Computing Hessenberg Matrix associated to self-similar measures

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Happy sixties Guillermo!
Summary

Objective: The obtention of the Hessenberg matrix associated to a self-similar measure with compact support in the complex plane in two different ways.
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3. Hessenberg matrix of a sum of measures (generalization of Mantica’s spectral techniques). Hessenberg matrix associated to a self-similar measure.
4. Examples.
Let $\mu$ be a \textbf{positive measure in $\mathbb{C}$ with compact support $\Omega$.}
Moment and Hessenberg matrices

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1. The **hermitian moment matrix** $M = (c_{jk})_{j,k=0}^{\infty}$ given by

$$c_{jk} = \int_{\Omega} z^j \overline{z}^k d\mu, \quad j, k \in \mathbb{Z}_+$$

is the matrix of the inner product in the canonical basis.
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2. Let $D$ be the infinite upper Hessenberg matrix of the multiplication by $z$ operator in the basis of ONPS $\hat{P}_n(z)$ in the closure of the polynomials.
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2. Let $D$ be the infinite upper Hessenberg matrix of the multiplication by $z$ operator in the basis of ONPS $\hat{P}_n(z)$ in the closure of the polynomials.

3. The Hessenberg matrix $D$ is the natural generalization to the hermitian case of Jacobi matrix.
Self-similar Measures

An Iterated Functions System (IFS) (M. Barnsley 1988) is a family of contractive maps \( \{\varphi_s\}_{s=1}^k \) on a complete metric space. In all this work, assume that \( \varphi_s \) (\( s = 1, \ldots, k \)) are contractive similarities (\( |\varphi(x) - \varphi(y)| = r|x - y|, 0 \leq r < 1 \), for all \( x, y \)). The family \( \{\varphi_s\}_{s=1}^k \) then, will be called an **Iterated Functions System of Similarities (IFSS)**.
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Given an IFSS \( \{ \varphi_s \}_{s=1}^k \) on a complete metric space, there exists a unique compactum \( K \) (**self-similar set**) satisfying

\[
K = \bigcup_{s=1}^k \varphi_s(K).
\]
Self-similar Measures

Examples of self similar sets

\[ \cdots \cdots \\]

\[ \cdots \cdots \cdots \cdots \\]

\[ \cdots \cdots \cdots \cdots \\]

\[ \cdots \cdots \cdots \cdots \\]

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Computing Hessenberg Matrix associated to self-similar measures
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Consider a probability vector \( p = (p_s > 0)_{s=1}^k \) with \( \sum_{s=1}^k p_s = 1 \).
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Let \( T \) be the **Markov operator** defined over the set of Borel regular probability measures as \( T\nu = \sum_{s=1}^{k} p_s \nu \varphi_s^{-1} \). Then, there exists a unique probability **invariant measure** \( \mu \).
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We call $\mu$ the self-similar measure associated to the IFSS with probabilities $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k; p_1, p_2, \ldots p_k\}$. 
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The support of \( \mu \) is the self-similar set \( K \) and satisfies (Hutchinson, 1981, Mandelbrot, 1977)

\[
\mu = \sum_{s=1}^k p_s \mu \varphi_s^{-1}, \quad \int_{\text{Supp}(\mu)} f d\mu = \sum_{s=1}^k p_s \int_{\text{Supp}(\mu)} f \circ \varphi_s d\mu,
\]

for any continuous function \( f \) on \( K \).
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Let $M$ be the moment matrix of a measure $\mu$ in $\mathbb{C}$. Then, $M_{\phi} = A_H \phi M A_{\phi}$ where $A_{\phi}$ denotes the conjugated transposed matrix of $A_{\phi}$ given by:

$$
A_{\phi} = \begin{pmatrix}
(0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0) & \alpha \\
(1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0) & \alpha \\
(2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0) & \alpha \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(3 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0) & \alpha \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
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\end{pmatrix}
$$
Moment matrix of the image of a measure by a similarity

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Then,

$$M_\varphi = A_\varphi^H MA_\varphi$$

where $A_\varphi^H$ denotes the conjugated transposed matrix of $A_\varphi$ given by

$$A_\varphi =
\begin{pmatrix}
(0)^0 \alpha^0 \beta^0 & (1)^0 \alpha^0 \beta^1 & (2)^0 \alpha^0 \beta^2 & (3)^0 \alpha^0 \beta^3 & \cdots \\
0 & (1)^1 \alpha^1 \beta^0 & (2)^1 \alpha^1 \beta^1 & (3)^1 \alpha^1 \beta^2 & \cdots \\
0 & 0 & (2)^2 \alpha^2 \beta^0 & (3)^2 \alpha^2 \beta^1 & \cdots \\
0 & 0 & 0 & (3)^3 \alpha^3 \beta^0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$
Moment matrix of self-similar measures

