UNIVERSIDAD POLITÉCNICA DE MADRID
ESCUELA TÉCNICA SUPERIOR DE INGENIEROS AERONÁUTICOS

**Instability Analysis of Vortex Systems
in Lifting-Body Wakes**

Doctoral Thesis

by
Juan Ángel Tenders Ventanas
Master in Aerospace Engineering

MADRID, JULY 2016

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Vassilios Theofilis

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Abstract

The present work deals with vortical flow instabilities, in particular with those that appear in the wake left by aircraft, wind turbines or any other body surrounded by fluid, generating a force perpendicular to the flow, namely lift. The case of the aircraft wake destruction caused by the growth of vortex instabilities is of great importance, as the selection of safe distances between aircraft is in most of the cases very conservative due to the lack of knowledge of the phenomena involved in this process. Therefore, any improvement on their knowledge could help to reduce these distances without concerning the safety of the flight, which is of great importance when large aircraft, like Airbus 380 or Boeing 747, are flying followed by small aeroplanes. Other relevant case where the analysis of vortical wakes is crucial, is that of the wind farms, where the interaction of the wake left from a turbine upwind could degrade the performance of the wind turbine downstream by a great amount or even concern its structural safety.

A wide variety of methodologies are discussed and employed in the present thesis to perform the stability analysis of such flows. Two groups can be distinguished among these methods. These belonging to one of them, designed vortex methods, are, in most of the cases, incompressible and inviscid, in contrast with standard stability analysis methods, that are usually viscous. However, the main difference between these two approaches is the fact that vortex methods are grid-less (Lagrangian description), in contrast to standard methods, which need a grid (Eulerian description). Both, vortex and grid methods, could be used to solve the fluid motion as an evolution problem (Initial Value Problems, IVP). This approach is usually denominated DNS (Direct Numerical Simulation), although here this denomination will be used only for the grid IVP. The study of the flow behaviour to instabilities could be done using IVP of the two types, where imposed perturbations or random noise will activate the instabilities, whose growth or decay will be observed. However, in many cases, it is more interesting to perform the stability analysis by means of linearisation of the equations and analysis of the eigenvalues and eigenmodes of the resulting EVP (EigenValue Problem). This second approach could be also used with both of the aforementioned methods, based on the Lagrangian or the Eulerian descriptions of the flow equations, which constitutes the core of the present research work.

Some canonical cases are studied first, and the complexity is increased gradually to add more realistic features. However, even the results of the most simple cases have their application to explain phenomena related to the behaviour of vortices in the real world. The case of the counter-rotating vortex pair is a clear example. Despite its simplicity, the results extracted from it are of great relevance, as it is the instability discovered by Crow [35]. In this thesis, the movement of the vortices in a two-dimensional plane is studied: from the simplification of point vortices by a vortex simulation, to the scenario where the vortices are close and there is viscosity in the flow by means of a two-dimensional DNS. Homogeneous instabilities in the axial direction are also studied, first for two point

vortices, following the idea of Crow [35], and later for the viscous vortex dipole, using the BiGlobal methodology. Comparisons of these two approaches could be found in Tenero et al. [129, 130, 131]. In addition, axial diffusion could be added and the resulting problem studied employing the three dimensional Parabolised Stability Equations (PSE-3D) analysis concept. All the methodologies considered have their range of validity. Nevertheless, for a true counter-rotating vortex pair with a great amount of axial diffusion, only the PSE-3D analysis proves that can give trustworthy results, as the axial diffusion leads to saturation of the modal growth, in contrast to parallel vortex results.

This thesis collects results from several publications: Paredes et al. [113], Tenero et al. [129, 130, 131], He et al. [63], and Suryanarayanan et al. [128], as well as some original work.

Resumen

El presente trabajo trata inestabilidades en flujos con vórtices, en particular, aquellas que aparecen en la estela dejada por aeronaves, aerogeneradores o cualquier otro cuerpo inmerso en un fluido que genere una fuerza perpendicular al mismo, conocida como sustentación. El caso de la destrucción de las estelas de aviones causadas por el crecimiento de inestabilidades es de gran importancia, ya que la selección de distancias de seguridad entre aeronaves es en la mayoría de los casos muy conservadora debido a la falta de conocimiento de los fenómenos involucrados en este proceso. Por lo tanto, cualquier mejora en su conocimiento podría ayudar a reducir esas distancias sin afectar la seguridad del vuelo, lo que es de gran importancia cuando grandes aviones, como el Airbus 380 o el Boeing 747, vuelan seguidos de aviones pequeños. Otro caso relevante en el que el análisis de estelas de vórtices es crucial, es el de los campos de aerogeneradores, donde la interacción de la estela dejada por un aerogenerador aguas arriba podría degradar de una forma importante las actuaciones de una turbina aguas abajo, incluso llegando a afectar a su integridad estructural.

Una gran cantidad de metodologías se discuten y emplean en la presente tesis para proceder con el análisis de estabilidad de este tipo de flujos. Pueden diferenciarse dos grupos entre estos métodos. Los pertenecientes a uno de ellos, denominados métodos de vórtices, son, en la mayoría de los casos, incompresibles y no viscosos, al contrario que los métodos típicos para el análisis de estabilidad, que normalmente son viscosos. Sin embargo, la principal distinción entre estas dos aproximaciones es el hecho de que los métodos de vórtices son sin malla (descripción lagrangiana), al contrario que los métodos tradicionales, que sí la necesitan (descripción euleriana). Ambos, métodos de vórtices y con malla, pueden usarse para resolver el movimiento del fluido como un problema de evolución (Problema de Valores Iniciales, IVP en inglés). Este enfoque es normalmente denominado DNS (Simulación Numérica Directa por sus siglas en inglés), aunque aquí se usará esa denominación sólo para los IVP con malla. El estudio del comportamiento del fluido en cuanto a inestabilidades puede hacerse usando IVP de los dos tipos analizados, donde la imposición de perturbaciones o el ruido aleatorio activarían las inestabilidades, cuyo crecimiento o amortiguamiento sería observado. Sin embargo, en muchos casos es más interesante ejecutar el análisis de estabilidad por medio de la linealización de las ecuaciones y el análisis de autovalores y autovectores del resultante EVP (Problema de AutoValores, por sus siglas en inglés). Esta segunda estrategia puede utilizarse con los dos métodos mencionados previamente, tanto con los basados en la descripción lagrangiana como en la euleriana de las ecuaciones del fluido, lo que constituye el núcleo del presente trabajo de investigación.

Primero se estudian algunos casos canónicos y luego se va incrementando la complejidad gradualmente para ir añadiendo características más realistas. Sin embargo, incluso los resultados de los casos más simples tienen su aplicación para explicar fenómenos relativos

al comportamiento de vórtices en casos reales. El caso del par de vórtices contra-rotatorios es un claro ejemplo. A pesar de su simplicidad, los resultados que se extraen de él son de gran relevancia, como lo es la inestabilidad descubierta por Crow [35]. En esta tesis se estudia el movimiento de los vórtices en un plano bidimensional: desde la simplificación de vórtices puntuales mediante una simulación de vórtices, hasta el escenario donde los vórtices son cercanos y hay viscosidad en el flujo mediante una simulación DNS bidimensional. Se estudian inestabilidades con homogeneidad en la dirección axial, primero para dos vórtices puntuales, siguiendo la idea de Crow [35], y luego para el dipolo de vórtices, usando la metodología BiGobal. Se pueden encontrar comparaciones entre estas dos aproximaciones en Tendero et al. [129, 130, 131]. Además, puede añadirse difusión axial y estudiarse el problema resultante empleando el concepto de análisis por medio de las Ecuaciones de Estabilidad Parabolizadas en tres dimensiones (PSE-3D por sus siglas en inglés). Todas las metodologías consideradas tienen su propio rango de validez. No obstante, para un verdadero par de vórtices contra-rotatorios con una gran cantidad de difusión axial, sólo el análisis por medio de PSE-3D prueba que puede proporcionar resultados de confianza, ya que la difusión axial provoca una saturación del crecimiento del modo, en contra de lo que ocurre en el caso de vórtices paralelos.

Esta tesis recopila resultados de varias publicaciones: Paredes et al. [113], Tendero et al. [129, 130, 131], He et al. [63], y Suryanarayanan et al. [128], así como trabajos originales.

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Introduction

Vortical flows, and particularly, vortex instabilities, have been a topic of interest for human kind since it began to pay attention to fluids behaviour. In this sense, the famous pictures of Leonardo da Vinci (1452-1519), which represent the formation of vortices in a flow, and how these break apart, are most probably the first study of vortex instabilities. However, despite the amount of research undertaken, the topic is not fully understood yet, and, therefore, research in the area is still of interest.

From its title, it can be read that this thesis deals with vortex instabilities, particularly with those that refer to systems of vortices that appear in the wake of lifting bodies, which could be from two-dimensional wing profiles, to full aeroplanes. Despite the enormous research efforts along history, the phenomena involving such instabilities is not fully understood. Therefore, this thesis tries to shed more light on this topic, which is still the subject of many investigations, and involves a great deal of phenomena that should yet to be understood. This work is therefore, a small step in the understanding of these phenomena, whose outcomes will be settled on later.

In spite of the variety of configurations analysed here, an outstanding case among all of them is the vortical aircraft wake, whose understanding is critical for the aeronautical field, see Spalart [125]. The underlining factor is that the natural mechanisms of wake dissipation are very slow to act, and, therefore, aircraft separation distances in commercial aviation are very large. It is known that various mechanisms could accelerate the wake destruction due to the growth of instabilities, but the actual process is not very well understood, and, therefore, these distances must be very conservative in order to avoid any possible hazard. This point is of special relevance for take-off and landing because the aeroplanes must be aligned one after another maintaining very long distances, which reduces the amount of airport operations and, therefore, their capacity. In order to avoid any possible danger, regulations determine that this distance must be bigger when the preceding aircraft has a larger size compared to the following, which is a disadvantage for the operation of large size aircraft. In fact, the development of large aircraft as Boeing 747 and Airbus 380 has been accompanied by enormous amount of research in the understanding of wake disintegration mechanisms. As part of one of these research peaks, Crouch et al. [34] and Crouch and Spalart [33] proposed a method to accelerate the wake destruction. As there is not any new of such developments in the present time, the activity in the field has decreased, but nevertheless, as the aircraft density is always increasing [23], the motivation of such studies remains.

Although the focus is on aircraft wake instabilities, the methodology studied here is applicable to any other vortical wake or flow in general. A relevant example outside the aeronautical field is the wind turbine wake. The case has some similarities with the aircraft wake, as in the operation of wind farms, the interaction of a wake left from a

turbine upwind could degrade the performance of the wind turbine downstream by a great amount or even concern its structural safety [37].

The diverse methodologies that could be applied to the study of such flows are mainly of two categories. On the one hand the most standard and widely used (but not the oldest) set of methodologies is based on the Navier-Stokes equations, or a simplification of them, and uses a grid to discretise the fluid volume. It uses an Eulerian specification of the flow field, the magnitudes are calculated over points where time and position are two independent variables. On the other hand, the vortex methods are based on the Biot-Savart law for point vortices or vortex filaments, which is a consequence of the fluid equations under some severe restrictions (potential and irrotational flow), restrictions that could be relaxed in some scenarios. In contrast to grid methods, vortex methods are grid-less, because they are based on a Lagrangian specification of the flow field, where the particles are moving with the flow, following definite trajectories, and therefore, their positions are a function of time. In general vortex methods are simpler and faster, while grid methods are more precise, but this will depend on the particular methodology.

Both of these methodologies could also be classified by a different criterion, attending to the type of mathematical or numerical problem solved. The equations can be studied as an IVP (Initial Value Problem) or as an EVP (EigenValue Problem). The two approaches have their own areas of applicability for the instability analysis of vortical flows, IVP are more specific to observe the actual behaviour of the flow in some particular conditions, while EVP give a broader information, allowing for the prediction of a great amount of instabilities and whether they will or will not appear.

The vortex methods seen as IVP are a mature computational tool with decades of proven reliable results [91], but the EVP counterpart is even previous, as it gave solution to the von Kármán [139] vortex street more than one century ago. Maybe more relevant for the aeronautical wake problem is the work of Crow [35] using the same methodology but letting homogeneous perturbations to act in the direction of the vortex axes.

Grid methods, although more standard, are much more modern. They base their good performance on computer capabilities, so it was not until two or three decades ago, when many relevant investigations appear. IVP based on grid methods are basically DNS (Direct Numerical Simulations), being one example of vortex simulation that of Abid and Brachet [1]. When the grid methods are solved as an EVP, the analysis is usually based on Modal Linear Stability Theory (MLST), which is said to be local when the basic flow has only one relevant direction, as in Mayer and Powell [105], and global when the basic flow has more than one relevant direction, as in Hein and Theofilis [66].

The motivation of the work is been now clarified: the improvement or progress in the understanding of the diverse instabilities that appear in vortical flows, with focus on their application to the aircraft wake problem, among others, as the same time that the suitability of the application of the different methodologies to this problem is studied. This is completely linked with the main objective of the thesis, which could be summarised as an effort to improve the understanding of vortical flow instabilities, while the methodologies available are analysed in detail.

The structure of this thesis, summarised in the Contents, is as follows:

- Introduction: The topic and motivation are introduced in this chapter. The importance of the aircraft wake analysis is highlighted, as well as the most relevant works in the area are reviewed.
- Chapter 1 (Background): This chapter is divided in three parts. The first of them deals with vortices, it gives several vortex definitions as well as exposes the vortex models and the most relevant vortex systems. Next, a wide literature survey of the vortex instabilities is carried out, and finally the most relevant to the field numerical tools are also reviewed.
- Chapter 2 (Theoretical Instability Analysis Concepts): The theoretical fundamentals behind all the methodologies used to address the vortex instabilities are collected in this chapter. Here, it is also highlighted the duality between vortex and grid methods on one side, and between the IVP and the EVP on the other side.
- Chapter 3 (Numerical Considerations): The numerical aspects that are relevant to accomplish the implementation of the algorithms based on the ideas exposed in Chapter 2 are shown in Chapter 3.
- Chapter 4 (Vortex Recognition): This chapter exposes few results of the application of vortex recognition criteria to some flow fields.
- Chapter 5 (Vortex Movement): A collection of IVP results of different vortical configurations is gathered here. Some comparisons are made between results obtained from the two approaches, standard DNS and point vortices, as well as comments of the most relevant aspects are given.
- Chapter 6 (Vortex Instabilities): This chapter deals with vortical flow instabilities. Again, relevant results of both standard and vortex methods are shown, compared and properly discussed.
- Chapter 7 (Conclusions): Finally, the most important outcomes of this thesis are summarised and commented in the last chapter.

This thesis collects information from the list of publications given below:

- Journal papers:
 - J. A. Tendero, V. Theofilis, M. Roura, and R. Govindarajan. On vortex filament methods for linear instability analysis of aircraft wakes. *Aerosp. Sci. Technol.*, 44:51–68, 2015
 - W. He, J. A. Tendero, P. Paredes, and V. Theofilis. Linear instability in the wake of an elliptic wing. *Theor. Comp. Fluid Dyn.*, (accepted), 2016
 - S. Suryanarayanan, J. A. Tendero, R. Govindarajan, and V. Theofilis. On the linear and nonlinear instability of viscosity stratified shear layers. *J. Fluid Mech.*, (in preparation), 2016

- Conference papers:
 - P. Paredes, V. Theofilis, D. Rodríguez, and J. A. Tendero. The PSE-3D instability analysis methodology for flows depending strongly on two and weakly on the third spatial dimension. In *6th Theoretical Fluid Mechanics Conference, Honolulu, HI, USA, June 27-30*. AIAA Paper 2011-3752, 2011
 - J. A. Tendero, P. Paredes, M. Roura, R. Govindarajan, and V. Theofilis. BiGlobal and point vortex methods for the instability analysis of wakes. In *43rd AIAA Fluid Dynamics Conference and Exhibit, San Diego, CA, USA, June 24-27*. AIAA Paper 2013-2820, 2013
- Book chapters:
 - J. A. Tendero, P. Paredes, M. Roura, R. Govindarajan, and V. Theofilis. Filament vortex and global instability analysis of the Crow mode. In V. Theofilis and J. Soria, editors, *Fluid Mechanics and its Applications series, Instability and Control of Massively Separated Flows*, volume 107, pages 229–234, 2014

Chapter 1

Background

Before entering into the details of the theory behind the different numerical approaches, the vortex concept and the variety of models of vortices and vortex systems are reviewed. Later a literature survey of previous works dealing with vortex instabilities is carried out, focusing on the physical phenomena, and last, the diversity of algorithms used in each one of the different studies is also reviewed.

1.1 Vortices

It is agreed that a vortex consists in flow rotating and vorticity concentrated, but these are very subjective definitions, not valid for every single case. An objective definition of vortex is then needed. The next section addresses this issue, which is essential for a work dealing with vortices. Once it is specified what is a vortex, there are a wide variety of flows that can be considered vortices. Then, particular vortices need to be characterised by means of vortex models, which are vortex idealizations that can be expressed from an analytical form of the flow velocity and vorticity components, and are also explained subsequently.

1.1.1 Vortex definition

To find an objective definition of a vortex is not an easy task. Several illustrious personalities in the field have proposed definitions with variable success. Lugt [100] proposed two important definitions, one for a vortex: “A vortex is the rotating motion of a multitude of material particles around a common center” and another for vorticity: “The angular velocity of matter at a point in continuum space is called vorticity”. However, the vortex definition proposed is quite subjective, and it is needed to find an objective definition of it. This idea is based on the principle that regions with located vorticity are vortices, but there are examples when this idea is not clear. The Couette flow has uniform vorticity different from zero and the flow is parallel. And the opposite occurs in a region of potential flow rotating around a single point, where vorticity is zero everywhere except in the point vortex, where it is infinity.

The vortex definition is linked to flow topology studies, as vortices are maybe the most important structures of the flow. The studies of the flow topology started from a classification of the singular points in the flow, as done by Reyn [117] for a system of three linear PDE (Partial Differential Equations). Later Bakker and Winkel [10] elaborated on the topic and applied their analysis to fluid flows. Dallmann [38] proposed a vortex definition based on the second flow invariant, Q . Using this definition, a region is a vortex

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if $Q = [|\mathbf{\Omega}|^2 - |\mathbf{S}|^2]/2 > 0$, where $\mathbf{S} = [\nabla\mathbf{U} + (\nabla\mathbf{U})^T]/2$ is the strain tensor and $\mathbf{\Omega} = [\nabla\mathbf{U} - (\nabla\mathbf{U})^T]/2$ the vorticity tensor, being $\nabla\mathbf{U}$ the velocity gradient. This is one of the earliest precise vortex definitions, which will identify successfully vortices in most of the cases. Later, Chong et al. [30] proposed another definition, $\Delta = (Q/3)^3 + (R/2)^2 > 0$, where $R = \det \nabla\mathbf{U}$. One more vortex definition followed by Jeong and Hussain [75], where they proposed the famous λ_2 vortex definition, that defines a vortex as the region where the second eigenvalue of the matrix $\mathbf{S}^2 + \mathbf{\Omega}^2$ is negative. Wu et al. [146] analysed these and other definitions and provided with a criteria for a better vortex definition.

All the previous definitions give quite good results in most of the cases, but they have a problem. Some definitions are not Galilean invariant, that is, the result changes if the frame of reference moves. Furthermore, none of them is invariant to rotations. Haller [58, 59] proposed a new vortex definition that overcomes these issues, both for inertial systems and for rotating systems, and is really objective, the $\mathbf{M}_Z$ vortex definition. He defined vortices as sets of fluid trajectories with indefinite $\mathbf{M}_Z$, where, first, $\mathbf{M} = \partial_t\mathbf{S} + \mathbf{U} \cdot \nabla\mathbf{S} + \mathbf{S}(\nabla\mathbf{U}) + (\nabla\mathbf{U})^T\mathbf{S}$ is the strain acceleration tensor, and, second, Z is a zero strain region such as $Z = \{\xi \mid \langle \xi, \mathbf{S}\xi \rangle = 0\}$. There are two sets of perpendicular directions that satisfy the previous equation, $\xi^\pm$. Therefore, the region where $\mathbf{M}_Z$ is indefinite can be written as $\langle \xi^-, \mathbf{M}\xi^- \rangle \langle \xi^+, \mathbf{M}\xi^+ \rangle < 0$. As mentioned above, the relevance of this vortex definition is that it is not only Galilean invariant, but also invariant for rotations, which is very important, as it gives a vortex definition independent of the frame of reference, in contrast to the other vortex definitions, which vary the results when one of such changes occur.

A few simple cases are analysed in Chapter 4, to compare results from the various definitions. However, as all the cases are two dimensional, the true advantages of the more elaborated criteria are not demonstrated.

1.1.2 Vortex models

Despite the subjectivity of the general definition from Lugt [100], “A vortex is the rotating motion of a multitude of material particles around a common center,” it could be precise enough for the general purposes of this work, unless stated otherwise. Basically a vortex will be defined as a region of concentrated vorticity.

However, when dealing with vortex instabilities, a vortex definition is not enough, a definite vorticity or velocity flow field is needed in order to study its instabilities. There are several models used for that purpose, some of which are two-dimensional, while others incorporate axial diffusion along the vortex axis. The most relevant of these models are explained hereinafter.

The notation, unless stated otherwise, will be to consider the plane perpendicular to the vortex axis as the x - y plane for Cartesian coordinates and r - θ for cylindrical coordinates, being z the perpendicular or axial direction. However, this is not the most common notation, especially in the application to aircraft wakes, where x is usually the direction of the vortex axis. Nevertheless, this notation is selected here for consistency among the different cases.

Batchelor vortex

The most relevant vortex model over all the vortex models that accounts for axial flow and has a dependence on the axial direction of the vortex size due to diffusion is that proposed by Batchelor [11], shown in Eqs. (1.1),

$$U_r = 0 \quad (1.1a)$$

$$U_\theta = \frac{C_0}{r} (1 - e^{-\eta}) \quad (1.1b)$$

$$U_z = U_\infty - \frac{C_0^2}{8\nu z} \ln \frac{zU_\infty}{\nu} e^{-\eta} + \frac{C_0^2}{8\nu z} Q(\eta) - L \frac{U_\infty^2}{8\nu z} e^{-\eta} \quad (1.1c)$$

where

$$\eta = \frac{U_\infty r^2}{4\nu z} \quad (1.2)$$

$$Q(\eta) = e^{-\eta} (\ln \eta + \text{ei}(\eta) - 0.807) + 2\text{ei}(\eta) - 2\text{ei}(2\eta) \quad (1.3)$$

$$\text{ei}(\eta) = \int_\eta^\infty \frac{e^{-\xi}}{\xi} d\xi \quad (1.4)$$

and (U_r, U_θ, U_z) denote the velocity components in the radial r ($r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$ with (x_c, y_c) the vortex axis' coordinates), azimuthal θ and axial z , directions respectively. Here U_∞ is the axial velocity far from the vortex, ν the fluid kinematic viscosity and C_0 and L are constants.

