el análisis de inestabilidad del flujo en la estela de un ala tridimensional, de plataforma elíptica, con bajo ratio entre la cuerda y la envergadura construido a partir de perfiles Eppler E387 escalados apropiadamente. El flujo sobre el ala y sobre la estela fueron computados mediante simulación numérica directa tridimensional a número de Reynolds basado en la cuerda igual a 1750 y dos ángulos de ataque, $AoA = 0^\circ$ y 5° . El problema de valores propios espacial que gobierna las perturbaciones globales lineales de pequeña amplitud superpuestas al flujo base fue resuelto, y los resultados fueron usados para inicializar el PSE-3D linear el cuál utiliza un procedimiento de “marching” en el que no se realizaron ninguna hipótesis de simplificación en cuanto a la forma del sistema de vórtices de arrastre, que no sea la de la débil dependencia de todas la cantidades fluidas con respecto a la dirección espacial axial. De esta manera se identificaron dos clases de perturbaciones inestables, a saber, modos simétricos fuertemente amplificados y perturbaciones anti-simétricas débilmente amplificadas. Ambas alcanzan el máximo en la zona de la función de amplitud que conecta los vórtices de arrastre. Dichas funciones de amplitud han sido documentadas y las predicciones del “N-factor” para la ruptura potencial del flujo laminar fueron calculadas.

Acknowledgements

To my mother and father

First of all I would like to give my appreciations to Professor Vassilis Theofilis for advising my Ph.D studies. With his guidance leading me to engage the world of flow instability. His idea always shed a light on me and I benefit uncountable from his enthusiasm in research and the free atmosphere he created. I appreciate him for enrolling me into the program *IRSES* and supporting me to study in Brazil. I would also like to thank my co-advisor Professor José Miguel Pérez for discussing the instability and helping me to solve the programming bugs in the first stage when I was starting to learn CFD. His humor makes the life in Madrid more colorful. Without supervising from both of them with great patience, it would not be able to finish this thesis.

I want to show my gratitude to all the people I met here. Especially my colleagues and friends Dr. Daniel Rodríguez, Dra. Soledad Le Clainche, Dr. Francisco Gómez, Dra. Qiong Liu, Dr. Pedro Paredes and Juan Ángel Tendaro. The scientific discussions and culture exchanges with them broaden my outlook. The warmth from them accompany with me in these years. I benefit a lot in the last semester working with Pedro and Juan Ángel. Also I am lucky that have Chinese friends studied together in Madrid. Here I would thank Dr. Rui Sun, Dr. Changhong Fu, Dr. Siwei Dong, Yongjun Pan, Chao Zhang and Baojun Sun.

During these years I had the pleasure to meet many people, the discussions with who are kindly appreciated. I would like to thank Professor Marcello Augusto Faraco de Medeiros and Professor Gustavo Roque de Silva Assi in the Universidad de São Paulo (USP) for holding me to study in their groups. The three-month working on the aeroacoustics under the advise of Marcello buried a seed in my brain. During my stay in Brazil, I had an opportunity to interact with the people from the two groups and make many friends, Elmer, Daniel, Andres, Leandro, João, Sérgio, etc. In particularly the collaboration with Professor Rafael dos Santos Gioria in USP on the topic of airfoil separation instability.

I would also like to thank Professor Spencer Sherwin in Imperial College London and his Ph.D students Gian and Chris for answering my questions about using the code *Nektar++* and Professor Hugh Blackburn in Monash University for helping to fix the problem arisen in *Semtex* and *dog*. I would thank Professor Bruno Souza Carmo in USP for sharing his *Nektar* experience.

Once again I would like to mention my advisor during my master study at Xiamen University, Professor Qi Lin, who opened the door and brought me into the experimental aerodynamics field and encouraged me to pursue my doctor study abroad. Last but not least, I must to thank my family for their encourage and understanding during these years

Acknowledgements

hard work and far away from home.

This work has been supported by China Scholarship Council (CSC) and nuModelling SL.

Contents

Abstract	iii
Resumen	v
Acknowledgements	vii
Contents	ix
1 Introduction	1
1.1 Background	1
1.2 Massive Separation Airfoil Flow	2
1.3 Wingtip Vortex Stability Investigations	9
1.4 Scope of the Thesis	12
2 Hydrodynamic Instability	15
2.1 Linearized Perturbation Equations	17
2.2 Local Linear Instability	18
2.2.1 Orr-Sommerfel and Squire Equations	18
2.2.2 Spatial Orr-Sommerfeld equation	21
2.2.3 Local PSE	22
2.2.4 Gaster Relation	22
2.2.5 Secondary instability	23
2.2.6 Transient growth	23
2.3 BiGlobal Instability	24
2.3.1 Temporal BiGlobal analysis	24
2.3.2 Spatial BiGlobal analysis	25
2.4 Linear PSE-3D	26
2.5 TriGlobal Instability	28

3	Analysis of Flow Topology	29
3.1	Critical Point Theory	29
3.2	Vortex Identification	32
4	Instability of Separated Airfoil Flow	35
4.1	NACA Airfoils Base Flow	35
4.1.1	Code Employed	35
4.1.2	Meshing strategies	36
4.1.3	Boundary conditions	37
4.1.4	Convergence	39
4.1.5	Effect of trailing-edge geometry on the NACA 0015 airfoil	40
4.1.6	Unsteadiness on the three airfoils, as a function of Re and AoA . . .	43
4.1.7	Flow topology of 2D airfoil flow	45
4.2	BiGlobal Stability Analysis of Separated Flow	46
4.2.1	Effect of trailing-edge geometry on the primary stability	46
4.2.2	Spatial shape of eigenvector	47
4.2.3	Effect of Re on the modal stability	49
4.2.4	Effect of AoA on the modal stability	50
4.2.5	Effect of geometry on the modal stability	50
4.2.6	Modal stability of the unstable steady flow	53
4.3	Local Stability Analysis of 2D Wake	55
4.4	Nonmodal Linear Stability Analysis	56
4.4.1	Effect of wavenumbers on transient growth	57
4.4.2	Effect of Re and AoA on the transient growth	61
4.4.3	Effect of geometry on the transient growth	61
4.5	Instability Analysis of Periodic Wakes	61
4.6	A New Path for the Generation of Stall Cells	69
5	Instability of the Laminar Trailing Vortices	75
5.1	Wingtip Vortex Formation	75
5.1.1	3D elliptical wing model	75
5.1.2	Base state computation	75

5.2	Spatial BiGlobal Analysis in the Wingtip Vortex	83
5.3	PSE-3D Analysis in the Wake of a Trailing Vortices	84
5.4	Comparisons with Local Linear Theory	88
5.4.1	Wake profiles at the plane of symmetry	88
5.4.2	Local analysis near the vortex core	90
6	Discussions and Future Work	93
6.1	Discussion	93
6.2	Future Work	95
A	OpenFOAM Validation	97
A.1	OpenFOAM: A quick glance	97
A.2	Circular cylinder steady flow	97
B	Flow Unsteadiness Validation	101
B.1	Immersed boundary method	101
B.2	Spectral element method	102
C	Convergence Studies	105
C.1	Base flow	105
C.2	Modal stability analysis	105
C.3	Non-modal stability analysis	106
C.4	Floquet analysis	108
C.5	Spatial BiGlobal instability analysis validation on the q -vortex	109
C.6	PSE-3D analysis validation on the Batchelor vortex	111
D	E-387 Geometry	113

List of Figures

2.1	Sketch of the laminar-to-turbulence transition in a boundary layer. Taken from [89].	15
2.2	Sketch of absolute (left) and convective (right) stability.	21
3.1	Critical points classification in the plane $[p, q]$ (Extracted from [42]).	30
3.2	Flow patterns in the compressible flow of an evolution hemisphere-cylinder at higher angle of attack. Extracted from [133].	31
3.3	Stream surfacelines on the hemisphere-cylinder in the flow of $Re = 200$ at $AoA = 45^\circ$: The critical points are visualized on the head (<i>Left</i>) and body tail bottom (<i>Right</i>).	31
3.4	The isosurface of vorticity at $Q = 0.01$ (left) and $\lambda_2 = -0.01$ (right) in the flow pasts the hemisphere-cylinder at $Re = 200$ and $AoA = 45^\circ$	33
3.5	The evolution of vortex core in the flow pasts the hemisphere-cylinder at $Re = 200$ and $AoA = 45^\circ$	34
3.6	Vortex advanced along the hemisphere-cylinder at $Re = 200$ and $AoA = 45^\circ$, in the $yz - plane$ perpendicular to the body central line.	34
4.1	2D simulation mesh generated by <i>blockMesh</i> . <i>Top Row</i> : 3D hexahedral mesh. <i>Low Row</i> : Full domain (left) and closed view near the boundary layer (right).	36
4.2	Mesh topologies employed for the baseflow computation and instability analyses of the NACA airfoils. <i>Top Row</i> : Finite element mesh used for the 4415 in the matrix-forming computations based on FreeFEM++. <i>Middle Row</i> : Spectral element mesh used for the 0015 in the time-stepping computations employing Nektar++ or Nek5000. <i>Lower Row</i> Spectral element mesh generated by Gambit for Nektar++ and Semtex. <i>Left Column</i> : Part of the full domain. <i>Right Column</i> : Close-up near the airfoil.	38
4.3	Detail of the spectral element mesh in figure 4.2 near the leading edge (left) and the trailing edge (right)	39
4.4	Trailing edge geometries of the NACA 0015 airfoil	40

4.5	Convergence history of the residual of C_d and C_l for the four NACA 0015 trailing edge geometries, using OpenFOAM at $Re = 200, AoA = 18^\circ$: Top-left: <i>baseline</i> profile; top-right: <i>sharp TE</i> , lower left: <i>circular TE</i> , lower right: <i>modified</i> profile.	42
4.6	Pressure distribution on the NACA 0015 airfoil with various trailing edge: <i>Baseline</i> (—), <i>sharp-TE</i> (— —), <i>circular-TE</i> (· · ·) and <i>Modified</i> (— · —).	42
4.7	Vorticity distribution around the NACA 0015 airfoil and the streamlines visualization of the separation bubble behind the airfoil at $Re = 200$ and $AoA = 18^\circ$	43
4.8	Geometries of the three NACA airfoils: NACA 0009 (<i>left</i>), NACA 0015 (<i>middle</i>) and NACA 4415 (<i>right</i>).	44
4.9	Instantaneous vortex distribution for the flow past NACA 4415 at $Re = 1000$ and $AoA = 20^\circ$ (upper) and its flow topology explanation (lower)	45
4.10	Comparison of primary perturbation modes in two-dimensional flow for the three airfoils flow at $Re = 200$ with vorticity ω_z contour visualized over -5 (blue) to 5 (red). From left to right are NACA 0009, NACA 0015 and NACA 4415.	48
4.11	NACA 0009 at $Re = 200, AoA = 18^\circ$. Amplitude functions of the least-damped traveling (<i>upper</i>) and stationary (<i>lower</i>) perturbation velocity components at $\beta = 1$ and $\beta = 3.5$, respectively. Normalized by the highest velocity component, from -0.1 (blue) to 0.1 (red)	48
4.12	NACA 0015 at $Re = 200, AoA = 18^\circ$. Amplitude functions of the least-damped traveling (<i>upper</i>) and stationary (<i>lower</i>) perturbation velocity components at $\beta = 1$ and $\beta = 3.5$, respectively. Normalized by the highest velocity component, from -0.1 (blue) to 0.1 (red).	48
4.13	NACA 4415 at $Re = 200, AoA = 18^\circ$. Amplitude functions of the least-damped traveling (<i>upper</i>) and stationary (<i>lower</i>) perturbation velocity components at $\beta = 1$ and $\beta = 3.5$, respectively. Normalized by the highest velocity component, from -0.1 (blues) to 0.1 (red)	49
4.14	Stability analysis of flow around the three airfoils at $AoA = 18^\circ$. <i>Left</i> : Growth rate of the leading eigenvalue, as function of spanwise wavenumber β . <i>Right</i> : Strouhal number dependence on wavenumber. $\circ$: 0009 at $Re = 200$ and 220; $\square$: 0015 at $Re = 200, 220$ and 230; $\diamond$: 4415 at $Re = 150$ and 200.	51
4.15	Stability analysis of flow around the three airfoils at $Re = 200, AoA = 15^\circ$ and $AoA = 18^\circ$. <i>Left</i> : Growth rate of the leading eigenvalue, as function of spanwise wavenumber β . <i>Right</i> : Strouhal number dependence on wavenumber. $\circ$: 0009; $\square$: 0015; $\diamond$: 4415.	52
4.16	Effect of thickness and camber on modal results of the three airfoils at $Re = 200, AoA = 18^\circ$: Growth rate (<i>left</i>) and Strouhal number (<i>right</i>) of the leading eigenvalue, as function of the spanwise wavenumber. $\circ$: 0009; $\square$: 0015; $\diamond$: 4415.	53

4.17	Vorticity and streamlines of the unsteady flow (left) and its associated steady solution (right) for NACA 4415 at $Re=500$ and $AoA = 20^\circ$, respectively.	54
4.18	Stability analysis of the flow around NACA 4415 at $Re=500$ and $AoA = 20^\circ$. <i>Left</i> : growth rate, <i>Right</i> : Strouhal number.	54
4.19	Amplitude functions of the primary (<i>left column</i>) and secondary (<i>right column</i>) oscillation modes at $\beta = 0$ and $Re = 500$. Normalized by the highest velocity component, from -0.1 to 0.1. The associated eigenvalues are (0.47476, 2.0236) and (0.12069, 2.19388), respectively.	55
4.20	Amplitude functions of the primary stationary (<i>Left column</i>) and secondary travelling (<i>Right column</i>) modes at $\beta = 5$ and $Re = 500$. Normalized by the highest velocity component, from -0.1 to 0.1. The associated eigenvalues are $(-4.88364 \times 10^{-4}, 0)$ and $(-0.154204, 0.276417)$ respectively.	55
4.21	Local spatial stability analysis using the wake profile $u(y)$ at different location. Wavenumbers versus of ω of real part α_r (left) and imaginal part α_i (right).	56
4.22	Wake modes further verified by local spatial stability analysis. Amplitude function of $\hat{u}$ -component (left) and $\hat{v}$ -component (right).	56
4.23	Optimal gain $G(\tau)$ as a function of wavenumber β at $Re = 200, AoA = 18^\circ$ for the three airfoils: 0009 ($\circ$), 0015 ($\square$) and 4415 ($\diamond$) airfoils. Upper-to-lower curves: $\beta = 0, \pi/8, \pi/4, \pi/2, \pi$ and 2π	58
4.24	Maximum gain $G(\tau)$ at $\beta = 0$ (—), $\beta = \pi/4$ (— —), $\beta = \pi/2$ (— · —) and $\beta = \pi$ (···) for the 0015 airfoil at $Re = 200, AoA = 18^\circ$ at short time-horizons in the range $0 \leq \tau \leq 4$	59
4.25	Sequence of leading optimal perturbations vorticity countous developed from the two-dimensional flow past the NACA 0015 airfoil at $Re = 200, AoA = 18^\circ$	59
4.26	Three-dimensional optimal growth as a function of τ at $Re = 200$ and $AoA = 18^\circ$ for the NACA 0015 with wavenumber $\beta = 1$. Compared with $\beta = 0$ (dashed line).	60
4.27	Optimal perturbations of the flow past the NACA 0015 airfoil at $Re = 200, AoA = 18^\circ, \beta = 1$. From upper to lower, $\tau = 0, 10, 20, 30, 40, 50$ and 60 . Scaled at $[-0.5, 0.5]$ after normalizing with respect to each maxima velocity component in time interval.	60
4.28	Maximum gain, $G(\tau)$ at $\beta = 0$ for the 0009 ($\circ$), 0015($\square$) and 4415($\diamond$) airfoils. Left column: Effect of Reynolds numbers, $Re = 150, 200$ and 220 at $AoA = 18^\circ$. Right column: Effect of angle of attack, $AoA = 10^\circ, 15^\circ$ and 18° at $Re = 200$	62
4.29	Optimal gain at $Re = 200, AoA = 18^\circ, \beta = 0$ as a function of airfoil thickness and camber.	63

4.30	Dependence of the Strouhal number of the wake behind the NACA 0009 ($\circ$), 0015 ($\square$) and 4415 ($\diamond$) airfoils as function on the Reynolds number considered in this work.	64
4.31	Maximum Floquet multiplier, μ , as function of wavenumber β for a range of Reynolds numbers at $AoA = 20^\circ$. <i>Upper</i> : NACA 0009 at (upper-to-lower) $Re = 600, 500, 442$ and 400 . <i>Middle</i> : NACA 0015 at (upper-to-lower) $Re = 600, 500, 474$ and 400 . <i>Lower</i> : NACA 4415 at (upper-to-lower) $Re = 600, 500, 435$ and 400	65
4.32	Effect of angle of attack on Floquet multiplier, μ , for the NACA 0009 ($\circ$), 0015 ($\square$) and 4415 ($\diamond$) airfoils at $Re = 500$ and $AoA = 18^\circ$ (full symbols) and 20° (open symbols)	66
4.33	Ratio $\theta/(2\pi)$ between the secondary and primary periods, T_s and T as function of β at $AoA = 20^\circ$. In all three plots lower to upper set of results correspond to $Re = 400, 500$ and 600	68
4.34	Isosurfaces of $\hat{\omega}_z$: NACA 4415 airfoil at $Re = 500$, $AoA = 20^\circ$. <i>Upper</i> : Long-wavelength instability at $\beta = 3$. <i>Lower</i> : Short-wavelength instability at $\beta = 11$	69
4.35	Three-dimensional reconstruction of the results of Figure 4.34. <i>Upper</i> : LW instability at $\beta = 3$. <i>Lower</i> : SW instability at $\beta = 11$	70
4.36	Three-dimensional reconstruction of the results of superposing perturbations with LW instability at $\beta = 3$, and <i>Lower</i> : SW instability at $\beta = 11$ on base flow at the level of $\epsilon = 5 \times 10^{-4}$	71
4.37	Wall streamlines of the reconstructed flow field composed of four periods of the $\beta = 3$ LW eigenmode (<i>upper</i>) and the $\beta = 11$ eigenmode (<i>lower</i>) perturbation superposed at $\epsilon = 5 \times 10^{-4}$ upon the $O(1)$ time-periodic base state. The critical points of saddles and nodes are mutual distributed periodically along the spanwise of the wing.	72
5.1	Perspective view of the E387 elliptic wing and its airfoil profile.	75
5.2	Full view of the computational domain, showing the macro-elements structure.	76
5.3	Streamlines show laminar separation on the wing surface and trailing vortices. The critical points are marked on the surface.	78
5.4	Isosurface of wingtip vortex at the level of $\lambda_2 = -0.02$	79
5.5	<i>Left</i> : Three-dimensional perspective of the DNS-obtained steady laminar base flow over the wing at $Re_c = 1750$, $AoA = 5^\circ$. <i>Right</i> : Same for the flow obtained by inviscid point-vortex methods, vortex core lines being those obtained in the DNS, and inviscid instability developing in the wake at $x = 40$ (<i>right insert</i>)	80

5.6	Dependence of the normal, y_c , and lateral, z_c , spatial coordinates of the trailing vortex centroids on the streamwise coordinate, x . Symbols correspond to calculations using equation (5.4) applied to the DNS data and lines to calculations using equation (5.6) applied to data generated by the inviscid point-vortex method.	82
5.7	Dependence of the polar moments, a_y, a_z , and vortex radius, a , on the streamwise coordinate, x . Symbols correspond to calculations using equation (5.3) applied to the DNS data and lines to calculations using equation (5.7) applied to data generated by the inviscid point-vortex method.	82
5.8	Spatial amplification $-\alpha_i$ as a function of the wavenumber α_r for the two classes of boundary conditions, $\bullet$: symmetric; $\circ$: anti-symmetric, obtained by solution of the spatial BiGlobal EVP (2.45). Results of the spatial Orr-Sommerfeld equation are also shown. In all three sets of results the axial range analyzed is $x \in [6, 12]$. The most-/least-amplified results correspond to $x = 6$ and $x = 12$, respectively.	84
5.9	Amplitude functions of the perturbation velocity components $\hat{\mathbf{q}}_1(y, z)$ resulting from solution of (2.45) at $x = 10$ and imposition of symmetric boundary conditions at $\omega = 5.5$ (upper) and the antisymmetric boundary conditions at $\omega = 5$ (lower). In the background isolines of the streamwise component of the base flow velocity are shown	85
5.10	Kinetic energy of symmetric (left) and anti-symmetric (right) perturbations obtained with linear PSE-3D on the AR=1.7 wing as a function of the axial coordinate x for a range of amplified frequencies ω	86
5.11	Dependence on the frequency ω of spatial amplification rates, $-\alpha_i$ (spatial BiGlobal EVP – lines) and σ (PSE-3D – symbols), for both symmetric (s) and antisymmetric (a) boundary conditions. Upper to lower curves correspond to the locations $x = 6, 8, 10$ and 12 in both sets of results. PSE-3D computations have been initialized with results at $x = 6$	86
5.12	Evolution of the amplitude functions $\hat{u}_1(y, z)$ with axial distance. <i>Upper to lower:</i> $x = 6, 8, 10$ and 12 . <i>Left column:</i> symmetric perturbations at $\omega = 6$. <i>Right column:</i> antisymmetric perturbations at $\omega = 5$	87
5.13	Perspective view of isosurfaces of streamwise perturbation velocity of symmetric (<i>left</i>) and anti-symmetric (<i>right</i>) disturbances obtained by linear PSE-3D analysis, superposed upon the image of base flow vortex core positions.	88
5.14	<i>Left:</i> Comparison of the amplitude function $\hat{u}_0(y)$ of the leading eigenmodes delivered by the Orr-Sommerfeld equation at $x = 10$ with the profile extracted from the the leading two-dimensional eigenfunction $\hat{u}_1(x = 10, y, z = 0)$ obtained by solution of (2.45). <i>Right:</i> Instability in the wake, predicted by local analysis.	89

5.15	<i>Left:</i> Leading eigenmodes of local analysis in the neighborhood of the vortex. <i>Right:</i> Amplitude functions of the leading $m = 2$ eigenmode around the vortex core at $x = 10, \omega = 5.85; r = [(y - y_c)^2 + (z - z_c)^2]^{\frac{1}{2}}$	90
A.1	The grid used for computing the cylinder flow	98
A.2	The residual of drag for the circular cylinder flow at $Re = 40$	98
B.1	Convergency history of force in IBM at $Re = 200$ with $AoA = 18^\circ$. From left to right, top to down: mesh 1, 2, 3 and 4, respectively.	102
B.2	Convergence history of forces on the NACA 0015 at $Re = 200, AoA = 18^\circ$, using Nektar++ (left) and Nek5000 (right) at polynomial orders $p = 5, 7$ and 9.	103
C.1	Damping rates and Strouhal number of the least-damped modes of the NACA 0015 airfoil at $Re = 200, AoA = 18^\circ$, obtained by the nektar++ and the FreeFEM++ methodologies	105
C.2	Three meshes used to investigate the effect of <i>inflow</i> and <i>outflow</i> extents on transient growth: D1 (top), D2 (middle) and D3 (bottom).	106
C.3	Dependence of $G(\tau)$ (left) and the maximum singular value $\sigma_{\max}$ (right) on the extent of the analysis domain for the 0015 airfoil at $Re = 200, AoA = 18^\circ$	108
C.4	Comparison of μ dependence on β , as obtained in the present work by Semtex for the NACA 0012 airfoil at $Re = 500, AoA = 20^\circ$, against literature results [166]	109
C.5	<i>Left:</i> Eigenspectrum of the q -vortex at $Re = 1200, q = 0.8$ and $\omega = -2.0$. Full symbol denotes the most unstable mode of this family that corresponds to a eight lobes ($m = 8$) eigenmode. <i>Right:</i> The pertinent eigenfunction of this mode is plotted with contours (-0.9:0.1:0.9) of normalized real part of axial velocity amplitude function. Dashed lines correspond to negative values	111
C.6	Evolution of amplification rate for the non-parallel isolated vortex flow for the leading eigenmode comparing local analysis, PSE and PSE-3D results	112

List of Tables

1.1	Stall cells observed in Blunt body and airfoil flow experiments	3
2.1	Linear stability analysis methodologies	17
4.1	Boundary conditions for basic flow $(\bar{u}, \bar{v})^T$, direct perturbation $(\hat{u}, \hat{v}, \hat{w})^T$ and adjoint perturbation $(\hat{u}^+, \hat{v}^+, \hat{w}^+)^T$ velocity components. D: homogeneous Dirichlet; N: homogeneous Neumann; U: uniform inflow	39
4.2	Three different grid sizes for the <i>sharp</i> – <i>TE</i> airfoil. N_R is the number of radial points and N_C is the number of circumferential points.	41
4.3	Coefficients of aerodynamics in the flow of NACA 0015 with four <i>TE</i> at $Re=200$ and $AoA=18^\circ$, computed by Nektar++.	43
4.4	Unsteadiness as a function of Re and AoA for three NACA airfoils. S : steady, Us : unsteady base flow	44
4.5	Maximum recirculation, $ \frac{\bar{u}_{min}}{\bar{u}_{max}} \times 100 $ as a function of Re and AoA	44
4.6	Growth rate of modal analysis of NACA 0015 airfoil at $Re = 200$ and $AoA = 18^\circ$ with different trailing edge geometries.	47
4.7	Period of base flow and maximum Floquet multiplier, μ , and wavenumber, β , as function of Re and AoA for the three NACA airfoils; dash denotes that no unstable μ has been found at the respective parameters	67
4.8	Critical conditions for secondary instability of Modes A and B in the wake of the three airfoils. SW: short-wavelength, LW: long-wavelength	67
5.1	Convergence study of vorticity components at the location $(x, y, z) = (12, 0, -0.85)$, close to the vortex core, at $Re_c = 1750$	77
A.1	Drag coefficient C_d and the length of the wake bubble L at Reynolds number 40	99
B.1	Convergence history corresponding to the Immersed Boundary Method code	102
C.1	Convergence history of base flow spanwise vorticity $\bar{\omega}_z = -\partial \bar{u} / \partial y + \partial \bar{v} / \partial x$ at probe locations, from left to right column, $P1$ (3.0, 0.5), $P2$ (2.0, 0.5) and close to the center of the recirculation zone $P3$ (1.0, 0.0).	107

List of Tables

C.2	Effect of domain extent on the modal analysis for the basic flow at $Re = 220$, $AoA = 18^\circ$ and wavenumber $\beta = 1$	107
C.3	Temporal BiGlobal instability analysis of the q -vortex flow at $q = 0.475$, $Re = 100$ and $\beta = 0.418$. MP denotes the result of Mayer & Powell [118] . .	110
D.1	Coordinates defining the geometry of E-387	113

Chapter 1

Introduction

1.1 Background

One of the most important limitations of aerodynamic performance is stall, a massive separation bubble covering the suction surface of the wing or blunt body, which leads to reduction of lift and increase of drag, respectively. Besides the stall will affect the flight safety, the unsteady wake at high inclined position will exert fluctuating pressure on the wing, through which induce the wing vibration, even resonance and shorten the structure expectancy. The wingtip vortex washing down the flow around the wing will reduce the effect angle of attack and lift, further will affect the flight after the large airplane since it can keep high-strength and long-distance in down stream. These phenomena arise by the vortex shedding over and behind the wing which clarifies depend on the angle of attack and Reynolds numbers (Huang et al. [77]). The trailing edge and leading edge vortex shedding will emergence before stall, while the separation will dominate the flow over the suction surface sequential the post stall flow and the vortex coherent structure will interact each other which emanate from the leading and trailing edges at the very inclined angle to the incoming flow.

One interesting phenomenon is the massive coherent structures appearing in the separation flow at conditions close to stall, commonly known as *stall cells*, have been found on the suction side of wings surfaces when the wing is or around stall condition (Bippes and Turk 1980 [16], Winkelmann and Barlow 1980 [172]) at flight Reynolds numbers. More detailed investigations reveal that the stall cells have a pair of counter-rotating swirls in small aspect ratio (AR) wing, which emanate from the surface and travel into the wake (Yon and Katz 1998 [176]), or more counter-rotating swirls in large AR wings (Schewe 2001). Similar cellular structures associated with massive separation have been observed in flow around a circular cylinder, (Humphreys 1960 [79], Gölling 2001 [65]) and obstacles, such as automobile after windows (Délery [43]).

The onset of stall cells are usually accompany with large separation bubbles and vortex shedding immediately behind the lee side surface of wing. Table 1.1 shows the experimental results of stall cells in different blunt body and finite AR wing flow at high Reynolds number and varying angle of attack. The oil and turf wind tunnel experiments visualize schematically the streamlines closed to the surface pre- and post-stall. The mechanisms of vortex shedding correlation with stall cells involves the interaction of the boundary layer flow, free shear layer separation and wake flow, which is depend on the certain flow conditions (Williamson 1996 [171]).