Consider:

\[ \Phi = \{ \phi_1, \phi_2, \ldots, \phi_k; p_1, p_2, \ldots, p_k \} \]

an IFSS with probabilities. Then, the sections the moment matrix \( M \) of \( \mu \) satisfy the following matricial relation

\[ M = \sum_{s=1}^{k} p_s A H \phi_s MA \phi_s \]
Moment matrix of self-similar measures

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1. $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_k; p_1, p_2, \ldots p_k\}$ an IFSS with probabilities.
2. $K_\Phi$ and $\mu_\Phi$ the self-similar set and measure, respectively.
3. $f$ a similarity map.
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3. \( f \) a similarity map.

Then the set \( f(K) \) and the measure \( \mu_\Phi \circ f^{-1} \) are self-similar for the IFSS

\[
f \Phi f^{-1} = \{ f \circ \varphi_1 \circ f^{-1}, f \circ \varphi_2 \circ f^{-1}, \ldots, f \circ \varphi_k \circ f^{-1}; p_1, p_2, \ldots p_k \},
\]

and

\[
\mu_{f \Phi f^{-1}} = \mu_\Phi \circ f^{-1}.
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Moment matrix of self-similar measures

Remark
When the measure is supported in the unit ball, the moments are bounded. In any other case they are not bounded.
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Consider the complete metric spaces

\[ M_\infty = \{(m_{ij})_{i,j=0}^\infty | \sup_{i,j} |m_{ij}| < \infty \} \quad M_1 = \{ M \in M_\infty | m_{00} = 1 \} . \]
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\[ \mathcal{M}_\infty = \left\{ (m_{ij})_{i,j=0}^\infty \mid \sup_{i,j} |m_{ij}| < \infty \right\} \quad \mathcal{M}_1 = \left\{ M \in \mathcal{M}_\infty \mid m_{00} = 1 \right\} . \]

If \( \text{Supp}(\mu) = K \not\subset B_1(0) \), there exists a contractive map \( f(z) = \alpha z \) such that \( f(K) \subset B_1(0) \). Then,

\[ \mathcal{M}_f = \{ M \mid A_f^H M A_f \in \mathcal{M}_1 \} \text{ with } ||M||_f = \|A_f^H M A_f\|_{\text{sup}}, \]

is a complete metric space.
Theorem

Let $\Phi = \{\varphi_s; p_s\}_{s=1}^k$ be an IFSS with probabilities. Let $K_\Phi$ and $\mu_\Phi$ be the self-similar set and measure, respectively. Let $f(z) = \alpha z$ be a contractive central dilation such that $f(K) \in B_1(0)$. Let $T_{f\Phi f^{-1}} : (\mathcal{M}_f, \| \cdot \|_f) \rightarrow (\mathcal{M}_f, \| \cdot \|_f)$ be the transformation defined as

$$T_{f\Phi f^{-1}}(M) = \sum_{s=1}^k p_s A_{f\varphi_s f^{-1}}^H MA_{f\varphi_s f^{-1}}.$$

Then $T_{f\Phi f^{-1}}$ is a contractive map with the moment matrix of the self-similar measure $\mu_\Phi$ as unique fixed point. Moreover, the ratio of this contractive map is $r = \sup \{ |\alpha_s|, s = 1, 2, \ldots, k \}$. 
Fixed point theorem for moment matrix of self-similar measures

**Theorem**

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$$r = \sup\{ |\alpha_s|, s = 1, 2, \ldots k \}$$
Then we have the following algorithm

\[
\begin{align*}
\nu & \rightarrow T(\nu) \rightarrow T^2(\nu) \rightarrow \cdots \rightarrow T^n(\nu) \rightarrow \mu \\
M_\nu & \rightarrow T_\Phi(M_\nu) \rightarrow T_\Phi^2(M_\nu) \rightarrow \cdots \rightarrow T_\Phi^n(M_\nu) \rightarrow M_\mu
\end{align*}
\]
Hessenberg Matrix. Cholesky Factorization