However, the complexity of this formulation is usually simplified. The asymptotic behaviour $z \rightarrow \infty$ is usually selected, and then, the term in Q of Eq. (1.1c) can be dropped:

$$U_r \approx 0 \quad (1.5a)$$

$$U_\theta \approx \frac{C_0}{r} (1 - e^{-\eta}) \quad (1.5b)$$

$$U_z \approx U_\infty - \frac{C_0^2}{8\nu z} \ln \frac{zU_\infty}{\nu} e^{-\eta} - L \frac{U_\infty^2}{8\nu z} e^{-\eta} \quad (1.5c)$$

The previous dimensional equations are non-dimensionalized and simplified using for that purpose the velocity far from the vortex U_∞ and the length scale ν/U_∞ . The following non-dimensional variables appear: $\tilde{U}_r, \tilde{U}_\theta, \tilde{U}_z, \tilde{z}, \tilde{r}$.

$$\frac{U_r}{U_\infty} \approx 0 \quad (1.6a)$$

$$\frac{U_\theta}{U_\infty} \approx \frac{C_0}{\nu(U_\infty r/\nu)} \left(1 - e^{-\frac{(U_\infty r/\nu)^2}{4(\tilde{U}_\infty z/\nu)}} \right) \quad (1.6b)$$

$$\frac{U_z}{U_\infty} \approx 1 - \frac{C_0^2 \ln \frac{zU_\infty}{\nu} + LU_\infty^2}{8\nu^2(U_\infty z/\nu)} e^{-\frac{(U_\infty r/\nu)^2}{4(\tilde{U}_\infty z/\nu)}} \quad (1.6c)$$

$$\tilde{U}_r \approx 0 \quad (1.7a)$$

$$\tilde{U}_\theta \approx \frac{C_0}{\nu \tilde{r}} \left(1 - e^{-\frac{\tilde{r}^2}{4\tilde{z}}} \right) \quad (1.7b)$$

$$\tilde{U}_z \approx 1 - \frac{C_0^2 \ln \tilde{z} + LU_\infty^2}{8\nu^2 \tilde{z}} e^{-\frac{\tilde{r}^2}{4\tilde{z}}} \quad (1.7c)$$

To simplify the notation, the constant swirl parameter $\kappa_0 = C_0/\nu$, the non-dimensional constant $l = LU_\infty^2/\nu^2$ based on the constant L from Batchelor [11], the local core radius $\delta(\tilde{z}) = 2\sqrt{\tilde{z}}$ and the axial velocity defect $\gamma(\tilde{z}) = -[(\kappa_0^2 \log \tilde{z}) + l]/(8\tilde{z})$ are defined. Also the tildes are removed following the notation of Heaton et al. [64] and the simplified Batchelor equations for the asymptotic limit $z \rightarrow \infty$ are deduced:

$$U_r \approx 0 \quad (1.8a)$$

$$U_\theta \approx \frac{\kappa_0}{r} \left\{ 1 - e^{-\left[\frac{r}{\delta(z)}\right]^2} \right\} \quad (1.8b)$$

$$U_z \approx 1 - \gamma(z) e^{-\left[\frac{r}{\delta(z)}\right]^2} \quad (1.8c)$$

***q*-vortex**

The *q*-vortex is one of the most common vortex models for two-dimensional flow. Despite its two-dimensionality, it has axial flow. It could be obtained from the Batchelor vortex, making negligible the term in Q , as in Eq. (1.5c), which is reasonable, particularly for large free flow Reynolds number, $Re = zU_\infty/\nu$. To obtain it, the variables must be non-dimensionalized using U_s and r_s as the characteristic length and distance as in Lessen et al. [94]:

$$U_s = \frac{C_0^2}{8\nu z} \ln \frac{zU_\infty}{\nu} + L \frac{U_\infty^2}{8\nu z} \quad \text{and} \quad r_s = \sqrt{\frac{4\nu z}{U_\infty}} \quad (1.9)$$

Doing so, the *q*-vortex is obtained, which do not depend upon the axial distance z :

$$\frac{U_r}{U_s} = 0 \quad (1.10a)$$

$$\frac{U_\theta}{U_s} = \frac{C_0}{U_s r_s (r/r_s)} \left[1 - e^{-(r/r_s)^2} \right] \quad (1.10b)$$

$$\frac{U_z}{U_s} = \frac{U_\infty}{U_s} - e^{-(r/r_s)^2} \quad (1.10c)$$

Now calling tilde variables those without dimensions, and $\tilde{q} = C_0/(U_s r_s)$, the equations read:

$$\tilde{U}_r = 0 \quad (1.11a)$$

$$\tilde{U}_\theta = \frac{\tilde{q}}{\tilde{r}} \left(1 - e^{-\tilde{r}^2} \right) \quad (1.11b)$$

$$\tilde{U}_z = \tilde{U}_\infty - e^{-\tilde{r}^2} \quad (1.11c)$$

Eqs. (1.11) are only function of the radial distance, as the vortex is axisymmetric and does not have axial variation. These equations are commonly written without tildes and

the axial velocity is usually changed to W_∞ , see Eqs. (1.12). The only parameters are q and W_∞ , which are measures of the intensity and form of the vortex and jet. If q is zero there is no vortex, only jet, if it is of order one, vortex and jet have similar order of magnitude and if it is infinity, the vortex does not have axial flow. The parameter W_∞ measures the external axial flow.

$$U_r = 0 \quad (1.12a)$$

$$U_\theta = \frac{q}{r} \left(1 - e^{-r^2}\right) \quad (1.12b)$$

$$U_z = W_\infty - e^{-r^2} \quad (1.12c)$$

Note that if in Eqs. (1.8), a particular constant z_0 is taken, then, doing $q = \kappa_0/(\gamma_0\delta_0)$, taking r as r/δ_0 , and dividing all velocities by γ_0 , being $W_\infty = 1/\gamma_0$, the q -vortex equations, Eqs. (1.12), are recovered.

Eqs. (1.13) are the equations of the q -vortex in Cartesian coordinates. This time U_x , U_y and U_z are the velocities in x , y and z respectively, being z the axial direction. This equations are function of x and y , where $r = \sqrt{x^2 + y^2}$.

$$U_x = -\frac{qy}{r^2} \left(1 - e^{-r^2}\right) \quad (1.13a)$$

$$U_y = \frac{qx}{r^2} \left(1 - e^{-r^2}\right) \quad (1.13b)$$

$$U_z = W_\infty - e^{-r^2} \quad (1.13c)$$

Eqs. (1.14) are obtained from the non-dimensional equations, and do not have a vortex size parameter. In order to write the equations in a truly dimensional form, the exponent should be divided by a distance a , and a new velocity should appear in the exponential term of the axial velocity:

$$U_r = 0 \quad (1.14a)$$

$$U_\theta = \frac{q}{r} \left[1 - e^{-\left(\frac{r}{a}\right)^2}\right] \quad (1.14b)$$

$$U_z = W_\infty - W_0 e^{-\left(\frac{r}{a}\right)^2} \quad (1.14c)$$

And the same in Cartesian coordinates:

$$U_x = -\frac{qy}{r^2} \left[1 - e^{-\left(\frac{r}{a}\right)^2}\right] \quad (1.15a)$$

$$U_y = \frac{qx}{r^2} \left[1 - e^{-\left(\frac{r}{a}\right)^2}\right] \quad (1.15b)$$

$$U_z = W_\infty - W_0 e^{-\left(\frac{r}{a}\right)^2} \quad (1.15c)$$

Lamb-Oseen or Gaussian vortex

A further simplification is to make zero the axial flow. Then, Eqs. (1.14) are transformed into Eqs. (1.16). Note that they are written in dimensional form, and a vortex core

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parameter a should be used.

$$U_r = 0 \quad (1.16a)$$

$$U_\theta = \frac{q}{r} \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] \quad (1.16b)$$

$$U_z = 0 \quad (1.16c)$$

As there is not axial flow, the Gaussian vortex is a true two-dimensional vortex. In Cartesian coordinates it is written as follows:

$$U_x = -\frac{qy}{r^2} \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] \quad (1.17a)$$

$$U_y = \frac{qx}{r^2} \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] \quad (1.17b)$$

$$U_z = 0 \quad (1.17c)$$

Rankine vortex

The Rankine vortex is a vortex where the fluid is rotating as a rigid solid if $r < a$, and outside the flow does not have vorticity. It is the simplest vortex model for a definite radius, see Green [57]. Eqs. (1.18) give it shape, where it is observed how there is a jump in the derivative of the velocity for $r = a$.

$$U_r = 0 \quad (1.18a)$$

$$U_\theta = \frac{qr}{a^2} \quad (r \leq a) \quad U_\theta = \frac{q}{r} \quad (r > a) \quad (1.18b)$$

$$U_z = 0 \quad (1.18c)$$

The Rankine vortex is written in Cartesian coordinates by Eqs. (1.19):

$$U_x = -\frac{qy}{a^2} \quad (r \leq a) \quad U_x = -\frac{qy}{r^2} \quad (r > a) \quad (1.19a)$$

$$U_y = \frac{qx}{a^2} \quad (r \leq a) \quad U_y = \frac{qx}{r^2} \quad (r > a) \quad (1.19b)$$

$$U_z = 0 \quad (1.19c)$$

Potential or point vortex

If a is taken to be zero for any of the vortex models without axial flow (Lamb-Ossen or Rankine vortices), the resulting vortex model is equivalent. The result is a singularity at a single point and potential flow around it, Eq. (1.20b), which is also obtainable from any other vortex model if the velocity is sought far from it, meaning that the distance from the vortex is much larger than size of it. In this case, the azimuthal is the only component of the velocity, being zero the radial and axial components. A point vortex is, therefore, an idealization of a vortex with vorticity concentrated at a single point. This is physically the limit when the core of the vortex is much smaller than the distance to the center and Eq. (1.20b) is consequently applicable because $r \gg a$.

$$U_r = 0 \quad (1.20a)$$

$$U_\theta = \frac{q}{r} \quad (1.20b)$$

$$U_z = 0 \quad (1.20c)$$

The point vortex, is also defined in potential theory as one of the elemental singularities. There, it is usually most common to define the point vortex from its circulation, $\Gamma = 2\pi q$:

$$U_r = 0 \quad (1.21a)$$

$$U_\theta = \frac{\Gamma}{2\pi r} \quad (1.21b)$$

$$U_z = 0 \quad (1.21c)$$

In Cartesian coordinates:

$$U_x = -\frac{\Gamma y}{2\pi r^2} \quad (1.22a)$$

$$U_y = \frac{\Gamma x}{2\pi r^2} \quad (1.22b)$$

$$U_z = 0 \quad (1.22c)$$

Effective core

In order to be able to compare different parameters of vortex models, and basically to be able to define a measure of the vortex core that has the account of the effect of the axial flow in it, Widnall [141] defined the vortex core, from the values of the azimuthal and axial velocities:

$$A = \lim_{\tilde{r} \rightarrow \infty} \left[\int_0^{\tilde{r}} \tilde{r}' \tilde{U}_\theta(\tilde{r}')^2 d\tilde{r}' - \ln(\tilde{r}) \right] \quad (1.23a)$$

$$C = \int_0^\infty \tilde{r} \tilde{U}_z(\tilde{r})^2 d\tilde{r} \quad (1.23b)$$

$$a_e = ae^{1/4-A+C} \quad (1.23c)$$

The definition is not valid for vortex definitions that include axial diffusion, and therefore, the Batchelor vortex is not considered in the following. It should be also noted that in these definitions, the non-dimensionalization should be done from the azimuthal velocity, and not the axial velocity.

- *q*-vortex

Eqs. (1.11) are non-dimensionalised by the axial velocity, and not by the parameter q . As the intention here is to use q for the non-dimensionalisation, Eqs. (1.14) are taken, and they are non-dimensionalised by the azimuthal velocity, then, $\tilde{q} = 1$ and the exponential term of the axial velocity is not multiplied by one, but by aW_0/q , while the other term is taken to be zero without loss of generality, being the last

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valid since the vortex core is independent form the actual external axial velocity. If they are introduced into Eqs. (1.23), the following is obtained:

$$A = \frac{1}{2}[\gamma - \ln(2)] \quad (1.24a)$$

$$C = \frac{a^2 W_0^2}{2q^2} \quad (1.24b)$$

$$a_e = ae^{\frac{1}{4} + \frac{a^2 W_0^2}{2q^2} + \frac{1}{2}[-\gamma + \ln(2)]} \quad (1.24c)$$

where $\gamma \approx 0.577$ is the Euler constant.

- Lamb-Oseen or Gaussian vortex

The Lamb-Oseen of Gaussian vortex core size, Eq. (1.25c) is simply derived from Eq. (1.24c) making $W_0 = 0$:

$$A = \frac{1}{2}[\gamma - \ln(2)] \quad (1.25a)$$

$$C = 0 \quad (1.25b)$$

$$a_e = ae^{\frac{1}{4} + \frac{1}{2}[-\gamma + \ln(2)]} = 1.36065a \quad (1.25c)$$

- Rankine vortex

The effective core definition is such that the vortex size should coincide with the Rankine vortex size, as Eqs. (1.26) demonstrate:

$$A = \frac{1}{4} \quad (1.26a)$$

$$C = 0 \quad (1.26b)$$

$$a_e = a \quad (1.26c)$$

- Potential or point vortex

The effective core is not really well defined for the asymptotic case of the point vortex. However, from the definition of effective core, as well as for the limit of the other vortex models for $a \rightarrow 0$ the same is obtained, basically $a_e = 0$:

$$A = \infty \quad (1.27a)$$

$$C = 0 \quad (1.27b)$$

$$a_e = 0 \quad (1.27c)$$

1.1.3 Vortex systems

The vortex models established previously in subsection 1.1.2 are solutions of the flow equations. Now a set of these solutions could be joined together to have a vortex system. As all the vortex model velocity fields go to zero as the radius goes to infinity, the vortex system will also be a solution of the equations if the vortices are far apart. However, if the vortices are close, the superposition of vortices will not fulfil the flow equations. The

procedure should be in these cases to perform a DNS with the superposition of vortices that will not satisfy the equations, but that after a few iterations will reorganise and produce a solution that is really a solution of the flow equations. However, that will not always be necessary for practical reasons and even if the superposition of vortices will not be strictly a solution of the equations, its analysis may be performed giving good results in many cases.

Counter-rotating vortex-pair

The simplest vortex system is formed by two vortices. In this case, two vortices of the same intensity and different sense of rotation are separated by a distance b between their centroids. The counter-rotating vortex pair adds then one parameter to the vortex models previously studied. Polar coordinates are not convenient any more and Cartesian coordinates are used.

- Point vortices

The simplest approach to analyse the counter-rotating vortex pair assuming axial homogeneity, is directly to superpose two point vortices. This simplified approach, does not take vortex radius into account. The pair of counter-rotating point vortices, is a solution of the equations, because the vortex size tends to zero and the ratio distance between the vortices to vortex radius will be infinitely big. The flowfield will be given by the superposition of two point vortices:

$$U_x = -\frac{\Gamma y}{2\pi [(x - b/2)^2 + y^2]} + \frac{\Gamma y}{2\pi [(x + b/2)^2 + y^2]} \quad (1.28a)$$

$$U_y = \frac{\Gamma(x - b/2)}{2\pi [(x - b/2)^2 + y^2]} - \frac{\Gamma(x + b/2)}{2\pi [(x + b/2)^2 + y^2]} \quad (1.28b)$$

$$U_z = 0 \quad (1.28c)$$

- Rankine vortices

The next approach in simplicity, also assuming axial homogeneity, is to superpose two Rankine vortices. This choice will be valid when the vortices are distant and the individual velocity fields do not superpose, but it will fail as they are located closer, because the result of the superposition of vortices is not a solution of the flow equations. The equations are written in Cartesian coordinates, as the flow is not axisymmetric any more:

$$\begin{aligned} U_x &= 0 \quad [(x - b/2)^2 + y^2 \leq a \text{ and } (x + b/2)^2 + y^2 \leq a] \\ U_x &= -\frac{qy}{a^2} + \frac{qy}{(x + b/2)^2 + y^2} \quad [(x - b/2)^2 + y^2 \leq a \text{ and } (x + b/2)^2 + y^2 > a] \\ U_x &= -\frac{qy}{[(x - b/2)^2 + y^2]} + \frac{qy}{a^2} \quad [(x - b/2)^2 + y^2 > a \text{ and } (x + b/2)^2 + y^2 \leq a] \\ U_x &= -\frac{qy}{[(x - b/2)^2 + y^2]} + \frac{qy}{(x + b/2)^2 + y^2} \quad [(x - b/2)^2 + y^2 > a \text{ and } (x + b/2)^2 + y^2 > a] \end{aligned} \quad (1.29a)$$

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$$\begin{aligned}
U_y &= -\frac{qb}{a^2} \quad [(x - b/2)^2 + y^2 \leq a \text{ and } (x + b/2)^2 + y^2 \leq a] \\
U_y &= \frac{q(x - b/2)}{a^2} - \frac{q(x + b/2)}{(x + b/2)^2 + y^2} \quad [(x - b/2)^2 + y^2 \leq a \text{ and } (x + b/2)^2 + y^2 > a] \\
U_y &= \frac{q(x - b/2)}{[(x - b/2)^2 + y^2]} - \frac{q(x + b/2)}{a^2} \quad [(x - b/2)^2 + y^2 > a \text{ and } (x + b/2)^2 + y^2 \leq a] \\
U_y &= \frac{q(x - b/2)}{[(x - b/2)^2 + y^2]} - \frac{q(x + b/2)}{(x + b/2)^2 + y^2} \quad [(x - b/2)^2 + y^2 > a \text{ and } (x + b/2)^2 + y^2 > a] \quad (1.29b)
\end{aligned}$$

$$U_z = 0 \quad (1.29c)$$

- Lamb-Oseen or Gaussian vortex

The same is done for Gaussian vortices, and Eqs. (1.30) are obtained:

$$U_x = -\frac{qy}{(x - b/2)^2 + y^2} \left[1 - e^{-\frac{(x-b/2)^2 + y^2}{a^2}} \right] + \frac{qy}{(x + b/2)^2 + y^2} \left[1 - e^{-\frac{(x+b/2)^2 + y^2}{a^2}} \right] \quad (1.30a)$$

$$U_y = \frac{q(x - b/2)}{(x - b/2)^2 + y^2} \left[1 - e^{-\frac{(x-b/2)^2 + y^2}{a^2}} \right] - \frac{q(x + b/2)}{(x + b/2)^2 + y^2} \left[1 - e^{-\frac{(x+b/2)^2 + y^2}{a^2}} \right] \quad (1.30b)$$

$$U_z = 0 \quad (1.30c)$$

- q -Vortices

The q vortices are equivalent to the Gaussian vortices, but adding axial flow, therefore, it will be the only change:

$$U_x = -\frac{qy}{(x - b/2)^2 + y^2} \left[1 - e^{-\frac{(x-b/2)^2 + y^2}{a^2}} \right] + \frac{qy}{(x + b/2)^2 + y^2} \left[1 - e^{-\frac{(x+b/2)^2 + y^2}{a^2}} \right] \quad (1.31a)$$

$$U_y = \frac{q(x - b/2)}{(x - b/2)^2 + y^2} \left[1 - e^{-\frac{(x-b/2)^2 + y^2}{a^2}} \right] - \frac{q(x + b/2)}{(x + b/2)^2 + y^2} \left[1 - e^{-\frac{(x+b/2)^2 + y^2}{a^2}} \right] \quad (1.31b)$$

$$U_z = W_\infty - W_0 \left[e^{-\frac{(x-b/2)^2 + y^2}{a^2}} + e^{-\frac{(x+b/2)^2 + y^2}{a^2}} \right] \quad (1.31c)$$

- Batchelor vortices

The same can be done with two Batchelor vortices, obtaining a 3D basic flow. The result, similar to that of the q vortices, has a variation in z direction, as the radius grows as z advances. That will provoke that for long distances, finally the two vortices will be very close, even if they were far apart, and the result will not be a solution of the equations. This time, the equations are not written, as they become too long in Cartesian coordinates and do not introduce anything significant.

Co-rotating vortex-pair

The vortex models for two co-rotating vortices are the same as for the counter-rotating vortex pair, but with a change of sign for one of the vortices and they are not repeated for simplicity.

Other vortex systems

Following the same strategy, any number of vortices of different sizes and intensities could be added, being the only problem that if the vortices are very close to each other, the superposition of them will not be a solution of the flow equations. Having that in mind, many other vortical configurations could be thought of, being an example of these the Rennich and Lele [116] four aligned vortex configuration, which consist of a double set of counter rotating vortices that move maintaining their relative distances.

1.2 Vortex instabilities

This section intends to summarise the historical advances in the understanding of vortex instabilities in a very concise manner. There are instabilities that affect a single vortex (core instabilities), as well as others that appear when there exist a system of vortices (cooperative instabilities). In addition, three-dimensional vortices, will have axial variation and can develop instabilities in this direction, which usually end up causing the vortex to be unstable at some axial distance, producing the effect denominated vortex breakdown.

1.2.1 Single vortex instabilities

A single potential vortex in two-dimensions cannot have instabilities, as it is known that the vortex does not induce velocity over itself, so there is no mechanism to affect it. Three lines can be started from that point to find instabilities. First of all is to change the vortex model keeping it two-dimensional. Lamb-Oseen or Rankine vortices might have instabilities due to its shape. The second direction is to look at the vortex from a three-dimensional perspective, even when it has not variation along the axis coordinate. In that sense, homogeneous perturbations in this axial direction may be imposed, leading to instabilities. And the last alternative is to seek for instabilities of vortices that have changes in this axial direction, where for example a sudden break could be found for some cases, as it is the vortex breakdown.

Traditionally, isolated vortices such as the q -vortex approximation of the classic Batchelor [11] vortex have been used as basic flow models. The analysis of a single vortex by means of viscous local modal stability theory commenced by Lessen and Paillet [93] and continued in the works of Khorrami et al. [82], Khorrami [80, 81], and Mayer and Powell [105] who found a variety of vortex core instabilities. Delbende et al. [40] also encountered absolute instabilities. In such vortices both the axial and the azimuthal spatial directions can be taken as homogeneous and are resolved via EVP using local analysis. If the whole

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two-dimensional vortex is analysed by the BiGlobal methodology, the results are identical to those of the local theory [19, 112], although in BiGlobal analysis the assumption of azimuthal homogeneity is not invoked and all different azimuthal eigenmodes of local analysis are obtained at once.

Axial diffusion is not included in the previous modal analyses, which assume axial perturbations as homogeneous. However, if this axial homogeneity assumption is relaxed, the whole physics of vortex instability changes, which is something to consider when performing analysis of axially homogeneous flows. If that variation of the flow variables in the axial direction is assumed to be slow, the parabolised stability equations (PSE) can be used without further simplifications, see Herbert [69]. The importance of axial homogeneity is revealed by the results of Broadhurst et al. [19] and Broadhurst [20]. In particular, Fig. (1.1) shows two direct numerical simulations, which only differ in the assumption of axial homogeneity versus outflow boundary conditions, and lead to very different results for the two cases, highlighting the importance of axial variation. A systematic attempt to understand this phenomenon has been undertaken by Paredes [112].

Figure 1.1: *Upper:* Linear and nonlinear instability development of a q -vortex at $Re = 1000$, $q = 0.8$ in an axially periodic DNS. *Lower:* Same in a fully 3D DNS without imposing axial periodicity. Broadhurst et al. [19].

1.2.2 Vortex systems instabilities

After discussing the possible instabilities for a single vortex, the systems of vortices are analysed. In this case, even for punctual vortices in two-dimensions, cooperative instabilities might appear. In addition, all the effects that can cause instability for a single vortex, from different vortex models, axial homogeneous perturbations, and vortices with axial diffusion, can also be relevant for vortex systems.

There are a wide variety of different vortex systems, but the review is started from

the counter-rotating vortex pair, because its simplicity and importance at the same time. This flow develops several instabilities, among which the most relevant are the long wave or Crow [35] instability and the short wave, elliptic or also called Widnall [141] instability, which can be included into the category of cooperative instabilities, because they are based on the interaction between vortices. Other instabilities that appear in the vortex pair, as the core instabilities, can also appear for an isolated vortex.

Apart from the counter-rotating vortex pair, other vortex configurations can be analysed, as the co-rotating vortex pair studied by Jiménez [76] or four vortex systems as those analysed by Crouch [32], Rennich and Lele [116] and Fabre and Jacquin [43], which are very unstable to instabilities for the inner vortices that may end up eliminating them, but may not affect significantly to the outer vortices. These works, including Crow [35], have used vortex filament methods to analyse the instabilities triggered by interaction between neighbouring vortices. This inviscid linear instability analysis of vortex systems by point-vortex methods was introduced by von Kármán [138, 139], where the analysis of vortex street carrying his name was studied.

Systems of vortices can only be analysed with respect to their viscous linear stability by global linear theory, on account of the inapplicability of the assumption of azimuthal spatial homogeneity, as required by local theory, in the presence and proximity of multiple vortices. Further, when performing viscous BiGlobal instability analysis, the assumption needs to be made that the base flow is homogeneous along the axial spatial direction. This latter assumption may occasionally be approximately correct in high Reynolds numbers, but is certainly inapplicable at moderate and low Reynolds numbers owing to viscous diffusion and interaction; in the latter case, a window of applicability of the assumptions underlying BiGlobal must then be sought prior to the analysis. Several authors have used these methodologies to study the vortex systems mentioned above [22, 53, 54, 66, 74, 113, 124]. Brion et al. [17], Benton and Bons [13], and Tendero et al. [129, 130] offer comparisons between this viscous approach and the inviscid vortex filaments analysis.

A realistic vortex system has instabilities that depend inhomogeneously on two and weakly on the third spatial direction, which is where diffusion takes place and can be obtained by application of the PSE-3D analysis concept. Instability analysis of flows depending strongly on two and weakly on the third spatial direction must commence with the specification of the base flow to be analysed. Suppression of potentially developing instabilities is one means of obtaining base states, as done by Heaton et al. [64] in the isolated Batchelor vortex.

After this demonstration of the enormous amount of works related to vortex instabilities, it is important to highlight that the present work is focused on linear instability. The reader might rightly think that the non-linear analysis will be more accurate, but it is also true that for understanding the non-linear behaviour is usually needed to understand first the linear one. In addition, the linear analysis provides in general more information about how to actuate on these instabilities to control the system, as well as non-linear analysis is usually orders of magnitude more expensive in computational terms.

1.3 Numerical tools for vortex instability analysis

The methodologies that can be used to study the vortical flows presented in the previous section are exposed here. However, the aim of this section is not to explain them in detail, which is the objective of Chapter 2. Instead, they are reviewed showing how they have been successfully applied to the analysis of instabilities of a variety of flows, mostly, but not only, involving vortices or vortex systems.

Before the exposition of the enormous amount of works related to vortex instabilities given below, it is important to highlight that the present work is focused on linear instability. It is true that non-linearities offer a wider spectrum of possibilities, but it is also true that for understanding the non-linear behaviour is usually needed to understand first the linear one. In addition, the linear analysis provides in general more information about how to actuate on these instabilities to control the system, as well as non-linear analysis is usually orders of magnitude more expensive in computational terms.

1.3.1 Direct Numerical Simulations

Direct Numerical Simulations (DNS) reproduce the actual fluid behaviour. Sometimes a particular set of perturbations could be imposed on the flow, as well as random noise could be left to actuate. If particular constraints are not imposed on the numerical algorithms, this type of analysis will show whether the flow is stable or unstable. If the flow is unstable, the growth of the most unstable mode will be seen, as well as non-linear behaviour and in many cases transition to turbulence, as it would happen in a real experiment. Therefore, this methodology is the most realistic, but it only solves a particular case at a time, and the response of the system against different perturbations has to be studied separately. Furthermore, it is the most computationally expensive, and should be avoided if other methodologies are available, but used if possible for validations.