The vortex shedding after the wing will interact with the separation bubble when the wing resolving in a situation at higher angle of attack. The separation flow affected by the

interaction with wingtip vortex will lead the flow becoming more complex. To examine the stability mechanisms of the wingtip vortex and investigate the trailing vortices formation, a steady laminar flow is established. Unlike the larger aspect ratio straight uniform wing, the elliptical platform wing has a improved stalling characteristics Anderson [6].

1.2 Massive Separation Airfoil Flow

Our present concern is with three-dimensional linear global instability mechanisms of spanwise-homogeneous steady and time-periodic laminar separated flows around three unswept airfoils placed at high angle of attack to the oncoming stream. The objective of the work is to provide a complete description of all modal and non-modal instability mechanisms which lead the laminar oncoming flow on these configurations to transition and turbulence.

The complete definition of the problem at hand requires specification of two geometrical parameters, namely the thickness and camber of the airfoil profile, as well as two additional independent flow parameters, namely the Reynolds number, Re , based on free-stream conditions and airfoil chord, and the angle of attack, AoA . Different flow instability mechanisms associated with the attached or separated state in which the flow is found at any one combination of these four parameters may be identified and possibly co-exist. Discussion in the literature mostly focuses on the classic Tollmien-Schlichting instability, active in the attached boundary layer on the airfoil, and on the Kelvin-Helmholtz instabilities in the shear layer associated with the relatively narrow leading- and trailing-edge laminar separation bubbles formed on the suction side of the airfoil. Both mechanisms are already present at low angles of attack, such that less is known presently about the physics of linear instability mechanisms once massive flow separation has set in. The present contribution aims at filling this gap by identifying all of the different linear instability mechanisms that lead separated laminar flows on airfoils to transition at conditions at which stall is approached.

Motivation for the present work is provided by the ongoing quest for theoretical description of experimental observations of the (relatively) low Reynolds number flow around an airfoil and in its wake at finite angles of attack. Understanding the underlying physical instability mechanisms can provide handles for theoretically-founded flow control via control of flow instabilities, an aspect that has recently become increasingly interesting in terms of description of airfoil performance in oscillatory motion [34]. Existing literature has dealt with flows at particular combinations of the above mentioned four parameters, which has resulted in fragmentary evidence being presented regarding the nature and physical origins of the linear instability mechanisms at play in separated flows around airfoils; this is briefly reviewed in what follows.

The concept of low- Re flow is somewhat ill-defined in the literature on flow around airfoils, since it is typically intended to describe flows at which laminar open or closed separation occurs. However, depending on the angle of attack, laminar separation can occur over a (chord-) Reynolds number range comprising up to five orders of magnitude. Consequently, flow features other than the existence of a separating laminar shear layer can be sufficiently different at different Reynolds numbers, which may limit the universality

Table 1.1: Stall cells observed in Blunt body and airfoil flow experiments

Geometry	Experiments	
Cylinders	$Re \approx 1.0 \times 10^5$ (Humphreys, 1960 [79])
	$Re = 4.1 \times 10^5$ (Gölling, 2001 [65])
	$Re = 1.5 \times 10^6, \alpha = 25^\circ$ (Fairlie, 1980 [8])
Airfoils	Clark Y, 14%, AR=3.5 $Re = 2.6 \times 10^5, \alpha = 22.8^\circ$ (Winkelmann & Barlow, 1980 [172])
	NACA4415, 15%, AR=2.33 $Re = 2.1 \times 10^6, \alpha = 22.5^\circ$ (Bippes & Turk, 1980 [16])
	NACA 0015, 15%, AR=6 $6.2 \times 10^5, \alpha = 17^\circ$ (Yon & Katz, 1998 [176])
	Growian, 27%, AR=4 $10^4 \leq Re \leq 10^7, \alpha = 12^\circ$ (Schewe, 2001 [145])
	NUTA, 18%, AR=2 $Re = 1.0 \times 10^6, \alpha = 9^\circ$ (Manolesos & voutsinas, 2013 [115])

1. Introduction

of physical instability scenarios identified at any given set of parameters. Out of the large body of work on the topic of separated flow around a two-dimensional airfoil, a selected number of theoretical works and recent experiments, representative of the current state of knowledge on the subject, are discussed first.

In an important theoretical contribution to the understanding of linear instability mechanisms in separated flows, Rist and Maucher [140] performed direct numerical simulations in a laminar separation bubble set up by an adverse pressure gradient in a flat plate boundary layer, and analysed profiles in the separated flow region using classic linear theory. These authors showed that two instability mechanisms exist in a separated shear layer. The first mechanism, denoted as *outer*, is associated with the outer portion of the shear layer that is unstable to inviscid Kelvin-Helmholtz modes due to the inflectional shape of the velocity profile. The second, *inner* mechanism, is driven by viscous instability in the reversed-flow region near the wall and may grow in strength and potentially lead to absolute instability, as was predicted in the earlier work of Hammond and Redekopp [68]. Alam and Sandham [4] performed direct numerical simulations of flow in a separation bubble and observed that the viscous instability is the dominant mechanism when the level of flow reversal exceeds approximately 15 – 20% of the local freestream velocity. While the previous analyses were performed on flat plates, more recently Jones et al. [87] performed direct numerical simulations to describe unsteadiness in a separation bubble formed on a NACA 0012 airfoil at $Re = 50 \times 10^3$, $AoA = 5^\circ$. At these conditions the flow transitions to (well resolved in the simulations) turbulence, and attention was paid to the potential of the time-averaged flow to sustain turbulence through an absolute instability of the separated region. No such mechanism was found but, by contrast, three-dimensional absolute instability was found in the braid region of vortices developing in the near-wake of the airfoil. Three-dimensional secondary instability in the wake of bluff bodies has been studied by means of Floquet theory by Barkley and Henderson [10] in the wake of bluff bodies. Brinkerhoff and Yaras [25] performed direct numerical simulations at conditions typical of low pressure turbines and monitored the growth of Tollmien-Schlichting waves in a boundary layer subjected to an adverse pressure gradient. They found that a viscous instability precedes the laminar separation zone and interacts with the inviscid instability that predominates in the latter region. No evidence of absolute instability was found, which was justified by the authors on account of the reverse flow level in their simulation being below than 8% of the free-stream value, in line with the predictions of Rist and Maucher [140].

Experimental work at low Reynolds numbers has been presented by a number of authors. As recently as the turn of the century Huang et al. [77], who studied flow around a NACA 0012 airfoil of aspect ratio (span/chord) five in the ranges of $0 \leq Re \leq 3 \times 10^3$ and $0^\circ \leq AoA \leq 90^\circ$, stated rather vaguely that:

The wake behind an airfoil usually consists of instability waves and coherent structures with periodic unsteady motions, depending on the Reynolds number and the angle of attack.

They went on to classify five flow regimes according to the nature of the shed vortices and used the critical point theory [109] as previously applied to bluff bodies by Perry et al. [135] to discuss the evolution of vortex shedding. It should be noted that no reference

to flow three-dimensionality is made, implying that the analysis presented holds for the spanwise averaged flow.

At Reynolds numbers up to one order of magnitude larger, Elimelech [51], Elimelech et al. [52] studied airfoils at $5 \times 10^3 \leq Re \leq 50 \times 10^3$, while Yarusevych et al. [174] performed experiments in a Reynolds number range $55 \times 10^3 \leq Re \leq 150 \times 10^3$. The latter authors made the distinction between phenomena occurring at Reynolds numbers up to $Re = 100 \times 10^3$, where open separation is observed, and $Re = 150 \times 10^3$, where a separation bubble regime is documented. Both sets of experiments dealt with low angles of attack, $AoA = 3^\circ$ and 5° , respectively, but addressed different cross-sectional profiles, namely the NACA 0009 and NACA 0025, respectively. Flow at $Re = 100 \times 10^3$ was later further investigated experimentally with respect to its linear instability by Boutilier and Yarusevych [22] albeit on an airfoil of yet another profile, namely the NACA 0018, at $AoA = 0^\circ(5^\circ)15^\circ$. Keeping in mind the disparity of the experimental conditions at which the above mentioned works were performed, their findings can be summarized as follows.

As Huang et al. [77] showed that, starting at $Re = 1200$, the frequency of the vortex shedding decreases with an increase of the Reynolds number, up to the highest value examined, $Re = 3000$. The shedding frequency was also found to decrease with an increase of the angle of attack. No attempt to analyze the flows studied by the then available linear theory approaches was attempted by these authors. Yarusevych et al. [174] documented coherent structures in the separated flow region and the wake of the airfoil in two flow regimes of open separation and separation bubble formation. The argued that roll-up of vortices in the separated shear layer could be predicted by inviscid classic linear theory, the results of which point to the existence of a Kelvin-Helmholtz instability. Boutilier and Yarusevych [22] further analyzed shear layer transition by application of both the Orr-Sommerfeld and the Rayleigh equations at selected profiles extracted from the flowfield at different chordwise locations within the examined range of $0^\circ \leq AoA \leq 15^\circ$ and showed that, within experimental uncertainty, the measured instability frequencies coincide with the predictions of either equation. This result supported the claim that disturbance development over the majority of the shear layer associated with the laminar separation as the AoA is increased is primarily governed by a linear inviscid Kelvin-Helmholtz mechanism.

In an analogous manner, Elimelech [51] and Elimelech et al. [52] show that classic linear stability analysis employing the Orr-Sommerfeld equation and analyzing experimentally measured velocity profiles over the suction side of the NACA 0009 wing, predicts fairly well both the most unstable disturbance modes and their growth rate. By contrast to earlier interpretations, though, this linear theory is found to fail in its prediction of the growth rates of low-frequency disturbances observed in the vicinity of the wing leading edge. The authors suggested that the last stage of the transition process may be governed by global linear flow instability mechanisms.

Instability analysis of a time-periodic wake behind a bluff body commenced with the celebrated works of Barkley and Henderson [10] and Henderson and Barkley [73] on the secondary instability in the wake of the circular cylinder. These works presented the appropriate theoretical framework, based on temporal Floquet theory, to analyze two-dimensional time-periodic flows that are homogeneous along the third spatial direction, and explained the experimentally observed Mode A and B structures in the wake of the cylinder [171]. Abdessemed et al. [2] applied Floquet theory as part of their analysis

of instability in the wake of a cascade of Low Pressure Turbine blades, while Tsiloufas et al. [166] were the first to perform three-dimensional temporal Floquet analysis of the time-periodic two-dimensional flow in the wake of a NACA 0012 airfoil at $AoA = 20^\circ$ and $400 \leq Re \leq 550$. The latter authors found that at those conditions, flow would become unstable and three-dimensional at two characteristic periodicity lengths, $L_{z1} = 2\pi/\beta_1$ and $L_{z2} = 2\pi/\beta_2$, corresponding to $\beta_1 \approx 2.5$ and $\beta_2 \approx 11$. These wavenumbers are the analogs on the stalled airfoil of those pertinent to the classic Modes A and B in the circular cylinder, although on the airfoil it is the large wavenumber / short wavelength that first becomes unstable. Brehm and Fasel [23] performed direct numerical simulations and global instability analysis of flow around the NACA 0015 airfoil at $AoA = 18^\circ$ and a range of Reynolds numbers $200 \leq Re \leq 10^4$ and asserted that the first linear instability mechanism to be encountered is associated with the KH mechanism discussed by Tsiloufas et al. [166], followed by linear amplification of a three-dimensional Floquet mode superposed upon the unsteady two-dimensional base flow behind the airfoil.

As the angle of attack is increased from small values and stall is approached, qualitatively different phenomena arise. In the last thirty years experiments have been performed at high Reynolds numbers using rectangular wings of finite-span ($\text{span/chord} \geq 1$) and have shown that for a small range in angle of attack, $17^\circ \leq AoA \leq 19^\circ$, depending on the free stream Reynolds number, a plateau in the lift coefficient C_l vs. AoA curve exists, followed by a sudden decrease in lift and a sudden change in the pitching moment. Concurrently with the lift plateau, a three-dimensionalization of the flow on the lee-side of the airfoil appears. This three-dimensionalization was first made evident by using oil-streak visualization: cellular patterns, resembling *owl-faces* or *mushrooms*, appeared and were repeated along the spanwise direction with a fixed periodicity length. Inside these structures, the separation line is broken periodically and the surface streamlines fold around focal points. These cellular patterns, which appear in a very narrow range of AoA around stall are commonly referred to as *Stall Cells*.

Bippes and Turk [16] used surface paint, pressure probe, hot-film and velocity measurements to document this phenomenon in wings of $1.55 \leq \text{span/chord} \leq 3.1$ and put forward for the first time the idea of the stall cells giving rise to a longitudinal vortex, whose axis is normal to the wing surface and terminates at focal point on the wing surface. They went on to measure low frequencies inside the separation zone, $O(10Hz)$ at the free-stream velocity and chord length used in their experiment, while outside the stall cells a flat spectrum was seen. Using tufts and pressure probes on the wing surface, Yon and Katz [176] observed an analogous behavior: while the flow field was relatively quiet outside the cells – strong fluctuations were absent – it was found to be highly unsteady inside the cellular patterns. The examination of the pressure spectrum also revealed the presence of two dominant frequencies, as had been previously found by Bippes and Turk [17]. The larger of these frequencies was associated by Yon and Katz [176] with the Kelvin-Helmholtz instability of the shear layer, while the smaller frequency was attributed to flapping of the separated layer. Most importantly, Yon and Katz [176] concluded that the oscillatory motions coexist, but are not causally related to the stall cells. Broeren and Bragg (2001), in their experiments at relatively low $Re = 300 \times 10^3$ and turbulence level $Tu < 0.1\%$, did not report any stall cell motion as such, but reported non-symmetrical separation in the spanwise direction.

A key finding regarding the potential association of stall cells with an instability mechanism was reported by Schewe [145], who found the number of cells to be a function of the model span, actually decreasing as the span of the model decreased, in agreement with the earlier results of Winkelmann and Barlow [172] and Yon and Katz [176]. Manolesos and Voutsinas [115] carried out experiments at three Reynolds numbers, $Re = 0.5 \times 10^6, 1.0 \times 10^6$ and 1.5×10^6 , and two rectangular wing aspect ratios, 1.5 and 2.0. They found that the prerequisite for exciting dynamic stall cells is that the spanwise flow conditions are uniform (fully tripped or fully un-tripped) and therefore open to self-excited perturbations. These authors studied the effects of Reynolds number and wing aspect ratio on the stall cells and, most importantly, found that the angle at which a stall cell is created does not depend on the aspect ratio, but was considered to be a profile characteristic, while the AoA at which stall cells are created decreases linearly with Re . More recently, Manolesos and Voutsinas [116] performed experiments and direct numerical simulations with an 18% thick cambered airfoil at $Re = 0.87 \times 10^6, 12^\circ \leq AoA \leq 16^\circ$ on a finite aspect-ratio wing of $1.6 \leq \text{span/chord} \leq 2$. Partial reconciliation of previously proposed models for the vortical systems found on the wing was offered by these authors, who documented the existence of both the counter-rotating pair of vortices defining the stall cells and originating normal to the airfoil surface, as well as a system of vortices having their axes parallel to the trailing edge of the airfoil, identified as the separation line vortex and the trailing edge line vortex.

While the understanding of the effect of Reynolds number and AoA on the stall cell formation is still incomplete, uncertainty also exists and conflicting evidence has been reported regarding the role of linear instability as regards the formation of the large-scale separation cells seen in the work of Elimelech [51] and Elimelech et al. [52]. Experiments were performed with the NACA 0009 airfoil in the ranges $10^4 \leq Re \leq 2 \times 10^4$ and $3.5^\circ \leq AoA \leq 5^\circ$. Despite the relatively low angle of attack, far from conditions of stall, cellular patterns reminiscent of the stall cells discussed by Yon and Katz [176] were also found on both this and the same-thickness Eppler-61 airfoil. These authors performed local inviscid linear analysis focusing on the separated shear layer [121], these obtained results are disagreed with the experimental observations. This led Elimelech et al. [52] to conclude that the cellular pattern observed was the result of the amplified three-dimensional stationary global mode discussed by Rodríguez and Theofilis [143] despite the very different ranges of parameters between the respective works.

Attempts to explain the origin of stall cells, on occasion using linear stability theory, commenced with the model of Winkelmann and Barlow [172], according to which a loop vortex system exists, composed of vortices of opposite-sign vorticity that run along the trailing edge of the airfoil. By contrast, Weihs and Katz [170] put forward the idea of a three-dimensional vortex ring emanating from the surface at the foci of the stall cells a conjecture later strengthened by Yon and Katz [176] who disputed the existence of a spanwise vortex and suggested that instead counter-rotating vortices start from the surface and extend downstream, aligned with the flow.

Dallmann and Schewe [40] were the first to conjecture the existence of a global instability mechanism in laminar separation bubbles that intrinsically – without external excitation – results in three-dimensionalization of the flow field. As they noted, the lack of an appropriate analysis methodology precluded this possibility from being studied at

that time. It was not until the work of Theofilis et al. [162] that the existence of the three-dimensional instability global mode of a model laminar separation bubble was demonstrated, using a partial-derivative-based linear instability eigenvalue problem, which for the first time broadened the scope of instability analyses in use at that time. However, the computational requirements of such analysis methodology were too restrictive to permit studying the same problem of global mode amplification on a laminar separation bubble on the complete airfoil. It was in the follow-up work a decade later [96], that both a parallel algorithm and the necessary hardware were available to be able to employ global theory to analyze linear instability in a model separation bubble on a flat-plate [142] and in a stalled airfoil [143]. In the latter case analysis was performed on the NACA0015 at $Re = 200$, $AoA = 18^\circ$ and delivered both KH-like and stationary three-dimensional perturbations. The latter class of linear perturbations was predicted to be unstable at the conditions examined by Rodríguez and Theofilis [143] and was used to reconstruct a flow field composed of linear superposition of the three-dimensional stationary mode upon the spatially-homogeneous steady two-dimensional flow. Wall-streamline patterns of this flow field were described by critical point theory and the patterns revealed were strongly reminiscent of the stall cells appearing on airfoils at conditions close to stall, despite the orders-of-magnitude different Reynolds numbers.

The three-dimensional direct numerical simulations and analysis of Taira and Colonius [153] focused on the question of the minimum aspect ratio necessary for stall cells to appear, and showed that the flow behind a flat plate of aspect ratio (AR) equal to 1 does not sustain these structures although wings of higher aspect ratio (AR=2-4) did show stall cells. This indicated that a minimum spanwise space (or, equivalently, in linear stability nomenclature, a lower bound in wave length) is needed, beyond which multiple stall cells can form. However, the relationship of the fully three-dimensional results of Taira and Colonius [153] and those obtained by imposing spanwise periodicity is yet to be explored in the literature.

The first global linear instability analysis of a laminar separation bubble embedded in an adverse pressure gradient flat-plate boundary layer was performed by Theofilis et al. [162]. It was demonstrated that two independent modal linear instability mechanisms co-exist: strong amplification of incoming disturbances, which can be identified as the known Kelvin-Helmholtz instability in the shear layer, and a previously unknown mechanism of self-excitation of the laminar separation bubble, which manifests itself as a stationary three-dimensional global eigenmode. These mechanisms manifest themselves as different members of the spectrum obtained by solution of the pertinent partial-derivative eigenvalue problem, in which the entire separation region is taken as the base flow, without need to resort to the (near-)parallel flow assumption made by earlier analyses of absolute instability [68] or the classic linear stability theory based on solutions of the Rayleigh or the Orr-Sommerfeld equation.

Soon after the global instability analysis of the laminar separation bubble on the flat plate, the first application of global linear instability analysis to flow around an airfoil was reported on the NACA0012 profile at $Re = 10^3$, $AoA = 5^\circ$ [161, 163]. The stationary three-dimensional global mode associated with laminar separation at the trailing edge of the airfoil was also identified in this configuration, albeit only damped three-dimensional global eigenmodes were found. However, encouraged by the capability of the analysis to

identify linear instability in the wake of the airfoil as a global eigenmode without resorting to the approximations of weakly-nonparallel flow used by classic linear theories based on the Rayleigh or the Orr-Sommerfeld equation, linear global instability analyses in a periodic cascade of T106/300 Low Pressure Turbine blades was performed by Abdessemed et al. [2]. Four possible classes of linear instability mechanisms were identified: the first manifests itself through two-dimensional unsteadiness of the wake, associated with the Kelvin-Helmholtz mode identified in the wake of the NACA0012 airfoil [161]. The second is a three-dimensional stationary amplified eigenmode of the steady two-dimensional flow, akin to that discussed by Theofilis et al. [162] in the flat plate. The third mechanism develops upon the two-dimensional time-periodic flow ensuing amplification of the Kelvin-Helmholtz mode in the wake. It takes the form of linearly unstable three-dimensional Floquet eigenmodes akin to those discovered in the wake of the circular cylinder by Barkley and Henderson [10]. However, the significance of all of these three modal (asymptotic/long-time) linear mechanisms was put in perspective by the fourth linear instability scenario identified in the same work of Abdessemed et al. [2], namely strong transient energy amplification of several orders of magnitude developing upon the steady laminar two-dimensional flow within short time-horizons. The latter finding gave rise to the subsequent demonstration of transient growth in the wake of the circular cylinder by Abdessemed et al. [1] and the further quantification of this phenomenon in the LPT passage in the work of Sharma et al. [148]. Finally, transient growth analysis of separated flow around two-dimensional airfoils has been reported recently, regarding trailing-edge separation on the NACA 0015 airfoil at $Re = O(10^2)$ at several stalling angles [64], and also regarding leading-edge separation bubbles on the NACA 0012 at $Re = O(10^4)$ and a low $AoA = 5^\circ$ [112].

The above discussion suggests that ample motivation exists to revisit the problem of instability of flow over airfoils at low-Reynolds number and high angles of attack. The criterion to define the range of low Reynolds numbers to be addressed is that the two-dimensional flows analyzed be exact stationary or time-periodic solutions of the equations of motion. The multi-parametric nature of the problem suggests that a relatively sparse discretization of the parameter space can be addressed around values of physical significance. The Reynolds number is chosen to be sufficiently low, such that, at a given angle of attack, steady laminar two-dimensional flow results around a given airfoil. The angle of attack is chosen at values around stall, when massive separation in the form of a closed recirculation bubble can be observed in the base flow, in the neighborhood of the trailing edge of the airfoil.

1.3 Wingtip Vortex Stability Investigations

Instability analysis of vortical flows has been at the center of Fluid Mechanics interests for decades, both from a fundamental and from an applied research point of view. Traditionally, two different paths are followed in the analysis. The first approach, collectively referred to as *point-vortex* methods, dates back to the classic works of Von Kármán Kármán [90, 91] and is based on the study of small-amplitude deviations of vortices from their undisturbed equilibrium position Batchelor [14], Leonard [102]. Crow [39] and Jiménez [86] introduced this approach to the inviscid analysis of instability in the counter- and co-

rotating pairs of vortices, respectively. Crouch [38] extended this methodology to address modal and non-modal linear instability of a four-vortex system and showed that transient growth is the dominant linear instability mechanism for this class of flows; results were later confirmed by Fabre and Jacquin [54] and others. In the second approach the viscous equations of motion are linearized about a nominally steady laminar base flow and the linear eigenvalue problem is solved either by neglecting viscosity [104] or at a given finite value of the flow Reynolds number [103, 118]. The limited number of exact solutions of the equations of motion corresponding to viscous vortices led the majority of the analyses available in the literature to monitor idealized vortex solutions, such as the classic Batchelor model [13] and its simplifications, or various models of isolated vortices [83]. In most analyses no axial flow is considered and, in addition, viscous diffusion of the vortex along the axial direction is neglected.

The discovery of long [39] and short-wavelength instabilities [105, 164] in systems of counter-rotating vortices and the association of these mechanisms with those observed in the trailing-vortex system of commercial transport aircraft in cruise provided a boost to efforts aiming at the destruction of the coherence of trailing vortex systems based on identification and exploitation of physics-based linear instability mechanisms. Linear global instability analysis of a counter-rotating viscous pair of vortices commenced with the work of Hein & Theofilis [72], who recovered the eigenmodes of a vortex system initialized using the so-called q -vortex simplification of the Batchelor vortex model. This work demonstrated that knowledge gained from instability analysis of isolated viscous vortices [59, 118] may be of limited use for the description of instability of vortical systems, except in the limit of the separation between the cores of the counter-rotating vortices becoming very large in terms of the wing span. The subsequent work of Jacquin *et al.* Jacquin *et al.* [83] showed that vortex-core instabilities pertinent to either of the inner or the outer pair of a four-vortex system may be recovered as distinct eigenmodes in BiGlobal linear instability analysis considering the entire vortex system as its base flow. González *et al.* [66] demonstrated that elliptic instability of a pair of counter-rotating viscous vortices with axial flow may be analyzed using BiGlobal linear analysis. The eigenfunctions obtained bore no resemblance to those pertinent to analysis of isolated vortices based on classic local linear theory [59], thus making global analysis the method of choice to address linear instability of systems of viscous vortices in close proximity.

Theoretical efforts to analyze the instability of models of closely-spaced viscous vortices continued with the work of Meunier & Leweke [120] on a co-rotating vortex pair, and that of Lacaze *et al.* Lacaze *et al.* [97] who employed direct numerical simulation to study instability of a Batchelor vortex subject to the far-field strain of a second vortex, the latter not explicitly considered in the analysis. Duck [47] solved the BiGlobal analogue of the Rayleigh equation pertinent to systems of two or four counter-rotating vortices and went on to describe their breakdown process. Interestingly, at conditions at which the base flows considered in the viscous analysis of Lacaze *et al.* [97] and the inviscid work of Duck [47] were closely related, the instability results of the two works were also qualitatively analogous, despite the large disparity in magnitude of axial flow considered in the respective works. In a similar spirit, Brion *et al.* [26], Tenders *et al.* [154, 155] and Benton & Bons [15] have demonstrated that point-vortex and viscous BiGlobal instability analyses of a pair of axially-homogeneous counter-rotating vortices deliver equivalent results, provided the Reynolds number and the spacing of the vortices are taken to be sufficiently high.

Global linear theory [160] has now advanced sufficiently in order to address instability in the wake of a three-dimensional wing without the need to resort to the severe approximations of classic (local) analysis. The assumption of homogeneity along the azimuthal spatial direction, which underpins local linear theory applied to analyze instability of isolated vortices, is no longer needed (or indeed appropriate) when dealing with a system of vortices emanating from a finite low aspect-ratio wing. Instead, BiGlobal linear theory can be used to fully resolve any distribution of vorticity on a given plane Oyz normal to the axial spatial direction, x , considering the latter direction as homogeneous. Application of viscous BiGlobal linear modal analysis to an isolated q -vortex [28, 66, 129] has been shown to deliver identical results with those of the local analysis [118]; this point is further addressed in what follows.

Applying global linear theory to the system of multiple vortices formed in the near-wake behind a real wing, a number of flow properties inaccessible to classic analysis based on inviscid methods for the computation of the base flow or linear stability analyses of one-dimensional profiles can be taken into account. First, the interaction of trailing vortices due to their proximity introduces a downwash velocity which needs to be taken into account. This was first accomplished by Tendero *et al.* [154], who showed that the eigenvalue spectrum obtained by BiGlobal analysis of a pair of descending counter-rotating vortices is displaced compared to that delivered by analysis of a frozen base flow. Viscous diffusion along the axial direction is also relevant for the base flow definition at all but the highest Reynolds numbers; this flow property was introduced in the analysis of an isolated non-parallel Batchelor vortex by Heaton *et al.* [70]. Of course, any global linear instability analysis of vortex systems behind realistic wings should also account for a non-negligible amount of axial velocity component, related with the flight velocity, as well as with flow turbulence.

The present work revisits the issue of linear global instability of a system of counter-rotating vortices by addressing, for the first time, instability of the entire trailing vortex system behind a three-dimensional finite aspect-ratio wing of elliptical planform, placed at zero- and a small finite angle of attack to the oncoming flow. The wing considered is based on the Eppler-387 airfoil and has an elliptical planform defined in the recent experiments of Xue *et al.* [173]. This airfoil was chosen on account of the absence of laminar separation at low Reynolds numbers and low angles of attack, an assumption which has only been verified in the $AoA = 0^\circ$ basic flow results obtained herein. The full roll-up process in the near-wake as well as the formation of the trailing vortex system are described by three-dimensional direct numerical simulation. In addition, point-vortex methods have been used to predict the evolution of the inviscid counterpart of the flow field obtained in the DNS, compare integral flow quantities obtained by either the viscous or the inviscid analysis at the same parameters and predict the long-time behavior of the inviscid flow in an efficient manner.