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\[\downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow \]

\[M_\nu \rightarrow \mathcal{T}_\Phi(M_\nu) \rightarrow \mathcal{T}_\Phi^2(M_\nu) \rightarrow \cdots \rightarrow \mathcal{T}_\Phi^n(M_\nu) \rightarrow M_\mu\]

Since \(M\) and \(D\) are related (even for every PDH matrix \(M\)) by the formula

\[D = T^H S_R T^{-H}\]

where \(M = TT^H\) is the **Cholesky factorization** and \(S_R\) is the shift-right matrix; we can approximate the \(n\)-section of \(D_\mu\)

\[M_{\mu, n} \rightarrow M_{\mu, n} = T_n T_{n}^H \rightarrow D_{\mu, n} = T_n^{-1} M'_{\mu, n} T_{n}^{-H}\]
Hessenberg Matrix associated to a sum of measures

From now on, we use the following notation.

1. $\mu$ sum of measures, i.e., $d\mu = \sum_{i=1}^{m} p_i d\mu_i$, where $\sum_{i=1}^{m} p_i = 1$.
2. every measure $\mu_i$ has compact support on the complex plane.
3. $D = (d_{ij})_{i,j=1}^{\infty}$ the Hessenberg matrix associated to $\mu$.
4. $\{D^{(i)}\}_{i=1}^{m}$ its Hessenberg matrices of $\mu_i$. 
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We will give a technique to calculate \( D \) in terms of \( \{D^{(i)}\}_{i=1}^{m} \).
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Remark

First note that the matrices \( D^{(i)} \) are bounded in \( \ell^2 \) because the support of every \( \mu_i \) is compact; second, remark that every matrix defines a subnormal operator in \( \ell^2 \) (Atzmon, 1975, Torrano-Guadalupe, 1993, and Tomeo, 2003), due to the fact that the matrix of the inner product is a moment matrix. These two properties allow us to extend the spectral Mantica’s techniques (2000).
Large recurrence formula

\[ D = (d_{jk})_{j,k=1}^{\infty} \text{ upper Hessenberg matrix. The ONPS satisfy} \]

\[ z\hat{P}_{n-1}(z) = \sum_{k=1}^{n+1} d_{k,n}\hat{P}_{k-1}(z), \quad n > 1. \]

with \( \hat{P}_1(z) = 0 \) and \( \hat{P}_1(z) = 1 \) when \( c_{00} = 1 \). Then

\[ d_{n+1,n}\hat{P}_n(z) = (z - d_{nn})\hat{P}_{n-1}(z) - \sum_{k=1}^{n-1} d_{k,n}\hat{P}_{k-1}(z), \quad n > 1. \]

with \( d_{2,1}\hat{P}_1(z) = (z - d_{11})\hat{P}_0(z), \) for \( n = 1 \).
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For \( D \) subnormal, we can write

\[ d_{n+1,n} \hat{P}_n(D) = (D - d_{nn} I) \hat{P}_{n-1}(D) - \sum_{k=1}^{n-1} d_{k,n} \hat{P}_{k-1}(D), \quad n > 1 \]
**Theorem (EST 2006, NTCAT06-ICM)**

Let $\mu, \{ \hat{P}_n \}_{n=1}^{\infty}, D = (d_{jk})_{j,k=1}^{\infty}$ and $\{ D^{(i)} \}$ be as above.

### Details

- The elements of $D = (d_{ij})_{i,j=1}^{\infty}$ can be calculated recursively by:
  \[
  d_{k,n} = \sum_{i=1}^{m} p_i \langle D^{(i)} v^{(i)}_{n-1}, v^{(i)}_{k-1} \rangle, \quad i = 1, \ldots, m, \quad k = 1, \ldots, n
  \]

- When $n = 1$ we have:
  \[
  w^{(i)}_1 = \left[ D^{(i)} - d_{11} I \right] v^{(i)}_0, \quad d_{11} = \sum_{i=1}^{m} p_i d^{(i)}_{11} d_{n+1}, \quad n = \lfloor \sqrt{m} \sum_{i=1}^{m} p_i \langle w^{(i)}_n, w^{(i)}_n \rangle \rfloor
  \]