The analysis of laminar flows could be simply analysed by DNS, however, for turbulent flows, DNS becomes prohibitive (only affordable in supercomputers and for simple geometries), due to the small size of the Kolmogorov scale. In such situations, Large Eddy Simulations (LES) are a better choice, as they model the small scales of turbulence and calculate the bigger ones. In the other extreme are the Reynolds Averaged Navier Stokes equations (RANS), which solve the averaged values of the velocities and uses models for the perturbations. This approach is very appreciated in industrial calculations, where the complexity of the geometries is high and results are needed in reasonable time with limited resources. In these applications, accurate mean values are important, but the understanding of the phenomena of the small scales is of least importance.

A different approach is that of the Parabolic Navier Stokes (PNS), which are marched in space instead of time, and are only valid for flows that have small variations in that marching direction. The computational cost of these equations is much cheaper than the equivalent DNS, while the results have a similar accuracy as far as there is slow dependence of the flow in the marching direction, see Paredes et al. [115].

In addition, there exists another important usage of direct simulations when dealing with instability analysis, which is the obtaining of what are called basic flows. A basic

flow is a steady solution of the flow equations upon which the instability analysis could be performed. In many cases, analytical solutions are used as basic flows, but in more general cases, these basic flows should be obtained numerically.

Direct Numerical Simulations (DNS) and Large Eddy Simulations (LES) have been used to perform viscous instability analysis of vortical flows: Abid and Brachet [1] analysed a single vortex, Laporte and Corjon [89] studied a vortex pair, Rennich and Lele [116] a four vortex wake, Bristol et al. [18] investigated a vortex pair of unequal strength, and Nybelen and Paoli [109] studied co-rotating vortices. Others analyse instabilities solving initial value problems (IVP) as it is done by means of DNS or LES, but based on the linearised Navier-Stokes equations (LNSE), as in the work of Billant et al. [15] for a particular vortex pair. In addition, DNS of vortex particles, available from almost four decades ago, see Leonard [91], offers exceptionally good performance for vortical flows, particularly for wakes [28, 29, 144] and for helical vortices [140].

Although not being numerical methods, due to the similarities with direct simulations, experiments are just mentioned here. Some of them, focused on the stability of vortical wake flows, have provided with key information on novel instability mechanisms of vortex systems: the works of Leweke and Williamson [96], Laporte and Leweke [90] and the review of Williamson [143] for the cylinder wake offer examples of the impact that experimentation on the instability of vortex flow systems has had on the understanding of vortex dynamics and instabilities. It is worth mentioning that 3D DNS, LES and the experiments, despite the quality of results that they produce, may require expensive equipment or large computational resources and, as such, can be performed only for a limited number of parameters.

1.3.2 Modal Linear Stability

Modal Linear Stability Theory (MLST), could be inviscid or viscous, being many of the old works for inviscid flow, while most of the recent works are viscous. It is based on a linearisation of the flow equations (Euler equations for inviscid and Navier-Stokes for viscous), and a subsequent introduction of homogeneous perturbations of the time and some directions to achieve an eigenvalue problem. The theory is explained with more detail in section 2.3.

A variety of local analysis studies of the isolated vortex [80–82, 93, 105] have already been mentioned in section 1.2. Global analysis, Theofilis [132, 133] and Gómez et al. [52], is an excellent tool when the flow is not axisymmetric. Earlier works, Sipp and Jacquin [124], Hein and Theofilis [66] and Jacquin et al. [74], were based on Chebyshev spectral methods, later, Broadhurst et al. [19] employed high-order spectral/hp element spatial discretisations, while González et al. [53, 54] are based on finite elements, while more recent works as Paredes et al. [113] and Tendero et al. [129, 130, 131] base their BiGlobal results in compact finite differences, Paredes et al. [114].

1.3.3 Non-Modal Linear Stability

Non-modal results are much more difficult to find in the literature if the results from direct simulations are excluded. For linear equations, modal analysis provides with information of the long term behaviour. If the real equations were to be linear, that will be the actual behaviour for long times. However, due to the non-normality of the modes, a transient growth of the instabilities could appear at earlier times. This growth should always be damped for long times in the linear case. However, as the true equations are not linear, a high growth could trigger non-linear instabilities and move the flow far from its initial state, invalidating the modal analysis.

Antkowiak and Brancher [4] and Brion et al. [17] performed non-modal BiGlobal instability analysis of vortical flows. In the latter of them, structural sensitivity analysis of a pair of counter-rotating vortices was performed and the spatial location at which forcing could promote algebraically growing instabilities in the vortex system was predicted.

Crouch [32] demonstrated for the first time the role that transient growth can play in the destruction of vortex system coherence by a non-modal vortex filaments instability analysis of a four-vortex system.

1.3.4 Parabolised Stability Equations

Parabolised Stability Equations (PSE) offer the possibility to analyse the streamwise growth of linear and non-linear disturbances in slowly varying flows in a similar fashion to what Modal Linear Stability achieves for homogeneous perturbations. Growth mechanisms include both algebraic transient growth and exponential growth through primary and higher instabilities. In contrast to the eigensolutions of traditional linear stability equations, PSE solutions incorporate inhomogeneous initial and boundary conditions as do numerical solutions of the Navier-Stokes equations. The PSE are valid for convectively unstable flows in which the root-mean-square of the variables profiles vary slowly in the streamwise direction, i.e. $\partial/\partial z = \mathcal{O}(\varepsilon)$, with $\varepsilon \ll 1$, for example being $\varepsilon = \mathcal{O}(Re^{-1})$ in boundary-layer flows.

Instability analysis of flows depending strongly on two and weakly on the third spatial direction can be analysed by the PSE-3D concept. The idea is similar to PSE, but starting from a 2D flow and marching along the perpendicular direction. Examples of successful application include vortical flows [19, 21, 64, 113], the wake of an isolated roughness element in supersonic flow [39], streaks in a boundary layer [104], corner flows [50], fully-turbulent rectangular ducts [14] and models of duct intakes [51]. The near and mid-field wake of the wing, where strong lateral flow gradients, in addition to a relatively milder flow dependence along the axial direction, make PSE-3D the method of choice for the instability analysis of this inhomogeneous flow field.

1.3.5 Stability by vortex methods

Following the same idea of subsection 1.3.2, but applied to point vortices, the inviscid linear modal stability analysis by point vortices or vortex filaments is presented here. This

inviscid modal linear stability theory was introduced by von Kármán [138, 139], where the vortex street of a wake that is named after him is analysed and has been extremely useful in providing qualitative and on occasion quantitative information on linear flow dynamics of vortical systems.

If the point vortex instability analysis is done in the plane, the two-dimensional limit is considered, as in the mentioned works of von Kármán [138, 139]. The vortical flowfield is reconstructed by a number of straight and parallel vortex filaments, which can be viewed as (two-dimensional) point vortices in equilibrium, whose stability is analysed. In an aeronautical context, Donaldson and Bilanin [41] have used this methodology to analyse different configurations of four aligned point vortices and assess whether placement of vortices of a given circulation at specific distances results in a system of vortices in equilibrium.

Other approach is to consider the three-dimensional limit, which means that the otherwise point vortices, can have homogeneous perturbations along their axes. Examples of this methodology were already given in subsection 1.1.3, when dealing with vortex systems instabilities [32, 35, 43, 76, 116].

However, beyond the simplified configurations of two or more aligned filaments exposed previously, a wide variety of other vortical configurations have been analysed with respect to their instability using the same methodology, such as the arrays of vortices situated at the vertices of a regular polygon, a configuration referred to as n -gon, originally studied by Kelvin [78] and Thomson [135]. Havelock [62] presented a consistent analysis of the same configuration, while half a century later, Campbell [26] reviewed the topic and added more results. However, it was not up until recent times that the $N = 7$ configuration was shown to be unstable by Kurakin and Yudovich [86]. An analogous configuration, the instability of which was first analysed by Joukowski [77] using the vortex filament method, is the vortex n -gon to which an additional vortex is added at its center and is taken to rotate in the opposite sense as compared with the sense of rotation of the vortices of the n -gon. This model is the cornerstone of analyses of instabilities in the downwash of a helicopter rotor, as well as those in the wake behind wind turbines. Additional interesting examples of application of point-vortex or vortex-filament instability analysis include the three vortices discussed by Aref [5, 7], or other configurations involving a larger number of vortices Aref [6]. Modifying the instability analysis methodology for straight vortex filaments, instability in a vortex helix can also be analysed. This was first studied by Kelvin [79], and Levy and Forsdyke [95], while Ricca [118] and Kuibin and Okulov [85] added more light to the problem. Recently, instability in the wake of wind turbines was analysed using an extension of the helical vortex filament method, in which multiple vortex helices with a common center are considered to model the tip vortices of the wind turbine blade, in the absence [110] or in the presence [111] of an additional counter-rotating vortex modelling the wake of the turbine nacelle.

As it cannot be otherwise, consequently, in the point vortex stability analysis, the only relevant parameter of the vortices is their position, which will be the centroid of the real vortices. On the contrary, the integrals that appear in the 3D formulation may diverge and the analysis needs some modification to overcome this idealisation. The simplest remedy is to use cutoff integrals, which consists in removing a small part from the integral in the process of integration. The length cut from the integral is what is called the cutoff

parameter, or cutoff length, and its value will be a measure of the vortex size. Crow [35] suggested a proportionality relation between cutoff parameter and vortex radius, assuming that the filament is actually a Rankine vortex. Previously, Hama [60] and Arms and Hama [8] have already introduced the localized-induction concept, which is based on an expansion of the cutoff integral where the leading term is of order $\log(1/\varepsilon)$ being ε a small non-dimensional value proportional to the cutoff length. Several later works (Widnall et al. [142], Moore and Saffman [107] and Widnall [141]) have exposed a more general relationship for different vortex core shapes, which even includes axial flow, by means of matching expansions between inner and outer velocity fields, see inside subsection 1.1.2. Callegari and Ting [25], using the same methodology, obtained a formula for a vortex filament with viscous core, while Fukumoto and Miyazaki [49] extended the expansion to second order in the inviscid case, as well as Klein and Majda [83] derived a formula that includes self-stretching. The localized-induction concept, referred in more recent papers as local induction approximation (LIA), is the base of all the works mentioned above, and, additionally, is still leading to new developments as that from Segalini and Alfredsson [122], who derived a higher order formula for their particular case, a model of propeller/wind-turbine rotor wake. Most of these works, based on matching expansions, relate strictly the vortex core size and cutoff length, which couples the vortex core shape and the solution of the problem. In order to obtain more general results, this approach is not followed here, and the results are function of cutoff length instead of vortex core size. The choice of only using a cutoff, without an inner expansion matching, has an important disadvantage that the previous approaches that include an inner expansion do not have, which is that wavelengths shorter than the cutoff length cannot be analysed, since they are removed. However, the results are independent of vortex core shape, and therefore more general for an unknown vortex shape. Only for comparison with other methodologies based in vortex size, the approach of Widnall [141] is chosen, where a relationship between vortex core and cutoff length is given for a general vortex shape. Additional approaches could be used satisfactorily to study the short-wave instabilities as shown by Tsai and Widnall [137].

From the previous assumptions, core size and vortex shape are analysed as a single parameter, the cutoff. Axial flow is another relevant feature for real flows that is not included in filament methods, although it could be taken as a further modification of the cutoff length. By definition, viscosity is not considered, and although there exist approaches permitting the addition of viscosity to time evolving vortex filament methods Chatelain et al. [28, 29], Walther et al. [140], Winckelmans et al. [144], application of viscous linear modal to stability analysis by vortex filaments is not straightforward.

By contrast inviscid linear modal stability analysis by vortex filaments can be solved analytically in most of the cases or computed using negligible computing resources and, as such, it is well suited to perform parametric studies of the different values of the variables involved in the instability problem.

Chapter 2

Theoretical Instability Analysis Concepts

This chapter exposes the theoretical basis of most of the methodologies that could be used to perform the fluid mechanics instability analysis. This analysis, as well as any other study of Physics, is based on a model of the reality by means of equations that must reproduce the actual behaviour of some phenomena, while the magnitudes involved are kept in some range of validity. Later, the analysis of these equations will lead to some conclusions and the subsequent applications. The theoretical concepts and models are accordingly translated into mathematical language in form of equations, whose derivation comprises the basis of the whole work, and, in consequence, this chapter comprises the essential part of the whole thesis. Despite of that, in most of the cases, the reader may be more interested in the results because they are the novelty, the outcome and the final goal of this investigation, but one must remember that without a theory behind that support them, results are meaningless.

Focusing on instability analysis, there are two types of approaches to follow. The more direct is to make a model of the physical problem and introduce perturbations to seek whether they grow or decay in time or space. This could be done analytically or by means of computation or experimentation. The other option is to study instabilities of the problem by a direct analysis of the equations. There are many ways of doing so, for example, modal instability analysis is done by a linearisation of the previous equations and obtaining an eigenvalue problem. This approach will produce the full spectrum of the problem, showing the stable and unstable modes and therefore, it gives more information. However, the linearisation may kill some non-linear mechanisms that could be essential to understand the response of the system to some perturbations, because it must not be forgotten that modal analysis is done for behaviour at infinite time, only valid if non-linearities do not saturate the system response previously. In addition to the non-linear behaviour of the equations, also non-modal analysis of the linear equations may be important, because the initial transient of the instabilities may end in a different state invalidating the long term, modal, analysis. All that implies that there is not an unique methodology for instability analysis, instead, each of them gives useful information about the problem that should be handled with care.

Dealing with fluid dynamics, particularly with vortex flows, there are two directions to take. Maybe the most straightforward, is that based on the solution of the Navier-Stokes equations, which uses an Eulerian coordinate system and requires of a grid when calculated numerically. The other approach is based on a simplification of these equations using a Lagrangian coordinate system, leading to the Biot-Savart law for vortices. It does not need a grid, as the positions of the fluid particles are the points where the flow variables are calculated.

2. Theoretical Instability Analysis Concepts

Attending to the last two paragraphs, two classifications could be established. That means two times two equal four different methodologies: Direct Numerical Simulations, Vortex Methods, Modal Linear Stability and Vortex Methods for Stability. According to that, the chapter is consequently divided in four parts, whose similarities and differences are summarized next. Direct Numerical Simulations are iterative simulations, in time or space, which are based on the Navier-Stokes equations or a simplification of them. The Vortex Methods are also iterative simulations, based this time on the Biot-Savart law for a set of vortex elements. Modal Linear Stability is based on the Linearised Navier-Stokes equations, but instead of a iterative process, the solution is given by eigenvalue problems. Finally, the Vortex Methods for Stability, also solve an eigenvalue problem, but are based on the Biot-Savart law. Non-modal linear Stability is not explained in detail, as it is not analysed further in this thesis. There are variations of these techniques that do not exactly fill in these scheme. In this case, the Parabolic Navier-Stokes are explained as a subsection inside the Direct Numerical Simulations, and Parabolised Stability Equations as a subsection inside Modal Linear Stability.

Once the similarities and differences between them are clear, it can be deduced that, concerning to the actual subjects of study, these four methodologies could be joined in groups of two following two different approaches. One of them regarding the equations in which the method is based, where Vortex Methods and Vortex Methods for Stability, based on the solution of the Biot-Savart law for a set of vortex elements, form one group, and Direct Numerical Simulations and Modal Linear Stability, based on the Navier-Stokes equations or a simplification of them, form another group. Alternatively, other classification may be done attending to the kind of problem to be solved, in that case, Direct Numerical Simulations and Vortex Methods solve iterative problems or initial value problems (IVP), while Modal Linear Stability and Vortex Methods for Stability are eigenvalue problems (EVP).

Figure 2.1: Comparison of similarities and differences amongst the four methodologies.

This work deals with the instability analysis of vortical flows, which can be performed by all the methodologies presented. However, attending to the last classification of the two introduced, that emphasizes the kind of solution attempted (EVP or IVP), a few more words can be stated about which methodology is better in which case. The iterative

methods, although usually represent the real world with more fidelity, are more demanding of computational resources and give the solution for only one particular configuration or set of parameters. They show the real behaviour of the flow, how it actually evolves and whether it is stable or unstable for the initial set of conditions given, but do not give much information of other mechanisms that are not acting for that particular situation, and may be excited for other frequencies. In addition to that, the iterative methods, when stable, provide for what is called a basic flow, a steady solution of the equations, which is needed if one desires to perform the modal stability analysis by means of eigenvalue problems and do not know an analytic steady solution of the equations. Several modifications of the iterative methods may be performed to obtain growth of stable modes or stable basic flows from unstable configurations, but they are in general complicated. However, this work will focus on the modal stability analysis, which can be performed with less computational resources and gives the full spectrum of modes that grow or decay, providing with more practical information to understand the instability mechanisms and, if it is the case, for a further control of the flow.

2.1 Direct Numerical Simulations theory

DNS stand for Direct Numerical Simulations, because it is a numerical method that solves the full Navier-Stokes Equations (NSE) directly. These equations are exact while the hypothesis of the Fluid Mechanics are valid: continuum hypothesis, local thermal equilibrium, and many more for particular cases as incompressible flow, Newtonian flow and others. This thesis deals only with Newtonian incompressible flows in the absence of massic forces, and in this case, the Three-dimensional NSE take the form of Eqs. (2.1):

$$\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0, \quad \rho \frac{\partial \tilde{\mathbf{u}}}{\partial t} + \rho \tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{\mathbf{u}} = -\tilde{\nabla} \tilde{p} + \mu \tilde{\nabla}^2 \tilde{\mathbf{u}}, \quad (2.1)$$

where tilde is used for dimensional variables, as later most variables will be non-dimensional, except for the density, ρ and the viscosity, μ . For Newtonian incompressible viscous flow, these two values are given for a given fluid, and will be taken as parameters. The other variables, $\tilde{\mathbf{u}}$, the velocity and $\tilde{p}$, the pressure, are the unknowns of the equations.

Usually, the NSE are written in non-dimensional form:

$$\nabla \cdot \bar{\mathbf{u}} = 0, \quad \frac{\partial \bar{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = -\nabla \bar{p} + \frac{1}{Re} \nabla^2 \bar{\mathbf{u}}, \quad (2.2)$$

where bar is used for non-dimensional velocity, $\bar{\mathbf{u}}$ and pressure, $\bar{p}$, and nabla operator, ∇ and time, t are written without bar meaning non-dimensional. The equations are then much cleaner, and only a new non-dimensional parameter appears, the Reynolds number, Re , defined as $Re = UL/\nu$ being U and L the velocity and length scale respectively, and $\nu = \mu/\rho$ the kinematic viscosity.

Given initial and boundary conditions (IC, BC), these equations will provide the solution. To solve the problem numerically, the domain must be discretised, as well as the derivative operators, and the time step chosen. Done that, together with the IC, BC and the value of Re , the solution is straightforward. However, for turbulent problems (usually

for moderate and high Re), there are many considerations about the mesh size that lead to huge amount of computational resources even for simple geometries. Here, however, DNS is used to solve laminar problems, in order to have basic flows for modal linear stability analysis and solutions are much computationally cheaper, as well as easier to obtain.

2.1.1 DNS 2D theory

The non-dimensional NSE for incompressible flow, Eqs. (2.2), can be rewritten for a two-dimensional flow in Cartesian coordinates as Eqs. (2.3):

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (2.3a)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) \quad (2.3b)$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} = -\frac{\partial \bar{p}}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right) \quad (2.3c)$$

where bar is used again for non-dimensional velocity components, $\bar{u}$ and $\bar{v}$, and pressure, $\bar{p}$, and positions, x and y , and time, t are written without bar meaning non-dimensional. The Reynolds number, Re is again the only parameter in the equations.

As long as $\partial/\partial z = 0$, even when the velocity component in z is different from zero, $\bar{w} \neq 0$, these equations remain the same, and the third momentum equation, not shown, will provide for $\bar{w}$, but the flow can be calculated as two-dimensional.

The continuity equation for two-dimensional incompressible flow, Eq. (2.3a), implies the existence of the streamfunction ψ :

$$\frac{\partial \psi}{\partial x} = -\bar{v}, \quad \frac{\partial \psi}{\partial y} = \bar{u} \quad (2.4)$$

Vorticity, defined in general as a vector, $\boldsymbol{\omega} = \nabla \times \bar{\mathbf{u}}$ is a scalar in two dimensions:

$$\omega = \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \quad (2.5)$$

Defined these two variables, ψ and ω , the continuity equation Eq. (2.3a) is automatically satisfied. Operating the momentum equations, Eqs. (2.3b, 2.3c), Eq. (2.6a) is obtained. Then, Eq. (2.6b) is a direct consequence of the vorticity definition, Eq. (2.5). Two equations are therefore obtained for the two variables ψ and ω , which are the 2D-NSE written in vorticity-streamfunction form:

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{1}{Re} \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) = 0 \quad (2.6a)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \omega = 0 \quad (2.6b)$$

To solve this problem numerically, the derivative operators must be discretised, as well as a time evolution solver used. In order to leave the equations written in a language closer to that, previous Eqs. (2.6) are written as follows:

$$\frac{\partial \omega}{\partial t} + \mathcal{D}_y \psi \mathcal{D}_x \omega - \mathcal{D}_x \psi \mathcal{D}_y \omega - \frac{\mathcal{K} \omega}{Re} = 0 \quad (2.7a)$$

$$\mathcal{K} \psi + \omega = 0 \quad (2.7b)$$

Here, $\mathcal{D}_x = \partial/\partial x$ and $\mathcal{D}_y = \partial/\partial y$ are derivative operators in directions x and y respectively, as well as $\mathcal{K} = \mathcal{D}_x^2 + \mathcal{D}_y^2$ is an operator function of the operators $\mathcal{D}_x^2 = \partial^2/\partial x^2$ and $\mathcal{D}_y^2 = \partial^2/\partial y^2$. These operators will have to be discretised and the evolution problem solved, but that will be explained when talking about numerics, in Chapter 3.

2.1.2 DNS 3D theory

This time the non-dimensional NSE for incompressible flow, Eqs. (2.2), are rewritten for a three-dimensional flow in Cartesian coordinates as Eqs. (2.8):

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad (2.8a)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right) \quad (2.8b)$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} = -\frac{\partial \bar{p}}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2} \right) \quad (2.8c)$$

$$\frac{\partial \bar{w}}{\partial t} + \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} = -\frac{\partial \bar{p}}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2} \right) \quad (2.8d)$$

where bar is used one more time for non-dimensional velocity components, $\bar{u}$, $\bar{v}$ and $\bar{w}$, and pressure, $\bar{p}$, and positions, x , y and z , and time, t are written without bar meaning non-dimensional. The Reynolds number, Re is again the only parameter in the equations. This is a system of four equations with four unknowns. The three-dimensionality of the problem makes it much more demanding of resources. This fact, together with some issues that are common with the boundary conditions for the pressure, lead to the common approach of eliminating the pressure from a combination of the momentum equations, reducing the number of equations to three. However, this increases the order of the derivatives, and in many cases, this leads to many other issues. Therefore, this approach is not followed here.

Once again, to solve the problem numerically, the derivative operators must be discretised, and for that purpose, the problem is written in terms of these operators:

$$\mathcal{D}_x \bar{u} + \mathcal{D}_y \bar{v} + \mathcal{D}_z \bar{w} = 0 \quad (2.9a)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \mathcal{D}_x \bar{u} + \bar{v} \mathcal{D}_y \bar{u} + \bar{w} \mathcal{D}_z \bar{u} = -\mathcal{D}_x \bar{p} + \frac{\mathcal{K}_{3D} \bar{u}}{Re} \quad (2.9b)$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u} \mathcal{D}_x \bar{v} + \bar{v} \mathcal{D}_y \bar{v} + \bar{w} \mathcal{D}_z \bar{v} = -\mathcal{D}_y \bar{p} + \frac{\mathcal{K}_{3D} \bar{v}}{Re} \quad (2.9c)$$

$$\frac{\partial \bar{w}}{\partial t} + \bar{u} \mathcal{D}_x \bar{w} + \bar{v} \mathcal{D}_y \bar{w} + \bar{w} \mathcal{D}_z \bar{w} = -\mathcal{D}_z \bar{p} + \frac{\mathcal{K}_{3D} \bar{w}}{Re} \quad (2.9d)$$

2. Theoretical Instability Analysis Concepts

Here, apart from $\mathcal{D}_x$ and $\mathcal{D}_y$ defined before, a new derivative operator appears, $\mathcal{D}_z = \partial/\partial z$ in direction z , and instead of $\mathcal{K}$, the three-dimensional form of it appears, $\mathcal{K}_{3D} = \mathcal{D}_x^2 + \mathcal{D}_y^2 + \mathcal{D}_z^2$, function also of the operator $\mathcal{D}_z^2 = \partial^2/\partial z^2$. As stated previously, these operators will have to be discretised and the evolution problem solved, which will be explained later in Chapter 3.