Subsequently, instability of the trailing vortex system is analyzed by solving the spatial BiGlobal linear viscous eigenvalue problem [41] at selected locations along the axial direction, to unravel the response of the steady laminar flow to small-amplitude convectively unstable perturbations of constant frequency introduced into the flow. Finally, the axial viscous diffusion and downwash of the base flow are simultaneously taken into account by a linear PSE-3D (also known as *plane-marching PSE* [7]) analysis of the entire

trailing-vortex system. The PSE-3D analysis is the three-dimensional extension of the parabolized stability equations [29, 41, 129] and is initialized at a given axial location by solution of the spatial BiGlobal eigenvalue problem. This provides a first approximation of the streamwise wavenumber of the linear perturbations; once converged, the linearized PSE-3D equations are marched along the axial spatial direction and the results obtained are compared with those previously delivered by solution of the spatial BiGlobal eigenvalue problem in an attempt to quantify the differences in flow stability characteristics predicted by the respective approaches.

An alternative experimental work on which the present analysis could have focused is that by Higuchi *et al.*[76]. These authors have provided a wealth of information on flow development on a elliptic wing of aspect ratio 3 constructed on the basis of the NACA 66₂ – 415 chord section. However, interpretation of those results suggests that several linear instability mechanisms are active at the Reynolds number $Re \approx 5 \times 10^4$ at which that experiment was performed, related to flow separation on the wing and the subsequent roll-up of the vortex sheet into laminar transitional or turbulent vortices. Consequently, it was decided to first perform analyses at the low Reynolds numbers at which the Xue *et al.*[173] experiments are performed, identify the linear perturbations that lead steady laminar flow to unsteadiness and systematically advance to instability analyses at higher Reynolds numbers at a later stage.

1.4 Scope of the Thesis

The present thesis is organized in the following chapters containing:

- Chapter 2: This chapter introduces the derivations of the fundamental linear instability theories. Classical one-dimensional local stability established from parallel flow, as well as BiGlobal, PSE and TriGlobal stability theories are reviewed.
- Chapter 3: This chapter introduces the critical point theory, a tool can be used to depict the flow patterns qualitatively. The type of critical points is classified by the characteristics of eigenvalues of the flow on the surface of the objectives. In the three-dimensional flow, the parameters to evaluate the vortex structures quantitatively by Q - and λ_2 -criteria are introduced, and an example of hemisphere-cylinder flow is given here.
- Chapter 4: One of the main constructs of this thesis focuses on analyzing the modal and non-modal instability for the steady two- and three-dimensional and periodic flow past NACA airfoils as functions of angle of attack, Reynolds number and geometries. Steady flows are analyzed with respect to their potential to sustain modal and non-modal instabilities first. The relative effect of airfoil thickness on primary instability analysis results are examined by monitoring the symmetric NACA 0009 and NACA 0015 profiles, while the effect of curvature are assessed by comparing results on the latter airfoil by those obtained flow around the same-thickness cambered NACA 4415 profile. Subsequently, time-periodic flows ensuing amplification of the two-dimensional global mode, as either or both of the Reynolds number and

angle of attack parameters are increased, are analyzed by Floquet theory. The existence and nature of spanwise periodic secondary instabilities akin to the well-known Modes A, B and quasi-periodic (QP) in the wake of the circular cylinder is discussed. Subsequently, a link between the large-scale separation structures and linear flow instabilities is sought, be it at small angles of attack, or at conditions near and above stall. The base flows analyzed are discussed briefly in Section 4.1. Primary modal linear global instability analysis results are reported in Section 4.2, followed by non-modal linear analysis results presented in Section 4.4. Results of the Floquet analysis and the generation of stall cells is discussed are reported in Section 4.5.

- Chapter 5: Instability analysis in the wake of a laminar finite elliptical wing flow is performed in this chapter. The spatial BiGlobal, inviscid point vortex and PSE-3D methodologies are adopted to analyze the wingtip vortex and trailing vortex systems. Basing on the theoretical background analysis presented in Chapter 2, the three-dimensional wingtip vortex flow is defined in 5.1.1, followed in section 5.1.2 by a brief introduction of the direct numerical simulation work employed to compute the base state analyzed and the point vortex method used to predict the inviscid evolution of this flow. A discussion follows of the global instability analysis concepts employed, spatial BiGlobal linear modal analysis in 2.3.2 and the PSE-3D concept in 2.4. Results of these global instability analysis methodologies are presented in section 5.1.2 and are compared with results obtained by classic linear stability theory at appropriate portions of the wake in Section 5.3.

1. Introduction

Most of the material in the present thesis, along with additional specific details, has been presented in the following list of publications:

- Journal Articles

- Wei He, Juan Ángel Tendero, Pedro Paredes, Vassilis Theofilis 2016 “Linear stability in the wake of an elliptic wing”, *Theoretical and Computational Fluid Dynamics*, Published online, DOI: 10.1007/s00162-016-0400-2
- W. He, R. S. Gioria, J. M. Pérez, V. Theofilis 2016 “Linear instability of low-Re massively separated flow around three NACA airfoils” *Journal of Fluid Mechanics* (revised)

- Book Chapters

- R. S. Gioria, W. He, V. Theofilis 2015 “On Global Linear Instability Mechanisms of Flow Around Airfoils at Low Reynolds Number and High Angle of Attack” In: *Procedia IUTAM* (ed. M. A. Medeiros & J. R. Meneghini), Vol. 14, pp 88-95, Río de Janeiro, Brazil: Elsevier.
- W. He, F. Gómez, D. Rodríguez, V. Theofilis 2015 “Effect of the Trailing Edge Geometry on the Unsteadiness of the Flow Around a Stalled NACA 0015 Airfoil” In: *Instability and Control of Massively Separated Flows* (ed. V. Theofilis & J. Soria). Fluid Mechanics and Its Applications Series, Vol. 107, pp 45-50, Prato, Italy: Springer.

- Conference Papers

- R. S. Gioria, W. He, J. M. Pérez, V. Theofilis 2016 “Modal and non-modal global instability analyses of low-Re massively separated flow around a NACA 0015 airfoil” *46th AIAA Fluid Dynamics Conference, Washington D. C., USA, 13-17 June, 2016. AIAA paper 2016-3778.*
- C. Zhang, C. Rossi, W. He, J. Corado 2016 “Virtual-work-based Optimization Design on Compliant Transmission Mechanism for Flapping-wing Aerial Vehicles” *International Conference on Manipulation, Automation and Robotics at Small Scales, Paris France, 18-22 July, 2016*
- S. Le Clainche, M. Schlapkohl, V. Theofilis, W. He, J. A. Tendero, Q. Liu, J.M. Pérez 2014 “Wind Tunnel Experiments to Teach Physics” *6th International Conference on Education and New Learning Technologies, Barcelona, Spain, 7-9 July 2014*
- F. Gómez, V. Theofilis, P. Paredes, Q. Liu, W. He 2012 “On the Role of Global Flow Instability Analysis in Closed Loop Flow Control” *6th AIAA Flow Control Conference, New Orleans, LA, USA, 25-28 June 2012. AIAA paper 2012-2679.*

Chapter 2

Hydrodynamic Instability

Hydrodynamic stability has been recognized as one of the central problems of fluid mechanics. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence [46]. In 1883, Reynolds [139] carried out the classic experiment of flow in a pipe. He found that when the velocities were sufficient low, the flow is *laminar*. As the velocity was increased, firstly, there arose *transition* phenomenon and the flow was not stable anymore, then after the velocity exceed a critical value, the flow became *turbulent*. The initial state of the natural transition process is known as the *receptivity* (Morkovin [123]) and consists of the transformation of environmental disturbances (such like acoustic and turbulence) into small perturbations within the boundary layer. The mechanisms of these disturbances are varied with the enviroment, *e.g.* freestream sound, turbulence interacting with surface curvature, shape discontinuities, surface roughness and suchlike. These initial conditions are small, often immeasurable perturbations to the basic state flow. When the environmental disturbances are much strong, which will directly active the transition, usually called *bypass*. Hereafter the receptivity stage, the growth of these disturbances depends on the nature of the disturbance and the nature of the basic state. Acoustic disturbances tend to influence the growth of two-dimensional instabilities such as Tollmien-Schlichting (T-S) waves, while vortical disturbances tend to lead to the growth of three-dimensional phenomena such as the cross flow instability documented by Saric et al. [144].

The theoretical work should be back to Helmholtz [168], Kelvin [93] and Rayleigh[137, 138] on the incompressible inviscid flow. Rayleigh pointed out that in the non-viscous rotational flow, its stable or not depends on the square of the circulation whether increasing monotonically outwards. And if the parallel flow is unstable, there must be a reflection point in the velocity distribution. Stimulated by Reynolds's experiment, Orr [124, 125]

Figure 2.1: Sketch of the laminar-to-turbulence transition in a boundary layer. Taken from [89].

2. Hydrodynamic Instability

and Sommerfeld [150] independently analyzed the small and traveling wave disturbances flow and derived the Orr-Sommerfeld equation, which has become the essential basis in the theory of hydrodynamic stability. Many authors have been studying the stability of plane Poiseuille flow with respect to two-dimensional disturbances. Heisenberg was the first to conclude the instability of the flow at sufficiently high Reynolds numbers, but he did not arrive at a critical value beyond which instability begins. Tollmien [165] calculated the first neutral eigenvalues for Poiseuille flow and gave a critical value at certain Reynolds number.

Early prediction of solving the Orr-Sommerfeld equation for the plane Couette flow is stability for all Reynolds number. Which was apparently disagree with the experiment results. The reason is there exists viscosity in the real flow. Lin [31, 110] solved the conflict between prediction and observation and obtained the analytical solution of stability flow. Meksyn and Stuart [119], Benney [82] and Eckhaus [49] were all contributors to the theory of weakly nonlinear stability [37]. Recently, Theofilis [158–160] presents a detail description about the global linear stability analysis and its applications along the last two decades..

Numerous experiments in recent decades have revealed that the extent of the amplification region, and hence the location of the transition point on the body surface, is strongly dependent not only upon the amplitude or the spectrum of external disturbances but also on their physical nature. Some of the disturbances easily penetrate into the boundary layer whilst others do not. Consequently, the concept of boundary layer transition is a complex one and still lacks a complete theoretical exposition. If the initial, environmentally-generated disturbance is small enough, the next stage of the transition process is that of primary mode growth. In this stage, the initial disturbances grow or decay in a manner described by linear stability theory [114]. The specific instabilities that are exhibited in reality depend on the geometry of the problem and the nature and amplitude of initial disturbances. Across a range of Reynolds numbers in a given flow configuration, the most amplified modes can and often do vary.

There are several major types of instability which commonly occur in boundary layers (see Figure 2.1). In subsonic and early supersonic flows, the dominant two-dimensional instabilities are T-S waves. For flows in which a three-dimensional boundary layer develops such as a swept wing, the cross flow instability becomes important. For flows navigating concave surface curvature, Görtler vortices may become the dominant instability. Each instability has its own physical origins and its own set of control strategies, some of which are contraindicated by other instabilities adding to the difficulty in controlling laminar-turbulent transition.

The primary modes themselves do not actually lead directly to breakdown, but instead lead to the formation of secondary instability mechanisms. As the primary modes grow and distort the mean flow, they begin to exhibit nonlinearities and linear theory no longer applies. Complicating the matter is the growing distortion of the mean flow, which can lead to inflection points in the velocity profile a situation shown by Lord Rayleigh to indicate absolute instability in a boundary layer. These secondary instabilities behaving much higher frequency than their linear precursors will lead the flow rapidly to breakdown. In the case of non-orthogonal modes, these will arise a strong energy amplified in a short time during the evolution of perturbation, and will be force the flow into transition or bypass. This phenomenon usually called *transient growth*. One of application of linear

Table 2.1: Linear stability analysis methodologies

	Basic state	Perturbation
TriGlobal	$\bar{q}(x,y,z)$	$\hat{q}(x,y,z)e^{-i\omega t}$
PSE-3D	$\bar{q}(x^*,y,z)$	$\hat{q}(x,y,z)e^{\int_{x_0}^x \alpha(\xi)d\xi - \omega t}$
Floquet-2D	$\bar{q}(x,y,t)=\bar{q}(x,y,t+T)$	$e^{\sigma t} \sum_{n=-N}^N \hat{q}(x,y)e^{i\delta t}, \delta = 2\pi/T$
BiGlobal	$\bar{q}(x,y)$	$\hat{q}(x,y)e^{i(\beta z - \omega t)}$
PSE	$\bar{q}(x^*,y)$	$\hat{q}(x,y)e^{\int_{x_0}^x \alpha(\xi)d\xi + \beta z - \omega t}$
Floquet-1D	$\bar{q}(x,t)=\bar{q}(x,t+T)$	$e^{\sigma t} \sum_{n=-N}^N \hat{q}(x)e^{i\delta t}, \delta = 2\pi/T$
OSE	$\bar{q}(y)$	$\hat{q}(y)e^{i(\alpha x + \beta z - \omega t)}$

stability theory (LST) is analyzing and predicting the onset of instability, of which contains varied analysis methodologies as addressed in Table 2.1. The expressions of base flows and perturbations are depended on the very flow stability properties, which are reviewed in the next sections.

2.1 Linearized Perturbation Equations

Before analyzing the stability of any laminar flow one must find the velocity and pressure at each point and time. Considering an incompressible viscous flow without external forces in a given domain. Then flow is governed by the dimensional Navier-Stokes equations,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\frac{1}{\rho^*} \nabla p^* + \nu \Delta \mathbf{u}^*, \quad (2.1)$$

and

$$\nabla \cdot \mathbf{u}^* = 0. \quad (2.2)$$

where $\mathbf{u}^*$ and p^* are the velocity vector and pressure of the fluid, respectively, and ρ^* is the density and ν is the kinematic viscosity of the fluid. For a given boundary conditions and initial values, in the small Reynolds number flow, the above equations will describe a steady base flow. Let we choose characteristic length scale as L and characteristic velocity scale V , define dimensionless variables such as $\mathbf{x} = \mathbf{x}^*/L$, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^*/V$, $p(\mathbf{x}, t) = p^*/\rho^*V^2$, $t = Vt^*/L$, and Reynolds number $Re = VL/\nu$, and then the momentum equation 2.1 and continue equation 2.2 will become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \quad (2.3)$$

and the equation of continuity,

$$\nabla \cdot \mathbf{u} = 0 \quad (2.4)$$

Following the theory of *Liapounov*, the base flow is *stable* if, for any $\varepsilon > 0$, there exists a positive number δ such that if

$$\|\mathbf{q}(\mathbf{x}, 0) - \tilde{\mathbf{q}}(\mathbf{x}, 0)\| < \delta, \quad (2.5)$$

2. Hydrodynamic Instability

then, for all $t \geq 0$,

$$\|\mathbf{q}(\mathbf{x}, t) - \tilde{\mathbf{q}}(\mathbf{x}, t)\| < \varepsilon, \quad (2.6)$$

where $\mathbf{q} = \mathbf{q}(\mathbf{x}, t) = (\mathbf{u}, p)^T$ are the variables of the flow field and its corresponded reference state are $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}(\mathbf{x}, t) = (\tilde{\mathbf{u}}, \tilde{p})^T$.

Assuming $\mathbf{q}(\mathbf{x}, t) = \bar{\mathbf{u}}, \bar{p}$ define the *base flow* (or basic flow) field, and $\mathbf{q}'(\mathbf{u}, p)$ are the perturbation quantities, then $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ and $p = \bar{p} + p'$. Substituting the previous decomposition into equation (2.3) and (2.4), then obstructing the equations corresponding to the base flow and neglecting the terms multiplied by powers of ε higher than order $O(1)$. Then the *linearized Navier-Stokes equations* (LNSE) are obtained,

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}' \cdot \nabla \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \mathbf{u}' = -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}' \quad (2.7)$$

and

$$\nabla \cdot \mathbf{u}' = 0 \quad (2.8)$$

This is a set of four partial differential equations in four unknown functions (u, v, w, p) , each defined in three spatial dimensions x, y, z , and time dimension t . If the base flow is steady, the amplitude function of the perturbations are independent of t , and its solution could be considered has a form of $\hat{u}(x, y, z)e^{-i\omega t}$, where $\hat{u}$ is the its amplitude function, and ω is the complex eigenvalues. The real part is the frequency of the perturbation and the imaginary part is its growth or damping rate.

2.2 Local Linear Instability

Local linear stability theory (LST), based on variants of the Navier-Stokes equations linearized around one-dimensional planar or axisymmetric profiles has been employed to identify instabilities at appropriately chosen locations in the wake. The Ansatz for the local linear perturbations, $\mathbf{q}'$, is

$$\mathbf{q}'(x_1, x_2, x_3, t) = \hat{\mathbf{q}}(x_2)e^{i(\alpha x_1 + \beta x_3 - \omega t)} + c.c., \quad (2.9)$$

with $\hat{\mathbf{q}} = (\hat{u}, \hat{v}, \hat{w}, \hat{p})^T$ representing the vector of one-dimensional complex amplitude functions and *c.c.* denoting complex conjugation, such that the left-hand-side of equation (2.9) be real. In a Cartesian coordinate system the coordinates (x_1, x_2, x_3) are respectively identified as the axial, normal and lateral spatial directions, (x, y, z) , while in cylindrical coordinates they represent the axial, radial and azimuthal directions, (x, r, θ) .

2.2.1 Orr-Sommerfel and Squire Equations

Now assuming that for the parallel viscous flow, it can be decomposed into a steady laminar base flow $\bar{u}$ only depending on normal direction y and a disturbance part u' and p' are fluctuating around the base flow. Then we have

$$\begin{aligned} u &= \bar{u}(y) + u', \\ \bar{v} &= \bar{w} = 0, \\ p &= \bar{p}(x) + p' \end{aligned} \quad (2.10)$$

with boundary conditions $u' = v' = w' = 0$ on the wall. Substituting above equations into equation (2.7) and (2.8), we have

$$\begin{aligned}\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + v' \frac{d\bar{u}}{dy} &= -\frac{\partial p'}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} \right), \\ \frac{\partial v'}{\partial t} + \bar{u} \frac{\partial v'}{\partial x} &= -\frac{\partial p'}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2} \right), \\ \frac{\partial w'}{\partial t} + \bar{u} \frac{\partial w'}{\partial x} &= -\frac{\partial p'}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} \right), \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} &= 0.\end{aligned}\tag{2.11}$$

Since the variables x, z, t are periodic, the system has an exponential function in x, z and t as well. The form of an expression of disturbance are

$$\begin{aligned}u' &= \hat{u}(y)e^{i(\alpha x + \beta z) - i\omega t}, \\ v' &= \hat{v}(y)e^{i(\alpha x + \beta z) - i\omega t}, \\ w' &= \hat{w}(y)e^{i(\alpha x + \beta z) - i\omega t}, \\ p' &= \hat{p}(y)e^{i(\alpha x + \beta z) - i\omega t},\end{aligned}\tag{2.12}$$

where α, β are real wave numbers. Setting a complex number $c = \omega/\alpha$ presents the wave velocity, and these variables should satisfy the perturbation equations. After substituting them into equation (2.11), we have

$$\begin{aligned}\{D^2 - (\alpha^2 + \beta^2) - i\alpha Re(\bar{u} - c)\} \hat{u} &= Re(D\bar{u})\hat{v} + i\alpha Re\hat{p}, \\ \{D^2 - (\alpha^2 + \beta^2) - i\alpha Re(\bar{u} - c)\} \hat{v} &= ReD\hat{p}, \\ \{D^2 - (\alpha^2 + \beta^2) - i\alpha Re(\bar{u} - c)\} \hat{w} &= i\beta Re\hat{p}, \\ i(\alpha\hat{u} + \beta\hat{w}) + D\hat{v} &= 0.\end{aligned}\tag{2.13}$$

where the symbol D stands for the differentiation d/dy . If the imaginary part of c is positive, then the solution will increase exponentially with time, and the flow is unstable. If it is negative, the perturbation will be damped out with time, and the flow becomes steady. If c is real, there is a possible neutral oscillation. In the two-dimensional disturbances flow, $\beta = 0$, then the system can be conveniently reduced to a single differential equation for $\hat{v}(y)$

$$(D^2 - \alpha^2)^2 \hat{v} = i\alpha Re\{(\bar{u} - c)(D^2 - \alpha^2) - D^2 \bar{u}\} \hat{v}\tag{2.14}$$

The equation (2.14) is often referred to as the *Orr-Sommerfeld equation* ([124], [125], [150]). Here we introduce the vorticity variable η to expand the above equation into three-dimensional flow field, define the normal vorticity as

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\tag{2.15}$$

Through equation (2.11), we have the Squire equation [151]

$$\frac{\partial \eta}{\partial t} + \bar{u} \frac{\partial \eta}{\partial x} - \frac{1}{Re} \nabla^2 \eta = -\frac{d\bar{u}}{dy} \frac{\partial v}{\partial z}\tag{2.16}$$

2. Hydrodynamic Instability

Similarly, the vorticity disturbance has the expression of

$$\eta' = \hat{\eta}(y)e^{i(\alpha x + \beta z) - i\alpha ct} \quad (2.17)$$

Combining (2.16) and (2.17), the format of Squire disturbed equation comes to be

$$\{D^2 - (\alpha^2 + \beta^2) - i\alpha Re(\bar{u} - c)\} \hat{\eta} = i\beta Re D\bar{u}\hat{v} \quad (2.18)$$

with the boundary conditions $\hat{v} = D\hat{v} = \hat{\eta} = 0$ at fixing wall and in the far field.

The above system can be defined by introducing a linear operator $\mathcal{L} = \mathcal{L}(\bar{\mathbf{q}}, \alpha, \beta)$ and a vector $\hat{\mathbf{q}} = (\hat{v}, \hat{\eta})^T$, then the linearized Navier-Stokes equations 2.7 transformed into

$$\mathcal{L}\hat{\mathbf{q}} = -i\omega\mathcal{M}\hat{\mathbf{q}} \quad (2.19)$$

where

$$\mathcal{L} = \begin{bmatrix} (D^2 - \alpha^2)^2 - i\alpha Re\{\bar{u}(D^2 - \alpha^2) - D^2\bar{u}\} & 0 \\ -i\beta Re D\bar{u} & D^2 - (\alpha^2 + \beta^2) - i\alpha Re\bar{u} \end{bmatrix} \quad (2.20)$$

and

$$\mathcal{M} = \begin{bmatrix} Re(D^2 - \alpha^2) \\ Re \end{bmatrix} \quad (2.21)$$

Thus lead to an *eigenvalue problem* (EVP) which dispersion relation is

$$\mathcal{F}(\alpha, \beta, \omega, Re) = 0 \quad (2.22)$$

Equation (2.19) is deduced into the Rayleigh equation [138], for invicid flow. Three-dimensional disturbances parallel flow can be transformed into the corresponding two-dimensional disturbances parallel flow (see Squire's theorem [151]) by taking

$$\begin{aligned} \tilde{\alpha}\tilde{u} &= \alpha\hat{u} + \beta\hat{w}, \quad \tilde{v} = \hat{v}, \quad \tilde{p}\tilde{Re} = pRe, \\ \tilde{\alpha}\tilde{Re} &= \alpha Re, \quad \tilde{c} = c, \quad \tilde{\alpha}^2 = \alpha^2 + \beta^2 \end{aligned} \quad (2.23)$$

into equation 2.13, then we can get

$$\begin{aligned} \left\{ D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{Re}(\bar{u} - \tilde{c}) \right\} \tilde{u} &= \tilde{Re}(D\bar{u})\tilde{v} + i\tilde{\alpha}\tilde{Re}\tilde{p}, \\ \left\{ D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{Re}(\bar{u} - \tilde{c}) \right\} \tilde{v} &= \tilde{Re}D\tilde{p}, \\ i\tilde{\alpha}\tilde{u} + D\tilde{v} &= 0. \end{aligned} \quad (2.24)$$

From equation (2.23) and (2.24), each three-dimensional problem can be reduced to the form of two-dimensional flow. Because of $\tilde{\alpha} > \alpha$, then $\tilde{Re} < Re$, which means that to an unsteady three-dimensional disturbance there corresponds a more unstable two-dimensional disturbance, and the minimum critical Reynolds number is given directly by the two-dimensional analysis. It should be noticed that Squire's theory does not valid for three-dimensional perturbations in a compressible flow because of temperature and velocity behave in non-identical manner during the transformation [48]. Furthermore,

the development of weak-nonlinear theories claim to take into account the dispersion characteristics of three-dimensional modes.

For a given dispersion relation, the above Ansatz classified as *temporal* stability analysis with searching the complex ω values corresponding to a real wave number $\alpha = 2\pi/\lambda$, where λ is the streamwise wavelength. The wave-like perturbations spread with *phase velocity* c valued as wave crest velocity in space and time. At mean time, $c_g = d\omega/d\alpha$ defined the *group velocity* of the wave envelope. At the critical value Re_c , the ω_i is tangent to the real wave number α , which leads $c_g = 0$. If $\omega_i > 0$, the perturbations will growth with time and traveling up- and down-stream from the location it periodically introduced. This kind of unstable case recognized as *absolute instability* [78]. While if the perturbation grows in space but traveling down-stream, it is the *convective instability* (see Figure 2.2). On the contrary, the perturbations decay in time when $\omega_i < 0$. The characteristic of absolute unstable flow responds the mode grows in space with zero group velocity and damps the other modes.

Figure 2.2: Sketch of absolute (left) and convective (right) stability.

2.2.2 Spatial Orr-Sommerfeld equation

Considering the convenience to experimentalists measuring the perturbation straightforward when dealing with monochromatic frequency wave. In the spatial framework employed, ω is taken to be a real angular frequency parameter, two-dimensional perturbations ($\beta = 0$) are assumed and the complex eigenvalue, α , and the associated eigenvectors, $\hat{\mathbf{q}}$, are sought. The equation (2.9) can be expressed as

$$\mathbf{q}'(x, y, z, t) = \hat{\mathbf{q}}(y)e^{i(\alpha x - \omega t)} + c.c., \quad (2.25)$$

The real part of the eigenvalue, α_r , is related with the wavenumber of the eigenmode along the streamwise direction, while its imaginary part, α_i , is the growth/damping rate of the perturbation.

Substituting (2.25) into the Navier-Stokes equations and linearizing leads to the spatial Orr-Sommerfeld equation and to the system solved by Mayer and Powell [118], in Cartesian and axisymmetric coordinates, respectively. Through integrating α_i along the streamwise

direction, we can obtain

$$N = \ln \frac{A}{A_0} = - \int_{x_0}^x \alpha_i(x) dx \quad (2.26)$$

which is called *N-factor* [149, 167]. A is the perturbation amplitude and A_0 is the initial perturbation where starts to grow. This semi-empirical method computes a group of frequencies whose amplitude will reach the preset value, and plot the envelope of e^N to estimate the location of transition. Since this method only concerns the perturbation amplification without the initial amplitude, on which depends the environment, such as, wall roughness, noise amplitude and etc., it need experimental results and include receptivity analysis to give a reliable transition location [88].

2.2.3 Local PSE

Assume a flow is weakly nonparallel in one spatial direction x , and the velocity growth in this direction is much less than in the other two directions, then the Ansatz (2.9) can be rewritten as listed in Table 2.1

$$\mathbf{q}'(x, y, z, t) = \hat{\mathbf{q}}(x, y) e^{i(\int_{x_0}^x \alpha(\xi) d\xi + \beta z - \omega t)} + c.c., \quad (2.27)$$

where $\hat{\mathbf{q}}(x, y)$ and $\alpha(x)$ scale with Re^{-1} . This local stability analysis, named as local Parabolized Stability Equations (PSE), is typically used to solve the 1D eigenvalue problems. Tollmien [165] succeeded in solving the Blasius boundary layer flow using this methodology.

2.2.4 Gaster Relation

The relation between temporal growth rate and spatial growth rate in the vicinity of neutral curve are manifested by Gaster [60] relation. For the temporal mode

$$\alpha = \alpha_r^{(T)}, \quad \omega = \omega_r^{(T)} + i\omega_i^{(T)} \quad (2.28)$$

and for the spatial case

$$\alpha = \alpha_r^{(S)} + i\alpha_i^{(S)}, \quad \omega = \omega_r^{(S)} \quad (2.29)$$

Considering a neutral normal mode (α_c, ω_c) such that $\alpha_i = \omega_i = 0$. Expanding ω in a Taylor series of α closed to the neutral curve, after dropping the higher order terms, we have

$$\omega = \omega_c + \frac{\partial \omega}{\partial \alpha} (\alpha - \alpha_c) \quad (2.30)$$

where assuming ω is a well-behaved function of $\omega(\omega, Re)$. According to the Cauchy-Riemann relation, the imaginary part of ω is

$$\omega_i = \frac{\partial \omega_i}{\partial \alpha_r} (\alpha_r - \alpha_c) + \frac{\partial \omega_r}{\partial \alpha_r} \alpha_i \quad (2.31)$$

From equations (2.28) and (2.31), we have

$$\omega_i^{(T)} = \frac{\partial \omega_i}{\partial \alpha_r} (\alpha_r - \alpha_c) \quad (2.32)$$

and substitute (2.32) into (2.31) leading to a transformation that from temporal growth to spatial growth at the given α_r :

$$\omega_i^{(T)} = -\frac{\partial \omega_r}{\partial \alpha_r} \alpha_i^{(S)}, \quad \omega_r^{(T)} = \omega_r^{(S)} \quad (2.33)$$

The Gaster relation dictates that in the weak disturbances from neutral curve, the temporal growth rate $\omega_i^{(T)}$ and the spatial growth rate $\alpha_i^{(S)}$ are related by the group velocity $c_g = \partial \omega_r / \partial \alpha_r$. Conversely, basing on the reference frame of group velocity, the spatial advected perturbation quantitatively equals to the temporal growth rate.