- Lastly, we have:
  \[
  v^{(i)}_n = w^{(i)}_n d_{n+1}, \quad v^{(i)}_0 = e_0, \quad i = 1, \ldots, m.
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Let $\mu$, $\{\hat{P}_n\}_{n=1}^{\infty}$, $D = (d_{jk})_{j,k=1}^{\infty}$ and $\{D^{(i)}\}$ be as above. Then the elements of $D = (d_{ij})_{i,j=1}^{\infty}$ can be calculated recursively by

$$d_{k,n} = \sum_{i=1}^{m} p_i \langle D^{(i)} v_{n-1}^{(i)}, v_{k-1}^{(i)} \rangle, \quad i = 1, \ldots, m, \quad k = 1, \ldots, n \quad (1)$$

$$w_{n}^{(i)} = \left[ D^{(i)} - d_{nn} I \right] v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n} v_{k-1}^{(i)}, \quad i = 1, \ldots, m \quad (2)$$
Let $\mu, \{\hat{P}_n\}_{n=1}^{\infty}, D = (d_{jk})_{j,k=1}^{\infty}$ and $\{D^{(i)}\}$ be as above. Then the elements of $D = (d_{ij})_{i,j=1}^{\infty}$ can be calculated recursively by

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$$w_{n}^{(i)} = \left[ D^{(i)} - d_{nn} I \right] v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n} v_{k-1}^{(i)}, \quad i = 1, \ldots, m \tag{2}$$

When $n = 1$ we take $w_1^{(i)} = \left[ D^{(i)} - d_{11} I \right] v_0^{(i)}, \quad d_{11} = \sum_{i=1}^{m} p_i d_{11}^{(i)}$

$$d_{n+1,n} = \sqrt{\sum_{i=1}^{m} p_i \langle w_n^{(i)}, w_n^{(i)} \rangle}, \tag{3}$$

$$v_n^{(i)} = \frac{w_n^{(i)}}{d_{n+1,n}}, \quad v_0^{(i)} = e_0 \quad i = 1, \ldots, m. \tag{4}$$
Recurrent algorithm

We have \( \{v_0^{(i)}, v_1^{(i)}, \ldots, v_{n-1}^{(i)}\}_{i=1}^m \), \( D_n = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix} \).
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\[
D_n \quad v_0^{(i)}, v_1^{(i)}, \ldots, v_{n-1}^{(i)}
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\end{pmatrix} \)

\[
\begin{align*}
D_n & \downarrow \\
\begin{array}{l}
\text{\(v_0^{(i)}, v_1^{(i)}, \ldots, v_{n-1}^{(i)}\)} \\
\text{\(w_n^{(i)}\)}
\end{array} & \quad = \left[D^{(i)} - d_{nn}I\right]v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n}v_{k-1}^{(i)} \quad (2) \\
\begin{array}{l}
d_{n+1,n} \\
\text{\(w_n^{(i)}\)}
\end{array} & \quad = \sqrt{\sum_{i=1}^{m} p_i \langle w_n^{(i)}, w_n^{(i)} \rangle} \quad (3)
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\]
Recurrent algorithm

We have \( \{v_0^{(i)}, v_1^{(i)}, \ldots, v_{n-1}^{(i)}\}_{i=1}^m \), \( D_n = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{nn} \end{pmatrix} \)

\[
D_n = \begin{bmatrix} v_0^{(i)}, v_1^{(i)}, \ldots, v_{n-1}^{(i)} \end{bmatrix}
\]

\[
\begin{align*}
\mathbb{w}_n^{(i)} &= \left[ D^{(i)} - d_{nn}I \right] v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n} v_{k-1}^{(i)} \quad (2) \\
\mathbb{d}_{n+1,n} &= \sqrt{\sum_{i=1}^{m} p_i \langle \mathbb{w}_n^{(i)}, \mathbb{w}_n^{(i)} \rangle} \quad (3) \\
\mathbb{v}_n^{(i)} &= \frac{\mathbb{w}_n^{(i)}}{\mathbb{d}_{n+1,n}} \quad (4)
\end{align*}
\]
Recurrent algorithm

We have \( \{v_0^{(i)}, v_1^{(i)}, \ldots, v_{n-1}^{(i)}\}_{i=1}^m \), \( D_n = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_{nn} \end{pmatrix} \)

\[
D_n w_n^{(i)} = \left[ D(i) - d_{nn} I \right] v_{n-1}^{(i)} - \sum_{k=1}^{n-1} d_{k,n} v_{k-1}^{(i)} \quad (2)
\]

\[
D_{n+1,n} = \sqrt{\sum_{i=1}^{m} p_i \langle w_n^{(i)}, w_n^{(i)} \rangle} \quad (3)
\]

\[
v_n^{(i)} = \frac{w_n^{(i)}}{d_{n+1,n}} \quad (4)
\]

\[
d_{k,n+1} = \sum_{i=1}^{m} p_i \langle D(i) v_{n-1}^{(i)}, v_{k-1}^{(i)} \rangle \quad (1)
\]
Hessenberg Matrix associated to a sum of measures

The theorem gains in interest if we realize that it can be written in a matricial way.