The incompressible 3D DNS equations, Eqs. (2.8), are also presented in cylindrical coordinates, as it is a more desirable form in many situations. In that cylindrical coordinates, $[r, \theta, z]$ are the radial, azimuthal and perpendicular coordinates and $[\bar{u}_r, \bar{u}_\theta, \bar{u}_z]$ the non-dimensional flow velocity components in each corresponding direction.

$$\frac{1}{r} \frac{\partial(r\bar{u}_r)}{\partial r} + \frac{1}{r} \frac{\partial\bar{u}_\theta}{\partial\theta} + \frac{\partial\bar{u}_z}{\partial z} = 0 \quad (2.10a)$$

$$\begin{aligned} \frac{\partial\bar{u}_r}{\partial t} + \bar{u}_r \frac{\partial\bar{u}_r}{\partial r} + \frac{\bar{u}_\theta}{r} \frac{\partial\bar{u}_r}{\partial\theta} + \bar{u}_z \frac{\partial\bar{u}_r}{\partial z} - \frac{\bar{u}_\theta^2}{r} \\ = -\frac{\partial\bar{p}}{\partial r} + \frac{1}{Re} \left(\frac{\partial^2\bar{u}_r}{\partial r^2} + \frac{1}{r} \frac{\partial\bar{u}_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2\bar{u}_r}{\partial\theta^2} + \frac{\partial^2\bar{u}_r}{\partial z^2} - \frac{\bar{u}_r}{r^2} - \frac{2}{r^2} \frac{\partial\bar{u}_\theta}{\partial\theta} \right) \end{aligned} \quad (2.10b)$$

$$\begin{aligned} \frac{\partial\bar{u}_\theta}{\partial t} + \bar{u}_r \frac{\partial\bar{u}_\theta}{\partial r} + \frac{\bar{u}_\theta}{r} \frac{\partial\bar{u}_\theta}{\partial\theta} + \bar{u}_z \frac{\partial\bar{u}_\theta}{\partial z} + \frac{\bar{u}_\theta\bar{u}_r}{r} \\ = -\frac{1}{r} \frac{\partial\bar{p}}{\partial\theta} + \frac{1}{Re} \left(\frac{\partial^2\bar{u}_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial\bar{u}_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2\bar{u}_\theta}{\partial\theta^2} + \frac{\partial^2\bar{u}_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial\bar{u}_r}{\partial\theta} - \frac{\bar{u}_\theta}{r^2} \right) \end{aligned} \quad (2.10c)$$

$$\begin{aligned} \frac{\partial\bar{u}_z}{\partial t} + \bar{u}_r \frac{\partial\bar{u}_z}{\partial r} + \frac{\bar{u}_\theta}{r} \frac{\partial\bar{u}_z}{\partial\theta} + \bar{u}_z \frac{\partial\bar{u}_z}{\partial z} \\ = -\frac{\partial\bar{p}}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2\bar{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial\bar{u}_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2\bar{u}_z}{\partial\theta^2} + \frac{\partial^2\bar{u}_z}{\partial z^2} \right) \end{aligned} \quad (2.10d)$$

2.1.3 Parabolic Navier-Stokes

The incompressible Parabolic Navier-Stokes (PNS) equations are derived from the three-dimensional steady incompressible Navier-Stokes equations in the limit case when the flow motion occurs predominantly along the streamwise direction z ¹, while exhibiting a slow, spatial variation in the same direction. These flows are usually associated with high Reynolds numbers, limiting the effective length of viscous forces along z . Therefore, the streamwise viscous derivative terms are negligible compared to the normal and transverse viscous derivative terms:

$$L_z \gg L_x, L_y, \quad \partial(\cdot)/\partial z \ll \partial(\cdot)/\partial x, \partial(\cdot)/\partial y, \quad \partial(\cdot)^2/\partial^2 z \ll \partial(\cdot)^2/\partial^2 x, \partial(\cdot)^2/\partial^2 y, \quad (2.11)$$

where L_x , L_y and L_z are the characteristic lengths on the directions x , y and z respectively. Hence, the PNS equations are obtained by dropping all viscous terms containing partial derivatives with respect to the streamwise direction from the steady Navier-Stokes equations. Under this conditions, Eqs. (2.8) result in a system of equations only contains first-order derivatives on the streamwise direction, and can be integrated in the downstream direction, Eqs. (2.12). Under circumstances for which there is no flow, or regions

¹Many works, as Paredes et al. [115] define this direction as x , but here z is chosen for consistency through the whole document.

of reversed flow along the slow direction exist, the downstream integration (towards increasing z -coordinates) of the Navier-Stokes equations is not correct, as the assumptions leading to the parabolisation of the equations do not hold; in such circumstances numerical instabilities will prevent the recovery of converged solutions.

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad (2.12a)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} = -\frac{\partial \bar{p}}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) \quad (2.12b)$$

$$\bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} = -\frac{\partial \bar{p}}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right) \quad (2.12c)$$

$$\bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} = -\frac{\partial \bar{p}}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} \right) \quad (2.12d)$$

The resulting set of equations is modified here to reduce the derivative order of $\bar{u}$ with respect to x and of $\bar{v}$ with respect to y from second order to first order using the total pressure $\bar{p}_t = \bar{p} + (\bar{u}^2 + \bar{v}^2)/2 + \partial \bar{w}/\partial z / Re$ and the streamwise vorticity $\omega = \partial \bar{v}/\partial x - \partial \bar{u}/\partial y$ [103]. This reduces also the number of boundary conditions to be imposed. The resulting set of equations for a Cartesian coordinate system is given by:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad (2.13a)$$

$$\bar{w} \frac{\partial \bar{u}}{\partial z} - \bar{v} \omega = -\frac{\partial \bar{p}_t}{\partial x} - \frac{1}{Re} \frac{\partial \omega}{\partial y} \quad (2.13b)$$

$$\bar{w} \frac{\partial \bar{v}}{\partial z} + \bar{u} \omega = -\frac{\partial \bar{p}_t}{\partial y} + \frac{1}{Re} \frac{\partial \omega}{\partial x} \quad (2.13c)$$

$$\bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} = -\frac{\partial \bar{p}_\infty}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} \right) \quad (2.13d)$$

being $\partial \bar{p}_\infty / \partial z$ the streamwise derivative of the free-stream pressure, which enter into the equations as a known function. The flows considered in this work are not subjected to a free-stream pressure gradient, so $\partial \bar{p}_\infty / \partial z = 0$. The PNS Eqs. (2.13) have to be complemented with appropriate boundary condition depending on the particular problem.

The incompressible PNS equations, Eqs. (2.13), are also presented in cylindrical coordinates. The result is based on Eqs. (2.10), imposing the conditions for the PNS: steady flow, neglecting the second order derivatives in the axial direction, as well as imposing axisymmetry, that implies neglecting the azimuthal derivatives. In that cylindrical coordinates, $[r, \theta, z]$ are the radial, azimuthal and perpendicular coordinates and $[\bar{u}_r, \bar{u}_\theta, \bar{u}_z]$ the non-dimensional flow velocity components in each corresponding direction.

$$\frac{\bar{u}_r}{r} + \frac{\partial \bar{u}_r}{\partial r} + \frac{\partial \bar{u}_z}{\partial z} = 0 \quad (2.14a)$$

$$\bar{u}_r \frac{\partial \bar{u}_r}{\partial r} + \bar{u}_z \frac{\partial \bar{u}_r}{\partial z} - \frac{\bar{u}_\theta^2}{r} = -\frac{\partial \bar{p}}{\partial r} + \frac{1}{Re} \left(\frac{\partial^2 \bar{u}_r}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_r}{\partial r} - \frac{\bar{u}_r}{r^2} \right) \quad (2.14b)$$

$$\bar{u}_r \frac{\partial \bar{u}_\theta}{\partial r} + \bar{u}_z \frac{\partial \bar{u}_\theta}{\partial z} + \frac{\bar{u}_\theta \bar{u}_r}{r} = +\frac{1}{Re} \left(\frac{\partial^2 \bar{u}_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_\theta}{\partial r} - \frac{\bar{u}_\theta}{r^2} \right) \quad (2.14c)$$

$$\bar{u}_r \frac{\partial \bar{u}_z}{\partial r} + \bar{u}_z \frac{\partial \bar{u}_z}{\partial z} = -\frac{\partial \bar{p}}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 \bar{u}_z}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_z}{\partial r} \right) \quad (2.14d)$$

2.2 Vortex methods theory

Vortex methods solve, in general, unsteady, incompressible, non viscous and unbounded flows using Lagrangian coordinates, which means using elements that move with the flow. Only the evolution of the vorticity field is needed to determine the behaviour for the previously mentioned flows. This vorticity field is calculated from the position of vortex elements, which are only located in the irrotational part of the flow and contain all the vorticity of the fluid. This is an important advantage against Eulerian (grid) methods that cover all the domain, because often, the vorticity is concentrated over a small portion of it. The review of Leonard [92], more than three decades ago, shows how mature was already the development of this methodology at that time.

The previous paragraph introduced three very restrictive constraints: incompressible, inviscid and unbounded flows. There are several strategies that can be employed to overcome these constraints, which are not new, as the works of Winckelmans [145] and Koumoutsakos [84] demonstrate. However, this thesis focuses precisely on the solution of flows that are incompressible and unbounded, and do not require the use of these strategies. The viscous case analysis will be performed by standard DNS, already implemented, not needing to derive a complex tool to achieve the same result.

Two sections follow, where the deduction of the equations, as well as some particularities of the methodology, are exposed both for two and three dimensions. However, although both cases are exposed, the methodology has been implemented only in two dimensions, as in three dimensions many computational issues arise (usage of supercomputers, implementation of fast algorithms, computation by Graphics Processing Units, and many more), which are themselves the topic of full theses.

2.2.1 Two-dimensional vortex methods (point vortices)

Point Vortex Methods are based on a model of the flow where the vorticity is zero in all the domain, except a number of points, where it is infinity. The circulation is the magnitude that measures the amount of vorticity in these points.

A point vortex produces a rotating velocity field around it, Eq. (1.21b), which gives the induced velocity by the point vortex over any other point in the plane. Following this idea, the velocity can be calculated for a finite number of point vortices given its circulations and coordinates. Once these quantities are known, the calculation of the induced velocities is straightforward following Eqs. (2.15-2.17), where also Fig. (2.2) is presented to visualise these relationships.

$$\mathbf{u}_j = \sum_{i=1, i \neq j}^N \frac{\Gamma_i}{2\pi r_{ij}} \mathbf{t}_{ij} \quad (2.15)$$

$$\mathbf{t}_{ij} = -\frac{\mathbf{r}_{ij}}{r_{ij}} \times \mathbf{e}_z \quad (2.16)$$

$$\mathbf{r}_{ij} = \mathbf{x}_j - \mathbf{x}_i \quad (2.17)$$

Figure 2.2: Induced velocities scheme.

By this mean, the induced velocities are known at a specific time. If an evolution in time of the point locations is desired, the simple evolution problem given by Eq. (2.18) needs to be solved, where the term $\mathbf{u}_i(\mathbf{x}_j, t)$ is taken from Eq. (2.15).

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{u}_i(\mathbf{x}_j, t), \quad \text{where } \mathbf{x}_j \text{ means all } j \neq i \quad (2.18)$$

Here, Eq. (2.18) represents an incompressible and non viscous method, which will provide with reasonable results in the viscous case only in the limit of $Re \rightarrow \infty$, neglecting turbulence in the second case.

Eq. (2.15) could be used as a general formula to calculate the velocity field if it is particularized for the desired points instead of the j point vortex.

The simplicity of the formulation of the method is therefore exposed. However, there are several issues that increase the complexity of the methodology, but most of them are numerical or particularities of different cases.

Additionally, the former equations can be deduced in a more formal way, which is done below. The NSE are simplified for inviscid flow removing the viscous term from Eqs. (2.1). The same notation is employed except that dimensional terms are all written without tildes for clarity in the exposition. Then the continuity equation, Eq. (2.19a), which remains the same, and the incompressible Euler momentum equations, Eqs. (2.19b) are deduced:

$$\nabla \cdot \mathbf{u} = 0 \quad (2.19a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \quad (2.19b)$$

These three equations (two momentum equations and one continuity equation), together with some initial conditions, Eq. (2.20), set an initial value problem for $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$.

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad (2.20)$$

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Pressure is eliminated by the combination of the two momentum equations, Eqs. (2.19b). Then, by using the continuity equation, Eq. (2.19a), everything is written as a single equation, function of $\mathbf{u}$ and $\omega = v_x - u_y$, being u and v the components of $\mathbf{u}$ in directions x and y respectively and ω the only non-zero component of vorticity. Sub-index here represent derivatives. The resulting equation is the vorticity equation, Eq. (2.21):

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0 \quad (2.21)$$

In the case of incompressible flow, the continuity equation allows to write the velocity using a stream function ψ :

$$\mathbf{u} = [u, v] = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right] \quad (2.22)$$

Vorticity can then be expressed as a function of ψ :

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi \quad (2.23)$$

Eqs. (2.22, 2.23) can be integrated to obtain the Biot-Savart law:

$$\mathbf{u} = \frac{1}{2\pi} \int \int \omega(x', y') \frac{[y' - y, x - x']}{(x - x')^2 + (y - y')^2} dx' dy' + \nabla \phi \quad (2.24)$$

In Eq. (2.24), $\nabla \phi$ accounts for potential flow and represents the effects of the walls and the velocity at infinity, and is equal to zero in the case of study. Also, as vorticity is concentrated in points, the integral can be expressed as a sum. Each point i has a circulation $\Gamma_i = \int \int \omega_i(x', y') dx' dy'$. The velocity can be substituted by position derivatives, $\mathbf{u} = d\mathbf{x}/dt$ and then it can be calculated for the positions where point vortex are. Doing that, Eq. (2.24) can be rewritten as Eq. (2.25), which is basically the same as Eq. (2.18). The reader may be referred to introductory fluid dynamics books as Batchelor [12] for further details.

$$\frac{d\mathbf{x}_i}{dt} = \frac{1}{2\pi} \sum_{j=1, i \neq j}^N \Gamma_j \frac{[y_j - y_i, x_i - x_j]}{(x_i - x_j)^2 + (y_i - y_j)^2} \quad (2.25)$$

The movement of the points is totally governed by this equation. However, there are some magnitudes that are conserved. These invariants are: the total circulation, Γ , Eq. (2.26); the centroid, (x_c, y_c) , Eqs. (2.27, 2.28); the averaged distance to centroid, a , Eq. (2.29); and the Hamiltonian, H , Eq. (2.30). They are also analysed in subsection 3.5.2 of Chapter 3. Besides the circulation, Eq. (2.26), all the others impose a number of restrictions to the movement.

$$\Gamma = \sum_{i=1}^N \Gamma_i \quad (2.26)$$

$$x_c = \sum_{i=1}^N \frac{\Gamma_i x_i}{\Gamma} \quad (2.27)$$

$$y_c = \sum_{i=1}^N \frac{\Gamma_i y_i}{\Gamma} \quad (2.28)$$

$$a = \sqrt{\sum_{i=1}^N \frac{\Gamma_i (x_i - x_c)^2 + \Gamma_i (y_i - y_c)^2}{\Gamma}} \quad (2.29)$$

$$H = -\frac{1}{4\pi} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \Gamma_i \Gamma_j \log \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \quad (2.30)$$

The usage of the invariants leads to an additional way to obtain the same equations, followed by Marchioro and Pulvirenti [102]. They remark that the system is Hamiltonian and converted Eq. (2.25) to Eqs. (2.31).

$$\begin{aligned} \Gamma_i \frac{dx_i}{dt} &= \frac{\partial H}{\partial y_i} \\ \Gamma_i \frac{dy_i}{dt} &= -\frac{\partial H}{\partial x_i} \end{aligned} \quad (2.31)$$

2.2.2 Three-dimensional vortex methods (vortex filaments)

This section can be seen as a generalization of the previous section or the previous section as a particularisation of this. Here, Eq. (2.32) is the the Biot-Savart law in three dimensions:

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{[\mathbf{x} - \mathbf{x}'(s')] \times \boldsymbol{\omega}[\mathbf{x}'(s'), t] ds'}{|\mathbf{x} - \mathbf{x}'(s')|^3} + \nabla \phi \quad (2.32)$$

Assuming a finite number of vortex tubes with curvature radius and distance much bigger than its core, Eq. (2.33) gives the vorticity.

$$\boldsymbol{\omega}(\mathbf{x}) = \sum_{i=1}^N \Gamma_i \int \delta[\mathbf{x} - \mathbf{r}_i(s')] \frac{\partial \mathbf{r}_i}{\partial s'} ds' \quad (2.33)$$

Being $\delta(\mathbf{x})$ the Dirac delta function, which value is zero everywhere but when $\mathbf{x} = \mathbf{0}$, $\mathbf{r}_i$ the position of the filament i and s' the arc length parameter. The substitution of Eq. (2.33) into Eq. (2.32) leads to the general formula of vortex filaments, Eq. (2.34), where the potential is taken to be zero because there are no walls and velocity is zero at infinity. The details of the procedure are collected in Leonard [91]. This integral diverges near to the filament. The solution, similarly as done in the two dimensional case, is not to consider the influence from one point over the same point. However, here, only a small part of filament is excluded, as some parts of the filament may interact with other parts of the same filament. This is called cutoff integral and excludes only influences of one point over distances of the core of the vortex from the point. If the integral is discretised, again it will mean not to sum each point with itself, but if the integral is done analytically, the cutoff has to be applied in a distance of the order of the core of the vortex. Note also that

different sizes of cutoff lead to different results, so this cutoff parameter should be handled with care.

$$\mathbf{u} = - \sum_{i=1}^N \frac{\Gamma_i}{4\pi} \int_C \frac{[\mathbf{x} - \mathbf{r}_i(s')] \times \frac{\partial \mathbf{r}_i}{\partial s'} ds'}{|\mathbf{x} - \mathbf{r}_i(s')|^3} \quad (2.34)$$

2.3 Modal Linear Stability theory

Modal linear stability studies the asymptotic behaviour of the flow to small disturbances. The linearised problem is converted into an eigenvalue problem and stability is given by the sign of the real part of the eigenvalues.

Starting from the NSE of a viscous, incompressible fluid in dimensionless form, Eqs. (2.2), the vector of fluid variables $\bar{\mathbf{q}} = [\bar{\mathbf{u}}, \bar{p}]^T$ is decomposed into a steady mean flow $\mathbf{Q}$ and an unsteady small disturbance or perturbation $\varepsilon \mathbf{q}$:

$$\bar{\mathbf{q}}(\mathbf{x}, t) = \mathbf{Q}(\mathbf{x}) + \varepsilon \mathbf{q}(\mathbf{x}, t) \quad (2.35)$$

where $\varepsilon \ll 1$.

Introducing this decomposition of the perturbed flow into Eqs. (2.2), and subtracting the basic flow (as it must satisfy the Navier-Stokes and continuity equations to be a basic flow), the so-called perturbation equations or Linearised Navier Stokes Equations (LNSE) are obtained:

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u} \quad (2.36)$$

In these equations the $\mathcal{O}(\varepsilon)$ terms have been retained, while the non-linear perturbation term ($\mathbf{u} \cdot \nabla \mathbf{u}$) is $\mathcal{O}(\varepsilon^2)$ and can be neglected. The linearisation is correct while the disturbance amplitude remains small enough for the non-linear terms to be negligible. This statement should be kept in mind as the true non-linear system can become unstable under finite-amplitude disturbances, under conditions in which the linearised system will stay stable. One physical example of this is the laminar-turbulent transition originated by the transient growth of initially small disturbances, that can be stable in a linear approach (see Butler and Farrell [24] for a review).

In Cartesian coordinates, Eqs. (2.36) have the form of Eqs. (2.37). Where $\mathbf{u} = [u, v, w]$, $\mathbf{U} = [U, V, W]$ and sub-index indicate derivatives.

$$\begin{aligned} u_t + Uu_x + Vv_x + Ww_x + uU_x + vU_y + wU_z &= -p_x + \frac{1}{Re}(u_{xx} + u_{yy} + u_{zz}) \\ v_t + Uv_x + Vv_y + Wv_z + uV_x + vV_y + wV_z &= -p_y + \frac{1}{Re}(v_{xx} + v_{yy} + v_{zz}) \\ w_t + Uw_x + Vw_y + Ww_z + uW_x + vW_y + wW_z &= -p_z + \frac{1}{Re}(w_{xx} + w_{yy} + w_{zz}) \\ u_x + v_y + w_z &= 0 \end{aligned} \quad (2.37)$$

In linear stability theory, the perturbation term is usually written as the product of an amplitude function and a phase function, $\mathbf{q} = \hat{\mathbf{q}}\Theta$. Table 2.1 summarises the different

instability approaches arranged by increasing constrains to the basic flow. These different approaches are explained in the next subsections in more detail.

Table 2.1: Classification of linear stability theory concepts for analysis of a steady state $\mathbf{Q}$.

Denomination	Assumptions	Basic State	Amplitude	Phase Function Θ
TriGlobal	-	$\mathbf{Q}(x_1, x_2, x_3)$	$\hat{\mathbf{q}}(x_1, x_2, x_3)$	$e^{-i\omega t}$
BiGlobal	$\partial_3 \mathbf{Q} = 0$	$\mathbf{Q}(x_1, x_2)$	$\hat{\mathbf{q}}(x_1, x_2)$	$e^{i(k_3 x_3 - \omega t)}$
OSE	$\partial_2 \mathbf{Q} = \partial_3 \mathbf{Q} = 0$	$\mathbf{Q}(x_1)$	$\hat{\mathbf{q}}(x_1)$	$e^{i(k_2 x_2 + k_3 x_3 - \omega t)}$

Using any of the given approaches, what is finally obtained is an eigenvalue problem for ω as eigenvalues and $\hat{\mathbf{q}}$ as eigenvectors. The eigenvalues will have a real part, that will be the growth or damping rate and an imaginary part that will be the frequency of the oscillations. In the notation used, ω is multiplied by $-i$, which means that the real part of ω will actually be the frequency changed of sign, and the growth or damping rate will be the imaginary part. This option here, is called temporal amplification theory, and it is the only one applicable to TriGlobal. For the other two approaches, however, there are also wavenumbers (k_i) that have to be taken into account. For temporal theory, they have to be all given and real, in order to have sinusoidal movement in the homogeneous directions. The temporal approach gives the growth or decay in time of the perturbations.

On the other hand, the spatial theory consists of analysing the problem using one of the wavenumbers as complex eigenvalues and leaving the frequency real. When spatial theory is selected, growth or decay may be found in the direction of the wavenumber that is calculated, given a frequency of oscillation. It cannot be used in TriGlobal, because it does not have any homogeneous direction. If the Orr-Sommerfeld equation (OSE) approach is taken, it needs to be given a real wavenumber for one of the homogeneous directions. Furthermore, it should be taken into account that directional derivatives are second order, and then, the eigenvalue problem to solve will have eigenvalues to the power of two, with the consequent difficulties of solution. For an interesting review of temporal and spatial theory, applied to local stability, see Mack [101].

2.3.1 One dimension, local theory

The OSE approach can be selected from table 2.1, which represent a basic flow varying only in one dimension. Perturbations will then be inhomogeneous in one dimension and homogeneous in the other two dimensions and time. In this case, the resulting equations will only have derivatives in one dimension, as the other derivatives will be converted into products. This analysis is also called local analysis, because it has only one inhomogeneous direction and the modes depend only on one dimension.

Cartesian coordinates: Orr-Sommerfeld equation

When the last line of table 2.1 is substituted into Eqs. (2.37), Eq. (2.38) are obtained. Some changes of notation are applied in that equation, x_1 , x_2 and x_3 are called y , z and

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x , while k_2, k_3 become β and α . Derivatives are shown as sub-index. This notation is used because it is the most commonly accepted among the scientific community, that use to think about y as the inhomogeneous direction, because the most common problems to solve are the channel and the boundary layer, where y is the perpendicular direction to the wall.

$$-i\omega\hat{u} + U\alpha\hat{u} + V\hat{u}_y + Wi\beta\hat{u} + \hat{v}U_y = -i\alpha\hat{p} + \frac{1}{Re}(-\alpha^2\hat{u} + \hat{u}_{yy} - \beta^2\hat{u}) \quad (2.38a)$$

$$-i\omega\hat{v} + U\alpha\hat{v} + V\hat{v}_y + Wi\beta\hat{v} + \hat{v}V_y = -\hat{p}_y + \frac{1}{Re}(-\alpha^2\hat{v} + \hat{v}_{yy} - \beta^2\hat{v}) \quad (2.38b)$$

$$-i\omega\hat{w} + U\alpha\hat{w} + V\hat{w}_y + Wi\beta\hat{w} + \hat{w}W_y = -i\beta\hat{p} + \frac{1}{Re}(-\alpha^2\hat{w} + \hat{w}_{yy} - \beta^2\hat{w}) \quad (2.38c)$$

$$i\alpha\hat{u} + \hat{v}_y + i\beta\hat{w} = 0 \quad (2.38d)$$

Adding the assumption of strictly parallel flow ($V = 0$), some algebra can be done to obtain the Orr-Sommerfeld equation, Eq. (2.39). That is, combine Eq. (2.38a) and Eq. (2.38c) into α Eq. (2.38a) + β Eq. (2.38c). Then, from Eq. (2.38d) and the resulting equation, the variable $\alpha\hat{u} + \beta\hat{w}$ can be eliminated, leaving one equation in $\hat{p}$ and $\hat{v}$. Pressure can be eliminated by differentiation, and Eq. (2.39) is finally achieved. The process can be followed in depth in Mack [101]. Eq. (2.39) is a four order ordinary differential equation for values of Re , α , β and ω given.

$$\begin{aligned} &\hat{v}_{yyyy} - 2(\alpha^2 + \beta^2)\hat{v}_{yy} + (\alpha^2 + \beta^2)^2\hat{v} \\ &= iRe \{ (\alpha U + \beta W - \omega) [\hat{v}_{yy} - (\alpha^2 + \beta^2)\hat{v}] - (\alpha U_{yy} + \beta W_{yy})\hat{v} \} \end{aligned} \quad (2.39)$$

Usually derivatives are written by the operator $\mathcal{D}$ instead of the sub-index y . Doing that, Eq. (2.39) can be written as Eq. (2.40). Using this notation, it is easy to see that the equation is actually an eigenvalue problem, where $\hat{v}$ are the eigenvectors and ω are the eigenvalues. This occurs when α , β and Re are fixed and it is called the temporal problem. In this case, the length of the perturbation is fixed and the eigenvalues are the frequencies of the problem. The problem can then be written to fit the general eigenvalue formula Eq. (2.41) when the temporal approach is taken. In this case, $\hat{v} = \mathbf{x}$, $\omega = \lambda$ and $\mathcal{A}$ and $\mathcal{B}$ are the operators that result from Eq. (2.40) function of $\mathcal{D}$, α , β , Re , U and W .

$$[\mathcal{D}^2 - (\alpha^2 + \beta^2)]^2\hat{v} = iRe[(\alpha U + \beta W - \omega)[\mathcal{D}^2 - (\alpha^2 + \beta^2)] - (\alpha U_{yy} + \beta W_{yy})]\hat{v} \quad (2.40)$$

$$\mathcal{A}\mathbf{x} = \lambda\mathcal{B}\mathbf{x} \quad (2.41)$$

The eigenvalue problem could also have been obtained directly from Eqs. (2.38), having $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}]^T$ as eigenvectors and ω as eigenvalues. Actually, this option may be preferred, because derivatives are only second order, instead of the fourth order of Eq. (2.40). The usage of that equation come from earlier times, where analytical work was better done when just an equation is obtained. Nowadays, with the use of computers, the four equations problem may be solved directly. In that case, matrices $\mathcal{A}$ and $\mathcal{B}$ are as follows:

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{1Dt} & U_y & 0 & i\alpha \\ 0 & \mathcal{L}_{1Dt} + V_y & 0 & \mathcal{D} \\ 0 & W_y & \mathcal{L}_{1Dt} & i\beta \\ i\alpha & \mathcal{D} & i\beta & 0 \end{pmatrix} \quad (2.42a)$$

$$\mathcal{B} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.42b)$$

where $\mathcal{L}_{1Dt} = i\alpha U + V\mathcal{D} + i\beta W - (-\alpha^2 + \mathcal{D}^2 - \beta^2)/Re$ is written as an operator to maintain the equations clean. Here, $V = 0$ for strictly parallel flows again, but has not been substituted in order to have more general equations. Eqs. (2.42), together with Eq. (2.41) define completely the eigenvalue problem.