2.2.5 Secondary instability

For the secondary instability analysis work the flow field is decomposed into a time-periodic two-dimensional base flow $\bar{\mathbf{q}}(x, y, t) = (\bar{\mathbf{u}}, \bar{p})^T$, which satisfies the two-dimensional incompressible equations of motion, and small-amplitude three-dimensional unsteady perturbations $\hat{\mathbf{q}}(x, y, z, t) = (\hat{\mathbf{u}}, \hat{p})^T$ superposed at small amplitude upon the base flow. The latter are governed by the linearized Navier-Stokes equations (2.7). Time-periodicity of the base flow manifests itself in the linear primary amplification of the $\beta = 0$ eigenmode - solution of (2.43).

Secondary instability of the time-periodic base flow $\bar{\mathbf{q}}(\mathbf{x}, t) = \bar{\mathbf{q}}(\mathbf{x}, t + T)$ is analyzed using temporal Floquet theory, for which equation (2.7) still holds, but now the operator $\mathcal{L}(\hat{\mathbf{u}})$ is T -periodic. Spanwise periodicity permits expanding the perturbation using a Fourier decomposition

$$\hat{\mathbf{u}}(\mathbf{x}, t) = \int_{-\infty}^{\infty} \hat{\mathbf{u}}(x, y, t; \beta) e^{i\beta z} d\beta, \quad (2.34)$$

treating the pressure perturbation $\hat{p}$ in an analogous manner. Floquet theory considers the decomposition of linear perturbations

$$\hat{\mathbf{u}}(\mathbf{x}, t) = e^{\sigma T} \hat{\mathbf{u}}(\mathbf{x}, t), \quad (2.35)$$

where σ is a complex number and the amplitude functions $\hat{\mathbf{u}}$ are T -periodic functions. Stability of the flow is determined by the Floquet multipliers $\mu = e^{\sigma T}$. If $|\mu| < 1$, then perturbations to the time-periodic solution are exponentially decaying and the time-periodic flow is asymptotically stable, otherwise for $|\mu| > 1$, the flow is unstable to three-dimensional perturbations. In the latter case the time-periodic perturbation grows in one period of time in amplitude a factor given the module of the multiplier.

2.2.6 Transient growth

For arbitrary time-dependence of the flow, the solution of (2.41) may be expressed as an *initial value problem* (IVP)

$$\hat{\mathbf{u}}(\mathbf{x}, t) = \mathcal{A}(Re, \beta, t) \hat{\mathbf{u}}_0, \quad (2.36)$$

with $\mathcal{A}$ as the fundamental solution operator which propagates the initial condition $\hat{\mathbf{u}}_0$ forward in time. Since the operator of the linearized Navier-Stoke equations is non-symmetric, as presented in (2.19) and (2.46), its eigenmodes are “non-normal” gernerally, which meaning that the energy inner products of distinct modes maybe nonzero. The energy growth of perturbations over a time interval τ is defined by the inner product as

$$\begin{aligned} E(\hat{\mathbf{u}}(\tau)) &= \frac{\langle \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle}{2} \\ &= \frac{\langle \mathcal{A}(\tau)\hat{\mathbf{u}}_0, \mathcal{A}(\tau)\hat{\mathbf{u}}_0 \rangle}{2} \\ &= \frac{\langle \mathcal{A}^+(\tau)\mathcal{A}(\tau)\hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0 \rangle}{2}, \end{aligned} \quad (2.37)$$

where $\mathcal{A}^+$ is the adjoint operator of $\mathcal{A}$ and $\mathcal{A}^+\mathcal{A}$ is a normal matrix. Here the inner product $\langle \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle = \int_{\Omega} \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} d\Omega$ is the integration of the velocity product in the control domain Ω . Solution of the initial value problem seeks to determine the maximum energy growth,

$$G(\tau) = \max_{\hat{\mathbf{u}}_0} \frac{\langle \mathcal{A}^+(\tau)\mathcal{A}(\tau)\hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0 \rangle}{\langle \hat{\mathbf{u}}_0, \hat{\mathbf{u}}_0 \rangle}, \quad (2.38)$$

which is decided by the dominant eigenvalues, σ , of the singular value decomposition problem [147]

$$\mathcal{A}^+\mathcal{A}\hat{\mathbf{u}}_0 = \sigma^2\hat{\mathbf{u}}_0, \quad (2.39)$$

subject to Dirichlet boundary conditions for the linear perturbations on all boundaries. Computation of (2.38) may be accomplished following a number of alternative procedures which are implemented in the codes employed for the present analysis, as discussed in detail by Barkley et al. [12].

2.3 BiGlobal Instability

2.3.1 Temporal BiGlobal analysis

When the base flow varies in two-dimensional plane and is independent of the third direction, the three-dimensional disturbances to be assumed containing a spanwise homogeneous variable defined as spanwise wave length $L_z = 2\pi/\beta$. The flow can be decomposed into the form of

$$\mathbf{q}(x, y, z, t) = \bar{\mathbf{q}}(x, y) + \varepsilon \hat{\mathbf{q}}(x, y) e^{i(\beta z - \omega t)} + c.c \quad (2.40)$$

On substitution of equation (2.40) into LNSE (2.7) and continuity equation (2.8) result

$$\begin{aligned} \hat{u}_x + \hat{v}_y + i\beta\hat{w} &= 0 \\ -i\omega\hat{u} + (\mathcal{L} + \bar{u}_x)\hat{u} + \bar{u}_y\hat{v} + \hat{p}_x &= 0 \\ -i\omega\hat{v} + (\mathcal{L} + \bar{v}_y)\hat{v} + \bar{v}_x\hat{u} + \hat{p}_y &= 0 \\ -i\omega\hat{w} + \mathcal{L}\hat{w} + \bar{w}_x\hat{u} + \bar{w}_y\hat{v} + i\beta\hat{p} &= 0 \end{aligned} \quad (2.41)$$

where the linear operator $\mathcal{L} = -(\partial_{xx} + \partial_{yy} - \beta^2)/Re + \bar{u}\partial_x + \bar{v}\partial_y + i\beta\bar{w}$, and subscript letters stand for partial difference on corresponding directions. Since in incompressible flow the pressure perturbation is a function of $\hat{\mathbf{u}}$,

$$\nabla^2 \hat{\mathbf{p}} = \nabla \cdot (\bar{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}}), \quad (2.42)$$

Substitute above equation into (2.41), this will be converted to the three-dimensional BiGlobal eigenvalue problem [159, 160].

$$\mathcal{A}(Re, \beta) \hat{\mathbf{u}} = \omega \hat{\mathbf{u}}, \quad (2.43)$$

In order to solve this systems, adaptive boundary conditions should be imposed to set up an automatic consistency equations, such as no-slip boundary on the wall, Dirichlet or Neumann boundary conditions are recommended for the far field.

2.3.2 Spatial BiGlobal analysis

Spatial BiGlobal analysis is the analog of classic local spatial LST in which two inhomogeneous spatial directions, y and z , are resolved simultaneously while the axial direction, x , is considered locally homogeneous. The perturbation vector is assumed to be of the form

$$\mathbf{q}'(x, y, z, t) = \hat{\mathbf{q}}(y, z) e^{i(\alpha x - \omega t)} + c.c., \quad (2.44)$$

with $\hat{\mathbf{q}} = (\hat{u}, \hat{v}, \hat{w}, \hat{p})^T$ representing the vector of two-dimensional complex amplitude functions and *c.c.* denoting complex conjugation, such that the left-hand-side of equation (2.44) be real. A spatial framework has also been used in a global context: ω is taken to be a real angular frequency parameter, while the complex eigenvalue α , and the associated eigenvectors $\hat{\mathbf{q}}$ are sought. In a manner analogous with local theory, the real part of the eigenvalue α_r of the global mode is related with its wavenumber along the axial direction x , $\alpha_r = 2\pi/L_x$, while its imaginary part is the growth/damping rate. A negative value of α_i indicates exponential growth of $\mathbf{q}'$ in space, while $\alpha_i > 0$ denotes decay of $\mathbf{q}'$ in space.

By substitution of the decomposition of flow variables along with definition (2.44) into the LNSE equations, the two-dimensional partial-differential equation based eigenvalue problem is obtained. The EVP is quadratic in the eigenvalue α , and is converted into a linear eigenvalue problem, which is larger in size by a factor equal to the degree of non-linearity (see [157]), using the companion matrix method [24], in which an auxiliary vector is defined, $\hat{\mathbf{q}} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \alpha\hat{u}, \alpha\hat{v}, \alpha\hat{w}]^T$. The resulting EVP is

$$\mathbf{A}\hat{\mathbf{q}} = \alpha\mathbf{B}\hat{\mathbf{q}}, \quad (2.45)$$

where the entries of matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} = \begin{pmatrix} \mathcal{L} & \bar{u}_y & \bar{u}_z & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L} + \bar{v}_y & \bar{v}_z & \mathcal{D}_y & 0 & 0 & 0 \\ 0 & \bar{w}_y & \mathcal{L} + \bar{w}_z & \mathcal{D}_z & 0 & 0 & 0 \\ 0 & \mathcal{D}_y & \mathcal{D}_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} \end{pmatrix}, \quad (2.46)$$

$$\mathbf{B} = \begin{pmatrix} -i\bar{u} & 0 & 0 & -i & -\frac{1}{Re} & 0 & 0 \\ 0 & -i\bar{u} & 0 & 0 & 0 & -\frac{1}{Re} & 0 \\ 0 & 0 & -i\bar{u} & 0 & 0 & 0 & -\frac{1}{Re} \\ -i & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\mathcal{L} = -i\omega + \bar{v}\mathcal{D}_y + \bar{w}\mathcal{D}_z - \frac{1}{Re}(\mathcal{D}_{yy}^2 + \mathcal{D}_{zz}^2)$, $\mathcal{D}_y = \partial/\partial y$, $\mathcal{D}_{yy}^2 = \partial^2/\partial y^2$, $\mathcal{D}_z = \partial/\partial z$ and $\mathcal{D}_{zz}^2 = \partial^2/\partial z^2$.

The results of spatial BiGlobal analysis are interesting in their own right, and are also used to obtain the shape function, wavenumber, and damping/growth-rate, required as initial conditions of the PSE-3D marching integration.

2.4 Linear PSE-3D

Parabolized stability equations (PSE) can incorporate varies frequencies and wavenumbers simultaneously to trace the evolution of single and multimodes and evaluates different disturbance levels effects on transition in engineering flow. This method is suitable for evaluating the early stages of transition. This complement to DNS provides reliable growth modes curves for the e^N method and can capture the development of nonlinear instability in the process of transition. This method has an advantage of low-cost computer resources, even using a desktop can plot amplitude growth curves by engineers.

The linear PSE-3D instability analysis methodology has been discussed in detail by Broadhurst & Sherwin [27] in the incompressible and by De Tullio *et al.* [41] in the compressible flow regime. PSE-3D extends the classical PSE for basic states with a single strongly inhomogeneous direction to basic states with a mild variation in the streamwise coordinate and strong gradients in the other two spatial directions. This situation is encountered in the near and mid-field wake of the wing, where strong lateral flow gradients, in addition to a relatively milder flow dependence along the axial direction, make PSE-3D the method of choice for the instability analysis of this inhomogeneous flow field.

The PSE-3D instability analysis methodology is gaining acceptance as an tool for the prediction of linear and nonlinear instability of basic states with a mild variation in the streamwise coordinate and strong base flow gradients on the plane normal to the stream-wise direction. Examples of successful application include vortical flows [28, 29, 129], the

wake of an isolated roughness element in supersonic flow [41], streaks in a boundary layer [117], corner flows [57] and models of duct intakes [58]. The near and mid-field wake of the wing, where strong lateral flow gradients, in addition to a relatively milder flow dependence along the axial direction, make PSE-3D the method of choice for the instability analysis of this inhomogeneous flow field.

The Ansatz utilized to decompose the flow perturbations is based on the assumption that $\partial/\partial x \ll \partial/\partial y, \partial/\partial z$ and replaces (2.44) by

$$\mathbf{q}'(x, y, z, t) = \hat{\mathbf{q}}(x, y, z) e^{i \int_{x_0}^x \alpha(\xi) d\xi - \omega t} + c.c.. \quad (2.47)$$

Here $\hat{\mathbf{q}}$ is the amplitude function and $\alpha(x)$ is the corresponding wavenumber, both varying slowly along the axial spatial direction.

By substitution of the decomposition (2.47) into the LNSE equations, neglecting viscous terms with streamwise derivatives of $O(Re^{-2})$ and higher, the linear PSE-3D equations can be obtained, written in a compact form as

$$\left(\mathbf{M} + \mathbf{N} \frac{\partial}{\partial x} \right) \hat{\mathbf{q}}(x, y, z) = 0. \quad (2.48)$$

The entries of the matrices $\mathbf{M}$ and $\mathbf{N}$ are

$$\mathbf{M} = \begin{pmatrix} \mathcal{M} + \bar{u}_x & \bar{u}_y & \bar{u}_z & i\alpha \\ \bar{v}_x & \mathcal{M} + \bar{v}_y & \bar{v}_z & \mathcal{D}_y \\ \bar{w}_x & \bar{w}_y & \mathcal{M} + \bar{w}_z & \mathcal{D}_z \\ i\alpha & \mathcal{D}_y & \mathcal{D}_z & 0 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} \bar{u} & 0 & 0 & 1 \\ 0 & \bar{u} & 0 & 0 \\ 0 & 0 & \bar{u} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (2.49)$$

where $\mathcal{M} = \bar{u}i\alpha + \bar{v}\mathcal{D}_y + \bar{w}\mathcal{D}_z - \frac{1}{Re}(\mathcal{D}_{yy}^2 - \mathcal{D}_{zz}^2 - \alpha^2) - i\omega$ and, as in spatial BiGlobal theory, $\omega \in \mathbb{R}$ is a real frequency parameter and $\alpha \in \mathbb{C}$.

An ambiguity arises regarding the normalization condition which needs to be imposed in order to ensure, firstly, that the variation of the shape function along the streamwise direction be small and, secondly, that all amplitude growth be captured by the phase function. This issue has been discussed and resolved by Herbert [74, 75] in the context of the classic parabolized stability equations. The same concept is extended to PSE-3D, where the normalization condition

$$\int_{\Omega} \hat{\mathbf{q}}^\dagger \frac{\partial \hat{\mathbf{q}}}{\partial x} d\Omega = 0 \quad (2.50)$$

is used, where Ω denotes the extent of the computational domain on the plane normal to the axial spatial direction and superscript $\dagger$ denotes complex conjugate. Much like in the classic PSE, upstream propagating disturbances which are inherent to the parabolized (but not strictly parabolic) nature of the disturbance equations, are also found to be present in PSE-3D [27]. Here the remedy proposed by Li and Malik [106–108] is used, namely streamwise spatial integration of (2.48) using a marching step $\Delta x > 1/|\alpha|$.

A spatial amplification rate, σ , may be defined in the PSE-3D analysis as

$$\sigma = -\alpha_i + \frac{1}{2} \frac{d}{dx} \log E(x). \quad (2.51)$$

Here $E = \int_{\Omega} (\hat{u}\hat{u}^{\dagger} + \hat{v}\hat{v}^{\dagger} + \hat{w}\hat{w}^{\dagger})d\Omega$ is the kinetic energy of the perturbations, the latter being three-dimensional functions of the spatial coordinates that vary slowly along the axial direction. In a manner analogous with the classic PSE, the amplification rate in (2.51) is composed of two contributions, the local $-\alpha_i$ obtained at convergence of solution of (2.48) at each x -location to which the term corresponding to the perturbation kinetic energy is added.

2.5 TriGlobal Instability

As for the three-dimensional eigenvalue problem without homogeneity assumptions on the base flow, then addressed TriGlobal instability perturbations Ansatz as

$$\mathbf{q}'(x, y, z, t) = \hat{\mathbf{q}}(x, y, z)e^{i\omega t} + c.c. \quad (2.52)$$

This is the most universal linear stability theory that works in all the cases discussed above. Tezuka and Suzuki [156] firstly analyzed the three-dimensional stability of a prolate flow with a angle of attack. Later TriGlobal stability analysis is performed in the cross jet [9] and lid-drive cavity [63] flows. the Generally, the *Global LST* covers the areas of BiGlobal, PSE-3D and TriGlobal modal linear stability theories.

Chapter 3

Analysis of Flow Topology

3.1 Critical Point Theory

In two-dimensional flow, several concepts, such as separation point, reattachment streamlines and separation bubble etc. are adopted to describe the flow structure. It should be carefully in applying these concepts on describing the three-dimensional flow, since the lacking of adequate concepts interpretation in the three-dimensional case.

The investigation did by Poincaré on the singular points of differential equations inspired Legendre to publish the first articles in 1950s [136][101]. In which he realized that this mathematical theory was the only rational tool that could work in understanding the topology of three-dimensional separated flows. Later Lighthill revisited Legendre's abstract and depicted the separation of a three-dimensional boundary layer. The other experimental contribution was from Werlé's water tunnel visualizations, which also motivated Legendre's publications.

In three-dimensional flow, the direction of wall shear stress are defined as the differential system of

$$\frac{dx}{\tau_x(x, z)} = \frac{dz}{\tau_z(x, z)} \quad (3.1)$$

where τ_x and τ_z are the wall shear stress in the streamwise x and spanwise z flow directions, respectively. Another more general concept adopted to call shear stress vectors is wall surface streamlines. In general, only one surface streamline passes through one point on the surface, the singular solution of the system 3.1 are the *critical points*, where the general rule fails. Introducing the Jacobian matrix of this system in the vicinity of one critical point P_0

$$J = \begin{vmatrix} \frac{\partial \tau_x}{\partial x} & \frac{\partial \tau_x}{\partial z} \\ \frac{\partial \tau_z}{\partial x} & \frac{\partial \tau_z}{\partial z} \end{vmatrix}_{P_0} \quad (3.2)$$

The solution can be achieved by deriving the first-order differential equations near the critical points and computing its eigenvalues equation

$$S^2 + pS + q = 0, \quad (3.3)$$

where p and q are given by,

$$p = -\text{trace}(J) = -\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_z}{\partial z}\right)_{P_0},$$

$$q = \det(J) = \left(\frac{\partial \tau_x}{\partial x} \frac{\partial \tau_z}{\partial z} - \frac{\partial \tau_z}{\partial x} \frac{\partial \tau_x}{\partial z}\right)_{P_0}.$$

Since the eigenvalues $S_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$ only depend on the values of p and q at the critical point, and its corresponding eigenvectors could be reconstructed the form of surface streamlines in the vicinity of P_0 . These results can be acquired by means of solving the Navier-Stokes equations or carrying out an flow visualization experiment. Figure 3.1 shows the typical surface streamlines at the critical points. Further detailed descriptions on the classification of critical points can be referenced to [113], [134] and [43].

Figure 3.1: Critical points classification in the plane $[p, q]$ (Extracted from [42]).

The experimental research on three-dimensional flow topological structure in the evolved ogive-/hemisphere-cylinder are abundant to be referred [8, 40, 50, 99, 133]. Figure 3.2 shows a oil-visualized experiments surface trace of the hemisphere-cylinder in a compressible flow by Peak and Tobak [133] at $Ma = 1.2$. The free stream Reynolds number is $Re = 4.9 \times 10^6$ and angle of attack is 32.5° . Besides [133], in recent years, Le Clainche

et al. [99] investigated the low Reynolds number experiments of hemisphere-cylinder flow and carried out global instability analysis on this configuration [98, 100].

Figure 3.2: Flow patterns in the compressible flow of an evolution hemisphere-cylinder at higher angle of attack. Extracted from [133].

According the critical point theory, the Mach number and Reynolds number have no significant effect on the flow pattern, as long as the objectives have similar geometry and experimental setups, such as closed angel of attack, as shown in Figure 3.3 (also see [99]). The incompressible DNS was carried out under the conditions of $Re = 200$ and $AoA = 45^\circ$, which represents that the surface streamlines distribution has an analogous behaviors as in Figure 3.2.

Figure 3.3: Stream surfacelines on the hemisphere-cylinder in the flow of $Re = 200$ at $AoA = 45^\circ$: The critical points are visualized on the head (*Left*) and body tail bottom (*Right*).

Under the deduced Poincaré formula:

$$\Sigma(nodes + foci) - \Sigma saddles = 2 - 2p \quad (3.4)$$

where p is the surface complexity, estimated by the number of “holes” in the body. For a sphere or a hemisphere-cylinder, its complexity is 0. As shown in Figure 3.3, there are one node N_1 on the stagnation of nose, another node N_2 is located on the upper surface of the head. There are two saddles S_1 and S_2 located on the head and tail bottom of the body, separately. While two foci F_1 and F_2 generated on the tail bottom symmetrically along the xy -plane. Hence the following numbering is

$$\Sigma(nodes + foci) - \Sigma saddles = 4 - 2 = 2 - 2 \times 0 = 2 \quad (3.5)$$

3.2 Vortex Identification

In section 3.1 flow patterns on the surface is introduced, another important side to depict the three-dimensional flow is by means of not only plotting pathlines or streamlines but identifying the vortex structure developed in space. To complete the fully three-dimensional classify topology, an analogous Jacobian matrix to equation (3.6) can be defined

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad (3.6)$$

Assuming λ_1 , λ_2 and λ_3 are the eigenvalues of matrix J , then its characteristic equation

$$\det(J - \lambda I) = 0, \quad (3.7)$$

can be written as

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0, \quad (3.8)$$

where

$$P = -\text{trace}(J) = -S_{ii}, \quad (3.9)$$

$$Q = \frac{1}{2}(P^2 - S_{ij}S_{ji} - \Omega_{ij}\Omega_{ji}), \quad (3.10)$$

$$\Omega = -\det(J) = \frac{1}{3}(-P^3 + 3PQ - S_{ij}S_{jk}S_{ki} - 3\Omega_{ij}\Omega_{jk}S_{ki}). \quad (3.11)$$

The Kronecker subscripts i, j and k represent the variables in x, y and z directions, and the S_{ij} and Ω_{ij} are symmetric and antisymmetric parts of $\frac{\partial \dot{x}_i}{\partial x_j}$, respectively, which are give by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial \dot{x}_j}{\partial x_i}\right), \quad (3.12)$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial \dot{x}_i}{\partial x_j} - \frac{\partial \dot{x}_j}{\partial x_i}\right), \quad (3.13)$$

here the dot represent of derivative.

Q-criterion

Q criterion is defined as the region with positive second invariant of J

$$Q = \frac{1}{2}(\Omega^2 - S^2) = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3), \quad (3.14)$$

which represents the relation between shear strain rate and vorticity magnitude, is equivalent of the average value of eigenvalues [81].

λ_2 -criterion

One of the method to describe the vortex is λ_2 basing on the theory of computing the region where the second eigenvalue, λ_2 , of the symmetric tensor $S^2 + \Omega^2$ is negative. Unlike the vortex core defined as *a connected region with two negative eigenvalues of $S^2 + \Omega^2$* [85], the vortex-core line computed in Tecplot are the lines extracted the one that run through the center of the region of negative eigenvalue λ_2 .

Example

Figure 3.4 presents the visualization of vorticity by the Q -criterion and λ_2 . Both show that there is a bubble closed to the top the nose and a pair vortices rolling up from the sides of the cylinder body and stretching into the wake. More clearly the vortex-core depicted in Figure 3.5.

Figure 3.4: The isosurface of vorticity at $Q = 0.01$ (left) and $\lambda_2 = -0.01$ (right) in the flow pasts the hemisphere-cylinder at $Re = 200$ and $AoA = 45^\circ$.

Extracted four planes along the hemisphere-cylinder at $x = 0.6$, 1.0 , 1.0 and 5.0 , the evolution of the two vortices are shown in Figure 3.6. At $x = 0.6$, which is closed to the saddle $S1$ in Figure 3.3, there is no separation. In the downstream, the vortex detached and developed, there are two symmetry foci placed opposite, one four-way saddle in the symmetry plane and the separation bubble starts from the two three-way saddles. Another one pair of three-way saddles exist in the up and down of the symmetry plane.

Figure 3.5: The evolution of vortex core in the flow pasts the hemisphere-cylinder at $Re = 200$ and $AoA = 45^\circ$.

Figure 3.6: Vortex advanced along the hemisphere-cylinder at $Re = 200$ and $AoA = 45^\circ$, in the yz - plane perpendicular to the body central line.

Chapter 4

Instability of Separated Airfoil Flow

4.1 NACA Airfoils Base Flow

4.1.1 Code Employed

All numerical work was performed using open source software. Several codes were used, as detailed below, in order to cross-verify the results obtained and also to optimize serial or parallel efficiency in the large number of parametric studies performed.

Base flow computations were initially performed using the finite-volume code OpenFOAM [84], the computational requirements of which were found to be excessively large for reliable results to be obtained in the subsequent linear instability analysis work. Since this solver discretizes space by a second-order accurate finite-volume scheme, a relatively large number of elements was needed to obtain grid-independent results, typically $O(4 \times 10^4)$ grid points. Take a two-dimensional low Reynolds number airfoil flow for example, to obtain a grid-independent results, the necessary wall-refinement of a mesh of this size leads to time-steps of $\Delta t = 10^{-3}$ which, combined with the need to run the unsteady code up to non-dimensional 300 time-units until convergence to steady state, results in serial computation costs to obtain a basic flow of about 600 hrs (see section 4.1.5). In addition, the linear modal stability analysis module developed and validated for OpenFOAM [111] was shown to require a rather high computational cost for reliable results to be obtained on the airfoils.

Modal stability analysis results obtained by the spectral solvers were independently verified by the finite-element package FreeFEM++ [71], in which a matrix forming approach was considered for the solution of the eigenvalue problem related with the modal analysis (see section C.2). The numerical work for the secondary instability analysis was performed using the Semtex and Nektar++ codes. Unlike Nektar++ or Nek5000, Semtex is a spectral direct numerical simulation code for the the solution of the Navier-Stokes equations in three-dimensional problems with one homogeneous spatial direction. Fourier expansion is used along the homogeneous direction while nodal Gauss–Lobatto–Legendre basis functions are used to discretize the plane normal to the homogeneous direction; details are provided by [18] and [19].

Consequently, base flow computations as well as modal and non-modal analyses of steady base states were obtained using the spectral-element time-stepping codes Nektar++ [33, 92], Nek5000 [45, 55] and Semtex [12, 18]. In the work presented in what follows, serial computations were performed for the base flows and parallel computations, using up to 64 CPUs, were employed for the instability analyses.

4.1.2 Meshing strategies

The structured meshes used by OpenFOAM were created by its inbuilt structured mesh generation routine *blockMesh*. Since this solver discretizes space by a second-order accurate finite-volume scheme, a relatively large number of elements was needed to obtain grid-independent results, typically $O(4 \times 10^4)$ hexahedral/quadrilaterals elements. It should be noted that, even for two-dimensional simulations, OpenFOAM requires a three-dimensional mesh of depth of one cell, as seen in Figure 4.1. An O-mesh topology is used, that is divided internally into several blocks to generate a structured grid. The outer boundary of the domain has a radius of 20 chords.

Figure 4.1: 2D simulation mesh generated by *blockMesh*. *Top Row:* 3D hexahedral mesh. *Low Row:* Full domain (left) and closed view near the boundary layer (right).

FreeFem++ [71] being a finite element solver using second-order P2P1 Taylor-Hood element, high-density meshes were also needed for the analyses performed with this soft-

ware. At convergence, an unstructured mesh of $O(5 \times 10^4)$ triangular elements, resulting in $O(9 \times 10^4)$ degrees of freedom per velocity component is used to discretize computational domain $\Omega = \{x \in [-32, 40] \times y \in [-32, 32]\}$ in chord units. Figure 4.2 shows a typical finite element discretization around the 4415 airfoil in the first row.