**Corollary**

Let $V^{(i)}$ denote the upper triangular matrix with the vectors $v_0^{(i)}$, $v_1^{(i)}$, $v_2^{(i)}$, ..., of $\ell^2$, as columns (i.e., $V^{(i)} = (v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, ...)$). Then, we have

$$D = \sum_{i=1}^{m} p_i [V^{(i)}]^H D^{(i)} V^{(i)}.$$
Hessenberg matrix associated to a self-similar measure

We use the following result of E. Torrano (1987) to apply the above result to self-similar measures.

1. Let $D$ be the Hessenberg matrix associated to a measure $\mu$.
2. Let $\varphi(z) = \alpha z + \beta$ be a similarity, where $\alpha, \beta \in \mathbb{C}$.
3. Let $\mu_\varphi$ be the transformation of this measure by $\varphi$.
4. Let $D^*$ be the Hessenberg matrix associated to $\mu_\varphi$. 

\[D^* = |\alpha| U H DU + |\beta| I,\] 

where 
\[U = \left(\delta_{jk} e^{(k-1)\theta_i}\right)_{j, k=1}^\infty,\] 

with 
\[|\alpha| = |\alpha| e^{\theta_i}.\]
Hessenberg matrix associated to a self-similar measure

We use the following result of E. Torrano (1987) to apply the above result to self-similar measures.

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$$D^* = \alpha \ U^H D U + \beta I,$$

where $U = (\delta_{jk} e^{(k-1)\theta i})_{j,k=1}^{\infty}$, with $\alpha = |\alpha| e^{\theta i}$. 

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Computing Hessenberg Matrix associated to self-similar measures
Hessenberg matrix associated to a self-similar measure

Corollary

Let $\Phi = \{\varphi_i(z) = \alpha_i z + \beta_i; p_i\}$ be an IFSS with probabilities. Let $\mu$ be the corresponding self-similar measure.
Corollary

Let $\Phi = \{\varphi_i(z) = \alpha_i z + \beta_i; p_i\}$ be an IFSS with probabilities. Let $\mu$ be the corresponding self-similar measure. Then, the Hessenberg matrix $D$ associated to the self-similar measure $\mu$ satisfies the following recurrent equation

$$
D = \sum_{i=1}^{m} p_i \left[ V^{(i)} \right]^H \left[ \alpha_i [U^{(i)}]^H DU^{(i)} + \beta_i I \right] V^{(i)},
$$

where $U = \left( \delta_{jk} e^{(k-1)\theta i} \right)_{j,k=1}^\infty$, with $\alpha = |\alpha| e^{\theta i}$. 
Convergence to Hessenberg matrix

Then we have the following algorithm

\[ \nu \rightarrow T(\nu) \rightarrow T^2(\nu) \rightarrow \cdots \rightarrow T^n(\nu) \rightarrow \mu \]

\[ M_\nu \rightarrow \Phi(M_\nu) \rightarrow \Phi(M_\nu) \rightarrow \cdots \rightarrow \Phi(M_\nu) \rightarrow M_\mu \]

\[ D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow D_n \rightarrow D_\mu \]
**Example 1.** Let $L$ be the normalized Lebesgue measure in the interval $[-1, 1]$. This is a self-similar measure for the IFSS

$$
\Phi = \{ \varphi_1(x) = 1/2x - 1/2, \varphi_2(x) = 1/2x + 1/2; p_1 = p_2 = 1/2 \}. 
$$
Examples

Example 1. Let $\mathcal{L}$ be the normalized Lebesgue measure in the interval $[-1, 1]$. This is a self-similar measure for the IFSS

$$\Phi = \{ \varphi_1(x) = \frac{1}{2}x - \frac{1}{2}, \varphi_2(x) = \frac{1}{2}x + \frac{1}{2}; p_1 = p_2 = \frac{1}{2} \}.$$

Algorithm 1.

$$T_\Phi(M_\nu) = \sum_{i=1}^{2} \frac{1}{2} A_{\varphi_i}^H M_\nu A_{\varphi_i}.$$
Example 1. Let $\mathcal{L}$ be the normalized Lebesgue measure in the interval $[-1, 1]$. This is a self-similar measure for the IFSS