Other view is possible. If ω is fixed real number, the frequency of the perturbation is fixed and the eigenvalues give the length of it. That is the spatial approach. In this case, Eqs. (2.38) are still holding and but they represent a second order eigenvalue problem, for α as eigenvalues. This second order eigenvalue problem can be converted to a bigger first order eigenvalue problem, where Eq. (2.41) will still be valid, but where $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \alpha\hat{u}, \alpha\hat{v}, \alpha\hat{w}]^T$ are the eigenvectors and α the eigenvalues. The resulting $\mathcal{A}$ and $\mathcal{B}$, function of $\mathcal{D}$, ω , β , Re , U and W are:

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{1Ds} & U_y & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}_{1Ds} + V_y & 0 & \mathcal{D} & 0 & 0 & 0 \\ 0 & W_y & \mathcal{L}_{1Ds} & i\beta & 0 & 0 & 0 \\ 0 & \mathcal{D} & i\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} \end{pmatrix} \quad (2.43a)$$

$$\mathcal{B} = \begin{pmatrix} -Ui & 0 & 0 & 0 & -\frac{1}{Re} & 0 & 0 \\ 0 & -Ui & 0 & 0 & 0 & -\frac{1}{Re} & 0 \\ 0 & 0 & -Ui & 0 & 0 & 0 & -\frac{1}{Re} \\ -i & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.43b)$$

where $\mathcal{L}_{1Ds} = -i\omega + V\mathcal{D} + i\beta W - (\mathcal{D}^2 - \beta^2)/Re$ is written as an operator to maintain the equations clean. Here, $V = 0$ for strictly parallel flows again, but has not been substituted in order to have more general equations. Eqs. (2.43), together with Eq. (2.41) define completely the eigenvalue problem.

Cylindrical coordinates

In this case, Eqs. (2.37) cannot be used, instead, Eqs. (2.44) are used. In that cylindrical coordinates, $[r, \theta, z]$ are the radial, azimuthal and perpendicular coordinates and $[u, v, w]$ and $[U, V, W]$ the perturbation and basic flow velocity components in each corresponding

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direction.

$$\begin{aligned}
& u_t + Uu_r + \frac{Vu_\theta}{r} + Wu_z + uU_r + \frac{vU_\theta}{r} + wU_z - \frac{2Vv}{r} \\
& \quad = -p_r + \frac{1}{Re} \left(\frac{u_r}{r} + u_{rr} + \frac{u_{\theta\theta}}{r^2} + u_{zz} - \frac{u}{r^2} - \frac{2v_\theta}{r^2} \right) \\
& v_t + Uv_r + \frac{Vv_\theta}{r} + Wv_z + uV_r + \frac{vV_\theta}{r} + wV_z + \frac{Uv}{r} + \frac{Vu}{r} \\
& \quad = -\frac{p_\theta}{r} + \frac{1}{Re} \left(\frac{v_r}{r} + v_{rr} + \frac{v_{\theta\theta}}{r^2} + v_{zz} + \frac{2u_\theta}{r^2} - \frac{v}{r^2} \right) \quad (2.44) \\
& w_t + Uw_r + \frac{Vw_\theta}{r} + Ww_z + uW_r + \frac{vW_\theta}{r} + wW_z \\
& \quad = -p_z + \frac{1}{Re} \left(\frac{w_r}{r} + w_{rr} + \frac{w_{\theta\theta}}{r^2} + w_{zz} \right) \\
& u + ru_r + v_\theta + rw_z = 0
\end{aligned}$$

Again the last line of table (2.1) is selected, but now, the equations to solve are Eqs. (2.44). Here, coordinates are r , θ and z corresponding to x_1 , x_2 and x_3 of table (2.1) and k_2 , k_3 are m and α . Note that because θ and $\theta + 2n\pi$, for n integer, have to give the same result, m has to be integer. Doing the substitution, Eqs. (2.45) are obtained, where, again, derivatives are shown as sub-index.

$$\begin{aligned}
& -i\omega\hat{u} + U\hat{u}_r + \frac{imV\hat{u}}{r} + i\alpha W\hat{u} + \hat{u}U_r - \frac{2V\hat{v}}{r} \\
& \quad = -\hat{p}_r + \frac{1}{Re} \left(\frac{\hat{u}_r}{r} + \hat{u}_{rr} - \frac{m^2\hat{u}}{r^2} - \alpha^2\hat{u} - \frac{\hat{u}}{r^2} - \frac{2im\hat{v}}{r^2} \right) \\
& -i\omega\hat{v} + U\hat{v}_r + \frac{imV\hat{v}}{r} + i\alpha W\hat{v} + \hat{v}U_r + \frac{U\hat{v}}{r} + \frac{V\hat{u}}{r} \\
& \quad = -\frac{im\hat{p}}{r} + \frac{1}{Re} \left(\frac{\hat{v}_r}{r} + \hat{v}_{rr} - \frac{m^2\hat{v}}{r^2} - \alpha^2\hat{v} + \frac{2im\hat{u}}{r^2} - \frac{\hat{v}}{r^2} \right) \quad (2.45) \\
& -i\omega\hat{w} + U\hat{w}_r + \frac{imV\hat{w}}{r} + i\alpha W\hat{w} + \hat{w}U_r \\
& \quad = -i\alpha\hat{p} + \frac{1}{Re} \left(\frac{\hat{w}_r}{r} + \hat{w}_{rr} - \frac{m^2\hat{w}}{r^2} - \alpha^2\hat{w} \right) \\
& \hat{u} + r\hat{u}_r + im\hat{v} + i\alpha r\hat{w} = 0
\end{aligned}$$

This set of equations can again be seen as an eigenvalue problem, when derivatives in r are substituted by operators and the form of Eq. (2.41) is used. $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}]^T$ are the eigenvectors and ω the eigenvalues, while the rest of values are known.

The system can be further reduced, but a single equation as Orr-Sommerfeld equation was for Cartesian coordinates, cannot be achieved in cylindrical coordinates, here the limit are two equations. $\hat{p}$ and $\hat{w}$ can then be eliminated, but the order of the resulting equations will also increase. See Mayer and Powell [105] for more details.

Finally, the the form of the matrices $\mathcal{A}$ and $\mathcal{B}$ for this case are:

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{1Dtc} + U_r + \frac{1}{r^2 Re} & -\frac{2V}{r} + \frac{2im}{r^2 Re} & 0 & \mathcal{D} \\ V_r + \frac{V}{r} - \frac{2im}{r^2 Re} & \mathcal{L}_{1Dtc} + \frac{U}{r} + \frac{1}{r^2 Re} & 0 & \frac{im}{r} \\ W_r & 0 & \mathcal{L}_{1Dtc} & i\alpha \\ 1 + r\mathcal{D} & im & i\alpha r & 0 \end{pmatrix} \quad (2.46a)$$

$$\mathcal{B} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.46b)$$

This time, $\mathcal{L}_{1Dtc} = U\mathcal{D} + imV/r + i\alpha W - (\mathcal{D}/r + \mathcal{D}^2 - m^2/r^2 - \alpha^2)/Re$ is the operator used in cylindrical coordinates. Again, the problem is completely defined.

Previous result is obtained for the temporal approach. Now the spatial approach is taken. The equations can again be seen as an eigenvalue problem, being $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \alpha\hat{u}, \alpha\hat{v}, \alpha\hat{w}]^T$ are the eigenvectors and α the eigenvalues, while the rest of values are known.

Finally, the the form of the matrices $\mathcal{A}$ and $\mathcal{B}$ for the spatial problem are:

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{1Dsc} + U_r + \frac{1}{r^2 Re} & -\frac{2V}{r} + \frac{2im}{r^2 Re} & 0 & \mathcal{D} & 0 & 0 & 0 \\ V_r + \frac{V}{r} - \frac{2im}{r^2 Re} & \mathcal{L}_{1Dsc} + \frac{U}{r} + \frac{1}{r^2 Re} & 0 & \frac{im}{r} & 0 & 0 & 0 \\ W_r & 0 & \mathcal{L}_{1Dsc} & 0 & 0 & 0 & 0 \\ 1 + r\mathcal{D} & im & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} \end{pmatrix} \quad (2.47a)$$

$$\mathcal{B} = \begin{pmatrix} -Wi & 0 & 0 & 0 & -\frac{1}{Re} & 0 & 0 \\ 0 & -Wi & 0 & 0 & 0 & -\frac{1}{Re} & 0 \\ 0 & 0 & -Wi & -i & 0 & 0 & -\frac{1}{Re} \\ 0 & 0 & -ir & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.47b)$$

This time, $\mathcal{L}_{1Dsc} = -i\omega + U\mathcal{D} + imV/r - (\mathcal{D}/r + \mathcal{D}^2 - m^2/r^2)/Re$ is the operator used in cylindrical coordinates. Again, the problem is completely defined.

2.3.2 Two dimensions, BiGlobal

If the dependence in the third direction is neglected, the two-dimensional parallel flow is assumed and the corresponding stability analysis is called BiGlobal instability (see Theofilis [132, 133] for a review). Assuming that the basic flow is now dependent on two out of the three spatial coordinates (see table 2.1):

$$\mathbf{Q} = [U, V, W, P]^T(x, y), \quad (2.48)$$

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the coefficients of the LNSE, Eqs. (2.36) are z -independent, and modal perturbations now get the form

$$\mathbf{q}(x, y, z, t) = \hat{\mathbf{q}}(x, y)e^{i(\beta z - \omega t)} \quad (2.49)$$

All this, is substituted into Eqs. (2.36), to obtain Eqs. (2.50). Again, some changes of notation are applied in that equation, x_1 , x_2 and x_3 are called x , y and z , while k_3 is called β . Derivatives are shown as sub-index, and now there are derivatives in two dimensions.

$$\begin{aligned} -i\omega\hat{u} + U\hat{u}_x + V\hat{u}_y + i\beta W\hat{u} + \hat{u}U_x + \hat{v}U_y &= -\hat{p}_x + \frac{1}{Re}(\hat{u}_{xx} + \hat{u}_{yy} - \beta^2\hat{u}) \\ -i\omega\hat{v} + U\hat{v}_x + V\hat{v}_y + i\beta W\hat{v} + \hat{u}V_x + \hat{v}V_y &= -\hat{p}_y + \frac{1}{Re}(\hat{v}_{xx} + \hat{v}_{yy} - \beta^2\hat{v}) \\ -i\omega\hat{w} + U\hat{w}_x + V\hat{w}_y + i\beta W\hat{w} + \hat{u}W_x + \hat{v}W_y &= -i\beta\hat{p} + \frac{1}{Re}(\hat{w}_{xx} + \hat{w}_{yy} - \beta^2\hat{w}) \\ \hat{u}_x + \hat{v}_y + i\beta\hat{w} &= 0 \end{aligned} \quad (2.50)$$

This can again be written as an eigenvalue problem with $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}]^T$ as eigenvectors and ω as eigenvalues.

The disturbances are three-dimensional, but a sinusoidal dependence is assumed only in the homogeneous z -direction, with the periodicity length $L_z = 2\pi/\beta$. Eqs. (2.50) can be again written in matrix form, using matrices $\mathcal{A}$ and $\mathcal{B}$.

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{2Dt} + U_x & U_y & 0 & \mathcal{D}_x \\ V_x & \mathcal{L}_{2Dt} + V_y & 0 & \mathcal{D}_y \\ W_x & W_y & \mathcal{L}_{2Dt} & i\beta \\ \mathcal{D}_x & \mathcal{D}_y & i\beta & 0 \end{pmatrix} \quad (2.51a)$$

$$\mathcal{B} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.51b)$$

where $\mathcal{L}_{2Dt} = U\mathcal{D}_x + V\mathcal{D}_y + i\beta W - \frac{1}{Re}(\mathcal{D}_{xx} + \mathcal{D}_{yy} - \beta^2)$. Notice that there are two different operators, $\mathcal{D}_x$ and $\mathcal{D}_y$. Some aspects about these operators and the methodology to solve these equations numerically will be given in Chapter 3. For this approach, β is the wavenumber parameter, related with the periodicity length along the homogeneous spatial direction, z , and should be given, done so, the eigenvalue problem is defined.

Previous equations represent the temporal problem. In order to study the spatial problem, ω must be given and real, while the sought complex eigenvalues are then β . The eigenvectors will be then $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \beta\hat{u}, \beta\hat{v}, \beta\hat{w}]^T$. The matrices $\mathcal{A}$ and $\mathcal{B}$ for the spatial BiGlobal problem are given by Eqs. (2.52).

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{2Ds} + U_x & U_y & 0 & \mathcal{D}_x & 0 & 0 & 0 \\ V_x & \mathcal{L}_{2Ds} + V_y & 0 & \mathcal{D}_y & 0 & 0 & 0 \\ W_x & W_y & \mathcal{L}_{2Ds} & 0 & 0 & 0 & 0 \\ \mathcal{D}_x & \mathcal{D}_y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{I} \end{pmatrix} \quad (2.52a)$$

$$\mathcal{B} = \begin{pmatrix} -Wi & 0 & 0 & 0 & -\frac{1}{Re} & 0 & 0 \\ 0 & -Wi & 0 & 0 & 0 & -\frac{1}{Re} & 0 \\ 0 & 0 & -Wi & -i & 0 & 0 & -\frac{1}{Re} \\ 0 & 0 & -i & 0 & 0 & 0 & 0 \\ \mathcal{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.52b)$$

where $\mathcal{L}_{2Ds} = -i\omega + U\mathcal{D}_x + V\mathcal{D}_y - \frac{1}{Re}(\mathcal{D}_{xx} + \mathcal{D}_{yy})$.

2.3.3 Three dimensions, TriGlobal

TiGlobal is the less restrictive approach of those studied and does not have any assumption of homogeneity in any direction. For this approach, Eqs. (2.36) remain almost unchanged, only derivatives in time are substituted by exponentials. The conversion of Eqs. (2.37) into Eqs. (2.53) is just done by substituting the time derivative by a multiplication by $-i\omega$. Again, some changes of notation are applied in the equations, x_1 , x_2 and x_3 are called x , y and z , and derivatives are shown as sub-indexes, as well as now there are derivatives in the three dimensions.

$$\begin{aligned} -i\omega\hat{u} + U\hat{u}_x + V\hat{u}_y + W\hat{u}_z + \hat{u}U_x + \hat{v}U_y + \hat{w}U_z &= -\hat{p}_x + \frac{1}{Re}(\hat{u}_{xx} + \hat{u}_{yy} + \hat{u}_{zz}) \\ -i\omega\hat{v} + U\hat{v}_x + V\hat{v}_y + W\hat{v}_z + \hat{u}V_x + \hat{v}V_y + \hat{w}V_z &= -\hat{p}_y + \frac{1}{Re}(\hat{v}_{xx} + \hat{v}_{yy} + \hat{v}_{zz}) \\ -i\omega\hat{w} + U\hat{w}_x + V\hat{w}_y + W\hat{w}_z + \hat{u}W_x + \hat{v}W_y + \hat{w}W_z &= -\hat{p}_z + \frac{1}{Re}(\hat{w}_{xx} + \hat{w}_{yy} + \hat{w}_{zz}) \\ \hat{u}_x + \hat{v}_y + \hat{w}_z &= 0 \end{aligned} \quad (2.53)$$

One more time, the equations are written as an eigenvalue problem, where $\mathbf{x} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}]^T$ are the eigenvectors and ω the eigenvalues. This time the matrices $\mathcal{A}$ and $\mathcal{B}$ take the most general form.

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_{3D} + U_x & U_y & U_z & \mathcal{D}_x \\ V_x & \mathcal{L}_{3D} + V_y & V_z & \mathcal{D}_y \\ W_x & W_y & \mathcal{L}_{3D} + W_z & \mathcal{D}_z \\ \mathcal{D}_x & \mathcal{D}_y & \mathcal{D}_z & 0 \end{pmatrix} \quad (2.54a)$$

$$\mathcal{B} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.54b)$$

The operator used here, $\mathcal{L}_{3D} = U\mathcal{D}_x + V\mathcal{D}_y + W\mathcal{D}_z - \frac{1}{Re}(\mathcal{D}_{xx} + \mathcal{D}_{yy} + \mathcal{D}_{zz})$, has derivative matrices in the three directions, $\mathcal{D}_x$, $\mathcal{D}_y$ and $\mathcal{D}_z$. Due to that, the problem is much bigger in practice, which leads to the fact that only very small resolutions can be done in standard computers. However, this discussion falls beyond the objectives of this chapter and will be discussed in Chapter 3.

2.3.4 Parabolised Stability Equations

Parabolised Stability Equations (PSE) are capable of studying slowly varying flows at a similar computational cost to the equivalent homogeneous analysis. They can analyse the streamwise growth of linear and nonlinear disturbances in slowly varying flows. Growth mechanisms include both algebraic transient growth and exponential growth through primary and higher instabilities. PSE solutions incorporate inhomogeneous initial and boundary conditions as do DNS, opposing to homogeneous conditions for MLS. The PSE are capable of performing the stability analysis of convectively unstable flows as long as the root mean square (RMS) of the variables profiles vary slowly in the streamwise direction² $\partial/\partial z = \mathcal{O}(\varepsilon)$, with $\varepsilon \ll 1$, being ε of the order of an inverse power of the Reynolds number, which is usually a large amount.

Standard Parabolised Stability Equations (PSE)

Standard PSE is based on a basic flow that varies significantly over one direction (which is usually designated as y), is homogeneous in a second direction (which is here selected as x , but that traditionally for boundary layers, the first field of application of this methodology, was taken to be z), and has a slow variation in the remaining direction.

The disturbance quantities are expanded in terms of their truncated Fourier components assuming that are periodic in time.

$$\mathbf{q}(x, y, z, t) = \hat{\mathbf{q}}(y, z) \exp \left[i \int_z \beta(z') dz' + i\alpha x - i\omega t \right] + \text{c.c.} \quad (2.55)$$

where $\hat{\mathbf{q}}(y, z)$ varies slowly with z . Within the PSE approximation, the axial derivatives of $\mathbf{q}$ take the form

$$\frac{\partial^k \mathbf{q}}{\partial z^k} = i^k \left[\beta^k \hat{\mathbf{q}} - ik\beta^{k-1} \frac{\partial \hat{\mathbf{q}}}{\partial z} - i \frac{k(k-1)}{2} \beta^{k-2} \frac{d\beta}{dz} \hat{\mathbf{q}} \right] \exp \left[i \int_z \beta(z') dz' + i\alpha x - i\omega t \right], \quad (2.56)$$

and for the first and second derivative, the above expression becomes

$$\frac{\partial \mathbf{q}}{\partial z} = \left(i\beta \hat{\mathbf{q}} + \frac{\partial \hat{\mathbf{q}}}{\partial z} \right) \exp \left[i \int_z \beta(z') dz' + i\alpha x - i\omega t \right], \quad (2.57)$$

$$\frac{\partial^2 \mathbf{q}}{\partial z^2} = \left(-\beta^2 \hat{\mathbf{q}} + 2i\beta \frac{\partial \hat{\mathbf{q}}}{\partial z} + i \frac{d\beta}{dz} \hat{\mathbf{q}} \right) \exp \left[i \int_z \beta(z') dz' + i\alpha x - i\omega t \right]. \quad (2.58)$$

Substituting Eq. (2.55) and the first and second streamwise (axial) derivatives Eqs. (2.57-2.58) into the LNSE, Eq. (2.36), and neglecting terms of $\mathcal{O}(\varepsilon^2)$ as $\beta_z \hat{\mathbf{q}}_z$, $\hat{\mathbf{q}}_{zz}$, β_{zz} and higher derivatives with respect to z or streamwise derivatives in the viscous part of the equations, the linear PSE is written in a compact form as

$$\left(\mathbf{M} + \mathbf{N} \frac{\partial}{\partial z} \right) \hat{\mathbf{q}}(y, z) = 0. \quad (2.59)$$

²Here this direction is the vortex axial direction and is taken as z , opposing to the most common notation of x for this direction.

The entries of the coefficient matrices for $\mathbf{M}$ and $\mathbf{N}$ can be found in Hanifi et al. [61], Hein [65], Hein et al. [67] for the non-linear compressible regime and are expressed as follow in the incompressible and linear limit:

$$\mathbf{M} = \begin{pmatrix} \mathcal{L}_{PSE} & U_y & U_z & i\alpha \\ 0 & \mathcal{L}_{PSE} + V_y & V_z & \mathcal{D}_y \\ 0 & W_y & \mathcal{L}_{PSE} + W_z & i\beta \\ i\alpha & \mathcal{D}_y & i\beta & 0 \end{pmatrix}, \quad (2.60)$$

$$\mathbf{N} = \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (2.61)$$

where $\mathcal{L}_{PSE} = -i\omega + i\alpha U + V\mathcal{D}_y + i\beta W - (-\alpha^2 + \mathcal{D}_{yy}^2 - \beta^2)/Re$. The linear PSE equations are recovered by setting zero forcing term, what is equivalent to neglect terms of $\mathcal{O}(\epsilon^2)$. Here, $\omega \in \mathbb{R}$ is a real frequency parameter and $\alpha \in \mathbb{R}$ is a real wavenumber, $\alpha = 2\pi/L_x$, while $\beta \in \mathbb{C}$ is sought.

An ambiguity exists in the PSE formulation, in which the changes in amplitude along the slow spatial direction can be contained both in the amplitude function $\hat{\mathbf{q}}$ or in the phase function of Eq. (2.55). A normalization condition is required in order to close the formulation of the problem (see Herbert [68, 69] for a review). In this work, the following normalization condition is used:

$$\int_{\Omega} \hat{\mathbf{u}}^* \frac{\partial \hat{\mathbf{u}}}{\partial z} d\Omega = \int_{\Omega} \frac{1}{2} \frac{\partial |\hat{\mathbf{u}}|^2}{\partial z} d\Omega = 0 \quad (2.62)$$

here $\hat{\mathbf{u}} = (\hat{u}, \hat{v}, \hat{w})^T$, and superscript $*$ accounts for transpose conjugate. This normalisation imposes that the kinetic energy of the shape functions remains independent of z . Thus, the amplitude growth is absorbed into the phase function.

The effective growth rate computed by PSE is formulated as

$$\sigma_{\hat{K}} = -\beta_i + \frac{1}{2} \frac{d \ln[\hat{K}(z)]}{dz}, \quad \text{with} \quad \hat{K}(z) = \int_{\Omega} (\hat{u}^* \hat{u} + \hat{v}^* \hat{v} + \hat{w}^* \hat{w}) d\Omega, \quad (2.63)$$

in order to take into account the residual slow variation of disturbance kinetic energy, $\hat{K}$, with z .

Inspection of this system suggests a parabolic set of equations that can be solved by treating the z direction as a pseudo-time, marching in this direction. The studies of Li and Malik [97, 99], found that these equations are mathematically elliptic, and are numerically ill-posed. Choosing a sufficiently large marching step, however, negates the upstream influence and was found to give accurate results. Li and Malik [97] found that, for a backward difference scheme in z , a limit of

$$\Delta z > \frac{1}{|\beta|} \quad (2.64)$$

needs to be placed on the marching step size.