The generation of spectral meshes are adjust to the employed spectral element codes. High-order grids, an example of which is shown in Figure 4.2 (middle row), were initially created by the open source *Gmsh* software [61], but were subsequently post-processed by Semtex-internal routines. The procedure used was to first generate within Gmsh a first-order approximation of each NACA airfoil, using a relatively large number of 200 points joined by straight-line segments and concentrated in the regions of high curvature. The solution domain was discretized entirely by a typical number of 2000 rectangular macro-elements. Subsequently, the (true, curved) NACA surface was approximated by a small number of $3 \sim 4$ high-order spline auxiliary curves. The first-order approximation and the high-order curves were provided to Semtex, which replaced the straight line side of any macro-element corresponding to the NACA surface by high-order line with the same local curvature as that at the auxiliary curve at the corresponding location. Finally, each physical element was mapped onto the canonical rectangular element which was then discretized using Gauss–Lobatto–Legendre points of typical order $p = 7$. Meshes for the airfoils at the different AoA values studied were constructed by analytically changing the coordinates of both the initial linear surface reconstruction and those of the auxiliary curves and following the same procedure discussed above.

Gambit is another powerful tool to generate two- and three-dimensional higher order structured and hybrid meshes, as shown in the third row of Figure 4.2. It can generate boundary layer mesh and adjust its quality automatically. It has a high efficient to generate local refinement structure mesh without increasing the computation in the field need low resolution. The mesh can be converted to Semtex and Nektar++.

In order to fit the boundary types available in the spectral elements method codes, the computational domain for the base flow in chord units has been taken to be $\Omega = \{x \in [-15, 50] \times y \in [-15, 15]\}$. Figure 4.2 shows full and detailed views of the domain and a typical discretization used when using these spectral solvers; Figure 4.3 presents a zoom of the latter near the leading- and trailing-edges of the airfoil. It should be noted that, for clarity, only the macro-element structure is shown in the lower rows of Figure 4.2 and in Figure 4.3; this structure is complemented by a high-degree polynomial, typically $p = 5 \sim 9$, in order to construct the actual mesh on which computations have been performed.

4.1.3 Boundary conditions

The boundary conditions employed to close the systems governing the base flow and its linear stability along the NORTH, WEST, SOUTH and EAST boundaries of the computational domain are shown in Table 4.1, with flow considered from left to right. The boundary conditions employed in the OpenFOAM solver are detailed described in section 4.1.5.

4. Instability of Separated Airfoil Flow

Figure 4.2: Mesh topologies employed for the baseflow computation and instability analyses of the NACA airfoils. *Top Row:* Finite element mesh used for the 4415 in the matrix-forming computations based on FreeFEM++. *Middle Row:* Spectral element mesh used for the 0015 in the time-stepping computations employing Nektar++ or Nek5000. *Lower Row:* Spectral element mesh generated by Gambit for Nektar++ and Semtex. *Left Column:* Part of the full domain. *Right Column:* Close-up near the airfoil.

Figure 4.3: Detail of the spectral element mesh in figure 4.2 near the leading edge (left) and the trailing edge (right)

Table 4.1: Boundary conditions for basic flow $(\bar{u}, \bar{v})^T$, direct perturbation $(\hat{u}, \hat{v}, \hat{w})^T$ and adjoint perturbation $(\hat{u}^+, \hat{v}^+, \hat{w}^+)^T$ velocity components. D: homogeneous Dirichlet; N: homogeneous Neumann; U: uniform inflow

	Wall	NORTH	SOUTH	EAST	WEST
$\bar{u}$	D	N	N	N	U
$\bar{v}$	D	N	N	N	U
$\hat{u}$	D	N	N	N	D
$\hat{v}$	D	N	N	N	D
$\hat{w}$	D	N	N	N	D
$\hat{u}^+$	D	D	D	D	D
$\hat{v}^+$	D	D	D	D	D
$\hat{w}^+$	D	D	D	D	D

4.1.4 Convergence

The numerical integrity of all base flow, modal and non-modal instability results presented in the next sections has been ensured in a threefold manner. First, for a given code choice and calculation domain, grid-independence was obtained by decreasing the characteristic element size, h , and, where appropriate, increasing the polynomial order, p . Second, grid independence was verified against changes in domain extent. Third, grid-independent results obtained by two different codes in a given domain were compared against each other; when large relative errors were observed, domain parameters were changed until the relative discrepancy dropped below a predetermined threshold, typically $O(1\%)$ or less for the base flow vorticity and $O(10\%)$ or less for the eigenvalues (see section C.2).

4.1.5 Effect of trailing-edge geometry on the NACA 0015 airfoil

A question that was addressed at the beginning of the present study was whether a steady flow exists in the wake of the NACA 0015 airfoil at conditions monitored in prior instability analyses, namely $Re = 200$, $AoA = 18^\circ$ [23, 96]. A feature of the finite-thickness airfoil is that their geometry at the trailing edge (TE) is provided by the coordinates of discrete points and several choices exist in order to close the airfoil. The question addressed was whether these different closures impact the steadiness of the massively separated two-dimensional steady laminar base flow at the AoA values of interest. Four TE closure configurations were considered, respectively denominated *baseline* (BL), *sharp-TE* (ST), *circular-TE* (CE) and *modified* airfoil (MD).

$$y_B = \pm 0.75(0.2969\sqrt{x} - 0.1260x - 0.3516x^2 + 0.2843x^3 - 0.1015x^4), x \in [0, 1], \quad (4.1)$$

$$y_S = \pm(-0.3503313x^2 + 0.5268143x - 0.1749080), x \in [1, 1.0089], \quad (4.2)$$

$$y_M = \pm 0.75(0.2969\sqrt{x} - 0.1260x - 0.3516x^2 + 0.2843x^3 - 0.1036x^4), x \in [0, 1]. \quad (4.3)$$

Figure 4.4: Trailing edge geometries of the NACA 0015 airfoil

The baseline configuration is that of the NACA 0015 airfoil, the coordinates (x, y_B) of which are described by (4.1), in which the points closest to the TE are simply joined together by a vertical straight line, leading to a non-zero trailing-edge thickness of 0.00315. The sharp-TE airfoil is the same as the baseline but replaces the trailing edge by extending the pressure and suction sides of the baseline configuration until they meet at a single point, using the parabolic function (4.2). By contrast, the circular-TE airfoil uses an arc-segment profile with diameter of $R \approx 0.001576875$ and center coordinate of $(0.9999923125, 0)$ to produce a trailing-edge closure. Figure 4.4 depicts the four different TE geometry closures of the NACA 0015 airfoil.

Two-dimensional incompressible transient laminar flow governed by Navier-Stokes equations are solved by OpenFOAM. OpenFOAM is an open-source finite volume method solver. The quality of the code has been demonstrated by some authors [30, 127]. The

code uses a second-order Crank-Nicholson method for time marching and PISO algorithm to couple velocity and pressure. The grid and computational domain for the sharp trailing-edge geometry are shown in Figure 4.1.

Boundary conditions adequate for external incompressible viscous flow are used. No-slip boundary condition $\mathbf{U} = \mathbf{0}$ is imposed at the solid walls. Inlet and outlet boundaries type are switched by the sign of the flux as

$$\begin{cases} \cos(\mathbf{n} \cdot \mathbf{u}) \geq 0, & \text{inlet,} \\ \cos(\mathbf{n} \cdot \mathbf{u}) < 0, & \text{outlet.} \end{cases}$$

$\mathbf{u} = \text{constant}$ and $\partial_n p = 0$ are imposed on inlet boundary. For outlet boundary, its boundary condition is $\partial_n \mathbf{u} = 0$ and $p = 0$, respectively.

In order to estimate the grid independence of the results, the grid convergence index (GCI) is calculated. Following Roache [141], GCI represents a simple method for uniform reporting of grid-convergence studies without any restriction to integer refinement. This index is defined as

$$GCI_{j+1,j} = 300 \times \left| \frac{f_{j+1} - f_j}{f_j(l^n - 1)} \right|,$$

where f_j is any chosen value in the $j - th$ point of the grid. The GCI is based on the generalized Richardson extrapolation which involves comparison of discrete solutions at two different grid spacings. Table 4.2 shows the grid information for the sharp trailing-edge geometry at three different grid size levels. The value of $GCI_{2,3}$ based on drag coefficient (C_d) is small enough to prove that medium mesh is closed to the grid independent solution.

Table 4.2: Three different grid sizes for the *sharp-TE* airfoil. N_R is the number of radial points and N_C is the number of circumferential points.

Mesh	$N_C \times N_R$	Grid points	C_d	$GCI_{j+1,j}(\%)$
Fine	260×212	$\sim 5.5 \times 10^4$	0.404166	0.09
Medium	194×150	$\sim 2.9 \times 10^4$	0.404270	0.23
Coarse	143×107	$\sim 1.5 \times 10^4$	0.404523	-

The residual of the discretized Navier-Stokes equations is employed to study the convergence of the solutions. It is defined as $Res = |C_c - C_t|$, where C_c and C_t denote the converged and time dependent values (e. g. drag or velocity), respectively. Figure 4.5 shows the residuals of aerodynamics in the four-geometry trailing edge flow, based on the analogous quality grid studied.

4. Instability of Separated Airfoil Flow

Figure 4.5: Convergence history of the residual of C_d and C_l for the four NACA 0015 trailing edge geometries, using OpenFOAM at $Re = 200$, $AoA = 18^\circ$: Top-left: *baseline* profile; top-right: *sharp TE*, lower left: *circular TE*, lower right: *modified* profile.

Figure 4.6: Pressure distribution on the NACA 0015 airfoil with various trailing edge: *Baseline* (—), *sharp-TE* (---), *circular-TE* (···) and *Modified* (- · -).

Figure 4.6 shows the pressure distributions on the NACA 0015 airfoil with different trailing edge geometries in the flow conditions as depicted before. There is no obvious

Table 4.3: Coefficients of aerodynamics in the flow of NACA 0015 with four *TE* at $Re=200$ and $AoA=18^\circ$, computed by Nektar++.

Geometry	C_d	C_l
<i>BL</i>	0.4023	0.5781
<i>ST</i>	0.4019	0.5846
<i>CE</i>	0.4022	0.5781
<i>MD</i>	0.4001	0.5781

change of pressure on the airfoils, except at the trailing edge where the pressure has a loss on the leeside of the single point cross type. Especially, the *baseline* and *circular-TE* airfoils have nearly the same pressure distributions which yield to the same values of integration of aerodynamics. This is verified in Table 4.3, after the convergence study by Nektar++ (see C.1).

The unsteadiness study is independently confirmed by Nektar++ and Nek5000 simulations with the *modified* airfoil, using 2291 and 3634 elements, respectively, as seen in figure B.2 and further discussed by [69]. It can be seen that all four TE geometries lead to the same result at these parameters, namely a steady laminar two-dimensional flow. Figure 4.7 shows the vorticity around the airfoil and the large separation bubble behind the airfoil.

Figure 4.7: Vorticity distribution around the NACA 0015 airfoil and the streamlines visualization of the separation bubble behind the airfoil at $Re = 200$ and $AoA = 18^\circ$.

4.1.6 Unsteadiness on the three airfoils, as a function of Re and AoA

The onset of flow unsteadiness has been investigated for each of the three profiles in the ranges $150 \leq Re \leq 300$ and $10^\circ \leq AoA \leq 20^\circ$, beyond which massive separation and unsteadiness are expected. Table 4.4 presents a qualitative description of the state of the flow. On all airfoils, an increase of Re or AoA promotes unsteadiness and a time-periodic flow is set up, the period of which will be characterized in Section 4.5. At a given Reynolds number and low angle of attack, thickness promotes unsteadiness, as seen at $Re = 300, AoA = 15^\circ$ for the NACA 0009 and 0015 airfoils. Close to stall, camber

Figure 4.8: Geometries of the three NACA airfoils: NACA 0009 (*left*), NACA 0015 (*middle*) and NACA 4415 (*right*).

Table 4.4: Unsteadiness as a function of Re and AoA for three NACA airfoils. S : steady, U_s : unsteady base flow

	NACA 0009					NACA 0015					NACA 4415				
	150	200	220	250	300	150	200	220	250	300	150	200	220	250	300
10°	S	S	S	S	S	S	S	S	S	S	S	S	S	S	S
15°	S	S	S	S	S	S	S	S	S	U_s	S	S	S	S	U_s
18°	S	S	S	U_s	U_s	S	S	S	U_s	U_s	S	S	U_s	U_s	U_s
20°	S	U_s	U_s	U_s	U_s	S	U_s	U_s	U_s	U_s	S	U_s	U_s	U_s	U_s

Table 4.5: Maximum recirculation, $|\frac{\bar{u}_{min}}{\bar{u}_{max}} \times 100|$ as a function of Re and AoA

	NACA 0009					NACA 0015					NACA 4415				
	150	200	220	250	300	150	200	220	250	300	150	200	220	250	300
10°	0.79	1.09	1.20	1.35	1.59	0.28	0.36	0.49	0.81	1.48	0.27	1.08	1.66	2.49	3.75
15°	2.29	2.92	3.73	4.97	6.80	2.40	3.96	4.77	5.89	–	3.14	5.47	6.29	7.39	–
18°	4.02	7.03	8.07	–	–	4.66	7.34	8.27	–	–	5.78	8.41	–	–	–
20°	6.51	–	–	–	–	6.72	–	–	–	–	7.61	–	–	–	–

of the airfoil also promotes unsteadiness, as seen at $Re = 220, AoA = 18^\circ$, where flow around the NACA 0015 is still steady while that in the wake of the NACA 4415 airfoil is time-periodic.

In view of the association of the level of recirculation in the nominally stationary laminar separation bubble (LSB) of the base flow with the appearance of absolute/global linear instability [2, 5, 53, 68, 140] the results of Table 4.4 are quantified in Table 4.5 in terms of the percentage of flow recirculation in the LSB. It is seen that in all airfoils an increase in Re or in AoA leads to an increased maximum recirculation level. At the same Re and AoA values, an increase in either airfoil thickness or camber also increases the level of recirculating flow. It is also noted that in most cases shown recirculation is less than the value of 7.5% found by [142] to be necessary for amplification of the stationary three-dimensional global mode in the LSB created by an adverse pressure gradient on a flat plate; further discussion of this point will be provided in the next section.

Figure 4.9: Instantaneous vortex distribution for the flow past NACA 4415 at $Re = 1000$ and $AoA = 20^\circ$ (upper) and its flow topology explanation (lower)

4.1.7 Flow topology of 2D airfoil flow

In two-dimensional incompressible flow the only critical points types are *centers* and *saddles* ([135]) and topologically the focus and center both belong to nodes. Figure 4.9 shows the two-dimensional vorticity in the wake of flow past NACA 4415 airfoil at $Re = 1000$ with $AoA = 20^\circ$. The vortex shedding has already reached a periodic oscillation process. In the flow topology analysis, the four-way critical points node and saddle denoted

as N and S , while the three-way node and saddle defined as N' and S' , respectively. A general formula obtained by [36, 80] is

$$(\Sigma N + \frac{1}{2}\Sigma N') - (\Sigma S + \frac{1}{2}\Sigma S') = -1 \quad (4.4)$$

By counting the numbers of the three-way and four-way nodes and saddles in Figure 4.9, there are four three-way saddles located at stagnation point on the nose, leading edge separation point trailing edge separation point and a reattachment point, respectively. Three centers in the wake behind the airfoil and two four-way saddles formulated in the wake. The numbers $\Sigma N = 3$, $\Sigma N' = 0$, $\Sigma S = 2$ and $\Sigma S' = 4$, meet the requirement of above formula.

4.2 BiGlobal Stability Analysis of Separated Flow

Results of modal instability analysis of flow around the three airfoils at selected combinations of the Reynolds number and angle of attack parameters shown in Tables 4.4 and 4.5 are discussed in this section. Steady base flows are analyzed by solution of the pertinent eigenvalue problem (2.43), whereby the spanwise spatial direction is considered homogeneous and is discretized by a Fourier Ansatz, which introduces the wavenumber parameter $\beta = 2\pi/L_z$, L_z being the spanwise periodicity length, as discussed in Section 2.3. The decomposition decouples the linearized problems to be solved for each β from each other, and the full three-dimensional space is covered by varying the wavenumber in a range from $\beta = 0$, i.e. $L_z \rightarrow \infty$, corresponding to two-dimensional perturbations, to a finite number of discrete $\beta \neq 0$ values corresponding to progressively smaller periodicity lengths, until $L_z = O(1)$ is reached. Modal instability analyses were performed using mesh sizes and spatial polynomial orders comparable to those used for the recovery of the base flow and convergence tests are shown in the Appendix.

4.2.1 Effect of trailing-edge geometry on the primary stability

Effect of trailing-edge geometry on the modal stability has been studied, following the flow unsteadiness aforementioned in Section 4.1. The investigation in Table 4.6 shows that the error in the modal stability analysis arisen by the trailing edge is less than $O(1\%)$. Both two- and three-dimensional modal analysis shown there is no difference to discretize the computational domain using the different trailing-edge geometry in the range of Reynolds number investigated.

Results have been obtained for all three airfoils, however discussion commences by focusing on the NACA0015 airfoil at $Re = 200$, $AoA = 18^\circ$, conditions at which previous analyses have produced contradictory predictions regarding which of the two instability scenarios prevails, namely *2D unsteadiness, followed by 3D instability* [23] or *3D instability, followed by 3D unsteadiness* [96, 143]. In order to cross-verify predictions, instability results are obtained at these conditions using the time-stepping code `nektar++` and the

Table 4.6: Growth rate of modal analysis of NACA 0015 airfoil at $Re = 200$ and $AoA = 18^\circ$ with different trailing edge geometries.

<i>Geometry</i>	$\beta = 0$	$\beta = 1$	$\beta = 3$
<i>BL</i>	-0.10103	-0.14860	-0.35156
<i>ST</i>	-0.10025	-0.14808	-0.34645
<i>CE</i>	-0.10106	-0.14862	-0.35071
<i>MD</i>	-0.10683	-0.15435	-0.35784

matrix-forming code FreeFEM++. The time-stepping technique [12] initializes linear perturbations with a normalized unit vector and evolves them during a non-dimensional time interval of unity, using the same mesh and time step employed for the base flow computations. By contrast, the matrix-forming technique uses a weak formulation of the linearized equations of motion, first to solve for the base flow on a high-density unstructured mesh and subsequently to form the matrix entries corresponding to the eigenvalue problem. Both approaches proceed by constructing Krylov subspaces of orthonormal vectors and an Arnoldi algorithm is used to recover a small number of physically most significant eigenmodes [160]. The leading two eigenmodes discovered in the wavenumber range $0 \leq \beta \leq 10$ by the two approaches are shown in appendix Figure C.1, from which results obtained using the time-stepping code nek5000 are omitted, since they were identical with those produced by the nektar++ code. Analogous results were obtained for the other two airfoils at the same parameters, but are not shown here for brevity.

The agreement between the predictions of the time-stepping and the matrix-forming approaches is remarkable, given that the results shown have been obtained using entirely different methodologies, both in terms of theoretical background and in terms of numerical implementation. Despite the quantitative differences in the damping rate results, both methodologies predict the existence of two instabilities, a traveling eigenmode which predominates at relatively small β (large L_z) values and a stationary eigenmode which is less damped than the traveling disturbance at larger β (small L_z) values. The cut-off point at which the nature of instability changes at these conditions is found to be $\beta \approx 2.5$.

4.2.2 Spatial shape of eigenvector

Examination of the spatial structure of the eigenfunctions reveals that these modes correspond to the instabilities known from the earlier analyses of Kitsios et al. [96] and Rodríguez and Theofilis [143]. Figure 4.10 shows the normalized vorticity two-dimensional perturbation amplitude functions for the NACA 0009, 0015 and 4415 airfoils at $AoA = 18^\circ$, which have a massive separation region in the wake of the base flow. The spatial distribution of perturbations have a similar structure as presented in circular cylinder stability results [1]. Figure 4.11, 4.12 and 4.13 present the amplitude functions of the perturbation velocity components at $\beta = 1$ and $\beta = 3.5$ for the three airfoils, respectively corresponding to a mode akin to Kelvin-Helmholtz instability in the wake of bluff bodies and the NACA0012 airfoil discussed in the Introduction [163], while at the higher wavenumber

4. Instability of Separated Airfoil Flow

Figure 4.10: Comparison of primary perturbation modes in two-dimensional flow for the three airfoils flow at $Re = 200$ with vorticity ω_z contour visualized over -5 (blue) to 5 (red). From left to right are NACA 0009, NACA 0015 and NACA 4415.

Figure 4.11: NACA 0009 at $Re = 200$, $AoA = 18^\circ$. Amplitude functions of the least-damped traveling (*upper*) and stationary (*lower*) perturbation velocity components at $\beta = 1$ and $\beta = 3.5$, respectively. Normalized by the highest velocity component, from -0.1 (blue) to 0.1 (red)

the mode corresponds to the stationary three-dimensional eigenmode of laminar separation bubbles discovered on the flat-plate [162] and in a multitude of massively separated laminar flow configurations [e.g. 2, 11, 96, 143].

However, the present results differ from those reported by Kitsios et al. [96] and Rodríguez and Theofilis [143] in two aspects. First, unlike the prediction of linear instability on the NACA 0015 airfoil at $Re = 200$, $AoA = 18^\circ$ in the earlier works, here only stable eigenvalues have been encountered at all β values examined. Second, it is found that at $\beta = 1$ the leading flow eigenmode is the unsteady KH instability and not the stationary three-dimensional mode, as reported in the earlier work. As a matter of fact,

Figure 4.12: NACA 0015 at $Re = 200$, $AoA = 18^\circ$. Amplitude functions of the least-damped traveling (*upper*) and stationary (*lower*) perturbation velocity components at $\beta = 1$ and $\beta = 3.5$, respectively. Normalized by the highest velocity component, from -0.1 (blue) to 0.1 (red).

Figure 4.13: NACA 4415 at $Re = 200$, $AoA = 18^\circ$. Amplitude functions of the least-damped traveling (*upper*) and stationary (*lower*) perturbation velocity components at $\beta = 1$ and $\beta = 3.5$, respectively. Normalized by the highest velocity component, from -0.1 (blues) to 0.1 (red)

the stationary mode did not become unstable in the range of Reynolds numbers examined, as will be discussed shortly. The stationary mode, which was shown by [143] to give rise to patterns akin to the stall-cells encountered at flight Reynolds numbers, does overtake the traveling instability at higher β values. This could be related with the appearance of stall-cell patterns in the low-Reynolds number simulations of Taira and Colonius [153] on $O(1)$ aspect-ratio wings. However, it is unclear whether results of instability analysis on finite-span wings in that work can be related with those obtained on the present spanwise-periodic geometries. It is also worth noting, that the numerical integrity of the results previously reported by Kitsios et al. [96] and Rodríguez and Theofilis [143] was already questioned by Kitsios [95], who reported that the instability analyses presented had not entirely converged. The present work in this thesis is currently underway in order to identify the origin of the discrepancy with the results reported in the earlier analyses.

4.2.3 Effect of Re on the modal stability

Attention is next turned to the dependence of the instability results on the Reynolds number, keeping a constant value of $AoA = 18^\circ$. Figure 4.15 shows results in the range $150 \leq Re \leq 230$. In this and all subsequent figures results for the NACA 0009, 0015 and 4415 airfoils will be presented using the circle, $\circ$, square, $\square$, and diamond, $\diamond$ symbols, respectively.

At all Re -values examined in the range $0 \leq \beta \leq 5$, the least-damped eigenmode corresponds to the Kelvin-Helmholtz instability, while the stationary three-dimensional eigenmode is substantially more damped. On all three airfoils KH instability in the wake is the strongest at $\beta = 0$ and more damped at $\beta > 0$. It is the strongest of the two instabilities until $\beta \approx 2.5$. Beyond this wavenumber the three-dimensional stationary global mode is the dominant instability, albeit damped at all combinations of the (Re, AoA) parameters examined.

At $\beta = 0$ the instability analysis results shown are in agreement with the threshold of unsteadiness obtained through two-dimensional numerical simulations of flow around airfoils. Instability analyses performed at $\beta \neq 0$ show that the Reynolds number parameter

has a quantitative but not a qualitative influence on the relative importance of the three-dimensional wake and stationary eigenmodes. Two sets of results, one at a lower and one at a higher Reynolds number are shown on the NACA 0009, 0015 and 4415 airfoils and there such analyses are shown for the NACA 0015 airfoil. As expected, increasing the Reynolds number leads to less stable flow, but does not change the qualitative picture described above, the KH mode being always less damped than the stationary eigenmode. The first unstable (KH) mode on the NACA 0015 airfoil is found at $Re = 230$. Figure 4.15 also documents the effect of Reynolds number on the frequency of the KH mode on all three airfoils within the β range in which the KH is the dominant flow instability. It can be seen that the frequency on each airfoil is practically independent of Re and only a weak function of β , decaying slightly as the wavenumber increases and the corresponding periodicity length shortens.

4.2.4 Effect of AoA on the modal stability

Figure 4.15 shows the effect on the global eigenmodes of changing the angle of attack while keeping a constant value of $Re = 200$. Results obtained at the value of $AoA = 18^\circ$ previously examined, as well as those at a lower value $AoA = 15^\circ$ are shown, since angles of attack beyond the higher of the two lead to time-periodic flow.

It can be seen that lowering the angle of attack strongly damps both the Kelvin-Helmholtz and the three-dimensional stationary mode, although the relative importance of the two mechanisms remains the same. It can be anticipated that in the limit of even lower AoA values, the KH instability found by global analysis on the NACA 0012 airfoil [163] will also be present on all three airfoils considered presently and will also be damped at this low Reynolds number value. It can further be anticipated that, in the limit of lower AoA values Tollmien-Schlichting instabilities in the boundary layer around the airfoil will emerge in the eigenspectrum. However, the Reynolds number range examined is too low for such an instability to be active.

With regards the three-dimensional stationary mode of laminar separation, it has been found to be damped on all three airfoils at all parameters examined, despite the existence of certain combinations of the (Re, AoA) parameters at which the level of recirculation in the primary separation bubble exceeds the value of 7.5%, predicted by [142] to be sufficient for this mode to be amplified on the flat plate. Present results suggest that higher levels of recirculation are required for instability of the stationary mode of laminar separation on any of the three airfoils analyzed.

4.2.5 Effect of geometry on the modal stability

Finally, the effect of thickness and camber on the characteristics of the eigenmodes discussed is examined on all three airfoils at $Re = 200$, $AoA = 18^\circ$. Figure 4.16 shows that changes in these parameters introduces quantitative but not qualitative differences to the instability results previously shown. Increasing the airfoil thickness in the symmetric airfoils while maintaining the same (zero) camber, results in a flow that is less stable against both KH and 3D stationary modes. The same effect is found by an increase in camber

Figure 4.14: Stability analysis of flow around the three airfoils at $AoA = 18^\circ$. *Left:* Growth rate of the leading eigenvalue, as function of spanwise wavenumber β . *Right:* Strouhal number dependence on wavenumber. $\circ$: 0009 at $Re = 200$ and 220 ; $\square$: 0015 at $Re = 200, 220$ and 230 ; $\diamond$: 4415 at $Re = 150$ and 200 .

4. Instability of Separated Airfoil Flow

Figure 4.15: Stability analysis of flow around the three airfoils at $Re = 200$, $AoA = 15^\circ$ and $AoA = 18^\circ$. *Left:* Growth rate of the leading eigenvalue, as function of spanwise wavenumber β . *Right:* Strouhal number dependence on wavenumber. $\circ$: 0009; $\square$: 0015; $\diamond$: 4415.

while the airfoil thickness is maintained, namely a destabilization of both classes of global instabilities.

Figure 4.16: Effect of thickness and camber on modal results of the three airfoils at $Re = 200$, $AoA = 18^\circ$: Growth rate (*left*) and Strouhal number (*right*) of the leading eigenvalue, as function of the spanwise wavenumber. $\circ$: 0009; $\square$: 0015; $\diamond$: 4415.

4.2.6 Modal stability of the unstable steady flow

The flow past an airfoil become unsteady when its conditions exceed some critical Reynolds number, angle of attack and depending on the airfoil geometry, etc. To determine a steady base flow, the Newton's method [18] has been successfully applied on the unsteady Navier-Stokes equations. However this method leads to solve the invert of matrix and requires a large memory storage. Another efficient algorithm has been introduced by [3], named *Selective Frequency Damping* (SFD) method. It is based on the roadmap of filtering the unstable temporal frequencies, by adding a forcing term and a time derivative term into the momentum equation, as shown in 4.5

$$\begin{aligned}\dot{\mathbf{q}} &= \mathbf{NS}(\mathbf{q}) - \chi(\mathbf{q} - \bar{\mathbf{q}}), \\ \dot{\bar{\mathbf{q}}} &= \frac{\mathbf{q} - \bar{\mathbf{q}}}{\Delta},\end{aligned}\tag{4.5}$$

where $\mathbf{q}$ represents the problem unknown, the dot represents the time derivative, $\mathbf{NS}$ represents the Navier-Stokes equations, χ is the control coefficient and Δ is the associated filter width, $\bar{\mathbf{q}}$ is a filtered version of $\mathbf{q}$. To achieve an effective convergence speed, χ can be chosen as two times of the growth rate of unsteady instability and $\Delta = \frac{1}{2\omega_{us}}$, where ω_{us} is the angular frequency of the unsteady flow. The steady solution is achieved when $\mathbf{q} = \bar{\mathbf{q}}$. The drawback of this method is it can not be applied for the flow which leading mode has zero frequency..