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Algorithm 1.

$$T_\Phi(M_\nu) = \sum_{i=1}^{2} \frac{1}{2} A_{\varphi_i}^H M_\nu A_{\varphi_i}.$$ 

If we iterate the transformation $T_\Phi$ 30 times starting with the sixth order identity matrix we obtain

$$\begin{pmatrix}
1.0 & 0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 \\
0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 \\
0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 \\
0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 \\
0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 \\
0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 & 0.09090909
\end{pmatrix}.$$
**Example I.** Let $\mathcal{L}$ be the normalized Lebesgue measure in the interval $[-1,1]$. This is a self-similar measure for the IFSS

$$\Phi = \{\varphi_1(x) = 1/2x - 1/2, \varphi_2(x) = 1/2x + 1/2; p_1 = p_2 = 1/2\}.$$ 

**Algorithm I.**

$$\mathcal{T}_\Phi(M_\nu) = \sum_{i=1}^{2} \frac{1}{2} A_{\varphi_i}^H M_\nu A_{\varphi_i}.$$ 

If we iterate the transformation $\mathcal{T}_\Phi$ 30 times starting with the sixth order identity matrix we obtain

$$
\begin{pmatrix}
1.0 & 0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 \\
0.0 & 0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 \\
0.33333333 & 0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 \\
0.0 & 0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 \\
0.20000000 & 0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 \\
0.0 & 0.14285714 & 0.0 & 0.11111111 & 0.0 & 0.09090909 \\
\end{pmatrix}.
$$

This matrix agrees with the 6th order moment matrix $M_\mathcal{L}$. 

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Then, the 5-section of Jacobi matrix $J_{L,5}$ is

$$
\begin{pmatrix}
0.0 & 0.5773502693 & 0.0 & 0.0 & 0.0 \\
0.5773502691 & 0.0 & 0.5163977795 & 0.0 & -0.7577722133 \cdot 10^{-9} \\
0.0 & 0.5163977796 & 0.0 & 0.5070925551 & 0.0 \\
0.3023715782 \cdot 10^{-9} & 0.0 & 0.5070925521 & 0.0 & 0.5039526136 \\
0.0 & -0.2639315569 \cdot 10^{-8} & 0.0 & 0.5039526419 & 0.0
\end{pmatrix}
$$
Examples

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\end{pmatrix}
$$

**Algorithm II.** $D = \sum_{i=1}^{m} p_i \left[ V^{(i)} \right]^H \left[ \alpha_i [U^{(i)}]^H D U^{(i)} + \beta_i I \right] V^{(i)}$
Then, the 5-section of Jacobi matrix $J_{L,5}$ is

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**Algorithm II.** $D = \sum_{i=1}^{m} p_i [V^{(i)}]^H \left[ \alpha_i [U^{(i)}]^H D U^{(i)} + \beta_i I \right] V^{(i)}$

$$
D_{5}^{30} = 
\begin{pmatrix}
0.0 & 0.5773502692 & 0.0 & -0.2133333332 \cdot 10^{-9} & 0.0 \\
0.5773502691 & 0.0 & 0.5163977796 & 0.0 & -0.1 \cdot 10^{-9} \\
0.0 & 0.5163977796 & 0.0 & 0.5070925526 & 0.0 \\
0.0 & 0.0 & 0.5070925529 & 0.0 & 0.5039526304 \\
0.0 & 0.0 & 0.0 & 0.5039526307 & 0.0 \\
\end{pmatrix}
$$
**Example II.** Let \( T \) be the Sierpinski triangle with basis on the \([-1, 1]\) interval. Consider the uniform measure \( \mu \) on \( T \), i.e., the \( \frac{\log 3}{\log 2} \)-dimensional Hausdorff measure on \( T \).
Example II. Let $T$ be the Sierpinski triangle with basis on the $[-1,1]$ interval. Consider the uniform measure $\mu$ on $T$, i.e., the $\frac{\log 3}{\log 2}$-dimensional Hausdorff measure on $T$.