Three dimensional Parabolised Stability Equations (PSE-3D)

PSE-3D is based on a basic flow that varies significantly over one plane (which is here designated as x - y) has a slow variation in the perpendicular direction (z). This methodology is particularly well suited to study vortex systems with axial diffusion.

The Ansatz utilized to decompose the flow perturbations is based on the assumption that $\partial/\partial z \ll \partial/\partial x, \partial/\partial y$ and replaces Eq. (2.49) by:

$$\mathbf{q}(x, y, z, t) = \hat{\mathbf{q}}(x, y, z) \exp \left[i \int_z \beta(z') dz' - \omega t \right] + \text{c.c.} \quad (2.65)$$

Here $\hat{\mathbf{q}}$ is the amplitude function and β is the corresponding wavenumber, both varying slowly along the axial spatial direction. The real parameter ω denotes the frequency of the perturbations.

By substitution of the decomposition Eq. (2.65) into the LNSE equations, neglecting viscous terms with streamwise derivatives of $\mathcal{O}(Re^{-2})$ and higher, the linear PSE-3D equations can be obtained, written in a compact form as

$$\left(\mathbf{M} + \mathbf{N} \frac{\partial}{\partial z} \right) \hat{\mathbf{q}}(x, y, z) = 0. \quad (2.66)$$

The entries of the matrices $\mathbf{M}$ and $\mathbf{N}$ are

$$\mathbf{M} = \begin{pmatrix} \mathcal{L}_{PSE3D} + U_x & U_y & U_z & \mathcal{D}_x \\ V_x & \mathcal{L}_{PSE3D} + V_y & V_z & \mathcal{D}_y \\ W_x & W_y & \mathcal{L}_{PSE3D} + W_z & i\beta \\ \mathcal{D}_x & \mathcal{D}_y & i\beta & 0 \end{pmatrix}, \quad (2.67)$$

$$\mathbf{N} = \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (2.68)$$

where $\mathcal{L}_{PSE3D} = -i\omega + U\mathcal{D}_x + V\mathcal{D}_y + i\beta W - \frac{1}{Re}(\mathcal{D}_{xx}^2 + \mathcal{D}_{yy}^2 - \beta^2)$ and, as in spatial BiGlobal theory, $\omega \in \mathbb{R}$ is a real frequency parameter and $\beta \in \mathbb{C}$.

An ambiguity arises regarding the normalization condition which needs to be imposed in order to ensure, firstly, that the variation of the shape function along the streamwise direction be small and, secondly, that all amplitude growth be captured by the phase function. This issue has been discussed and resolved by Herbert [68, 69] in the context of the classic parabolised stability equations. The same concept is extended to PSE-3D, where the normalization condition

$$\int_{\Omega} \hat{\mathbf{q}}^* \frac{\partial \hat{\mathbf{q}}}{\partial z} d\Omega = 0 \quad (2.69)$$

is used, where Ω denotes the extent of the computational domain on the plane normal to the axial spatial direction and superscript $*$ denotes transpose conjugate. Much like in the classic PSE, upstream propagating disturbances which are inherent to the parabolised (but

not strictly parabolic) nature of the disturbance equations, are also found to be present in PSE-3D Broadhurst and Sherwin [21]. Here the remedy proposed by Li and Malik [97, 98, 99] is used, namely streamwise spatial integration of Eq. (2.66) using a marching step $\Delta z > 1/|\beta|$.

A spatial amplification rate, $\sigma_{\hat{K}}$, may also be defined in the PSE-3D analysis as

$$\sigma_{\hat{K}} = -\beta_i + \frac{1}{2} \frac{d \ln[\hat{K}(z)]}{dz}, \quad \text{with} \quad \hat{K}(z) = \int_{\Omega} (\hat{u}^* \hat{u} + \hat{v}^* \hat{v} + \hat{w}^* \hat{w}) d\Omega, \quad (2.70)$$

Here $\hat{K}$ is the kinetic energy of the perturbations, the latter being three-dimensional functions of the spatial coordinates that vary slowly along the axial direction. In a manner analogous with the classic PSE, the amplification rate in Eq. (2.70) is composed of two contributions, the local $-\beta_i$ obtained at convergence of solution of Eq. (2.66) at each z -location to which the term corresponding to the perturbation kinetic energy is added.

2.4 Stability analysis by vortex methods

The aim of this section is to expose the procedure to perform the inviscid linear modal stability analysis by point vortices or vortex filaments. The idea is similar to standard linear modal stability, section 2.3, but, instead of the NSE used there, the vortex equations from section 2.2 are used here. These equations, the point or filament vortex motion equations, Eq. (2.32), relate the position derivatives (velocities) as function of the positions and intensities of all the vortices. The linearisation is performed by the introduction of small perturbations in the positions of the vortices, holding only linear terms of the perturbations, and neglecting higher order terms. Therefore the main differences with standard linear stability theory, section 2.3, are the equations (point or filament vortex motion equations, Eq. (2.32), instead of NSE, Eqs. (2.2) and the variables (vortex positions instead of flow variables). Once the differences are exposed, it must be mentioned that the procedure is quite similar. First the variables of the equations are perturbed and then, the perturbed equations are linearised. Later, the perturbations are written as complex exponentials of time and maybe some coordinates. And finally, an eigenvalue problem is achieved, whose eigenvalues will determine the linear stability of flow.

Two main different cases have been analysed. First of them, the two dimensional case, assumes point vortices located in a plane. In a three dimensional view, the points become straight, parallel lines, and the two dimensional array of point vortices can be recovered just by taking any section perpendicular to these lines. In this case, when points are perturbed, they remain as straight lines in the direction perpendicular to the plane, and the line is moved as a rigid solid. This method was first seen in the literature in the works of von Kármán [138, 139], where the vortex street of a wake that is named after him is analysed.

For the three dimensional case, the initial point vortex distribution will still be in a plane, but the perturbations deform the vortex filaments sinusoidally. This means that the cut of the three dimensional filaments by perpendicular planes will not be the same in every section and the problem should be regarded as three dimensional. In this case,

Eq. (2.32) is used instead of Eq. (2.24) because of the three-dimensional behaviour of the movement. The idea of this problem arose in Crow [35], where an aircraft wake is modelled by two counter-rotating vortices, and also the instability of the same name is presented. Jiménez [76] did the equivalent problem for co-rotating vortices and Crouch [32] introduced four vortices instead of only two. In these two works, as the initial point vortex distribution has a periodic movement rather than stationary or at constant velocity, the method can not be applied directly. However, Jiménez overcome that problem with a change in the axes of reference, while Crouch, on the other hand, had to use Floquet theory to analyse perturbations from the periodic movement.

The inviscid vortex method for stability is based on idealised vortices that have an infinite value of vorticity concentrated at points or along filaments, and therefore, vortex size and vorticity distribution are not taken into account in this idealization. The importance of these effects is more relevant in the case of self-induction (the influence of one vortex over itself) and is negligible for the interactions among different filaments. This may seem to contradict the fact that the core size becomes more relevant when the vortices are closer, but the point is that in the equations, see Eqs. (2.80), the vortex size only affects the self induction term, and the proximity effect is noticeable only because first the self induction acts, and then, the close vortex amplifies the effect. Actually, that is true in the 3D case, while in the 2D case, there is no mention at all to vortex size, the point vortices are infinitely small, and the results are completely independent of vortex size. Therefore, only the 3D case will be affected by these features, as self-induction does not exist in the 2D case.

2.4.1 Two dimensional perturbations procedure

In this case, the starting point is a configuration of N point vortices in a plane, $\mathbf{x}_p = [x_p, y_p]$, each of them denoted by the subindex p . This set of points will move following Eq. (2.71), which is the same as Eq. (2.25) seen in subsection 2.2.1, written in vectorial form, where $\mathbf{e}_z$ is the vector perpendicular to the plane.

$$\mathbf{u}(\mathbf{x}_p) = -\frac{1}{2\pi} \sum_{q=1, p \neq q}^N \Gamma_q \frac{(\mathbf{x}_p - \mathbf{x}_q) \times \mathbf{e}_z}{|\mathbf{x}_p - \mathbf{x}_q|^2} \quad (2.71)$$

Eq. (2.25) was used in section 2.2 to study the evolution of the points. However, in this section, the stability of the system is studied. Before that, an equilibrium configuration of point vortices has to be chosen, which, introduced in Eq. (2.71), leads to $\mathbf{u}(\mathbf{x}_p) = 0$. Later in the next paragraph it will be encountered that this is not strictly true, and a system of vortices where the distances among them remain constant can be also chosen, and then, the velocities $\mathbf{u}(\mathbf{x}_p)$ can be different from zero, but equal for all the points. This initial or equilibrium configuration, will be called here $\mathbf{X}_p = [X_p, Y_p]$, and perturbations of this position will be introduced, $\mathbf{x}'_p = [x'_p, y'_p]$. Then, $\mathbf{x}_p = \mathbf{X}_p + \varepsilon \mathbf{x}'_p$ ³ will be substituted into

³Note that strictly speaking, $\mathcal{O}(\varepsilon^2)$ and higher orders equations would imply constraints, but what it is done is a linearisation, and therefore, higher order terms are neglected. However, to be more precise, it could be stated that $\mathbf{x}_p = \mathbf{X}_p + \varepsilon \mathbf{x}'_p + \mathcal{O}(\varepsilon^2)$, and then everything is correct.

Eq. (2.71) obtaining Eqs. (2.72), where the assumption $\varepsilon \ll 1$ is made.

$$\mathcal{O}(1), \quad \frac{d\mathbf{X}_p}{dt} + \bar{\mathbf{U}} = -\frac{1}{2\pi} \sum_{q=1, p \neq q}^N \Gamma_q \frac{\mathbf{X}_p - \mathbf{X}_q}{|\mathbf{X}_p - \mathbf{X}_q|^2} \times \mathbf{e}_z \quad (2.72a)$$

$$\begin{aligned} \mathcal{O}(\varepsilon), \quad \frac{d\mathbf{x}'_p}{dt} = & -\frac{1}{2\pi} \sum_{q=1, p \neq q}^N \Gamma_q \left(\frac{\mathbf{x}'_p - \mathbf{x}'_q}{|\mathbf{X}_p - \mathbf{X}_q|^2} \right. \\ & \left. - \frac{2(\mathbf{X}_p - \mathbf{X}_q) \cdot (\mathbf{x}'_p - \mathbf{x}'_q)}{|\mathbf{X}_p - \mathbf{X}_q|^4} (\mathbf{X}_p - \mathbf{X}_q) \right) \times \mathbf{e}_z \end{aligned} \quad (2.72b)$$

$$\mathcal{O}(\varepsilon^2), \quad \text{Neglected} \quad (2.72c)$$

In the two dimensional case, $\mathbf{u}_p = [u_p, v_p] = d\mathbf{x}_p/dt = \mathbf{u}(\mathbf{x}_p) - \bar{\mathbf{U}}$, and therefore, $\mathbf{u}(\mathbf{x}_p) = d\mathbf{x}_p/dt + \bar{\mathbf{U}}$. The new term $\bar{\mathbf{U}}$ only appears when the basic state of the points is moving with constant speed, and it is necessary to change the reference system to one where the point vortices are stationary. Note that $\bar{\mathbf{U}}$ must be the same for all the vortices in order to have a reference system in which the set of points is in equilibrium.

The time derivative in Eq. (2.72a) vanishes when the initial points represent an equilibrium configuration. In fact, $\bar{\mathbf{U}}$ can be calculated from this equation. Eq. (2.72b) is the linearised equation of motion of the perturbations and will serve as the basis of the subsequent analysis. Eq. (2.72c) represents the neglected nonlinear terms that are neglected.

Eq. (2.72b) is an Initial Value Problem (IVP) for the linearised equations, based upon which non-modal analysis could be performed. In order to complete the modal analysis and obtain an EigenValue Problem (EVP), it is further assumed that the perturbation has a harmonic time variation that can be written in complex exponential form as $\mathbf{x}'_p = \hat{\mathbf{x}}_p e^{-i\omega t}$. An EVP for ω results, in which stability can be inferred from the sign of the imaginary part of the eigenvalues. If $\omega_i > 0$, the perturbation grows and the mode is unstable, and if $\omega_i < 0$, the perturbation is damped and the mode is stable. $\omega_i = 0$ means that the mode is neutrally stable. On the other hand, ω_r is the frequency of oscillation of one particular mode in time. Equations have been presented now in scalar form in order to show that this vectorial equation represents two scalar equations with a very similar form.

$$\begin{aligned} -i\omega \hat{x}_p = & \sum_{q=1, p \neq q}^N \frac{-\Gamma_q}{2\pi} \left(\frac{\hat{y}_p - \hat{y}_q}{(X_p - X_q)^2 + (Y_p - Y_q)^2} \right. \\ & \left. - \frac{2[(X_p - X_q)(\hat{x}_p - \hat{x}_q) + (Y_p - Y_q)(\hat{y}_p - \hat{y}_q)](Y_p - Y_q)}{((X_p - X_q)^2 + (Y_p - Y_q)^2)^2} \right) \end{aligned} \quad (2.73a)$$

$$\begin{aligned} -i\omega \hat{y}_p = & \sum_{q=1, p \neq q}^N \frac{\Gamma_q}{2\pi} \left(\frac{\hat{x}_p - \hat{x}_q}{(X_p - X_q)^2 + (Y_p - Y_q)^2} \right. \\ & \left. - \frac{2[(X_p - X_q)(\hat{x}_p - \hat{x}_q) + (Y_p - Y_q)(\hat{y}_p - \hat{y}_q)](X_p - X_q)}{((X_p - X_q)^2 + (Y_p - Y_q)^2)^2} \right) \end{aligned} \quad (2.73b)$$

The equations may be written in a more compact form:

$$-i\omega \hat{x}_p = \sum_{q=1, q \neq p}^N \frac{-\Gamma_q}{2\pi} \left(\frac{\hat{y}_p - \hat{y}_q}{\zeta_{pq}^2} - \frac{2[\mathcal{X}_{pq}(\hat{x}_p - \hat{x}_q) + \mathcal{Y}_{pq}(\hat{y}_p - \hat{y}_q)]\mathcal{Y}_{pq}}{\zeta_{pq}^4} \right) \quad (2.74a)$$

$$-i\omega\hat{y}_p = \sum_{q=1, q \neq p}^N \frac{\Gamma_q}{2\pi} \left(\frac{\hat{x}_p - \hat{x}_q}{\zeta_{pq}^2} - \frac{2[\mathcal{X}_{pq}(\hat{x}_p - \hat{x}_q) + \mathcal{Y}_{pq}(\hat{y}_p - \hat{y}_q)]\mathcal{Z}_{pq}}{\zeta_{pq}^4} \right) \quad (2.74b)$$

Eqs. (2.74) represent the general equations governing the stability of point vortices in Cartesian coordinates, where the variables $\mathcal{X}_{pq} = X_p - X_q$, $\mathcal{Y}_{pq} = Y_p - Y_q$ and $\zeta_{pq} = \sqrt{(X_p - X_q)^2 + (Y_p - Y_q)^2}$ are introduced to simplify the notation. Along this procedure of calculations, the point vortices are assumed to be in equilibrium and perturbations to their original positions $(\hat{x}_p, \hat{y}_p)^T$ are always on the plane on which point vortices are defined. Points are regarded as infinite straight lines perpendicular to the plane.

This subsection is basically a generalization of what von Kármán [138, 139] did. In those papers, Kármán went one step further and supposed $\hat{\mathbf{x}}_p = e^{ip\varphi}$, which helps a lot when the eigenvalue problem wants to be solved analytically for simple cases.

Along this procedure of calculations, the point vortices are assumed to be in equilibrium and perturbations are always inside the plane. Points are regarded as infinite straight lines perpendicular to the plane at all times.

2.4.2 Three dimensional perturbations procedure

The approach followed in this section is analogous to that adopted in the two dimensional case, but now, the previously considered straight parallel filaments can have homogeneous perturbations along the perpendicular z direction. This notation exposes the similarities with Eq. (2.71), showing that the only differences are the change of the direction of the filaments, that now is more generic than only $\mathbf{e}_z$, shown as $d\mathbf{l}_q$, and the integration along the filament, which involves some slight changes.

$$\mathbf{u}(\mathbf{x}_p) = -\frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_L \frac{(\mathbf{x}_p - \mathbf{x}_q) \times d\mathbf{l}_q}{|\mathbf{x}_p - \mathbf{x}_q|^3} \quad (2.75)$$

An important particularity of Eq. (2.75) is self-induction, namely that one filament can interact with itself, and therefore, the constraint $p \neq q$ has been removed. However, as mentioned earlier, these integrals diverge and cutoff integrals are used, as originally suggested by Crow [35].

As regards modal linear stability analysis, a similar procedure to that followed in the two dimensional case is carried out to obtain the matrix whose eigenvalues are sought. The positions of the vortices $(\mathbf{x}_p = [x_p, y_p, z_p])$ are decomposed into a basic state $(\mathbf{X}_p = [X_p, Y_p, Z_p])$ and a perturbation $(\mathbf{x}'_p = [x'_p, y'_p, 0])$ of order $\varepsilon \ll 1$.⁴ The filaments are perturbed only in their positions perpendicular to the vortex axis, so there are perturbations only for directions x and y , although their variation will depend upon the third direction as well. This time, $d\mathbf{l}_q = (\partial\mathbf{x}_q/\partial z_q)dz_q = (\mathbf{e}_z + \varepsilon\partial\mathbf{x}'_q/\partial z_q)dz_q$ as the filaments are now perturbed. Perturbations are only supposed to act in the perpendicular plane, while the z coordinate is only an integration parameter, as $z = z_p = Z_p$ is not

⁴The same that was said in a previous note for the 2D case, also applies here, and therefore, higher order terms are neglected.

any given coordinate. The substitution of $\mathbf{u}(\mathbf{x}_p)$ is not as straightforward as in the 2D case. Here, $\mathbf{u}_p = \mathbf{u}(\mathbf{x}_p) - \bar{\mathbf{U}} = [u_p, v_p, w_p]$, and the derivatives arise from the equation $d\mathbf{x}_p/dt + w_p d\mathbf{x}_p/dz_p = \mathbf{e}_x u_p + \mathbf{e}_y v_p + \mathbf{e}_z w_p$,⁵ where it is observed that z_p is the only directional dependence. It is that because $\mathbf{x}_p(z_p, t) = [x_p(z_p, t), y_p(z_p, t), z_p]$, as the direction z of the vortex p , z_p , is an independent variable, as for every filament p , x_p and y_p functions of this position in the z direction. Therefore, $d\mathbf{x}_p/dx_p = d\mathbf{x}_p/dy_p = dz_p/dt = 0$ and the terms $u_p d\mathbf{x}_p/dx_p$ and $v_p d\mathbf{x}_p/dy_p$ are null. The origin of the previous formula is due to the usage of a fix frame of reference (Eulerian), which is not following the particles, and therefore, the convective term appears. The reader can check that for the 2D case the same formula becomes $d\mathbf{x}_p/dt = \mathbf{e}_x u_p + \mathbf{e}_y v_p$, because there is not z dependence as it is for the 3D case. When this equation is linearised, the convective term is of higher order, because $\mathbf{u}_p$ and particularly w_p are of $\mathcal{O}(\varepsilon)$ and can be neglected. This is seen when the $\mathcal{O}(\varepsilon)$ equations are expanded for each of the three components in scalar form, because there exists a contribution in the z direction, which exposes that $w_p = \mathcal{O}(\varepsilon)$:

$$w_p = -\frac{\varepsilon}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \frac{(X_p - X_q) \frac{\partial y'_q}{\partial z_q} - (Y_p - Y_q) \frac{\partial x'_q}{\partial z_q}}{|\mathbf{X}_p - \mathbf{X}_q|^3} dz_q \quad (2.76)$$

The resulting equations take the form:

$$\frac{d\mathbf{X}_p}{dt} + \bar{\mathbf{U}} + \varepsilon \frac{d\mathbf{x}'_p}{dt} = -\frac{1}{4\pi} \sum_{q=1, p \neq q}^N \Gamma_q \int_{-\infty}^{\infty} \frac{[\mathbf{X}_p - \mathbf{X}_q + \varepsilon(\mathbf{x}'_p - \mathbf{x}'_q)] \times \left(\mathbf{e}_z + \varepsilon \frac{\partial \mathbf{x}'_q}{\partial z_q} \right)}{|\mathbf{X}_p - \mathbf{X}_q + \varepsilon(\mathbf{x}'_p - \mathbf{x}'_q)|^3} dz_q \quad (2.77)$$

Now, ordering by their order of magnitude:⁶

$$\mathcal{O}(1): \quad \frac{d\mathbf{X}_p}{dt} + \bar{\mathbf{U}} = -\frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \frac{(\mathbf{X}_p - \mathbf{X}_q) \times \mathbf{e}_z}{|\mathbf{X}_p - \mathbf{X}_q|^3} dz_q \quad (2.78a)$$

$$\mathcal{O}(\varepsilon): \quad \frac{d\mathbf{x}'_p}{dt} = -\frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \left[\frac{\left((\mathbf{x}'_p - \mathbf{x}'_q) \times \mathbf{e}_z + (\mathbf{X}_p - \mathbf{X}_q) \times \frac{\partial \mathbf{x}'_q}{\partial z_q} \right)}{|\mathbf{X}_p - \mathbf{X}_q|^3} - \frac{3 |\mathbf{X}_p - \mathbf{X}_q| \left((\mathbf{X}_p - \mathbf{X}_q) \cdot (\mathbf{x}'_p - \mathbf{x}'_q) \right) (\mathbf{X}_p - \mathbf{X}_q) \times \mathbf{e}_z}{|\mathbf{X}_p - \mathbf{X}_q|^6} \right] dz_q \quad (2.78b)$$

$$\mathcal{O}(\varepsilon^2): \quad \text{Neglected} \quad (2.78c)$$

The analysis of the $\mathcal{O}(\varepsilon)$ terms will lead to the eigenvalue problem that is used to study the stability of the desired configuration, but before, the satisfaction of the $\mathcal{O}(1)$ equation needs to be ensured. As seen in the 2D approach, $\bar{\mathbf{U}}$ might not be zero, but it has to be constant for all vortices. Examples of this kind of motion with constant velocity or angular velocity can be found in Crow [35] and Jiménez [76] respectively. In these examples, the system of reference is moving, and all the points are stationary in that frame. In the case of periodic motion, the stability of the system is studied by means of the Floquet theory as

⁵ $dx_p/dt = u(\mathbf{x}_p) - \bar{U}$, $dy_p/dt = v(\mathbf{x}_p) - \bar{V}$ and $w_p = w(\mathbf{x}_p)$

⁶Only terms of $\mathcal{O}(\varepsilon)$ are retained. Note that $\frac{1}{|\mathbf{a} + \varepsilon \mathbf{b}|^3} = \frac{1}{|\mathbf{a}|^3} - \varepsilon \frac{3\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^5} + \mathcal{O}(\varepsilon^2)$.

2. Theoretical Instability Analysis Concepts

discussed by Crouch [32]. The latter approach consists in the integration of the movement in one period to obtain the matrix of the integrated movement, which is the one used in the eigenvalue problem, but is not considered in the present work. The integration of Eq. (2.78a) results in exactly Eq. (2.72a) as the basic state is two dimensional and only the perturbations are allowed to change in the axial direction.

This equation can be expanded for each of the two components of $\mathbf{x}'_p = [x'_p, y'_p]$ in scalar form. The result are Eqs. (2.79), which reveals that only two components are present. Note that $(\mathbf{X}_p - \mathbf{X}_q) \times \partial \mathbf{x}'_q / \partial z_q$ has a contribution in the z direction, $(X_p - X_q) \partial y'_q / \partial z_q - (Y_p - Y_q) \partial x'_q / \partial z_q$, but only perturbations in x and y directions have been supposed and then, it has to be negligible, see Eq. (2.76). One more thing has to be noted and is introduced in Eqs. (2.79). $Z_q = z_q$, because there is not perturbation in that direction. Actually, the integration is done in that direction z , but z_q is used to make clear what is the variable of integration and that z_q is not known by the initial positions as X_q and Y_q are.

$$\frac{dx'_p}{dt} = \frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \left(\frac{(z_p - z_q) \frac{\partial y'_q}{\partial z_q} - (y'_p - y'_q)}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + (z_p - z_q)^2]^{\frac{3}{2}}} + \frac{3[(X_p - X_q)(x'_p - x'_q) + (Y_p - Y_q)(y'_p - y'_q)](Y_p - Y_q)}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + (z_p - z_q)^2]^{\frac{5}{2}}} \right) dz_q \quad (2.79a)$$

$$\frac{dy'_p}{dt} = \frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \left(\frac{-(z_p - z_q) \frac{\partial x'_q}{\partial z_q} + (x'_p - x'_q)}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + (z_p - z_q)^2]^{\frac{3}{2}}} - \frac{3[(X_p - X_q)(x'_p - x'_q) + (Y_p - Y_q)(y'_p - y'_q)](X_p - X_q)}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + (z_p - z_q)^2]^{\frac{5}{2}}} \right) dz_q \quad (2.79b)$$

The next step in the procedure is to assume exponential perturbations of the form of $\mathbf{x}'_p = \hat{\mathbf{x}}_p e^{ikz_p - i\omega t}$. Eqs. (2.79) become the eigenvalue problem for ω defined by Eqs. (2.80). The variable of integration z_q has been changed to the variable z that substitutes the term $(z_p - z_q)$.

$$-i\omega \hat{x}'_p = \frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \left(\frac{ikz \hat{y}'_q e^{-ikz} - (\hat{y}'_p - \hat{y}'_q e^{-ikz})}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + z^2]^{\frac{3}{2}}} + \frac{3[(X_p - X_q)(\hat{x}'_p - \hat{x}'_q e^{-ikz}) + (Y_p - Y_q)(\hat{y}'_p - \hat{y}'_q e^{-ikz})](Y_p - Y_q)}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + z^2]^{\frac{5}{2}}} \right) dz \quad (2.80a)$$

$$-i\omega \hat{y}'_p = \frac{1}{4\pi} \sum_{q=1}^N \Gamma_q \int_{-\infty}^{\infty} \left(\frac{-ikz \hat{x}'_q e^{-ikz} + (\hat{x}'_p - \hat{x}'_q e^{-ikz})}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + z^2]^{\frac{3}{2}}} - \frac{3[(X_p - X_q)(\hat{x}'_p - \hat{x}'_q e^{-ikz}) + (Y_p - Y_q)(\hat{y}'_p - \hat{y}'_q e^{-ikz})](X_p - X_q)}{[(X_p - X_q)^2 + (Y_p - Y_q)^2 + z^2]^{\frac{5}{2}}} \right) dz \quad (2.80b)$$

It is important not to forget that the cutoff is implicit in all these integrals. That is

because the integrals diverge when the integral is calculated along a filament over itself. Several approaches are made in literature, see Crow [35] and Widnall et al. [142].