Figure 4.17 shows the comparison of vorticity and streamlines of the the unsteady flow pasts NACA 4415 at high angle of attack 20° and $Re = 500$. Its associated steady solution is obtained by applying SFD method. The shedding vortex behind the foil has been killed

4. Instability of Separated Airfoil Flow

Figure 4.17: Vorticity and streamlines of the unsteady flow (left) and its associated steady solution (right) for NACA 4415 at $Re=500$ and $AoA = 20^\circ$, respectively.

Figure 4.18: Stability analysis of the flow around NACA 4415 at $Re=500$ and $AoA = 20^\circ$. *Left:* growth rate, *Right:* Strouhal number.

and a long separation bubble comprised by two circulation zones is generated which has a maximum reversed value 19.08%.

Figure 4.18 shows the eigenvalues given by modal instability analysis. The approximate critical wavenumbers for the travelling- and stationary mode are 2.8 and 5, respectively. The most unstable mode is two-dimensional perturbation travelling mode. The stationary mode includes a growing perturbation in a range of wavenumber $3.25 < \beta < 5$, and damping perturbation beyond the critical wavenumber.

The amplitude of primary and sub-primary modes for the two-dimensional perturbations are shown in Figure 4.19. The sub-primary travelling mode is a sort of Kelvin-Helmholtz mechanism as analyzed in section 4.2. Contrarily, the primary perturbation is self-excited, similarly to the discovery in the laminar separation bubble formed in the flat-plate flow [162],[142], will extent to about 15 times chord length in the wake. Figure 4.20 visualizes the three-dimensional mode at $\beta = 5$ which leading eigenvalue is very closed to the critical wavenumber with zero growth rate. Not like the stationary mode analyzed in the more slower Reynolds flow, the distribution of the perturbation emanates very over the airfoil and decreases with the increasing of wake. The damped sub-leading eigenvalue

Figure 4.19: Amplitude functions of the primary (*left column*) and secondary (*right column*) oscillation modes at $\beta = 0$ and $Re = 500$. Normalized by the highest velocity component, from -0.1 to 0.1. The associated eigenvalues are $(0.47476, 2.0236)$ and $(0.12069, 2.19388)$, respectively.

at $\beta = 5$ has a smaller frequency and evolved into the wake.

Figure 4.20: Amplitude functions of the primary stationary (*Left column*) and secondary traveling (*Right column*) modes at $\beta = 5$ and $Re = 500$. Normalized by the highest velocity component, from -0.1 to 0.1. The associated eigenvalues are $(-4.88364 \times 10^{-4}, 0)$ and $(-0.154204, 0.276417)$ respectively.

4.3 Local Stability Analysis of 2D Wake

To further demonstrate the discovery of traveling mode dominating the flow instability, local spatial instability is performed by using the streamwise velocity profile in vertical direction. The functions of $u(y)$ are extracted at $x = 2.5, 5$ and 10 from the DNS results for the flow past NACA 0015 at $Re = 200$ with $AoA = 18^\circ$. From Figure 4.21, it shows that

4. Instability of Separated Airfoil Flow

at $x = 2.5$, the highest amplitude wavenumber is $-\alpha_i = 2.5$. From the BiGlobal stability analysis results at $\beta = 0$, the periodic length of the most amplitude perturbations is $L_x \approx 2.23$. The local LST analysis result is very closed to the the BiGlobal analysis. In the downstream far away from the wing, the viscosity plays an more important role. Figure 4.22 shows the wave-like perturbations with $-\alpha = 1.7$ at $x = 10$.

Figure 4.21: Local spatial stability analysis using the wake profile $u(y)$ at different location. Wavenumbers versus of ω of real part α_r (left) and imaginal part α_i (right).

Figure 4.22: Wake modes further verified by local spatial stability analysis. Amplitude function of $\hat{u}$ -component (left) and $\hat{v}$ -component (right).

4.4 Nonmodal Linear Stability Analysis

Linear stability analysis of steady base flows is completed in this section by studying the short-time evolution of small-amplitude three-dimensional perturbations superposed upon the two-dimensional base flows over the three NACA airfoils. Ample evidence can

be found in the literature that substantial transient energy growth can be expected on both bluff-bodies [1] and lifting surfaces [2], while first nonmodal stability analysis results of flow around airfoils at large angles of attack have also been obtained recently [64, 112]. The steady base flows shown in Tables 4.4 and 4.5 are analyzed with respect to their capacity to sustain transient energy growth at Reynolds numbers slightly below the primary transition to a periodic wake. For the analysis that follows the choice of range of spanwise wavenumbers monitored, $0 \leq \beta \leq 2\pi$, was based on results of Gioria et al. [64] and the direct-adjoint iteration procedure [12] is performed for time-horizons τ long enough for the maximum of the gain function $G(\tau)$ to be well defined at all spanwise wavenumbers β .

4.4.1 Effect of wavenumbers on transient growth

Figure 4.23 presents $G(\tau)$ results at spanwise wavenumber values $0 \leq \beta \leq 2\pi$ for all three profiles. Higher β values were not analyzed, since it is clear from the results shown that the largest transient energy growth is attained for two-dimensional perturbations ($\beta = 0$). The results are qualitatively similar on all three airfoils and, in turn, also on those obtained on the circular cylinder [1]. Interestingly, flow corresponding to the smaller wavenumber values shown in the results on all three airfoils also exhibits relatively large transient growth, implying that low Reynolds number flow around large aspect ratio untapered and unswept wings may experience this linear instability mechanism if placed at high angles of attack to the oncoming flow. It should be noticed that the $\beta = 0$ instability only overtakes its three-dimensional counterparts after a certain time has elapsed. This result can be seen in Figure 4.24 and points to the fact that, when faced with a new non-modal analysis problem, long-time-horizon integration need to be performed in order to establish which class of perturbations grows at most transiently.

More details in the flow of NACA 0015 at $Re = 200$ with $AoA = 18^\circ$ transient growth comparisons in Figure 4.24 show for the two-dimensional perturbations its transient growth level is lower than gained by the three-dimensional perturbations, till the time-horizons reached approximately about $\tau = 2.0$. With the time advanced, the short wavelengths perturbations decay faster than long wavelengths. After $\tau \geq 4$, the two-dimensional optimal perturbations produce the larger transient growth and dominant the subsequent time-horizons, as shown clearly in Figure 4.23.

The leading optimal perturbations amplitude function at $\beta = 0$ are presented in Figure 4.25 by the normalized vorticity. The vorticity structure has a similar pattern as in modal analysis (see Figure 4.10). The transient growth has no significant difference in short time-horizons as gained by two- and three-dimensional perturbations. The maxima *Gain* is around about $\tau = 30$ at $\beta = 0$ and $\tau = 25$ for $\beta = 1$, respectively. The transient growth will decay after reaching its maxima, which are shown in Figure 4.26.

Figure 4.27 presents the initial optimal perturbation in terms of its three velocity components in the top row, followed by optimal linear perturbations at different, progressively larger time-horizons τ . The optimal initial conditions are strongly reminiscent of those found on the Low Pressure Turbine blade analysis [2, Fig. 17] where structures inclined against the main shear were also observed. As time progresses, the perturbations evolve

Figure 4.23: Optimal gain $G(\tau)$ as a function of wavenumber β at $Re = 200, AoA = 18^\circ$ for the three airfoils: 0009 ($\circ$), 0015 ($\square$) and 4415 ($\diamond$) airfoils. Upper-to-lower curves: $\beta = 0, \pi/8, \pi/4, \pi/2, \pi$ and 2π .

Figure 4.24: Maximum gain $G(\tau)$ at $\beta = 0$ (—), $\beta = \pi/4$ (---), $\beta = \pi/2$ (- · -) and $\beta = \pi$ (···) for the 0015 airfoil at $Re = 200$, $AoA = 18^\circ$ at short time-horizons in the range $0 \leq \tau \leq 4$.

Figure 4.25: Sequence of leading optimal perturbations vorticity contours developed from the two-dimensional flow past the NACA 0015 airfoil at $Re = 200$, $AoA = 18^\circ$.

4. Instability of Separated Airfoil Flow

Figure 4.26: Three-dimensional optimal growth as a function of τ at $Re = 200$ and $AoA = 18^\circ$ for the NACA 0015 with wavenumber $\beta = 1$. Compared with $\beta = 0$ (dashed line).

Figure 4.27: Optimal perturbations of the flow past the NACA 0015 airfoil at $Re = 200$, $AoA = 18^\circ$, $\beta = 1$. From upper to lower, $\tau = 0, 10, 20, 30, 40, 50$ and 60 . Scaled at $[-0.5, 0.5]$ after normalizing with respect to each maxima velocity component in time interval.

into the amplitude functions of the Kelvin-Helmholtz modal instability in the wake, shown in Figure 4.12.

4.4.2 Effect of Re and AoA on the transient growth

The dependence of the optimal gain on parameter changes is documented in Figure 4.28 for the three airfoils studied. The left column presents results of changes in Reynolds number at a fixed $AoA = 18^\circ$, while results obtained by keeping $Re = 200$ constant and varying the AoA parameter are shown in the right column of this figure. It can be seen that changes of $O(20\%)$ in either parameter lead to up to three orders of magnitude increase of the maximum value of $G(\tau)$. This is consistent with the $G \sim O(10^{10})$ prediction of Loh et al. [112] on the NACA 0012 airfoil at separated flow conditions, and analogously large gains found in other flows exhibiting large-scale laminar separation [e.g. 20, 21]. Such large transient energy amplification values are also known from the related earlier works of Abdessemed et al. [2] and Sharma et al. [148] in the wake of a Low Pressure Turbine blade cascade, as well as from the analogous analysis of Abdessemed et al. [1] in the circular cylinder wake. While the relevance of the strong transient growth in the cascade to the present open flow results may be questioned on account of the transverse periodicity of the domain analyzed, the results of Figures 4.23 and 4.28 are entirely consistent with those in the wake of the cylinder: two-dimensional perturbations experience the strongest transient growth, while the relevance of this scenario is diminishing for increasingly large wavenumbers / short spanwise domain lengths.

4.4.3 Effect of geometry on the transient growth

Finally, Figure 4.29 compares the maximum transient growth on the three airfoils at the respective optimal conditions. It can be seen that an increase in airfoil thickness for the two symmetric airfoils, as well as an increase in camber for the two thicker airfoils leads to increase in the maximal gain obtained. This result could be significant for applications such as wind-turbine blades, in which airfoils substantially thicker than the ones examined presently are utilized.

4.5 Instability Analysis of Periodic Wakes

Time-periodic base flows were obtained by running two-dimensional laminar simulations for sufficient time until well-defined periodic signals were obtained. The temporal dependence of integral quantities, such as the forces exerted on the airfoil surface, reached monochromatic sinusoidal temporal dependence, while probes at random locations in the wake identified the (same) frequencies and (different) amplitudes related to the passage of vortices related with the periodic shedding at the leading and trailing edge. The non-dimensional Strouhal number, St , associated with the shedding frequency of the periodic wave that arises from the first bifurcation examined in Section 4.2, is shown in Figure 4.30 in the range of Reynolds numbers considered in this study for the three different profiles. Error bars are added to the results, reflecting the different methods used to estimate the

4. Instability of Separated Airfoil Flow

Figure 4.28: Maximum gain, $G(\tau)$ at $\beta = 0$ for the 0009 ($\circ$), 0015($\square$) and 4415($\diamond$) airfoils. Left column: Effect of Reynolds numbers, $Re = 150, 200$ and 220 at $AoA = 18^\circ$. Right column: Effect of angle of attack, $AoA = 10^\circ, 15^\circ$ and 18° at $Re = 200$.

Figure 4.29: Optimal gain at $Re = 200, AoA = 18^\circ, \beta = 0$ as a function of airfoil thickness and camber.

signal frequency from the simulation results. The same pattern emerges for all three airfoils, namely that an increase of Re leads to almost quadratic increase of the perturbation frequency at the lower Reynolds numbers, while a trend of St to saturate is observed at the highest Re -values examined. Moreover, the thickness and camber of the airfoil do not appear to have a strong influence on the St values obtained at each Reynolds number, although a steeper increase can be seen in the curve corresponding to the cambered airfoil compared to its symmetric counterparts.

Thirty two snapshots are obtained from the periodic base flow in one period and are interpolated spectrally from the full domain onto a domain of smaller extent around the airfoil. The resulting high-resolution interpolated snapshots are then used in the subsequent linear temporal Floquet instability analysis, in which the wavenumber corresponding to the spanwise direction is fixed. The inflow and far-field boundary conditions on all perturbations are homogeneous Dirichlet for the velocity and homogeneous high-order boundary conditions for the pressure, while the outflow boundary conditions for velocity and pressure perturbations are homogeneous Neumann and homogeneous Dirichlet boundary conditions, respectively. The action of the linear operator over the amplitude function of the perturbations in the resolution of the Floquet analysis is obtained by direct-integration of the linearized version of the Navier-Stokes equations.

The secondary instability arising when the time-periodic flow becomes unstable to three-dimensional perturbations is considered by examining the dependence of the modulus of the Floquet multipliers, μ , on the flow parameters Re and AoA , as well as on the geometric parameters of airfoil thickness and camber. Monitored are high values of angle of attack and Reynolds numbers past the two-dimensional Hopf bifurcation, until two distinct three-dimensional perturbations have become unstable.

Figure 4.31 presents the dependence of the modulus of the Floquet multiplier μ on

Figure 4.30: Dependence of the Strouhal number of the wake behind the NACA 0009 ($\circ$), 0015 ($\square$) and 4415 ($\diamond$) airfoils as function on the Reynolds number considered in this work.

the spanwise wavenumber β for the three airfoils in the Reynolds number range $400 \leq Re \leq 600$ at $AoA = 20^\circ$. Starting from two-dimensional flow, $\beta = 0$, discrete values of the spanwise wavenumber parameter are examined until linear three-dimensional instability is encountered in the examined wavenumber range $0 \leq \beta \leq 20$. Any three-dimensional instability is expected to be encountered in this range, since at high β numbers the solution is dominated by viscous effects, for which multipliers are negative and proportional to $e^{-\beta^2}$.

Results obtained are qualitatively analogous on all three profiles. At the lowest Reynolds number shown, $Re = 400$, all Floquet multipliers remain $||\mu|| < 1$, implying stability of the corresponding periodic orbit: the two-dimensional time-periodic flow examined is also a solution of the three-dimensional equations of motion. Several intermediate Reynolds numbers were examined until the next value shown, $Re = 442, 474$ and 435 , for the NACA 0009, 0015 and 4415 profiles, respectively. At these Reynolds number values three-dimensional perturbations with $\beta \approx 11$ reach $\mu \approx 1$, namely the periodic orbit becomes unstable to three-dimensional perturbations that are spanwise periodic with a periodicity of $L_{z_1} \approx 2\pi/11 \approx 0.57$ chord lengths. The curves in Figure 4.31 where the maximum value of μ of the short-wavelength modal perturbations is equal to unity correspond with the critical Reynolds number of this secondary flow eigenmode. When the Reynolds number is increased to $Re = 500$ and 600 the instability of the mode peaking around $\beta = 11$ becomes stronger and at a yet higher Reynolds number value a second peak appears around $\beta \approx 3$, i.e. the two-dimensional periodic orbit becomes unstable to an additional spanwise periodic eigenmode having a longer wavelength of $L_{z_2} \approx 2$. At the same time, a shift towards lower wavenumber of the leading eigenmode is observed. This behavior, namely the existence on all three stalled airfoils of a strong three-dimensional secondary instability at larger wavenumbers (short spanwise wavelengths) and a weaker one at small wavenumber (long spanwise wavelengths) is the opposite to what is known about instability in the wake of the circular cylinder ([171];[10]) where also two spanwise

Figure 4.31: Maximum Floquet multiplier, μ , as function of wavenumber β for a range of Reynolds numbers at $AoA = 20^\circ$. *Upper:* NACA 0009 at (upper-to-lower) $Re = 600, 500, 442$ and 400 . *Middle:* NACA 0015 at (upper-to-lower) $Re = 600, 500, 474$ and 400 . *Lower:* NACA 4415 at (upper-to-lower) $Re = 600, 500, 435$ and 400 .

periodic modes are identified but the long-wavelength one dominates.

The magnitude of the Floquet multipliers on the three airfoils can be compared at the same Reynolds number and angle of attack parameters in the results of Figure 4.31. In all cases the short-wavelength mode is strongest on the NACA 4415 airfoil. Interestingly, the periodic wake of the thin symmetric NACA 0009 airfoil is less stable at $Re = 400$ and more unstable at $Re = 500$ than that of its thicker counterpart at the same angle of attack, although at the highest Reynolds number value examined, $Re=600$ both short- and long-wavelength instabilities on the NACA 0015 overtake their counterparts on the NACA 0009 airfoil. The strong stabilizing effect of a small decrease in AoA is shown in the results of Figure 4.32, where the dependence of μ on β for all three airfoils is shown at $Re = 500$ and two values of the angle of attack, $AoA = 18^\circ$ and 20° . The interpretation of this result is that at the low values of the angle of attack typically examined in the literature, the Reynolds number must be substantially larger in order for three-dimensionalization of the wake on account of secondary instability to set in.

Figure 4.32: Effect of angle of attack on Floquet multiplier, μ , for the NACA 0009 ($\circ$), 0015 ($\square$) and 4415 ($\diamond$) airfoils at $Re = 500$ and $AoA=18^\circ$ (full symbols) and 20° (open symbols)

Table 4.7 presents estimates of the period of the base flow obtained by using two distinct methodologies: discrete Fourier transform of the signal obtained from a single probe and tracking of the zero crossing of the relative lift force with respect to the mean lift. In both cases the time integration was sufficiently long for well-defined periodicity to set in. As mentioned earlier, the differences in these estimates give rise to the error bars shown in Figure 4.30. For these periodic-base flow Table 4.7 presents maximum values of μ and β for the leading short-wavelength (SW) and long-wavelength (LW) instabilities as function of Re and AoA for a number of the intermediate parameters at which secondary instability on the three NACA airfoils was examined. The critical conditions for both the dominant SW instability and the second LW three-dimensional perturbation on all three airfoils are shown in Table 4.8, calculated by cubic interpolation using the values described in

Table 4.7: Period of base flow and maximum Floquet multiplier, μ , and wavenumber, β , as function of Re and AoA for the three NACA airfoils; dash denotes that no unstable μ has been found at the respective parameters

Airfoil	Re	AoA	Period		Short-wavelength		Long-wavelength	
			lift	Fourier	β_{max}	μ_{max}	β_{max}	μ_{max}
0009	500	18	1.713	-	-	-	-	-
	300	20	2.136	2.174	-	-	-	-
	400	20	1.937	1.961	-	-	-	-
	450	20	1.890	1.887	10.66	1.06	2.81	0.84
	500	20	1.846	1.852	10.44	1.38	2.88	0.90
	550	20	1.810	1.818	10.07	1.56	2.91	0.95
	600	20	1.805	1.802	9.44	1.61	2.87	0.97
0015	500	18	1.905	-	-	-	-	-
	300	20	2.233	2.247	-	-	-	-
	400	20	2.016	2.020	-	-	-	-
	450	20	1.949	1.961	-	-	-	-
	500	20	1.897	1.905	11.03	1.17	2.96	0.92
	550	20	1.861	1.869	10.77	1.47	3.01	0.99
	600	20	1.836	1.852	10.45	1.70	3.05	1.04
4415	500	18	1.770	-	-	-	-	-
	300	20	2.222	2.222	-	-	-	-
	400	20	1.991	2.003	-	-	-	-
	450	20	1.922	1.941	11.13	1.12	-	-
	500	20	1.870	1.886	11.20	1.53	2.95	0.97
	550	20	1.827	1.839	11.25	1.92	3.03	1.04
	600	20	1.801	1.810	11.29	2.25	3.08	1.09

Table 4.7.

Table 4.8: Critical conditions for secondary instability of Modes A and B in the wake of the three airfoils. SW: short-wavelength, LW: long-wavelength

Mode	0009		0015		4415	
	Re_{crit}	β_{crit}	Re_{crit}	β_{crit}	Re_{crit}	β_{crit}
SW	632 ± 6	2.9 ± 0.1	561.0 ± 0.1	3.03 ± 0.01	523.2 ± 0.1	2.99 ± 0.01
LW	441.8 ± 0.1	10.68 ± 0.01	474.7 ± 0.1	11.08 ± 0.01	436.5 ± 0.1	11.12 ± 0.01

The representation of the Floquet multiplier as a complex number, $\mu = |\mu|e^{\pm i\theta}$, introduces the phase function θ . The dependence of the normalized phase function $\theta/(2\pi)$ on β is shown in Figure 4.33 for $Re = 400, 500$ and 600 , for the three airfoils considered in the present study. The dominant multiplier of the short-wavelength perturbations is real, while that corresponding to the long-wavelength perturbations is complex, appearing

Figure 4.33: Ratio $\theta/(2\pi)$ between the secondary and primary periods, T_s and T as function of β at $AoA = 20^\circ$. In all three plots lower to upper set of results correspond to $Re=400, 500$ and 600 .

Figure 4.34: Isosurfaces of $\hat{\omega}_z$: NACA 4415 airfoil at $Re = 500$, $AoA = 20^\circ$. *Upper* : Long-wavelength instability at $\beta = 3$. *Lower* : Short-wavelength instability at $\beta = 11$.

as a complex-conjugate pair. This corresponds to a Neimark–Sacker bifurcation which introduces a new temporal frequency, $1/T_s$, where $T_s = 2 (\pi/\theta)T$ and T is the period of the first bifurcation, namely the inverse of the Strouhal number shown in Figure 4.30. The corresponding three-dimensional long-wavelength solutions are quasiperiodic in time and the combination of the complex-conjugate pair could generate quasiperiodic standing or traveling-wave solutions. Since $2\pi/\theta > 2$, then $T_s > 2T$ and an interesting result obtained is that at low Reynolds numbers the value of $\theta/(2\pi)$ converges monotonically to zero at around $\beta \approx 5$, leading to very large associated frequency values in the vicinity of this value of the wavenumber parameter.

4.6 A New Path for the Generation of Stall Cells

Figure 4.34 shows the amplitude functions of the spanwise component, $\hat{\omega}_z$, of the perturbation vorticity of the unstable three-dimensional SW and LW Floquet eigenmodes on the 4415 airfoil at $Re=500$, $AoA=20^\circ$, while Figure 4.35 shows a perspective view of the three-dimensional reconstruction of the secondary perturbation. Similar reconstructions for these modes are obtained for the other two geometries considered in this study. The reconstruction of perturbations has used either of the following expressions

$$\hat{\mathbf{u}} = (\hat{u} \cos(\beta z), \hat{v} \cos(\beta z), \hat{w} \sin(\beta z))^T$$

or

$$\hat{\mathbf{u}} = (\hat{u} \sin(\beta z), \hat{v} \sin(\beta z), \hat{w} \cos(\beta z))^T$$

where a half-mode model approach is assumed. The maximum modulus of the linear perturbations used for this reconstructions is $\approx 0.05\%$. The resulting images are reminiscent

Figure 4.35: Three-dimensional reconstruction of the results of Figure 4.34. *Upper* : LW instability at $\beta = 3$. *Lower* : SW instability at $\beta = 11$.

of the well-known *Mode A* and *Mode B* instabilities in the wake of the circular cylinder [10, 171], however on all three airfoils the opposite instability scenario is seen to lead the two-dimensional wake to three-dimensional instability and transition: *short-wavelength instability precedes linear amplification of the long-wavelength eigenmode on all three airfoils*. In order to avoid confusion with the established literature on the modes of the cylinder, here reference is made to the SW and LW global modes.

Figure 4.36 presents the reconstruction of superposing secondary perturbations at a level $\epsilon = 5 \times 10^{-4}$ onto the time-periodic base flow. The *black-white* isosurface shows the streamwise vorticity ω_x and the *blue-red* color stands for the spanwise component vorticity ω_z . The *purple-green* color visualized the vertical vorticity ω_y . It can be found that a pair of massive separation bubble formed in the SW secondary perturbations.

Figure 4.37 shows streamlines of the composite field which may be reconstructed by linear superposition of the two-dimensional time-periodic base flow and either of the two unstable secondary eigenmodes at $\epsilon = 5 \times 10^{-4}$ a given time during the period of the

Figure 4.36: Three-dimensional reconstruction of the results of superposing perturbations with LW instability at $\beta = 3$, and *Lower* : SW instability at $\beta = 11$ on base flow at the level of $\epsilon = 5 \times 10^{-4}$.

base flow. Interestingly, at the high $AoA = 20^\circ$ value to which these results correspond, a pattern akin to the “owl face” structures seen at high Reynolds number turbulent flow may be seen to develop near the trailing edge of the airfoil on account of linear secondary amplification of the leading short-wavelength secondary perturbation. This pattern is present on all airfoils considered and is related to the fact that at this high AoA value the short-wavelength mode is the leading perturbation. This may not be the case at other combinations of the (Re, AoA) parameters, as can be seen in the results of Figure 4.32 corresponding to the slightly lower $AoA = 18^\circ$ value. At those conditions the peak of the short-wavelength eigenmodes has shifted toward higher β values, as has that of the long-wavelength mode, while a third instability is seen to develop on all three airfoils at intermediate β values. In other words, the visual analogy of the owl-face pattern resulting from linear secondary Floquet instability is valid for the high angles of attack examined. Analyses at higher Reynolds number will be necessary if the most amplified secondary mode is to be identified at lower values of the angle of attack.

Figure 4.37: Wall streamlines of the reconstructed flow field composed of four periods of the $\beta = 3$ LW eigenmode (*upper*) and the $\beta = 11$ eigenmode (*lower*) perturbation superposed at $\epsilon = 5 \times 10^{-4}$ upon the $O(1)$ time-periodic base state. The critical points of saddles and nodes are mutual distributed periodically along the spanwise of the wing.

Nevertheless, the scenario identified at $AoA = 20^\circ$ is a new linear mechanism of modal growth, qualitatively delivering analogous structures to the observed owl-face flow patterns. This mechanism is distinct to the previously reported linear amplification of the three-dimensional stationary spanwise periodic eigenmode superposed upon the steady primary flow [143]. The present secondary instability arises on account of a succession of two-dimensional unsteadiness of the primary steady laminar base flow, followed by three-dimensional exponential amplification of a spanwise periodic secondary instability. Both of these scenarios are still a long way from explaining the separated flow patterns in turbulent flow, or answering questions regarding their occasional appearance as steady flow structures. However the presently identified mechanism can be operative at substantially larger Reynolds numbers than those at which linear three-dimensional stationary mode of the primary laminar separation bubble would survive as the only amplified linear instability. Work is underway to further quantify the presently observed transition scenario at $AoA < 20^\circ$ and (substantially) higher values of the Reynolds number, at which secondary instability of the time-periodic base flow can be identified.

Chapter 5

Instability of the Laminar Trailing Vortices

5.1 Wingtip Vortex Formation

5.1.1 3D elliptical wing model

An orthonormal Cartesian frame of reference is considered, in which x denotes the axial spatial direction, while y and z are the normal and spanwise spatial directions, respectively. A three-dimensional wing model is constructed based on the low-Reynolds number Eppler E-387 airfoil, as shown in Figure 5.1. The coordinates are listed in appendix D. The wing planform is elliptical, its major axis being the wing span, $b = 1.714$, while its minor axis is the chord $c = 1$ of the E-387 airfoil at the plane of symmetry, $z = 0$. Thus a low aspect ratio wing is obtained. The airfoils defined by cuts of the wing by planes normal to the Oz axis at locations $0 \leq z \leq b/2$ are also E-387 profiles, the respective chords of which are scaled such that they fit the elliptical planform at each z location.

Figure 5.1: Perspective view of the E387 elliptic wing and its airfoil profile.