This is a self-similar measure for the IFSS given by

$$\Phi = \left\{ \varphi_1(z) = \frac{1}{2z} - \frac{1}{2}, \varphi_2(z) = \frac{1}{2z} + \frac{1}{2}, \varphi_3(z) = \frac{1}{2z} + \frac{1}{2} \sqrt{3} \frac{1}{2i}; p_i = \frac{1}{3} \right\}$$
Examples

Algorithm I. Applying $T_\Phi$ 30 times starting with the identity matrix we obtain an approximation of the 4-section of the Hessenberg matrix of the measure $\mu$:

$$
\begin{pmatrix}
0 + 0.5773502693i & 0.3 \cdot 10^{-9} + 0i & 0 - 0.4182428890i & -0.2457739408 \cdot 10^{-8} + 0i \\
0.6666666673 + 0.0i & 0 + 0.5773502691i & 0.1267731382 \cdot 10^{-8} + 0i & 0 - 0.3487499915i \\
0 + 0i & 0.7888106373 + 0i & 0 + 0.5773502706i & 0.1292460659 \cdot 10^{-8} + 0i \\
-0.406877 \cdot 10^{-9} + 0i & 0 + 0.279363 \cdot 10^{-9}i & 0.7737179471 + 0i & 0 + 0.5773502588i
\end{pmatrix}
$$
Algorithm I. Applying $\mathcal{T}_\Phi$ 30 times starting with the identity matrix we obtain an approximation of the 4-section of the Hessenberg matrix of the measure $\mu$:

$$
\begin{pmatrix}
0 + 0.5773502693i & 0.3 \cdot 10^{-9} + 0i & 0 - 0.4182428890i & -0.2457739408 \cdot 10^{-8} + 0i \\
0.6666666673 + 0.0i & 0 + 0.5773502691i & 0.1267731382 \cdot 10^{-8} + 0i & 0 - 0.3487499915i \\
0 + 0i & 0.7888106373 + 0i & 0 + 0.5773502706i & 0.1292460659 \cdot 10^{-8} + 0i \\
-0.406877 \cdot 10^{-9} + 0i & 0 + 0.279363 \cdot 10^{-9}i & 0.773719471 + 0i & 0 + 0.5773502588i
\end{pmatrix}
$$

Algorithm II. With only seven iterations, we have

$$
\begin{pmatrix}
0 + 0.572839i & -0.410^{-9} + 0i & 0 - 0.418197i & -0.548635 \cdot 10^{-10} - 0.635737 \cdot 10^{-20}i \\
0.666692 & 0 + 0.572839i & -0.110^{-9} - 0.380415 \cdot 10^{-20}i & 0.106810 \cdot 10^{-19} - 0.348729i \\
0 & 0.788866 & -0.108689 \cdot 10^{-19} + 0.572839i & -0.1610^{-9} - 0.521858 \cdot 10^{-19}i \\
0 & 0 & 0.773830 - 0.258454 \cdot 10^{-21}i & -0.797017 \cdot 10^{-19} + 0.572839i
\end{pmatrix}
$$
**Example III.** Let $T$ be the Sierpinski triangle as above. Consider the invariant for the same IFSS with probabilities $p_1 = \frac{1}{10}, p_2 = \frac{1}{5}, p_3 = \frac{1}{7}$. 
Algorithm I. Applying $T_{\Phi}$ 7 times starting with the identity matrix we obtain an approximation of the 4-section of the Hessenberg matrix of the measure $\mu$:

$$
\begin{pmatrix}
0.0992 + 1.2029i & -0.2046 - 0.1459i & -0.1799 \cdot 10^{-4} - 0.3176i & -0.0123 + 0.0555i \\
0.5538 + 0.1359 \cdot 10^{-9}i & 0.1439 + 0.8415i & 0.0208 - 0.0718i & -0.0396 - 0.3027i \\
0.5688 \cdot 10^{-9} + 1.7342 \cdot 10^{-21}i & 0.6848 + 0.5367 \cdot 10^{-9}i & 0.0390 + 0.7027i & 0.0117 - 0.0461i \\
0.5398 \cdot 10^{-8} + 0.8097 \cdot 10^{-9}i & 0.7127 \cdot 10^{-8} - 0.2649 \cdot 10^{-9}i & 0.7116 - 0.2392 \cdot 10^{-9}i & 0.07365 + 0.6745i
\end{pmatrix}
$$
Algorithm I. Applying $T_{\Phi}$ 7 times starting with the identity matrix we obtain an approximation of the 4-section of the Hessenberg matrix of the measure $\mu$:

$$
\begin{pmatrix}
0.0992 + 1.2029i & -0.2046 - 0.1459i & -0.1799 \cdot 10^{-4} - 0.3176i & -0.0123 + 0.0555i \\
0.5538 + 0.1359 \cdot 10^{-9}i & 0.1439 + 0.8415i & 0.0208 - 0.0718i & -0.0396 - 0.3027i \\
0.5688 \cdot 10^{-9} + 1.7342 \cdot 10^{-21}i & 0.6848 + 0.5367 \cdot 10^{-9}i & 0.0390 + 0.7027i & 0.0117 - 0.0461i \\
0.5398 \cdot 10^{-8} + 0.8097 \cdot 10^{-9}i & 0.7127 \cdot 10^{-8} - 0.2649 \cdot 10^{-9}i & 0.7116 - 0.2392 \cdot 10^{-9}i & 0.07365 + 0.6745i
\end{pmatrix}
$$

Algorithm II. With seven iterations, we have

$$
\begin{pmatrix}
0.099218 + 1.202963i & -0.204629 - 0.145941i & -0.0000179 - 0.317680i & -0.012314 + 0.055542i \\
0.5538131313 & 0.143933 + 0.841541i & 0.020889 - 0.0718614i & -0.039695 - 0.302772i \\
0 & 0.684812 + 2.05958 \cdot 10^{-12}i & 0.0390029 + 0.702786i & 0.011747 - 0.046155i \\
0 & 0 & 0.711680 + 1.54964 \cdot 10^{-12}i & 0.0736565 + 0.674541i
\end{pmatrix}
$$
Example IV. Let $C$ be the plane Cantor set.

Consider the uniform measure $\mu$ on this set. This measure is self-similar for the following IFSS
\[
\Phi = \{ \phi_1(z) = \frac{1}{4}z + 1 + \frac{i}{2}z, \phi_2(z) = \frac{1}{4}z + 1 - \frac{i}{2}z, \phi_3(z) = \frac{1}{4}z - 1 + \frac{i}{2}z, \phi_4(z) = \frac{1}{4}z - 1 - \frac{i}{2}z; p_i = \frac{1}{4} \}
\]
Example IV. Let $C$ be the plane Cantor set.

Consider the uniform measure $\mu$ on this set.

This measure is self-similar for the following IFSS

$$\Phi = \left\{ \begin{array}{l}
\varphi_1(z) = \frac{1}{4}z + \frac{1+i}{2}z, \\
\varphi_2(z) = \frac{1}{4}z + \frac{1-i}{2}z, \\
\varphi_3(z) = \frac{1}{4}z + \frac{-1+i}{2}z, \\
\varphi_4(z) = \frac{1}{4}z + \frac{-1-i}{2}z; p_i = \frac{1}{4} \end{array} \right\}$$
Examples

*Algorithm I.* Applying $T_\Phi$ 10 times starting with the identity matrix we obtain an approximation of the 5-section of the Hessenberg matrix of the measure $\mu$:

$$
\begin{pmatrix}
0 & 0 & 0 & -0.5534617900 & 0 \\
0.7302967432 & 0 & 0 & 0 & -0.1728136409 \\
0 & 0.7720611578 & 0 & 0 & 0 \\
0 & 0 & 0.8042685429 & 0 & 0 \\
0 & 0 & 0 & 0.6168489579 & 0
\end{pmatrix}
$$
Examples

**Algorithm I.** Applying $\mathcal{T}_\Phi$ 10 times starting with the identity matrix we obtain an approximation of the 5-section of the Hessenberg matrix of the measure $\mu$:

$$
\begin{pmatrix}
0 & 0 & 0 & -0.5534617900 & 0 \\
0.7302967432 & 0 & 0 & 0 & -0.1728136409 \\
0 & 0.7720611578 & 0 & 0 & 0 \\
0 & 0 & 0.8042685429 & 0 & 0 \\
0 & 0 & 0 & 0.6168489579 & 0
\end{pmatrix}
$$

**Algorithm II.** With only seven iterations, we have

$$
\begin{pmatrix}
0 + 0i & 0 + 0i & 0 + 0i & -0.5534617832 + 0i & 0 + 0i \\
0.7302967446 & 0 + 0i & 0 + 0i & 0 + 0i & -0.1728136428 + 0i \\
0 & 0.7720611608 & 0 + 0i & 0 + 0i & -4.0 \cdot 10^{-11} + 0i \\
0 & 0 & 0.8042685477 & 0 + 0i & 0 + 0i \\
0 & 0 & 0 & 0.6168489731 & 0 + 0i
\end{pmatrix}
$$