These integrals are different for the vortex $p=q$, where a cutoff integral is used, and, therefore, the sum is separated in these cases. d_p is the value of this cutoff for the vortex p , which has to be of the order of the real vortex core size. The integrals are represented using the modified Bessel functions of the second kind $K_0(\alpha)$, $K_1(\alpha)$ and $K_2(\alpha)$ and the cosine integral, $Ci(\alpha)$. Crouch [32] already derived these equations, and Fabre and Jacquin [43] particularised them for aligned vortices. However, the present notation, involves the full equations, highlighting the different variables and parameters on which the equations depend, including the variables $\mathcal{X}_{pq} = X_p - X_q$, $\mathcal{Y}_{pq} = Y_p - Y_q$ and $\zeta_{pq} = \sqrt{(X_p - X_q)^2 + (Y_p - Y_q)^2}$, used to simplify the notation. The stability equations in the three-dimensional case read in Cartesian coordinates:

$$\begin{aligned}
 -i\omega\hat{x}_p = & \sum_{q=1, p \neq q}^N \frac{\Gamma_q}{2\pi} \left[-\frac{\hat{y}_p}{\zeta_{pq}^2} - \frac{k^2 K_2(k\zeta_{pq})(\mathcal{X}_{pq}\hat{x}_q + \mathcal{Y}_{pq}\hat{y}_q)\mathcal{Y}_{pq}}{\zeta_{pq}^2} \right. \\
 & + \left(k^2 K_0(k\zeta_{pq}) + \frac{kK_1(k\zeta_{pq})}{\zeta_{pq}} \right) \hat{y}_q + \left(\frac{2(\mathcal{X}_{pq}\hat{x}_p + \mathcal{Y}_{pq}\hat{y}_p)\mathcal{Y}_{pq}}{\zeta_{pq}^4} \right) \Big] \\
 & + \frac{\Gamma_p}{2\pi} \left[\frac{\cos kd_p - 1 + kd_p \sin kd_p - k^2 d_p^2 Ci(kd_p)}{2d_p^2} \hat{y}_p \right]
 \end{aligned} \tag{2.81a}$$

$$\begin{aligned}
 -i\omega\hat{y}_p = & \sum_{q=1, p \neq q}^N \frac{\Gamma_q}{2\pi} \left[\frac{\hat{x}_p}{\zeta_{pq}^2} + \frac{k^2 K_2(k\zeta_{pq})(\mathcal{X}_{pq}\hat{x}_q + \mathcal{Y}_{pq}\hat{y}_q)\mathcal{X}_{pq}}{\zeta_{pq}^2} \right. \\
 & - \left(k^2 K_0(k\zeta_{pq}) + \frac{kK_1(k\zeta_{pq})}{\zeta_{pq}} \right) \hat{x}_q - \left(\frac{2(\mathcal{X}_{pq}\hat{x}_p + \mathcal{Y}_{pq}\hat{y}_p)\mathcal{X}_{pq}}{\zeta_{pq}^4} \right) \Big] \\
 & - \frac{\Gamma_p}{2\pi} \left[\frac{\cos kd_p - 1 + kd_p \sin kd_p - k^2 d_p^2 Ci(kd_p)}{2d_p^2} \hat{x}_p \right]
 \end{aligned} \tag{2.81b}$$

It can be seen that these equations reduce to Eqs. (2.74) in the two-dimensional limit, $k = 0$. Additionally, the relation between the cutoff d and the size of the vortex a should be stated here, to prevent the reader from misleading the meaning of both of them. Crow [35] showed that the effective core size relation with the cutoff parameter was $d/a_e = 0.6420$ for a Rankine vortex, but other approximations were made by Widnall [141] to match cutoff with any vortex. Details are given inside subsection 1.1.2.

Chapter 3

Numerical Considerations

Chapter 2 exposes the theoretical concepts and equations to be solved. However, between the equations and the results, there is an essential part which is analysed in this chapter. It comprises a huge amount of numerical aspects that should be considered to develop the numerical techniques employed to find the solution of these problems. Some of the issues discussed here are particular of one or more methodologies, while other are applicable to all of them.

3.1 Temporal schemes

The numerical schemes used to find the solution of ordinary differential equations of problems defined by initial conditions are basic in any book or course about numerics and it is not intended here to review the topic. Only some of the basics are mentioned, as they are used to implement the evolution algorithms presented in this thesis.

Temporal numerical schemes are methods used to advance in time the solution of an evolution problem (IVP). The general aspect of the problems solved by temporal schemes is represented by Eq. (3.1):

$$\frac{\partial u(t)}{\partial t} = F[u(t), t] \quad (3.1)$$

These schemes use previously stored values of the variable of integration, and evaluations of the operator of the problem. If these evaluations are all done in earlier times than the time to be calculated, the scheme is said to be explicit. If the operator has to be evaluated in the same or posterior time to that of the calculation, the scheme is implicit, because the operator has to be evaluated in a point that is not known before. Sometimes the operator is divided in two parts and one is evaluated as implicit, while the other as explicit, in this case, the scheme is said to be semi-implicit.

Other classification attends to schemes that use only one time evaluation, single-step, and these that use more than one, multi-step. In addition, time evaluations could be defined only in times defined by the time step, while other schemes use intermediate steps of time in order to obtain a better approximation of the solution, these are called single-stage and multi-stage respectively.

The schemes used in the numerical implementation of the algorithms of this thesis are explained in the subsequent subsections. Further details on these schemes may be found in Hernández [71] and Lambert [88] among many others.

3.1.1 Euler

The Euler method is the most simple evolution method. It follows the scheme of Eq. (3.2).

$$u(t_{j+1}) = u(t_j) + \Delta t F[u(t_j), t_j] \quad (3.2)$$

This method is very simple and fast, but usually needs a very small time step (Δt) in order to converge, because its stability region is very small and its tangency with the imaginary axis in the origin is only of second order. However, its simplicity makes it always a good point to start with.

3.1.2 Runge-Kutta

Runge-Kutta are multi-step explicit algorithms. For moderate order, the tangency of their stability region in the origin is quite high, which makes them very good and reliable. Their order is the number of fractional steps that the method uses. For the case of order 4, and constant time step, the advancement in time is calculated as follows:

$$\begin{aligned} k_1 &= F[u(t_j), t_j] \\ k_2 &= F[u(t_j) + \Delta t k_1/2, t_j + \Delta t/2] \\ k_3 &= F[u(t_j) + \Delta t k_2/2, t_j + \Delta t/2] \\ k_4 &= F[u(t_j) + \Delta t k_3, t_j + \Delta t] \\ u(t_{j+1}) &= u(t_j) + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \quad (3.3)$$

3.1.3 Spalart-Moser-Rogers [126]

This is an example of semi-implicit scheme. It uses different approaches for the linear and nonlinear parts of the operator $F[u(t)] = L[u(t)] + N[u(t)]$.

The method's algorithm is now exposed in Eq. (3.4), where it should not be forgotten that the linear part is semi-implicit:

$$\begin{aligned} u' &= u(t_j) + \Delta t \{ L[\alpha_1 u(t_j) + \beta_1 u'] + \gamma_1 N[u(t_j)] \} \\ u'' &= u' + \Delta t \{ L[\alpha_2 u' + \beta_2 u''] + \gamma_2 N(u') + \delta_1 N[u(t_j)] \} \\ u(t_{j+1}) &= u'' + \Delta t \{ L[\alpha_3 u'' + \beta_3 u(t_{j+1})] + \gamma_3 N(u'') + \delta_2 N(u') \} \end{aligned} \quad (3.4)$$

Where the constants are defined by:

$$\begin{aligned} \alpha_1 &= \frac{29}{96} & \beta_1 &= \frac{37}{160} & \gamma_1 &= \frac{8}{15} \\ \alpha_2 &= -\frac{3}{40} & \beta_2 &= \frac{5}{24} & \gamma_2 &= \frac{5}{12} & \delta_1 &= -\frac{17}{60} \\ \alpha_3 &= \frac{1}{6} & \beta_3 &= \frac{1}{6} & \gamma_3 &= \frac{3}{4} & \delta_2 &= -\frac{5}{12} \end{aligned} \quad (3.5)$$

For further details, Spalart et al. [126] should be consulted.

3.2 Spatial discretisation

The formulation used in section 2.1 and section 2.3 uses continuous space, but numerical algorithms need discrete space to be able to handle the problems. A spatial discretisation is then needed. Furthermore, derivative operators also need to be discretised. When discretised, these operators will be seen as matrices and then, they will be called derivation matrices. In general these matrices are defined as follows:

$$\frac{\partial u_i}{\partial x} = \mathcal{D}_{ij} u_j \quad (3.6)$$

$$\frac{\partial^2 u_i}{\partial x^2} = \mathcal{D}_{ij}^2 u_j \quad (3.7)$$

Usually, derivatives of different orders can be found in the literature, but higher orders of derivation can be also calculated as matrix products by $\mathcal{D}^2 = \mathcal{D} \cdot \mathcal{D}$ and so on.

Various different discretisations may be done. The main and traditional division is between spectral and finite differences. As a general comment, the main practical difference between them is the amount of points that are used to calculate a derivative. Spectral methods use all the points to calculate the derivative of each point, while finite differences use only some of them. This fact lead to sparse matrices when using finite differences, in opposition to spectral methods matrices that are full. However, for the same number of nonzero elements, spectral methods use to give more accurate results in general, although there are cases where that is not true.

3.2.1 Spectral collocation

These methods offer an optimal compromise between the highest accuracy possible and the necessity of reducing the amount of information to be stored. The reason of the high accuracy of (collocation) spectral methods, spectral or exponential convergence, lies on the use of high-order interpolating polynomials, comprising all the points in the discretisation domain, see Boyd [16] and Canuto et al. [27].

Two of these methods are used along this work, Fourier and Chebyhev-Gauss-Lobatto (CGL). Fourier automatically impose periodicity in the boundaries, and CGL is preferred in other cases. The derivative matrices of both of the methods are exposed in next sub-sections.

Fourier

Fourier is probably the most important spectral method. Its importance comes from the fact that Fourier expansion is widely used in most of the sciences. It will be used here to set periodic conditions.

The grid that has to be used for this method is:

$$x_i = \frac{\pi i}{N} \quad i = 0, \dots, 2N - 1 \quad (3.8)$$

The first and second derivative matrices are the following:

$$\mathcal{D}_{ij} = \begin{cases} \frac{(-1)^{i-j} \cot\left(\frac{x_i - x_j}{2}\right)}{2} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \quad (3.9)$$

$$\mathcal{D}_{ij}^2 = \begin{cases} \frac{(-1)^{i-j+1}}{2 \sin^2\left(\frac{x_i - x_j}{2}\right)} & \text{if } i \neq j \\ -\frac{1 + 2N^2}{6} & \text{if } i = j \end{cases} \quad (3.10)$$

More details on the formulation can be found in Boyd [16] that used the results presented by Gottlieb et al. [55].

Chebyshev-Gauss-Lobatto

CGL will in general be used for non-periodic conditions. It is maybe the most used spectral collocation method because it suits a wide variety of situations with very good accuracy.

The Chebyshev polynomials of degree n , $T_n(x)$, are given explicitly in terms of trigonometric functions by

$$T_n(x) = \cos(n\theta), \quad x = \cos(\theta) \quad (3.11)$$

From the trigonometric representation of this expression it is clear that the extrema of $T_N(x)$ are ± 1 . The collocation derivative matrix for the Gauss-Lobatto grid is defined by the set of zeros of T_{N+1} and is evaluated in the interval $[-1,1]$.

These are the Chebyshev Gauss-Lobatto collocation points:

$$x_i = \cos\left(\frac{\pi i}{N}\right) \quad (3.12)$$

with $i = 0, 1, \dots, N$.

Any function $f(x)$ can be expressed via the expansion:

$$f_N(x_i) = \sum_{j=0}^N a_j T_j(x_i) \quad (3.13)$$

where a_j are the series coefficients.

The derivative matrices found again in Boyd [16] with Gottlieb et al. [55] results are:

$$\mathcal{D}_{i,j} = \begin{cases} \frac{2N^2+1}{6} & \text{if } i = j = 0 \\ -\frac{x_i}{2(1-x_i^2)} & \text{if } i = j \neq 0, N \\ \frac{(-1)^{i+j}\bar{c}_i}{(x_i-x_j)\bar{c}_j} & \text{if } i \neq j \\ -\frac{2N^2+1}{6} & \text{if } i = j = N \end{cases} \quad (3.14)$$

where

$$\bar{c}_j = \begin{cases} 2 & \text{if } j = 0, N \\ 1 & \text{if } 1 \leq j \leq N-1 \end{cases} \quad (3.15)$$

An expression for the second derivative matrix can be found in the work of Ehrenstein and Peyret [42].

$$\mathcal{D}_{i,j}^2 = \begin{cases} \frac{(x_i^2 + x_i x_j - 2)(-1)^{i+j}}{(1-x_i^2)(x_i-x_j)^2 \bar{c}_j} & \text{if } 1 \leq i \leq N-1, 0 \leq j \leq N, i \neq j \\ -\frac{(N^2-1)(1-x_i^2)+3}{3(1-x_i^2)^2} & \text{if } 1 \leq i = j \leq N-1 \\ \frac{2(-1)^j[(2N^2+1)(1-x_j)-6]}{3\bar{c}_j(1-x_j)^2} & \text{if } i = 0, 1 \leq j \leq N \\ \frac{2(-1)^{j+N}[(2N^2+1)(1+x_j)-6]}{3\bar{c}_j(1+x_j)^2} & \text{if } i = N, 0 \leq j \leq N-1 \\ \frac{N^4-1}{15} & \text{if } i = j = 0, i = j = N \end{cases} \quad (3.16)$$

However, alternative expressions that reduce the impact of the round-off errors resulting from subtraction of nearly equal quantities are preferred. With this aim, the most obvious approach to reducing the impact of subtracting nearly equal numbers for the Chebyshev derivative matrices is to use trigonometric identities, giving the next derivative matrix Canuto et al. [27]:

$$\mathcal{D}_{i,j} = \begin{cases} \frac{2N^2+1}{6}, & i = j = 0 \\ -\frac{x_i}{2\sin^2(\pi i/N)}, & i = j \neq 0, N \\ \frac{(-1)^{i+j}\bar{c}_i}{2\sin[\pi(i+j)/2N]\sin[\pi(i-j)/2N]\bar{c}_j}, & i \neq j \\ -\frac{2N^2+1}{6} & i = j = N \end{cases} \quad (3.17)$$

3.2.2 Finite Differences

Finite Differences are one of the most widely used approaches to calculate derivatives and solve differential equations due to their simplicity and versatility. In general, they can be defined for any set of aligned points, although the majority of their implementations use equidistant grids. They can be defined for a particular order of the error, having a wide range of possibilities, choosing between accuracy and speed.

In general, derivative in each point is approximated by the derivative of the interpolation polynomial. For example, to obtain the formulas for two nodes, the interpolation polynomial of first degree between the points is taken, $p_1(x) = f(x_0) + f_d[x_0, x_1](x - x_0)$. Then, the derivative will be approximated by $f'(a) \approx p'_1(a) = f_d[x_0, x_1] = [f(x_1) - f(x_0)]/(x_1 - x_0)$. If the point a is between x_0 and x_1 , the formulas for central differences are found. Otherwise, if a coincides with x_0 or x_1 , formulas of forward and backward differences are obtained respectively. The same can be done for more nodes and higher derivatives order.

Standard Finite Differences

Here, the most common finite differences equations are exposed for the case of equidistant grid and second order. First and second derivatives are shown for central differences in Eqs. (3.18), where $\Delta x = x_{i+1} - x_i$ is constant for any i because nodes are equidistant.

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_{i+1}) - f(x_{i-1}))}{2\Delta x} \\ f''(x_i) &\approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{(\Delta x)^2} \end{aligned} \quad (3.18)$$

Equidistant forward differences of second order first and second derivatives are:

$$\begin{aligned} f'(x_i) &\approx \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2}))}{2\Delta x} \\ f''(x_i) &\approx \frac{2f(x_i) - 5f(x_{i+1}) + 4f(x_{i+2}) - f(x_{i+3}))}{(\Delta x)^2} \end{aligned} \quad (3.19)$$

And equidistant backward differences of second order first and second derivatives are:

$$\begin{aligned} f'(x_i) &\approx \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2}))}{2\Delta x} \\ f''(x_i) &\approx \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3}))}{(\Delta x)^2} \end{aligned} \quad (3.20)$$

All the information to build the first and second derivative matrices is plotted. Note that for big number of elements N , the matrices will have a lot of zeros. For a grid going

from $i = 1, \dots, N$, the derivatives will have the following shape:

$$\mathcal{D}_{ij} = \frac{1}{2\Delta x} \begin{cases} -3 & \text{if } i = 1, j = 1 \\ 4 & \text{if } i = 1, j = 2 \\ -1 & \text{if } i = 1, j = 3 \\ -1 & \text{if } 1 < i < N, j = i - 1 \\ 1 & \text{if } 1 < i < N, j = i + 1 \\ 1 & \text{if } i = N, j = N - 2 \\ -4 & \text{if } i = N, j = N - 1 \\ 3 & \text{if } i = N, j = N \\ 0 & \text{if any other case} \end{cases} \quad (3.21)$$

$$\mathcal{D}_{ij}^2 = \frac{1}{(\Delta x)^2} \begin{cases} 2 & \text{if } i = 1, j = 1 \\ -5 & \text{if } i = 1, j = 2 \\ 4 & \text{if } i = 1, j = 3 \\ -1 & \text{if } i = 1, j = 4 \\ 1 & \text{if } 1 < i < N, j = i - 1 \\ -2 & \text{if } 1 < i < N, j = i \\ 1 & \text{if } 1 < i < N, j = i + 1 \\ -1 & \text{if } i = N, j = N - 3 \\ 4 & \text{if } i = N, j = N - 2 \\ -5 & \text{if } i = N, j = N - 1 \\ 2 & \text{if } i = N, j = N \\ 0 & \text{if any other case} \end{cases} \quad (3.22)$$

Here, the order of the finite differences used for the two extremes is kept the same as it is for the internal points. However, it is common practice to lower one order the error in the extremes, to keep the same number of points used for the calculation of each derivative. Doing so, the order of the global method is maintained, because there are one order less of points in the extremes than in the interior, in one dimension, two points out of a total of N . Therefore, when the total error of the procedure is calculated, the error of the extremes accumulates one order less than the error of the interior points.

Finite Differences with uniform error (FDq)

This section exposes a particular case of finite differences that have uniform interpolation error. The method, developed by Hermanns and Hernández [70], improves the accuracy significantly compared to standard finite differences, even surpassing spectral methods in some cases. As differentiation matrices have a big sparsity pattern (a big number of zero elements), the usage of “sparse” codes that do not store the full matrices, but instead only the non-zero elements, can be of great use to decrease the amount of necessary resources. Some tips are offered below about this differential operators, for further details see Hermanns and Hernández [70].

Piecewise polynomial interpolation

This particular interpolation is performed by a piecewise interpolation polynomial, made from the union of several individual polynomials $I_i(x)$, only valid in small regions

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Ω_i , and whose discrete values $u_i(t)$ at the grid nodes x_i coincides with the function $u(x, t)$. The polynomial derivatives are computed to achieve the finite difference formulas. Each validity region of the individual interpolation polynomials, Ω_i includes its correspondent x_i and their union must be equal to the whole domain $[-1, +1]$ of the problem.

The expressions for each of the interpolants $I_i(x)$, given a set of grid nodes, is obtained from the Lagrange interpolation formula, available in Fornberg [47] and Hildebrand [72].

$$I_i(x) = \sum_{j=s_i}^{s_i+q} \ell_{ij}(x) u_j, \quad \ell_{ij}(x) = \prod_{\substack{m=0 \\ s_i+m \neq j}}^q \frac{x - x_{s_i+m}}{x_j - x_{s_i+m}}. \quad (3.23)$$

Eq. (3.23) is the Lagrange interpolation formula, being q the degree of the polynomial and s_i the seed index of the left most node x_i involved in the construction of the interpolation $I_i(x)$. Even polynomial degrees are chosen, for which the selection of values for s_i is:

$$\{s_i\} = \{\underbrace{0, \dots, 0}_{q/2 \text{ times}}, \underbrace{0, 1, \dots, N-q}_{\text{centered FD}}, \underbrace{N-q, \dots, N-q}_{q/2 \text{ times}}\}. \quad (3.24)$$

The situation of the seeds produces, therefore, different configurations, if it is far away from the boundaries, the formulas are equivalent to centred finite differences, while in the proximity to the extrema, the stencils are displaced towards the center of the domain in order to only make use of existing grid nodes.

The result obtained by this procedure, in virtue of the uniqueness of the interpolation polynomials, is equivalent to that achieved by classical means.

Uniform interpolation errors

The selection of the positions x_i of the grid nodes has not been fixed. This degree of freedom is taken to make uniform the interpolation errors in the polynomial domain, $[-1, +1]$. The resulting grid is unique for each pair of polynomial order q and total number of points N . The Chebyshev interpolation also makes uniform the interpolation error in the region $[-1, +1]$, and therefore, when the order of the polynomial q , coincides with N , the two methodologies must be the same. More details follow by Isaacson and Keller [73] and Boyd [16], where it is established that the resulting non-equispaced grid, known as the Chebyshev roots or Chebyshev-Gauss quadrature points, must be unique for each value of N , and then, both approaches achieve the same result, that is the suppression of the Runge phenomenon that spoils the accuracy of high order polynomial interpolations close to the ends of the interpolation interval.

Then, the case of $q = N$, could be obtained via a single interpolating polynomial, which have to be unique, as far as it is the polynomial that minimises the Runge phenomenon. In this sense, the Chebyshev polynomial minimises this same phenomenon, and therefore, the approaches should coincide. Therefore, this piecewise interpolation polynomial with uniform interpolation errors presents all the properties of spectral collocation methods, especially their spectral accuracy, see Boyd [16], Canuto et al. [27] and Fornberg [47]. Other limiting case is $q \ll N$, where only a few points $\mathcal{O}(q)$ are close to the boundaries and need to be clustered in order to control the interpolation error, while the majority of the points, which are far from the boundaries, are equally spaced.

3.3 Mapping

In most of the situations, the domain of the problem does not coincide with the grid domain that is prescribed by the method. One example could be the boundary layer flow computed by CGL. The domain goes from zero to infinity, while Chebyshev points are between -1 and 1. When such a situation is encountered, a mapping has to be done. In addition, some situations also require to concentrate points in some area of the domain, and then, a mapping should also be done.

The mapping will be represented by a change of reference of the form $\eta = \eta(x)$, where x are the standard collocation points and η the new variable that represent the values in the actual problem.

However, this change is not enough, as the derivative matrices $\mathcal{D}_{i,j}^m$ also change and have to be redefined to incorporate the stretching transformation. The new matrices $\hat{\mathcal{D}}_{i,j}^m$ are shown below in Eq. (3.25).

$$\begin{aligned}
\hat{\mathcal{D}}_{k,j}(\eta) &= \frac{dx}{d\eta} \mathcal{D}_{k,j}(x) \\
\hat{\mathcal{D}}_{k,j}^2(\eta) &= \left(\frac{dx}{d\eta}\right)^2 \mathcal{D}_{k,j}^2(x) + \frac{d^2x}{d\eta^2} \mathcal{D}_{k,j}(x) \\
\hat{\mathcal{D}}_{k,j}^3(\eta) &= \left(\frac{dx}{d\eta}\right)^3 \mathcal{D}_{k,j}^3(x) + 3 \frac{d^2x}{d\eta^2} \frac{dx}{d\eta} \mathcal{D}_{k,j}^2(x) + \frac{d^3x}{d\eta^3} \mathcal{D}_{k,j}(x) \\
\hat{\mathcal{D}}_{k,j}^4(\eta) &= \left(\frac{dx}{d\eta}\right)^4 \mathcal{D}_{k,j}^4(x) + 6 \frac{d^2x}{d\eta^2} \left(\frac{dx}{d\eta}\right)^2 \mathcal{D}_{k,j}^3(x) \\
&\quad + 3 \left(\frac{d^2x}{d\eta^2}\right)^2 \mathcal{D}_{k,j}^2(x) + 4 \frac{d^3x}{d\eta^3} \frac{dx}{d\eta} \mathcal{D}_{k,j}^2(x) + \frac{d^4x}{d\eta^4} \mathcal{D}_{k,j}(x)
\end{aligned} \tag{3.25}$$

This change allows to use $\hat{\mathcal{D}}_{i,j}^m$ in η coordinates as $\mathcal{D}_{i,j}^m$ was used in x coordinates. A collection of different mappings is given in Appendix B.