5.1.2 Base state computation

Direct numerical simulations

Incompressible fluid flow is considered in the present work, taking the kinematic viscosity, ν , as a free parameter. In this manner, two Reynolds numbers could be defined, that

Figure 5.2: Full view of the computational domain, showing the macro-elements structure.

based on mid-plane chord, Re_c , and the more commonly used in instability analysis of vortex systems, Reynolds number based on circulation, Re , respectively defined as

$$Re_c = \frac{U_\infty c}{\nu}, \quad \text{and} \quad Re = \frac{\Gamma}{\nu}. \quad (5.1)$$

Direct numerical simulations for the calculation of the basic state were performed using the nek5000 open source code [55]. The software *Gambit* was used to construct the three-dimensional meshes in which the half-wing model has been embedded, as shown in Figure 5.2. The blocks divide the domain into several macro-elements, in which discretized by a number of hexahedral elements. Within each of the element a high order polynomial was chosen in the spectral element code, until convergence was reached, as will be discussed shortly. The inflow boundary condition is imposed on the very left vertical surface as a uniform flow, the symmetry boundary condition is used in the oyz – plane, and the rest boundaries are set as outflow. The converged results of the simulations were post-processed in order to define the flow circulation at an axial location x_i at which the analysis was performed, using

$$\Gamma = \int \int \zeta(x_i, y, z) dy dz, \quad (5.2)$$

where $\zeta = -\partial_y \bar{w} + \partial_z \bar{v}$ is the axial vorticity component at x_i .

Table 5.1 presents results of the base flow around the wing, computed by direct numerical simulation using the nek5000 code at $Re_c = 1750$, $AoA = 0^\circ$ and $AoA = 5^\circ$. At these conditions, $\Gamma = 0.033$ and 0.12 , respectively; at the angle of attack $AoA = 5^\circ$ analyzed below, this corresponds to $Re = 210$. Shown in the table is the vorticity at a randomly selected location close to the trailing vortex core, as obtained on two meshes, $M1$ extending to $z = 2.5$ in the spanwise direction and comprising 16120 elements, and $M2$,

extending to $z = 5$ and comprising 25312 elements. Two computations are performed at each angle of attack, respectively using polynomial orders of $p = 4$ vs. 5 and $p = 5$ vs. 7. It can be seen that the relative error is of $O(1\%)$ in any of the three vorticity components, indicating a reasonable convergence of the base flow first derivatives, as required in the subsequent stability analysis. The vorticity distribution immediately behind the wing in the $AoA = 5^\circ$ case was used to initialize point-vortex simulations of the wake evolution. Equation (5.5) was advanced for forty non-dimensional time-units, which corresponds to an axial location well beyond that used in the DNS. This permitted comparisons between viscous and inviscid flow development within the domain resolved by the DNS and also revealed the long-time evolution of the inviscid counterpart of the flowfield.

Table 5.1: Convergence study of vorticity components at the location $(x, y, z) = (12, 0, -0.85)$, close to the vortex core, at $Re_c = 1750$

AoA		ω_x	ω_y	ω_z	ω_x	ω_y	ω_z
		$p = 5$			$p = 7$		
0°	$M1$	0.0287	-0.1559	-0.0306	0.0286	-0.1557	-0.0307
	$M2$	0.0289	-0.1565	-0.0299	0.0289	-0.1563	-0.0299
		$p = 4$			$p = 5$		
5°	$M2$	0.1392	-0.3310	-0.0718	0.1385	-0.3407	-0.0708

Figure 5.3 addresses three critical points on the surface of suction, two foci F_1 and F_2 located closed the wingtip and one saddle S_1 in the middle of the wing. There is one node N_1 invisible at this view perspective, but has the similar property as in Figure 3.3, according to the Poincaré formula 3.4. The streamwise velocity in the symmetry plane at $z = 0$ shows there is a laminar separation bubble covers one-third rear extent of wing. The flow streamlines rolling around the trailing-edge vortex core in the midst of wing. The streamlines past the wingtip rolling over the wing one part evolved in the wingtip vortex and another part will move forwards to the focus and finally embedded in to the separation bubble.

Figure 5.4 visualized the trailing vortices by λ_2 . The wingtip vortex can be identified more neatly from the wake than vorticity as shown in Figure 5.5. Besides the long-distance traveling vortex, there are two separated trailing-edge vortices visualized.

In order to compare the results of the present spatial BiGlobal analysis, as employed to an isolated Batchelor vortex, with those obtained by Mayer & Powell [118] using local instability analysis, a vortex radius $a = (a_y^2 + a_z^2)^{1/2}$ was defined using the polar moments

$$a_y = \left[\left(\int \int \zeta (y - y_c)^2 dy dz \right) / \Gamma \right]^{\frac{1}{2}}, \quad a_z = \left[\left(\int \int \zeta (z - z_c)^2 dy dz \right) / \Gamma \right]^{\frac{1}{2}}, \quad (5.3)$$

with (y_c, z_c) denoting the position of the vortex centroids [66],

$$y_c = \frac{1}{\Gamma} \int \int y \zeta dy dz, \quad z_c = \frac{1}{\Gamma} \int \int z \zeta dy dz. \quad (5.4)$$

Figure 5.3: Streamlines show laminar separation on the wing surface and trailing vortices. The critical points are marked on the surface.

Inviscid point vortex methods

The equations that govern motion of free point vortices can be found in classical fluid mechanics texts [14]. The ordinary differential equations describing evolution of the vortex coordinates without considering axial velocity and after imposing symmetry along the Oxy plane are [102]

$$\left[\frac{dy_i}{dt}, \frac{dz_i}{dt} \right] = \frac{1}{2\pi} \sum_{j=1, i \neq j}^N \Gamma_j \frac{[z_j - z_i, y_i - y_j]}{(y_i - y_j)^2 + (z_i - z_j)^2} - \frac{1}{2\pi} \sum_{j=1}^N \Gamma_j \frac{[-z_j - z_i, y_i - y_j]}{(y_i - y_j)^2 + (z_i + z_j)^2} \quad (5.5)$$

The initial vortex locations are extracted from the three-dimensional steady base flow computed in the DNS on a plane of constant axial coordinate x and a standard 4th order Runge-Kutta method is employed to advance (5.5) along t which, in the context of this simplified inviscid method, can be identified with the axial spatial coordinate, x , by assuming a constant unit axial free-stream velocity.

Circulation Γ_i is attached to the i -th point vortex, such that the initial circulation for N vortices can be computed as $\Gamma = \sum_{i=1}^N \Gamma_i$. The coordinates of the centroids y_c, z_c are then computed as

$$y_c = \sum_{i=1}^N \frac{\Gamma_i y_i}{\Gamma} \quad \text{and} \quad z_c = \sum_{i=1}^N \frac{\Gamma_i z_i}{\Gamma}, \quad (5.6)$$

Figure 5.4: Isosurface of wingtip vortex at the level of $\lambda_2 = -0.02$.

which permits calculation of the vortex radius, $a = (a_y^2 + a_z^2)^{1/2}$ again from the polar moments, now defined as

$$a_y = \left[\sum_{i=1}^N \Gamma_i (y_i - y_c)^2 / \Gamma \right]^{\frac{1}{2}}, \quad \text{and} \quad a_z = \left[\sum_{i=1}^N \Gamma_i (z_i - z_c)^2 / \Gamma \right]^{\frac{1}{2}}. \quad (5.7)$$

Figure 5.5 shows a perspective view of the base flow obtained in the DNS as well as the inviscid counterpart of this flow obtained by the point-vortex method. Four features are highlighted in the DNS result: First, the system of trailing vortices and the vortex parallel to the trailing edge are outlined by the position of the respective vortex cores. Second, a number of surface streamlines indicate the position of primary separation on the wing surface, while some of these streamlines are shown to continue in the wake. Third, five equally spaced isolines of the flow vorticity magnitude in the range $0 \leq \|\nabla \times \mathbf{u}\| \leq 1$ are plotted on four planes located at $x = 6, 8, 10$ and 12 . The subsequent global instability analysis will be performed on the planes shown in this figure for reasons that are explained shortly in the results of Figure 5.6. Finally, the left insert figure shows that steady laminar separation occurs on the wing at this set of parameters.

The result of the inviscid point vortex method is also shown in the right part of Figure 5.5. The vortex core lines in this image are those obtained in the DNS to which the result obtained by time-advancing equation (5.5) is added, using the constant free-stream velocity to convert the time- to axial coordinate. It can be seen that, while there certainly

Figure 5.5: *Left:* Three-dimensional perspective of the DNS-obtained steady laminar base flow over the wing at $Re_c = 1750$, $AoA = 5^\circ$. *Right:* Same for the flow obtained by inviscid point-vortex methods, vortex core lines being those obtained in the DNS, and inviscid instability developing in the wake at $x = 40$ (*right insert*)

exists a qualitative correspondence between the viscous and inviscid results, quantitative differences are also to be observed, especially as regards the vortex core location and size. The long-time evolution of the inviscid flow, obtained by integrating equation (5.5), reveals the development of instabilities in the entire wake. The result at $t = 40$, which approximately corresponds to a distance twice the axial length of the DNS computational domain, can be seen in the right insert figure and shows the development of instability in the vortex sheet as it rolls up. The complete time-evolution of the inviscid flow can be seen as supplementary material of this work.

A more quantitative description of the viscous base flow and its differences from its inviscid counterpart can be found in Figures 5.6 and 5.7. In the first, the dependence of the vertical y_c and lateral z_c coordinate of the trailing vortex core on the streamwise coordinate x is shown, as computed from equation (5.4). It can be seen that in the range $6 \leq x \leq 14$ the function $y(x)$ is approximately linear, indicating a constant downwash velocity. The latter quantity can be computed from these results and can be compared with analytic isolated vortex models on which the simpler (local) instability analysis is based. The point-vortex theory is precisely such a widely-used simplified model, the results of which for the vortex centroids, computed using equation (5.6), are included in Figure 5.6. Despite the relatively low Reynolds number of the DNS, a rather good agreement between the two sets of results can be observed.

Figure 5.7 shows results of application of equation (5.3) to compute the polar moments a_y and a_z using results of direct numerical simulation, as well as results obtained from

application of equation (5.7) to compute the polar moments in an inviscid framework. In both cases only the axial vorticity component has been used in the vicinity of the vortex. The agreement between viscous and inviscid polar moments is less favorable than that seen in the respective vortex centroid results of Figure 5.6. The values of the viscous polar moments can be seen to always exceed those of the respective inviscid quantities at all axial locations. As a consequence, at a given position x , the direct numerical simulation predicts a wider vortex than the one yielded by the inviscid analysis, differences increasing with increasing axial position. This result is not surprising, given the rather low Reynolds number value at which the DNS is performed.

An alternative approach for the generation of a fully consistent base flow which respects all of the flow properties of a trailing-vortex system has been discussed by Paredes *et al.* [131], who solve the Parabolized Navier Stokes equations in order to compute both viscous diffusion and downwash of a counter-rotating vortex system. This approach has not been followed in the present analysis. Instead, classic point-vortex [102] computations, initialized using the vorticity distribution extracted from the DNS results immediately downstream of the wing, have been performed in order to obtain a rapid estimate regarding the evolution of the wake behind the wing in the inviscid flow limit.

The instability analysis consists of first using the DNS to compute a steady laminar viscous basic flow solution of the incompressible Navier-Stokes and continuity equations around the three-dimensional wing, denoted as $\bar{\mathbf{q}} = (\bar{u}, \bar{v}, \bar{w}, \bar{p})^T$. It contains the basic flow velocity components and associated pressure, all of which are *inhomogeneous functions of all three spatial coordinates*. Subsequently, small-amplitude three-dimensional unsteady perturbations, denoted by $\mathbf{q}'(x, y, z, t) = (u', v', w', \hat{p})^T$, are introduced into the full equations of motion and the latter are linearized around the basic state, leading to the three-dimensional Linearized Navier Stokes equations (LNSE) that need to be solved in order to obtain the linear instability characteristics of the flow in question.

Three sets of assumptions on the form of linear perturbations $\mathbf{q}'$ are made in the present work. First, the DNS-obtained base flow is post-processed to extract one-dimensional velocity profiles amenable to local linear instability analysis. The inherent parallel flow assumption restricts the locations in the wake at which local analysis may be performed in a meaningful way to either the wake at large distances from the trailing vortices, or the immediate vicinity of the vortex cores. At these locations the spatial Orr-Sommerfeld equation and its axisymmetric analog will be solved in order to recover the characteristics of the unstable linear local perturbation, $\mathbf{q}'_0$.

The second set of assumptions leads to a class of perturbations, $\mathbf{q}'_1$, that comprise periodic disturbances along the axial spatial direction, implying that the x -derivatives of the basic state are neglected in the instability analysis. In addition, the perturbations are taken to be harmonic in time. The resulting theoretical framework employed is based on the spatial linear BiGlobal modal concept [160] and will be briefly discussed in what follows. It is clear that the assumption of axial homogeneity, invariably made in the vast majority of instability analyses of isolated vortices and literally all presently available literature dealing with linear instability analysis of systems of vortices in the wake of a wing, can only be justified at some distance downstream of the wing, in the so-called near mid-field of the wake [83]. In what follows, spatial BiGlobal analysis will be performed at a number of axial locations in the near mid-field in a range of (low) Reynolds numbers.

Figure 5.6: Dependence of the normal, y_c , and lateral, z_c , spatial coordinates of the trailing vortex centroids on the streamwise coordinate, x . Symbols correspond to calculations using equation (5.4) applied to the DNS data and lines to calculations using equation (5.6) applied to data generated by the inviscid point-vortex method.

Figure 5.7: Dependence of the polar moments, a_y, a_z , and vortex radius, a , on the streamwise coordinate, x . Symbols correspond to calculations using equation (5.3) applied to the DNS data and lines to calculations using equation (5.7) applied to data generated by the inviscid point-vortex method.

The third class of perturbations analyzed, $\mathbf{q}'_2$, will retain the harmonic assumption

in time but relax homogeneity of the base flow along the axial spatial direction in order to permit viscous vortex diffusion along x . This is accomplished by incorporating first derivatives of the base flow along this direction into the linear instability analysis framework. Short of resorting to full-scale direct numerical simulation, the form of linear (and non-linear) perturbations $\mathbf{q}'_2$ that is consistent with this assumption is that described by the PSE-3D concept [160]. Unlike the eigenvalue problem solution of the spatial BiGlobal approach, PSE-3D is based on space-marching of the appropriate LNSE on the Oyz plane along the axial spatial direction. The plane marching is initialized using results of the spatial BiGlobal approach at a given axial location and it is interesting to compare the instability results obtained by the space marching of the PSE-3D with those delivered by the spatial BiGlobal analyses at selected axial locations. In this manner, the appropriate linear instability analysis model for this class of flows can be established.

5.2 Spatial BiGlobal Analysis in the Wingtip Vortex

Having validated the algorithms on a relevant problem, attention is turned next to the main topic of the present article, namely instability analysis in the wake of a short aspect-ratio three-dimensional wing using global linear analysis methods. Planes of data at constant $x = 6, 8, 10$ and 12 in the region of constant downwash were extracted from the three-dimensional base flow fields computed by DNS in Section 5.1 and the spatial global EVP (2.45) was solved at non-dimensional frequencies in the band $\omega \in [0, 12]$. Half of the computational domain was resolved using FD-q6 spatial discretization, on rectangular domains discretized by $(Ny = 101) \times (Nz = 161)$ and $(Ny = 161) \times (Nz = 201)$ collocation points. All subsequent results were obtained using the lower resolution, which was found to be sufficient for convergence. Two classes of boundary conditions, symmetric and antisymmetric, were imposed at the symmetry boundary on all perturbation components, alongside homogeneous Dirichlet conditions on all other domain boundaries.

Figure 5.8 presents the dependence of the spatial amplification rate, $-\alpha_i$, on the wavenumber α_r at the conditions examined. Both symmetric and antisymmetric disturbances are found to be linearly unstable over a wide range of wavenumbers. In both classes of disturbances the entire amplified branch is damped as the downstream distance from the wing decreases. It can also be clearly seen that the symmetric perturbations are substantially more amplified than their antisymmetric counterparts. Results of the spatial Orr-Sommerfeld equation added to this figure will be discussed in detail in Section 5.4.1.

Figure 5.9 compares the amplitude functions of the linear symmetric and antisymmetric perturbations at a single downstream location, $x = 10$ and frequencies near the respective maximally amplified, $\omega = 5.5$ and $\omega = 5$. Shown are the real and imaginary parts of the amplitude functions of the perturbation velocity vector. In all plots the amplitude functions are normalized with respect to their maxima and are shown as greyscale isosurfaces superposed upon the respective underlying base flow. The latter is made visible through four isolines of the streamwise velocity component between the values $0.8 \leq \bar{u}(x_0; y, z) \leq 0.95$. It can be seen that both of the more feeble antisymmetric perturbations and the stronger amplified symmetric perturbations mainly affect the vortex sheet connecting the two trailing vortices.

Figure 5.8: Spatial amplification $-\alpha_i$ as a function of the wavenumber α_r for the two classes of boundary conditions, $\bullet$: symmetric; $\circ$: anti-symmetric, obtained by solution of the spatial BiGlobal EVP (2.45). Results of the spatial Orr-Sommerfeld equation are also shown. In all three sets of results the axial range analyzed is $x \in [6, 12]$. The most-/least-amplified results correspond to $x = 6$ and $x = 12$, respectively.

5.3 PSE-3D Analysis in the Wake of a Trailing Vortices

In the problem at hand, the PSE-3D marching procedure is initialized using results of the respective spatial eigenvalue problem at $x = 6$ for either symmetric or antisymmetric modes. The spatial integration is advanced until $x = 14$ for a frequency range of $\omega \in [3, 8]$ in steps of $\Delta\omega = 0.5$, which encompasses the maximally amplified modes predicted by the spatial BiGlobal analysis, previously shown in Figure 5.8. Figure 5.10 presents the streamwise evolution of the kinetic energy of symmetric and antisymmetric perturbations, respectively denoted by full and open symbols, normalized using the perturbation energy at $x = 6$. The respective frequencies of the maximally amplified perturbations can be seen in this figure. Results of Figure 5.10 show that linear PSE-3D analysis also predicts that symmetric disturbances are more strongly amplified than their antisymmetric counterparts. N-factors, not shown here for brevity, have been computed on the basis of these results and can be used to make predictions as regards linear breakdown and transition of the computed perturbations. An 8-fold increase in magnitude of the symmetric perturbation initial energy requires an axial distance $x = O(100)$ downstream of the wing.

Figure 5.9: Amplitude functions of the perturbation velocity components $\hat{\mathbf{q}}_1(y, z)$ resulting from solution of (2.45) at $x = 10$ and imposition of symmetric boundary conditions at $\omega = 5.5$ (upper) and the antisymmetric boundary conditions at $\omega = 5$ (lower). In the background isolines of the streamwise component of the base flow velocity are shown

Figure 5.10: Kinetic energy of symmetric (left) and anti-symmetric (right) perturbations obtained with linear PSE-3D on the AR=1.7 wing as a function of the axial coordinate x for a range of amplified frequencies ω .

Figure 5.11: Dependence on the frequency ω of spatial amplification rates, $-\alpha_i$ (spatial BiGlobal EVP – lines) and σ (PSE-3D – symbols), for both symmetric (s) and antisymmetric (a) boundary conditions. Upper to lower curves correspond to the locations $x = 6, 8, 10$ and 12 in both sets of results. PSE-3D computations have been initialized with results at $x = 6$

The distribution of the spatial amplification rate, σ , is compared with the imaginary part of the eigenvalue, $-\alpha_i$, obtained in the spatial BiGlobal analysis in Figure 5.11, in which the (identical with the spatial eigenvalue problem solution) initialization data for

Figure 5.12: Evolution of the amplitude functions $\hat{u}_1(y, z)$ with axial distance. *Upper to lower:* $x = 6, 8, 10$ and 12 . *Left column:* symmetric perturbations at $\omega = 6$. *Right column:* antisymmetric perturbations at $\omega = 5$

the PSE-3D at $x = 6$ is not included. Two noteworthy observations are made. First, the most amplified frequency range delivered by the linear PSE-3D analysis is slightly lower than that predicted by the spatial BiGlobal eigenvalue problem solution. Second, the difference in the predictions of the two methodologies diminishes as x increases. As discussed in sections 2.3.2 and 2.4, the global eigenvalue problem assumes independence of the base flow of the axial spatial direction, while PSE-3D assumes a weak viscous spreading. If the wake width, $\delta \propto \sqrt{x}$, is taken as a measure of viscous spreading [146], then $d\delta(x)/dx \propto x^{-\frac{1}{2}}$. The latter quantity diminishes as x increases, which is consistent with the diminishing of the differences in the predictions of the spatial BiGlobal eigenvalue problem and the linear PSE-3D marching procedure. Figure 5.12 compares the evolution of the magnitude of the symmetric and antisymmetric streamwise perturbation velocity obtained by linear PSE-3D analysis at the maximally amplified frequencies, $\omega = 6.0$ and $\omega = 5.0$, respectively. Consistently with the previous results the respective eigenfunctions peak at the vortex sheet between the two vortices over the entire range of axial locations examined.

There are two sorts of perturbation modes discovered: Symmetric perturbations corresponding to instability of the vortex sheet connecting the system of trailing vortices and antisymmetric perturbations peaking at the vortex sheet and also in the neighborhood of

the vortex cores. The global modes discovered are shown in Figure 5.13 in a perspective view showing isosurfaces of the streamwise perturbation velocity, alongside the base flow vortex core positions.

Figure 5.13: Perspective view of isosurfaces of streamwise perturbation velocity of symmetric (*left*) and anti-symmetric (*right*) disturbances obtained by linear PSE-3D analysis, superposed upon the image of base flow vortex core positions.

5.4 Comparisons with Local Linear Theory

5.4.1 Wake profiles at the plane of symmetry

Results of both of the spatial BiGlobal and the linearized PSE-3D analyses show that, at the conditions examined, a symmetric instability which peaks at the plane of symmetry is the leading linear perturbation of the steady laminar flow in the wake of the wing. It thus becomes interesting to examine whether application of classic linear local instability analysis to the streamwise velocity profile at the plane of symmetry can deliver reliable predictions of the physics of linear instability. This question has been answered by extracting streamwise velocity profiles $\bar{u}(y)$ from the steady laminar base flow computed by DNS. Profiles were extracted at the plane of symmetry $z = 0$ and the same streamwise locations $x = 6, 8, 10$ and 12 at which spatial BiGlobal analyses have also been performed. At those locations the spatial Orr-Sommerfeld equation has been solved to determine the most unstable local linear instabilities. The results have been shown as solid lines in Fig-

ure 5.8, where the solution of the spatial BiGlobal eigenvalue problem (2.45) at the same locations is also shown as symbols. Comparisons of the two sets of results at all axial positions leads to a two-fold conclusion.

Figure 5.14: *Left:* Comparison of the amplitude function $\hat{u}_0(y)$ of the leading eigenmodes delivered by the Orr-Sommerfeld equation at $x = 10$ with the profile extracted from the the leading two-dimensional eigenfunction $\hat{u}_1(x = 10, y, z = 0)$ obtained by solution of (2.45). *Right:* Instability in the wake, predicted by local analysis.

Qualitatively, the spatial Orr-Sommerfeld equation is capable of predicting the characteristics of the (convective) instability at this part of the domain. The range of amplified linear perturbation frequencies as well as the frequency of the most amplified modal perturbation are reasonably well captured by the simple (and substantially less expensive) local analysis method. However, if quantitative predictions are sought for the amplification rates, the $O(30\%)$ differences between the results of the local and global analyses suggests that the usefulness of the Orr-Sommerfeld equation may be limited in this respect. This may be attributed to the closeness of the vortex pair to the midplane of symmetry, which invalidates the assumption of spanwise base flow homogeneity, as employed in the Orr-Sommerfeld equation. Interestingly, the modulus of the amplitude function of the leading eigenmode delivered by local theory, $\hat{u}_0(y)$, and that of a cut through the two-dimensional streamwise component of the eigenvector delivered by the global analysis, $\hat{u}_1(x = 10, y, z = 0)$ are seen in Figure 5.14 to be in rather close agreement. Figure 5.14 also presents isosurfaces of the composite flow field at the symmetry plane, reconstructed according to the local linear theory Ansatz. To this end, the leading eigenmode of the Orr-Sommerfeld equation has been superposed to the underlying streamwise velocity component at a linearly small amplitude. The resulting flow shows the typical cats-eyes pattern of wake instability which, in the context of local theory, is periodic by construction along the axial direction. However, the rather large difference between the predictions on amplification delivered by spatial BiGlobal and linear PSE-3D analysis suggests that neither the local theory nor the spatial BiGlobal analysis should be employed to analyze instability in the near-mid-field of low-aspect ratio wings at low Reynolds numbers. Instead, the linear

Figure 5.15: *Left:* Leading eigenmodes of local analysis in the neighborhood of the vortex. *Right:* Amplitude functions of the leading $m = 2$ eigenmode around the vortex core at $x = 10, \omega = 5.85$; $r = [(y - y_c)^2 + (z - z_c)^2]^{\frac{1}{2}}$

PSE-3D methodology, which incorporates all relevant features of the underlying base flow should be employed if quantitative predictions are sought.

A final interesting observation can be made by comparing results presented in Figures 5.8 and 5.11, namely the approximate, but not exact as in temporal theory, proportionality of the frequency, ω , and α_r in spatial BiGlobal analysis, or σ in the linear PSE-3D. Much like the spatial local theory prediction of Michalke [122] in his classic work on the hyperbolic tangent profile, and unlike the corresponding temporal theories, in the wake configuration behind the wing too, both spatial BiGlobal and linear PSE-3D analyses deliver approximately, but not exactly constant phase velocities, ω/α_r .

5.4.2 Local analysis near the vortex core

Linear local instability of the vortex at the axial location $x = 10$ has been performed by solving the axisymmetric counterpart of the spatial Orr-Sommerfeld equation [118] for a base velocity profile centered at the vortex centroid and extending radially outward. A number of eigenmodes were identified in the frequency range $\omega \in [0, 8]$ and azimuthal wavenumbers $m \in [0, 10]$. The mode $m = 2$ is maximally amplified at a frequency $\omega \approx 6$, having a spatial amplification of $-\alpha_i \approx 3.2 \times 10^{-2}$, as can be seen in Figure 5.15. The amplitude functions also shown in this figure correspond by construction¹ to a purely axisymmetric eigenfunction with two positive and two negative peaks, as implied by the azimuthal wavenumber parameter.

Perhaps the most interesting aspect of these results is that the local analysis in the vicinity of the vortex core also predicts a maximally amplified frequency which peaks around $\omega = 6$, consistently with what has been found by the spatial Orr-Sommerfeld

¹c.f. Equation (2.9)

equation at the plane of symmetry and by both global approaches on the entire domain. However, unlike the result of the local analysis in the wake, an axisymmetric eigenmode structure near the vortex core is not to be seen in the amplitude functions of the global analyses shown in Figures 5.9 and 5.12. This may be attributed to the fact that instability in the vortex sheet is inviscid in nature and is expected to be the defining characteristic of the leading linear perturbation of the present base flow, independently of the analysis approach used. By contrast, the viscous instability in the vortex core has orders-of-magnitude lower amplification rates than the instability peaking at the symmetry plane and, as such, the global eigenvalue spectrum is dominated by instability in the wake.

Finally, it should be mentioned that the wake will not only develop inviscid instabilities in the Oxy plane due to the inflectional velocity profile in the x direction. This type of instabilities will also appear on Oyz planes too, as an inflectional velocity profile is also to be found along the z direction at some point between the axis of symmetry and the vortex core. This feature is captured by the point-vortex simulations as has already been seen in the insert Figure 5.5. However, complete description of all linear instability mechanisms beyond the leading perturbations identified by global methods is beyond the scope of the present work.

Chapter 6

Discussions and Future Work

6.1 Discussion

The global linear instability analysis of massive separated flows are studied using the state-of-the-art numerical methodologies. Modal and non-modal linear global instability analysis of steady laminar massively spanwise periodic three-dimensional separated flows around three airfoil profiles has been performed. Two physical mechanisms of linear modal instability have been identified around airfoils at low Reynolds numbers and large angles of attack. The first is a traveling Kelvin-Helmholtz mode which dominates flow instability at large spanwise periodicity wavelengths. The second is a three-dimensional stationary mode akin to that found in other laminar separation bubbles; it is active as the spanwise periodicity length diminishes and dominates flow instability at $O(1)$ airfoil aspect ratio. An increase of the angle of attack, while the flow is massively separated but still remains steady, promotes instability of both of the traveling and the stationary mode. On all three airfoils, independently of whether unsteadiness is the result of an increase in the flow Reynolds number or the angle of attack, the traveling KH mode becomes unstable first, with the $\beta = 0$ (two-dimensional) mode being the least stable linear perturbation under any combination of parameters examined. In other words, *flow becomes 2D unsteady prior to becoming 3D unstable*. Conditions at which the stationary three-dimensional mode is unstable have not been identified in the parameter range examined.

The modal stability has been performed in a unstable NACA 4415 airfoil flow at high AoA using the selective frequency damping method. Except the traveling KH mode discovered in the two-dimensional flow, the least-damped eigenmode mechanism is belong to self-excited, as found in the laminar flat-plate boundary layer flow. The short-wavelength stationary mode dominates the domain closed to the airfoil. Once the least-damped eigenmode of steady laminar two-dimensional flow around an airfoil at large angle of attack, which was identified as the $\beta = 0$ perturbation, becomes linearly unstable, the ensuing time-periodic flow remains nominally homogeneous along the spanwise direction in a range of Reynolds number and angle of attack parameters which depends on the airfoil thickness and camber. Secondary instability analysis of the laminar time-periodic nominally two-dimensional base flows around the NACA 0009, 0015 and 4415 airfoils has delivered two unstable modes with well-separated spanwise wavenumber regions. Unlike the wake of the cylinder, at high AoA values as the Reynolds number is increased a short spanwise wavelength instability becomes unstable first and is followed by an additional, long-wavelength unstable eigenmode. The critical conditions, $(Re_{crit}, \beta_{crit})$ have been determined for both modes on all three airfoils.

Earlier work has associated stall cells with linear amplification of the primary three-

dimensional stationary eigenmode of steady laminar separated flow [143]. If at all relevant to high-Reynolds number flow, this mechanism cannot explain the appearance of "owl-face" streamline patterns on thin airfoils at low angles of attack, away from stall conditions [51]. By contrast, the scenario identified in the present work, in which such streamline patterns arise from linear secondary amplification of a short-wavelength Floquet eigenmode, is related with the bluff-body nature of the airfoil wake and may be present at angles of attack lower than those at which an airfoil stalls. In addition, the way in which flow turbulence may modify the secondary instabilities discussed in the present work needs to be clarified; direct numerical simulation work is underway in order to address this point.

The trailing vortices in the laminar finite wing flow has been addressed in this thesis, combining of different techniques and codes. The present work addresses questions that were opened at the turn of the century, during the large-scale pan-European R&D efforts surrounding the introduction of the A380. At that time analysis approaches ranging from point vortices to direct numerical simulation were employed to understand the so-called 'near-mid-field' behind wings of various shapes. The then nascent global instability had been demonstrated for vortical flows but, at that time, had not reached the level of maturity in order to become an alternative to the established inviscid (and viscous) point-vortex methods and local linear instability analyses. The present work intends to revive some of these questions and demonstrate that global analysis is not only feasible for full wing configurations, but also delivers results that are qualitatively close but quantitatively different from those of the simplified approaches.

Here, linear global instability analysis of the trailing vortex system behind a low aspect ratio three-dimensional wing has been performed by solving the spatial BiGlobal eigenvalue problem, as well as by employing the linear PSE-3D disturbance equations. To the authors' knowledge both approaches are used for the first time to analyze a flow of this class. At the low Reynolds number examined, two classes of disturbances have been discovered, symmetric perturbations corresponding to instability of the vortex sheet connecting the system of trailing vortices and antisymmetric perturbations peaking at the vortex sheet and also in the neighborhood of the vortex cores. The global modes discovered are shown in Figure 5.13 in a perspective view showing isosurfaces of the streamwise perturbation velocity, alongside the base flow vortex core positions.

The global analysis results obtained are consistent with the physical understanding of instability in the wake of a short aspect ratio wing. A larger growth of linear instabilities at the plane of symmetry leads to earlier destruction of the coherence of the vortex sheet, while lower levels of the (two-dimensional) amplitude functions in the vicinity of the vortex suggest that the trailing-vortex system persists for axial distances longer than the $20b$ presently examined. In other words, the wake (shear) destruction, does not imply destruction of the trailing-vortex system, such that the vortex wake hazard persists after destruction of the vortex sheet by linear instability mechanisms. From a physical point of view the earlier destruction of the vortex sheet can be attributed to the inflectional nature of the profiles at the wake portion near the symmetry plane, while the persistence of the trailing vortex system is due to the viscous nature of vortex instabilities. The early destruction of the vortex sheet is corroborated in the inviscid limit by the point-vortex method employed, results of which show that the trailing vortex system persists for substantially longer axial distances beyond that at which linear instability of the wake

develops.

Qualitative agreement has been obtained between results delivered by global and local linear instability analyses regarding the dominant linear mechanism in the wake. However, the spatial Orr-Sommerfeld equation applied to wake velocity profiles at the plane of symmetry largely over predicts the amplification rates delivered by the spatial BiGlobal eigenvalue problem or the linear PSE-3D marching approach. On the other hand, more work will be necessary in order to identify instability in the vortex core which, owing to its viscous nature, will be less significant from a linear analysis perspective compared to the wake mode identified by solution of both the spatial BiGlobal eigenvalue problem and linear PSE-3D analysis.

Overall, the main contribution of the present work is the demonstration that application of inviscid point-vortex and viscous local, BiGlobal and linear PSE-3D analyses to analyze instability in the wake of a low aspect ratio wing at low Reynolds number are all capable of offering a qualitative consistent picture of the physics of active linear instability mechanisms, namely early destruction of the vortex sheet and persistence of the trailing vortex system. However, quantitatively different predictions of amplification rates of linear perturbations and their respective frequencies are obtained by each analysis methodology. Since only PSE-3D incorporates all necessary features of the underlying base flow field, namely viscosity, strong lateral velocity gradients and weak base flow development along the axial direction, while all other analysis methodologies (except DNS) need to make simplifying assumptions regarding the underlying base flow, it must be concluded that linear instability analyses in the problem at hand need to employ the PSE-3D methodology. The conclusions drawn are specific to the configuration chosen at the low Reynolds number addressed and should not be taken as a universal criterion for breakdown of the coherence of the wake behind the wing, but rather as a demonstration of the capability of the proposed analysis approach to deliver predictions without resorting to the simplifying assumptions on the form of the vortex system.

6.2 Future Work

There are still some open questions in analysis the NACA airfoils flow, for example, higher Reynolds number mean flow instability as reported by Brehm and Fasel [23] which will be carried out using the BiGlobal stability theory performed in this thesis; The TriGlobal stability analysis of the short aspect-ratio elliptical wing is undergoing; The work in progress addresses larger aspect-ratio wings and higher Reynolds numbers that are more relevant to flight conditions, with which is easier experimental comparisons; The instability of three-dimensional flow of the wing, such as swept-back which is commonly used in the commercial transport airplanes, will be analyzed under the help of numerical Large Eddy Simulation (LES) technology; The compressible flow instability of cascade and transonic will be reported.

Appendix A

OpenFOAM Validation

A.1 OpenFOAM: A quick glance

OpenFOAM is abbreviation for “Open Source Field Operation and Manipulation”, a free, open source CFD software package.

OpenFOAM uses Finite Volume Method (FVM) to solve the governing equations, the verification of the code has been demonstrated in many authors [30, 84, 126, 127]. In *fvSchemes* dictionary defines the time scheme, where we choose the second-order *Crank–Nicolson* method. There are multi-choice of linear solver in OpenFOAM, *PBiCG* (Preconditioned bi-conjugate gradient) and *GAMG* (Generalised geometric-algebraic multi-grid) are choosed in *fvSolution* dictionary to solve the velocity and pressure. The standard solver *icoFoam* is choosed to solve the incompressible transient laminar flow using the PISO algorithm. Rewritten the equations 2.4 and 2.3 again as

$$\nabla \cdot \mathbf{U} = 0 \quad (\text{A.1})$$

$$\frac{\partial \rho \mathbf{U}}{\partial t} + \nabla \cdot \rho \mathbf{U} \mathbf{U} - \nabla \cdot \mu \nabla \mathbf{U} = -\nabla p \quad (\text{A.2})$$

The above equations could be represented by OpenFOMA as

```
solve (
fvm::div(U)
==
0 );
```

```
solve (
fvm::ddt(rho,U)
+fvm::div(phi,U)
-fvm::laplacian(mu,U)
==
-fvc::grad(p) );
```

A.2 Circular cylinder steady flow

A classic cylinder flow [62] has been reproduced here. The computational domain is $[-25 : 50] \times [-20 : 20]$ with 300×300 grid points, as show in Figure A.1.

Figure A.1: The grid used for computing the cylinder flow

The inlet uniform flow boudary is:

$$\begin{aligned} U &= 1 \\ V &= 0 \\ p &= \text{zeroGradient} \end{aligned} \tag{A.3}$$

and at the outlet boudary, the boundary conditions is

$$\begin{aligned} \mathbf{U} &= \text{zeroGradient} \\ p &= 0 \end{aligned} \tag{A.4}$$

On the upper and lower surface, the boundary conditions are symmetric.

After running $t = 200s$ time units, the absolute tolerance is converged lower than $\mathcal{O}(10^{-8})$, as shown in Figure A.2, a steady state base flow is obtained at $Re = 40$, where Reynolds number characteristic length is the diameter of the cylinder.

Figure A.2: The residual of drag for the circular cylinder flow at $Re = 40$.

The DNS results are compared with the previous values documented in references [36, 44, 56, 94, 175], which are listed in Table A.1. Where length of bubble L defined as the distance between the stagnation and the minimum velocity in the centerline of the wake. The results obtained from OpenFOAM are very agreement with previous studies.

Table A.1: Drag coefficient C_d and the length of the wake bubble L at Reynolds number 40

	C_d	L
Dennis & Chang (1970)	1.52	2.35
Coutanceau & Bouard (1977)	...	1.89
Fornberg (1980)	1.50	2.24
Ye <i>et al</i> (1999)	1.52	2.27
Kim <i>et al</i> (2001)	1.51	...
Giannetti & Luchini (2007)	1.54	2.24
Current	1.5387	2.24

Appendix B

Flow Unsteadiness Validation

The primary work of investigating the effect of trailing edge geometries on flow unsteadiness has been done by OpenFOAM. The validation work has been carried out using other three codes, which adopt finite volume method and spectral element method, respectively. The complete convergence study are shown in C.1.

B.1 Immersed boundary method

In the immersed boundary method (IBM) the fluid is represented on an Eulerian coordinate and the structure is represented on a Lagrangian coordinate. Usually the flow is solved on an Eulerian grid (e.g. uniform Cartesian grid) that does not conform to the body geometry. The surfaces were viewed as flexible elastic membranes with a constitutive relation relating the forces to the motion of the Lagrangian points, and the boundary forces that exist as singular functions along the surface in the continuous equations are described by discrete delta functions that smear (regularize) the forcing effect over the neighboring Eulerian cells.

The IBM code from Taira and Colonius [152] and Colonius and Taira [35] was employed here. This code uses a second-order finite volume formulation on a staggered cartesian grid of the vorticity equation for incompressible two-dimensional open flows. A boundary force is applied at the immersed surface to satisfy the no-slip constraint. The NACA 0015 geometry is discretized using 199 Lagrangian points.

A nullspace projection method allows the divergence-free constraint to be automatically satisfied. The equations are marched in time using a fractional step approach, featuring an implicit Crank-Nicholson for the viscous terms and an explicit second-order Adams Bashforth for the convective terms. The immersed boundary treatment of the fixed airfoil geometry yields a Poisson-like equation that is efficiently solved using a triangular Cholesky decomposition.

Uniform flow is imposed at all boundaries except for the outflow, where the convective condition

$$\frac{\partial u}{\partial t} + U_{\infty} \frac{\partial u}{\partial x} = 0 \quad (\text{B.1})$$

is applied. The code uses a multidomain technique in order to improve the behavior of the far-field boundary conditions and account for the extensive potential flow induced by the airfoil. Consequently, a fine spatial resolution can be used in a relatively small domain centered in the airfoil. Few steps follow in which the domain is doubled in size while keeping the same number of points. In the results presented here, the smallest domain is

B. Flow Unsteadiness Validation

a square of side $7c$, where c is the chord, and 4 domains are used, resulting into a largest domain of size $28c$. Table B.1 summarizes the convergence history for the different meshes, for the NACA 0015 at $AoA = 18^\circ$ and $Re = 200$. Figure B.1 shows the convergence history of force in the four resolutions meshes.

Table B.1: Convergence history corresponding to the Immersed Boundary Method code

Case	Spatial resolution ($N_x \times N_y$)	C_L	C_D
1	800×600	0.5780	0.4057
2	1500×800	0.5654	0.4075
3	2000×1200	0.5433	0.4107
4	2500×1500	0.5226	0.4143

Figure B.1: Convergency history of force in IBM at $Re = 200$ with $AoA = 18^\circ$. From left to right, top to down: mesh 1, 2, 3 and 4, respectively.

B.2 Spectral element method

Spectral element method is a relatively new computational technique which is a high order finite element method that combines the generality of the finite element method with the accuracy of spectral techniques[132]. The spectral method expands the solution in trigonometric series, a chief advantage being that the resulting method is of very high

Figure B.2: Convergence history of forces on the NACA 0015 at $Re = 200$, $AoA = 18^\circ$, using Nektar++ (left) and Nek5000 (right) at polynomial orders $p = 5, 7$ and 9 .

order. The spectral element method using a high degree piecewise polynomials which are usually orthogonal Chebyshev polynomials or very high order Lobatto polynomials over non-uniformly spaced nodes. In the processing of discretization, the hyperbolic pieces of the governing equations is treated with an explicit collocation, while the pressure and viscous contributions are treated implicitly with a projection operator derived from a variational principle.

Nektar++ [33] is a suite of simulation codes. It is based on hp-finite element technology using polymorphic tessellation of non-overlapping grids of triangles, quadrilaterals, tetrahedra, pyramids, prisms and hexahedra. This code has been developed many years and utilized in many fields [169] [2] [32]. Spectral/hp methods could achieve more optimum solution by refining mesh and increasing the order of polynomials, or choosing certain evaluation strategies for a given numerical operator. Nek5000 [55] is an open source, highly scalable and portable spectral element code designed to simulate unsteady incompressible Navier-Stokes flow, heat transfer and species transport and so on. Nek5000 would work with quadrilateral and hexahedral mesh.

Appendix C

Convergence Studies

C.1 Base flow

Numerical solutions of the base flow spanwise vorticity, computed by Nek5000 for flow around the NACA 0015 at $Re = 200$, $AoA = 18^\circ$, is shown in Table C.1 at three probe locations. On account of such results, and analogous ones obtained using the Nektar++ code, the mesh comprising 1996 elements has been used at $p = 7$ for all base flow computations.

C.2 Modal stability analysis

Table C.2 presents the effect of changing the extent of the domain in which modal instability analysis is performed on the eigenvalues obtained for flow around the NACA 0015 airfoil at $Re = 220$, $AoA = 18^\circ$, $\beta = 1$. Shown are the damping rate and frequency of the leading (Kelvin-Helmholtz) mode at these conditions obtained using the nektar++ time-stepping code.

Figure C.1: Damping rates and Strouhal number of the least-damped modes of the NACA 0015 airfoil at $Re = 200$, $AoA = 18^\circ$, obtained by the nektar++ and the FreeFEM++ methodologies

It can be seen that an increase of the domain in the downstream direction in both the

x and the y spatial directions produces less stable flow. However, eigenvalue computations using the matrix-forming code FreeFEM++ and a fully unstructured grid, at convergence in any of the domains used by nektar++ code also produces a damped leading eigenvalue with characteristics analogous to those delivered by the time-stepping code.

C.3 Non-modal stability analysis

Figure C.2 shows the three meshes used to investigate the effectness of *inflow* and *outflow* extents on the adjoint sensitivity. The numerical integrity of the transient growth analysis results has been checked by repeating selected runs in domains of different extent. Three domains employed for this exercise are $D1 : [-15, 50] \times [-15, 15]$, $D2 : [-40, 50] \times [-15, 15]$ and $D3 : [-40, 100] \times [-15, 15]$. In chord length units $D1$ represents the domain on which most analyses were run with all airfoils, $D2$ is extended in the upstream spatial direction, in order to assess the degree by which the adjoint eigenvectors are captured and $D3$ is the largest domain, which is extended in both the upstream and the downstream spatial directions. New meshes were generated for each of these analyses, base flows were run in the new meshes and the adjoint looping was performed in the same mesh used for the base flow calculation.

Figure C.2: Three meshes used to investigate the effect of *inflow* and *outflow* extents on transient growth: D1 (top), D2 (middle) and D3 (bottom).

The results of Figure C.3, obtained on the NACA 0015 at $Re = 200$, $AoA = 18^\circ$ and

Table C.1: Convergence history of base flow spanwise vorticity $\bar{\omega}_z = -\partial\bar{u}/\partial y + \partial\bar{v}/\partial x$ at probe locations, from left to right column, $P1$ (3.0, 0.5), $P2$ (2.0, 0.5) and close to the center of the recirculation zone $P3$ (1.0, 0.0).

# of elements	p			p			p		
	5	7	9	5	7	9	5	7	9
1265	-0.3637	-0.3538	-0.3491	-0.4452	-0.4264	-0.4173	-0.6229	-0.6185	-0.6164
1996	-0.3643	-0.3540	-0.3492	-0.4468	-0.4267	-0.4173	-0.6012	-0.6061	-0.6079
2310	-0.3645	-0.3541	-0.3492	-0.4473	-0.4269	-0.4174	-0.6010	-0.6059	-0.6078

Table C.2: Effect of domain extent on the modal analysis for the basic flow at $Re = 220$, $AoA = 18^\circ$ and wavenumber $\beta = 1$

x	y	Damping Rate	Frequency	Residual $\times 10^6$	# of iterations
$[-5 : 15]$	$\times$ $[-5 : 5]$	-0.085053	2.3943	7.5	36
$[-5 : 20]$	$\times$ $[-5 : 5]$	-0.085051	2.3943	8.1	45
$[-5 : 30]$	$\times$ $[-5 : 5]$	-0.085728	2.3967	6.7	125
$[-5 : 40]$	$\times$ $[-5 : 5]$	-0.077320	2.4042	9.7	190
$[-5 : 30]$	$\times$ $[-10 : 10]$	-0.085683	2.3966	9.0	123
$[-10 : 20]$	$\times$ $[-5 : 5]$	-0.085024	2.3943	9.8	59
$[-10 : 40]$	$\times$ $[-10 : 10]$	-0.075904	2.4010	8.4	202
FreeFEM++		-0.072268	2.3962	9.7	—

being typical of those obtained on the other two airfoils, show that the maximum gain associated with the $\beta = 0$ wavenumber (two-dimensional flow) is somewhat reduced when the largest domain is used. However, the relative difference of the absolute values of G obtained at the same time τ in the smallest and the largest domains is less than 20% at its maximum, while the order-of-magnitude change in $G(\tau)$ at all times is well captured by the smallest domain. An analogous result (actually better in terms of maximum relative difference) is seen as regards the maximum singular values, $\sigma_{\max}$, obtained in the different domains. Since the cost of the numerical solutions obtained in domain $D3$ is more than double that in $D1$, the non-modal analyses discussed in this thesis were performed in domain $D1$.

Figure C.3: Dependence of $G(\tau)$ (left) and the maximum singular value $\sigma_{\max}$ (right) on the extent of the analysis domain for the 0015 airfoil at $Re = 200$, $AoA = 18^\circ$.

C.4 Floquet analysis

Floquet analysis of flow around a NACA 0012 airfoil was performed by [166], whose results at $Re = 500$, $AoA = 20^\circ$ are used to validate the present work. First a time-periodic flow at these conditions is computed and subsequently the Floquet multiplier μ is computed at each value of the spanwise wavenumber parameter β . The module of μ determines whether the periodic orbit is stable or unstable. The Semtex code was used for the base flow computation, while its linearized version was employed to perform secondary Floquet instability analysis. The base flow was computed in domains of size $x \in [-15, 100] \times y \in [-20, 20]$, the oscillation period at each set of parameters was identified and the base flow results for one period were interpolated in the domain $x \in [-3, 16] \times y \in [-3, 3]$ using n up to 32 planes in order to ensure convergence of the Floquet multipliers. Gauss-Lobatto-Legendre basis functions have been employed for the spatial discretization of the Oxy planes, while a Fourier expansion is used along the spanwise direction, as discussed in Section 4.1, along with a polynomial order $p = 7$ on each macro-element described previously, a Krylov dimension = 12, an eigenvalue convergence criterion of

$tol_{Arnoldi} = 10^{-6}$ and a maximum number of Arnoldi iterations $N_{Arnoldi} = 40$. Figure C.4 shows comparison of the results of [166] and the present computations. Note that the figure is symmetric with respect to a change of sign in β . The rather good agreement obtained, despite the different codes and meshing strategies used, verifies the present methodology which has subsequently been applied to analyze time-periodic flow instability on the three NACA airfoils using the same parameters as those employed to analyze instability in the wake of the NACA 0012 profile.

Figure C.4: Comparison of μ dependence on β , as obtained in the present work by Semtex for the NACA 0012 airfoil at $Re = 500$, $AoA = 20^\circ$, against literature results [166]

C.5 Spatial BiGlobal instability analysis validation on the q -vortex

Validations of the global instability analysis algorithms utilized have been presented in the incompressible [129] and the compressible [41] flow regime. Here the performance of the code employed for the solution of the spatial BiGlobal eigenvalue problem is assessed by comparing results obtained on an isolated q -vortex model against the classic work of Mayer & Powell [118]; the main results are summarized below.

The q -vortex flow proposed by Lessen *et al.* [103, 104] is a limit case of the Batchelor [13] vortex, when the dependence of the flowfield on the axial direction is neglected. In this case, the swirl parameter q , which is spatially-varying in the Batchelor vortex, is taken to be constant, $q = \Gamma/(2\pi U_c \delta)$, U_c being the core vortex velocity, U_0 the free-stream velocity and δ the core radius. The parallel-flow version of Batchelor vortex becomes

$$U_x = \frac{U_0}{U_c} + \exp(-r^2), \quad U_r = 0, \quad U_\theta = \frac{q}{r}\{1 - \exp(-r^2)\}, \quad (\text{C.1})$$

where r is the radial distance from the vortex center. The Reynolds number in this case is defined as

$$Re = \frac{U_c \delta}{\nu} \quad (\text{C.2})$$

Table C.3: Temporal BiGlobal instability analysis of the q -vortex flow at $q = 0.475$, $Re = 100$ and $\beta = 0.418$. MP denotes the result of Mayer & Powell [118]

Discretization	N^2	$\omega_r \times 10^{-2}$	$\omega_i \times 10^{-3}$
CGL	20^2	2.833	9.64
	30^2	2.8352	9.617
	50^2	2.8351897	9.6169098
	70^2	2.835189660	9.616909850
FD10	40^2	2.8353	9.617
	60^2	2.8351899	9.61698
	80^2	2.8351895	9.616913
	100^2	2.83518965	9.61690986
MP			9.61

The temporal stability problem pertinent to the base flow (C.1) was solved for the parameters addressed by Mayer and Powell [118], $q = 0.475$, $Re = 100$, $\frac{U_0}{U_c} = 0$ and axial wavenumber 0.418. The domain was truncated at $y_\infty = z_\infty = 40$ and the FD-q10¹ spatial discretization method was used, alongside the classic spectral collocation based on the Chebyshev Gauss-Lobatto (CGL) nodes. Table C.3 presents the convergence of the leading eigenvalue with increasing resolution and compares with the result of reference [118]. The eigenvalues converge to a value with absolute difference lower than 10^{-9} between the results of FD-q10 and CGL. Note that serial computation of the numerical solution of the temporal BiGlobal EVP using the highest resolution ($N = 71^2$) for CGL required 4935 Mbytes of memory, while in the case of FD-q10 ($N = 101^2$) the memory used was 1496 Mbytes, illustrating the performance improvement of the combination of high-order finite-difference schemes in conjunction with sparse storage and algebra [130].

These numerical efficiency improvements were crucial for the success of spatial BiGlobal EVP solutions, also performed for the q -vortex at $Re = 1200$ and $q = 0.8$. Temporal stability results for the same case were presented by González *et al.* [67]. Here, the spatial eigenvalue problem (2.45) was solved, employing FD-q10 spatial discretization method, alongside $N = 101^2$ collocation points along each of the simultaneously discretized spatial directions, y and z . Figure (C.5, left) shows the eigenspectrum at $\omega = -2$. A branch of eigenvalues appear for $\alpha_r = 0$ and $\alpha_i < 0$, which corresponds to a discretized spectrum of

¹defined by Paredes *et al.*[130]

upstream moving pressure waves. This branch will be suppressed in the subsequent PSE-3D analysis in order to permit a stable numerical integration along the axial direction, as discussed by Li and Malik [107].

Figure C.5: *Left:* Eigenspectrum of the q -vortex at $Re = 1200$, $q = 0.8$ and $\omega = -2.0$. Full symbol denotes the most unstable mode of this family that corresponds to a eight lobes ($m = 8$) eigenmode. *Right:* The pertinent eigenfunction of this mode is plotted with contours (-0.9:0.1:0.9) of normalized real part of axial velocity amplitude function. Dashed lines correspond to negative values

In addition, different branches of discrete eigenvalues are found, describing near-vertical lines. The discrete modes present a periodic structure in the azimuthal direction, with all the modes inside each branch having the same wavenumber m . Note that when local stability analysis based on the Orr-Sommerfeld equation in cylindrical coordinates is used, the wavenumber m is a parameter of the problem. The BiGlobal analysis recovers the results of several local analysis at the same time. The most unstable mode, which is found to be in good agreement with the result presented in [67], is noted by full symbol in the spectrum shown in Figure (C.5, left), while the spatial structure of its streamwise velocity component, $\hat{u}(y, z)$, is shown in Figure (C.5, right).

C.6 PSE-3D analysis validation on the Batchelor vortex

PSE-3D analysis permits considering a basic state which depends mildly on one and strongly on the other two spatial directions. An isolated non-parallel Batchelor vortex has been computed using the Parabolized Navier Stokes (PNS) formulation in cylindrical coordinates discussed by Paredes *et al.* [131]. The selected initial conditions are swirl strength parameters $\kappa_0 = 0.8$ and axial velocity defect $\gamma_0 = 0.8$. The Reynolds number based on the initial vortex radius and free-stream velocity is set to $Re = 3000$, and the

disturbance frequency is $\omega = 4.6$; additional details provided in [128, 131].

Figure C.6 shows the evolution of the spatial growth rate, $-\alpha_i$, along the axial direction computed using PSE-3D and conventional PSE with cylindrical coordinates [131]. Excellent agreement exists between the results delivered by both methodologies. Shown also in this figure are results of the classic local stability analysis, which are seen to strongly deviate from those of both PSE methodologies. For example at $x = 10$, a discrepancy in amplification rate larger than 50 % is observed. However, it is interesting to note that, although large discrepancies exist between the two sets of results, both curves cross the neutral stability threshold, $\alpha_i = 0$, at same axial position $x \approx 25.4$.

Figure C.6: Evolution of amplification rate for the non-parallel vortex flow ($\kappa_0 = 0.8$, $\gamma_0 = 0.8$) for the most unstable eigenmode at the selected parameters ($Re = 3000$, $\omega = 4.6$), comparing conventional PSE with cylindrical coordinates and PSE-3D results for a cross-validation test. Also, the leading eigenmode at each station is computed by local analysis (LST) [131]

Appendix D

E-387 Geometry

Eppler E387 airfoil is used to be a low Reynolds number flow. Its max thickness is 9.1% at 31.1% chord. The maxima camber is 3.2% at 44.8% chord.

Table D.1: Coordinates defining the geometry of E-387

x	y	x	y
1.00000	0.00000	0.00519	0.00931
0.99677	0.00043	0.00044	0.00234
0.98729	0.00180	0.00091	-0.00286
0.97198	0.00423	0.00717	-0.00682
0.95128	0.00763	0.01890	-0.01017
0.92554	0.01184	0.03596	-0.01265
0.89510	0.01679	0.05827	-0.01425
0.86035	0.02242	0.08569	-0.01500
0.82183	0.02866	0.11800	-0.01502
0.78007	0.03540	0.15490	-0.01441
0.73567	0.04249	0.19599	-0.01329
0.68922	0.04975	0.24083	-0.01177
0.64136	0.05696	0.28892	-0.00998
0.59272	0.06390	0.33968	-0.00804
0.54394	0.07020	0.39252	-0.00605
0.49549	0.07546	0.44679	-0.00410
0.44767	0.07936	0.50182	-0.00228
0.40077	0.08173	0.55694	-0.00065
0.35505	0.08247	0.61147	0.00074
0.31078	0.08156	0.66472	0.00186
0.26813	0.07908	0.71602	0.00268
0.22742	0.07529	0.76475	0.00320
0.18906	0.07037	0.81027	0.00342
0.15345	0.06448	0.85202	0.00337
0.12094	0.05775	0.88944	0.00307
0.09185	0.05033	0.92205	0.00258
0.06643	0.04238	0.94942	0.00196
0.04493	0.03408	0.97118	0.00132
0.02748	0.02562	0.98705	0.00071
0.01423	0.01726	0.99674	0.00021

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