3.4 Multidimensional differentiation matrices

When dealing with problems with more than one dimension, the variables depend upon more than one direction. In that cases, variables are not vectors as they were in one-dimension (1D). The variables will be multidimensional arrays, that is, two-dimensional (2D) arrays (matrices) in 2D and three-dimensional (3D) arrays in 3D.

However, a notation may be used where variables are still vectors and operators ma-

trices. This notation require to write the various rows one after another.

$$\begin{aligned}
 \mathbf{q}_m(\underbrace{N, \dots, N}_{m \text{ times}}) = \mathbf{q}(\underbrace{N \times N \times \dots \times N}_{m \text{ times}}) = [q_{0,\dots,0}, q_{0,\dots,0,1}, \dots, q_{0,\dots,0,N}, \\
 q_{0,\dots,1,0}, q_{0,\dots,1,1}, \dots, q_{0,\dots,1,N}, \\
 \dots \\
 q_{0,\dots,N,0}, q_{0,\dots,N,1}, \dots, q_{0,\dots,N,N}, \\
 \dots \\
 q_{N,\dots,N,0}, q_{N,\dots,N,1}, \dots, q_{N,\dots,N}]^T
 \end{aligned} \tag{3.26}$$

3.4.1 Two dimensional differentiation matrices

The two dimensional differentiation matrices can be formed from the 1D differential operator placing every coefficient in its respective row and column, which is the same that to use the Kronecker tensor product ($\otimes$). The Kronecker product of two matrices A and B of dimension $p \times q$ and $r \times s$ respectively is denoted by $A \otimes B$ of dimension $pr \times qs$, and it is exposed in Eq. (3.27) for two 2×2 matrices. See Trefethen [136] for more details.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{cc|cc} \alpha a & \alpha b & \beta a & \beta b \\ \alpha c & \alpha d & \beta c & \beta d \\ \gamma a & \gamma b & \delta a & \delta b \\ \gamma c & \gamma d & \delta c & \delta d \end{array} \right) \tag{3.27}$$

The differentiation matrices in a two dimensional problem, where a vector is written as:

$$\begin{aligned}
 \mathbf{q} = [q(x_0, y_0), q(x_0, y_1), \dots, q(x_0, y_N), \\
 q(x_1, y_0), q(x_1, y_1), \dots, q(x_1, y_N) \\
 \dots \\
 q(x_N, y_0), q(x_N, y_1), \dots, q(x_N, y_N)]^T
 \end{aligned} \tag{3.28}$$

are defined as:

$$\mathcal{D}_x = \mathcal{D} \otimes I =$$

$$= \begin{pmatrix} \begin{array}{ccc|ccc} d_{0,0} & & & & & \\ & d_{0,0} & & & & \\ & & \ddots & & & \\ & & & d_{0,0} & & \\ \hline d_{1,0} & & & & & \\ & d_{1,0} & & & & \\ & & \ddots & & & \\ & & & d_{1,0} & & \\ \hline & & & & & \\ & & & & \ddots & \\ & & & & & \\ \hline d_{N,0} & & & & & \\ & d_{N,0} & & & & \\ & & \ddots & & & \\ & & & d_{N,0} & & \end{array} & \begin{array}{ccc|ccc} d_{0,1} & & & & & \\ & d_{0,1} & & & & \\ & & \ddots & & & \\ & & & d_{0,1} & & \\ \hline d_{1,1} & & & & & \\ & d_{1,1} & & & & \\ & & \ddots & & & \\ & & & d_{1,1} & & \\ \hline & & & & & \\ & & & & \ddots & \\ & & & & & \\ \hline d_{N,1} & & & & & \\ & d_{N,1} & & & & \\ & & \ddots & & & \\ & & & d_{N,1} & & \end{array} & & \begin{array}{ccc|ccc} d_{0,N} & & & & & \\ & d_{0,N} & & & & \\ & & \ddots & & & \\ & & & d_{0,N} & & \\ \hline d_{1,N} & & & & & \\ & d_{1,N} & & & & \\ & & \ddots & & & \\ & & & d_{1,N} & & \\ \hline & & & & & \\ & & & & \ddots & \\ & & & & & \\ \hline d_{N,N} & & & & & \\ & d_{N,N} & & & & \\ & & \ddots & & & \\ & & & d_{N,N} & & \end{array} \end{pmatrix}$$

$$\mathcal{D}_y = I \otimes \mathcal{D} =$$

$$= \begin{pmatrix} \begin{array}{cccc|ccc} d_{0,0} & d_{0,1} & \dots & d_{0,N} & & & \\ d_{1,0} & d_{1,1} & \dots & d_{1,N} & & & \\ & & \ddots & & & & \\ d_{N,0} & d_{N,1} & \dots & d_{N,N} & & & \\ \hline & & & & d_{0,0} & d_{0,1} & \dots & d_{0,N} \\ & & & & d_{1,0} & d_{1,1} & \dots & d_{1,N} \\ & & & & & & \ddots & \\ & & & & d_{N,0} & d_{N,1} & \dots & d_{N,N} \\ \hline & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & \\ \hline & & & & & & & \\ & & & & & & d_{0,0} & d_{0,1} & \dots & d_{0,N} \\ & & & & & & d_{1,0} & d_{1,1} & \dots & d_{1,N} \\ & & & & & & & & \ddots & \\ & & & & & & d_{N,0} & d_{N,1} & \dots & d_{N,N} \end{array} \end{pmatrix}$$

where I is the $N \times N$ identity matrix.

Second order derivative matrices are built using the same technique $\mathcal{D}_{xx} = \mathcal{D}^2 \otimes I$, $\mathcal{D}_{yy} = I \otimes \mathcal{D}^2$, and $\mathcal{D}_{xy} = (I \otimes \mathcal{D})(\mathcal{D} \otimes I)$.

3.4.2 Three dimensional differentiation matrices

The differentiation matrices in a tridimensional problem are:

$$\mathcal{D}_x = \mathcal{D} \otimes I \otimes I \quad (3.29)$$

$$\mathcal{D}_y = I \otimes \mathcal{D} \otimes I \quad (3.30)$$

$$\mathcal{D}_z = I \otimes I \otimes \mathcal{D} \quad (3.31)$$

where I is the $N \times N$ identity matrix.

Second order derivative matrices are built using the same technique.

3.5 Point vortex implementation

The essence in the implementation of a 2D point vortex code is to solve Eq. (2.25). That is done by one of the schemes exposed in section 3.1. However, there are some aspects that have to be taken into account: the initial distribution of the point vortices and the procedure to obtain results that can be easily analysed. That is what this section explains. The complication in the point vortex initialization comes when a distribution of point vortices is used to represent a patch of vorticity. The process to go from a vorticity field to a initial distribution of point vortex coordinates and circulations, is explained below. Later, when the time of the analysis of the solution comes, it is needed to present the results in an understandable way. In order to do so, apart from the visualization of the point vortex field, some integral quantities may be very useful. Their meaning and the equations used to obtain them are also presented in this section.

3.5.1 Initial configuration of point vortices

Three ways to select that initial configuration are followed:

- **Locate points.** The easiest way of choosing a initial configuration is to simply place points and circulations. This is preferred when the configuration to simulate can be modelled as points with vorticity concentrated. In the problem of the simulation of wake, if the size of the vortex is negligible compared to their distance, the vortices can be assumed to be points and this way of positioning may be used.
- **Radial placement.** This method consists in situating point vortices in concentric circles. It is very useful when the initial distribution is axisymmetric and given analytically. The advantage is that it maintains the axisymmetry of the configuration. In the process, the number of vortices in each circle and the difference in radius between each pair of concentric circles are the variables. Once the number of point vortices is selected and the total circulation of the vortex to discretise is calculated, the circulation of each point vortex can be deduced. The number of vortices of each circle can be selected, and then the circulation inside the circle obtained. This has to be matched with the integral of the analytical shape of the vortex, leading to a fix radius. This can be repeated for each annulus until the totality of the point vortices have been placed. Fig. (3.1) schematises the point vortex positions.
- **Grid.** One general procedure exists to convert any vorticity field to point vortices. A grid has to be generated and vorticity values assigned. The vorticity values of the nodes of the mesh can be integrated in each small grid polygon to calculate the

circulation. Positive and negative values of circulation are added independently, and then, once the number of point vortices is selected, the circulation of each point vortex is calculated. A simple scheme of a cell of the grid is shown in Fig. (3.2) to facilitate the understanding of the procedure. Two difficulties have to be mentioned:

- First, as the multiplication of the total number of points with the ratio between cell circulation and total circulation will not be a integer number, the total summatory of point vortices of all cells will not coincide with the total amount of point vortices selected, and an error will appear. Three means to overcome that problem have been considered. One is to assign or suppress the vortices left or in excess to the cells with bigger errors. Other is to leave the vortices as they stay, which will not match the total initial distribution. And the last is to readjust the individual point vortex circulations to recover the initial circulation. Any of them solves the problem completely. In general, increase the number of point vortices and reduce the mesh size improves the results concerning to that matter, but they should be done following a ratio between points and number of cells in order to achieve the best performance. There is another method that is to vary the individual circulations of the vortex of each cell independently, which will solve this problem, but will lead to a new one because point vortex circulations will not be equal and hence post-processing will be much more complicated.
- Second, placement of vortices inside each cell has to be done. The location of points inside a region is not a straightforward task. There are algorithms of positioning like these of Fang and Lin [45], but one must think very much what variables intends to minimise, as there are a wide variety of possibilities. The positioning algorithms are topic of full theses (see Nuyens [108]), so in this work, a faster approach is taken, the position of the vortices inside the grid cell are done using a random algorithm. In this case, the size of the grid is very important, because the smaller the grid, the smaller the random behaviour, and hence the more accurate that the process will be.

The advantage of this method is that it can be used for any vorticity distribution, but the disadvantage is that for good precision, a very fine mesh has to be used, and hence computational time and memory will increase.

Figure 3.1: Radial placement of point vortices.

Figure 3.2: Calculation of cell circulation.

Each of these ways to select initial condition is better depending on how the data is given. All but locate the points manually involve vorticity. It is then worth to mention the relation between vorticity and circulation, as it reveals Eq. (3.32).

$$\Gamma = \int_{\Sigma} \omega dA \quad (3.32)$$

3.5.2 Point vortex integral variables

After the iteration of the evolution problem, a set of points and circulations is obtained. The simplest way to show them is just to plot the points in a 2D plane in two different colors if they are positive or negative. This method is not very good when point vortices of different amount of circulation are selected and that is why point vortices of the same circulation are chosen. In some circumstances, a field of velocities or vorticity is desired. In this case, the fact that the point vortices have or not the same circulation is irrelevant. In such a case, an inverse procedure to the one done to select the initial configuration when it is given by a grid has to be followed. However, not only field variables are interesting, also some integral variables give very important information. Some of these integral values are the invariants defined in subsection 2.2.1. These quantities are explained below.

- **Circulation.** The first integral variable to define is the circulation, Γ . It can be calculated just adding the circulations of all the point vortices, Eq. (3.33). In some cases this total circulation is zero, and it is more interesting to add separately positive and negative values, which will give more relevant information.

$$\Gamma = \sum_{i=1}^N \Gamma_i \quad (3.33)$$

- **Centroid.** The centroid is the first order moment of the circulation, divided by the total circulation. It is calculated as Eq. (3.34) and as Eq. (3.35) for x and y directions respectively, and Eq. (3.36) in vectorial form. It can be physically seen as the center of rotation of the whole of the vortices. If only one real vortex is calculated, the centroid will be the actual center of the vortex. When the total circulation is zero, its value is ∞ and it is interesting to take half of the domain, if it has some symmetry, to have a net nonzero circulation.

$$x_c = \sum_{i=1}^N \frac{\Gamma_i x_i}{\Gamma} \quad (3.34)$$

$$y_c = \sum_{i=1}^N \frac{\Gamma_i y_i}{\Gamma} \quad (3.35)$$

$$\mathbf{x}_c = \sum_{i=1}^N \frac{\Gamma_i \mathbf{x}_i}{\Gamma} \quad (3.36)$$

- **Second order moment.** The second order moment of the circulation is calculated by means of Eqs. (3.37, 3.38) for x and y directions respectively, and Eq. (3.39) gives its total value.

$$J_x = \sum_{i=1}^N \Gamma_i (x_i - x_c)^2 \quad (3.37)$$

$$J_y = \sum_{i=1}^N \Gamma_i (y_i - y_c)^2 \quad (3.38)$$

$$J = J_x + J_y \quad (3.39)$$

- **Full vortex radius.** This magnitude is a different way to plot the previous one. Eqs. (3.40, 3.41) show its formulation for x and y directions respectively, and Eq. (3.42) gives its total value. For a single vortex, it represents its physical radius, both in x and y directions, as well as the total magnitude.

$$a_x = \sqrt{\frac{J_x}{\Gamma}} \quad (3.40)$$

$$a_y = \sqrt{\frac{J_y}{\Gamma}} \quad (3.41)$$

$$a = \sqrt{a_x^2 + a_y^2} \quad (3.42)$$

- **Hamiltonian.** The Hamiltonian should be invariant. Its expression for N point vortices follows in Eq. (3.43), taken from Batchelor [12] and also found in many other references as Marchioro and Pulvirenti [102] and Shashikanth [123].

$$H = -\frac{1}{4\pi} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \Gamma_i \Gamma_j \ln r_{ij} \quad (3.43)$$

3.5.3 Symmetries and periodicity

The general algorithms for vortex methods are valid for unbounded flows, so they do not need boundary conditions. However, it is possible to add some kind of conditions as symmetry or periodicity for particular problems.

Symmetry

For point vortices, adding symmetry is like imposing a wall, because the wall is dealt with as a reflecting plane by the images method. The usage of this methodology is very recommendable, as reduces the number of points to be considered by a half, so the memory is reduced by the same amount. However, the speed of a general vortex method goes as N^2 , but in this case, even if the number of vortices considered is divided by two, the computational time is not reduced by four, as the number of operations to calculate the induced velocity over one point depends on $2N$ vortices, but has to be done only for N

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vortices. Therefore, an original problem with N vortices that took time proportional to N^2 will be reduced to $N/2$ vortices, and the computational time will be proportional to $N^2/2$, but not to $N^2/4$ as one could expect when using symmetry. If N is chosen as the number of points at one side of the symmetry plane, then the computational time will be proportional to $2N^2$, while the number of points without using symmetry conditions would be $2N$ and the number of operations proportional to $4N^2$.

The equation that governs the motion of free point vortices Eq. (2.25) is modified to impose symmetry along the y axis, and only positive x is then considered:

$$\frac{d\mathbf{x}_i}{dt} = \frac{1}{2\pi} \sum_{j=1, i \neq j}^N \Gamma_j \frac{[y_j - y_i, x_i - x_j]}{(x_i - x_j)^2 + (y_i - y_j)^2} - \frac{1}{2\pi} \sum_{j=1}^N \Gamma_j \frac{[y_j - y_i, x_i + x_j]}{(x_i + x_j)^2 + (y_i - y_j)^2} \quad (3.44)$$

Periodicity

Periodicity could be seen as an infinite sum of vortices located at both sides of the domain. Fortunately, the infinite sum could be reduced to a few, as the induced velocity decays as the inverse of the distance. The idea is then to introduce vortices outside the periodic domain in Eq. (2.25) in order to have periodic boundary conditions. These vortices induce a velocity, but their positions are already defined. If the periodic domain where vortices locate initially is $0 \leq x < l$, the formula is modified as follows:

$$\frac{d\mathbf{x}_i}{dt} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{j=1, i \neq j}^N \Gamma_j \frac{[y_j - y_i, x_i - (x_j + kl)]}{[x_i - (x_j + kl)]^2 + (y_i - y_j)^2}, \quad (3.45)$$

where the first summation is not strictly going from $k = -\infty$ to $k = \infty$, but instead performed while the whole number inside the summation is below the solution required precision. For $k = 0$ the summation is done for the points inside the domain. The number of stored points will then be N , but the number of operations, and then the computational time, will be proportional to $(2k_{max} + 1)N^2$.

Figure 3.3: Periodic array of vortices, where number k of the period is shown.

Following this approach, the points may leave the domain, and then, when one point leaves from the right (left), it should enter from the left (right), which is done simply by subtracting (adding) the value l to the calculated value of x_i .

This is not the only choice to calculate periodic domains, other more fancy solutions are available, as that of Crowdy [36]. Despite their simplicity, these type of solutions are: first more efficient, as they only need a number of operations proportional to N^2 , the

same as for the problem without periodicity, and then more accurate, as they match the solution for $k_{max} \rightarrow \infty$.

The closed form expression given by Lamb [87] and already used by Suryanarayanan et al. [127] for the point vortex problem with periodic conditions can be written as:

$$\frac{d\mathbf{x}_i}{dt} = \frac{1}{2l} \sum_{j=1, i \neq j}^N \Gamma_j \frac{\left\{ -\sinh \left[\frac{2\pi(y_i - y_j)}{l} \right], \sin \left[\frac{2\pi(x_i - x_j)}{l} \right] \right\}}{\cosh \left[\frac{2\pi(y_i - y_j)}{l} \right] - \cos \left[\frac{2\pi(x_i - x_j)}{l} \right]}, \quad (3.46)$$

3.5.4 Speed acceleration techniques

The usage of symmetry and the selection of Eq. (3.46) instead of Eq. (3.45) could be both considered speed acceleration techniques, as they reduce the computational time of computation. However, all of them have operations proportional to N^2 , only changing the multiplication factor.

Nevertheless, there are techniques, as that of Koumoutsakos [84], which make the number of operations proportional to N instead of N^2 . This is a great improvement, in particular when the computational time is big, usually for large number of particles. These have been explored, but not implemented, as for two dimensional point vortex simulations of the order of 1000 vortices, the computational time on a standard computer is affordable. For simulations of more than around 10000 vortices this techniques are essential, but such resolution of point vortices is not really needed here.

3.6 Eigenvalue computation

The purpose of this section is to show the approaches taken to solve the eigenvalue problems exposed in the different sections of Chapter 2. The eigenvalue problem is defined again, and then, two alternatives are proposed. The first of them, only valid for small problems or very big workstations is to obtain all the eigenvalues of the matrix by mean of some standard algebra package. The second, more interesting for big problems, only gives a few eigenvalues and eigenvectors, but makes the problem feasible with reasonable computational resources.

3.6.1 Matrix formulation and boundary conditions

Since matrix forming and storing approach is followed in this work, the next Generalized EigenValue Problem (GEVP) will be solved for every stability problem,

$$\mathcal{A} \cdot \hat{\mathbf{q}} = \omega \mathcal{B} \cdot \hat{\mathbf{q}}, \quad (3.47)$$

where $\hat{\mathbf{q}} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}]^T$ and the matrix operators $\mathcal{A}$ and $\mathcal{B}$ contain the discretised linearised Navier-Stokes equation, Eqs. (2.36) with $\mathcal{B}$ being singular due to the continuity equation. That is valid for modal linear stability. For stability by vortex methods, the matrices $\mathcal{A}$ and $\mathcal{B}$, will have a different form, and $\hat{\mathbf{q}} = [\hat{x}_p, \hat{y}_p]^T$, but the eigenvalue problem form will

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be equivalent and everything said here, except for the boundary conditions, is valid for both cases. The sought complex eigenvalue is $\omega = (\omega_r + i\omega_i)$, the real part being a circular frequency, while the imaginary part is the temporal amplification/damping rate.

The elliptic eigenvalue problem, Eq. (3.47), must be complemented with adequate boundary conditions for the disturbance variables. Different boundary conditions have to be imposed in these matrices. The procedure is to change one row of each matrix for each of the conditions. For instance, to impose Dirichlet boundary condition for $\hat{u}(x_i) = 0$, the same row of both $\mathcal{A}$ and $\mathcal{B}$ is set to zero and only the column i of the row of $\mathcal{A}$ that is set to zero is changed to 1. Doing that, what is actually done is to change one of the equations of the problem by the boundary condition. The same can be done for derivatives. If Newman boundary conditions for $\hat{u}'(x_i) = 0$ want to be imposed, the row of $\mathcal{B}$ is still set to zero, but the selected row of $\mathcal{A}$ is changed for the corresponding row of the derivative matrix. Imposed Boundary conditions for the disturbance pressure do not exist physically; instead on the boundaries, compatibility conditions are used derived from the Navier-Stokes equations at the boundary of the domain, see Theofilis et al. [134].

The above is valid for temporal problems, but also for spatial problems, where only the definition of the problem is different:

$$\mathcal{A} \cdot \hat{\mathbf{q}} = \alpha \mathcal{B} \cdot \hat{\mathbf{q}} \quad (3.48)$$

where $\hat{\mathbf{q}} = [\hat{u}, \hat{v}, \hat{w}, \hat{p}, \alpha \hat{u}, \alpha \hat{v}, \alpha \hat{w}]^T$ and the matrix operators $\mathcal{A}$ and $\mathcal{B}$ contain the discretised linearised Navier-Stokes for the spatial problem. Despite having the eigenvalue also inside the eigenvectors and a larger problem, the procedure for the solution is equivalent.

3.6.2 Direct numerical solution of the EVP

There are available a wide variety of libraries with numerical algorithms for the general solution of the eigenvalues and eigenvectors of the problem. *LAPACK* offers a reliable open source collection of subroutines for that purpose, as the ZGGEV, which computes the generalised eigenvalues and eigenvectors for a pair square complex nonsymmetric matrices. See *LAPACK Users' Guide* [3] for reference of the algorithm documentation.

It should be mentioned that this method is only valid for reasonably small matrices as it becomes very slow for very large matrices, and very demanding of memory. Its advantage is that it gives all the eigenvalues and eigenvectors, but usually not all of them are needed.

3.6.3 Arnoldi numerical solution of the EVP

From a linear stability analysis point of view, the most important eigenvalues are those closest to the axis $\omega_i = 0$ and this section exposes an iterative method that has been used for their determination. Specifically, the Arnoldi algorithm has been used in BiGlobal linear instability problems, see Theofilis [132]. Since the eigenvalues closest to the imaginary axis are sought, a simple transformation of the original eigenvalue problem, Eq. (3.47), is used in order to convert the original problem into one where the desired values have large

modules, which is usually called shift and invert:

$$\mu = \frac{1}{\omega - \sigma}, \quad \omega = \sigma + \frac{1}{\mu} \quad (3.49)$$

$$\mathcal{A}\Phi = \omega\mathcal{B}\Phi, \quad \mathcal{A}\Phi = \left(\sigma + \frac{1}{\mu}\right)\mathcal{B}\Phi, \quad \mu(\mathcal{A} - \sigma\mathcal{B})\Phi = \mathcal{B}\Phi \quad (3.50)$$

Note that the eigenvectors are not affected by this transformation. Specifically, the problem to solve is:

$$(\mathcal{A} - \sigma\mathcal{B})^{-1}\mathcal{B}\Phi = \mu\Phi, \quad (\mathcal{A} - \sigma\mathcal{B})^{-1}\mathcal{B} = \mathcal{C}, \quad \mathcal{C}\Phi = \mu\Phi \quad (3.51)$$

where Φ is the same vector as $\hat{\mathbf{q}}$, the vector of unknowns shown in Eq. (3.47). μ are the inverse of the ω desired eigenvalues. This transformation converts the original generalized EVP into a standard one. A finite but small (compared with the leading dimension of $\mathcal{A}$, $\mathcal{B}$) number of eigenvalues (equal to the Krylov subspace dimension) m is sought, which is obtained by application of the Arnoldi algorithm. Further details of the algorithm can be found in Saad [119].

3.7 EigenValue Decomposition algorithm

The basic idea of an EigenValue Decomposition algorithm (EVD) is to use the eigenvalue decomposition of a matrix in order to solve a problem of the form:

$$AX = B. \quad (3.52)$$

When the matrix A is decomposed, the following can be written:

$$A = M\lambda M^{-1}, \quad (3.53)$$

where M is a matrix which columns are the eigenvectors of A , while λ is a diagonal matrix formed by its eigenvalues. Eq. (3.53) is substituted into Eq. (3.52), and gives:

$$X = M\lambda^{-1}M^{-1}B. \quad (3.54)$$

It can be seen that this method is a straightforward way to calculate the problem defined in Eq. (3.52) for several different vectors B if the matrix A remains unchanged. In order to solve the problems, the eigenvalue decomposition of A has to be calculated only once. In addition, A usually represents a differentiation operator, which is constant for each problem. This is the reason to apply EVD in the DNS simulations.

The method can be generalized a further, for equations of the type of Eq. (3.55).

$$A_1X + XA_2 + cX = B \quad (3.55)$$

Two matrices exist in this case, A_1 and A_2 , being, M_1 and M_2 the matrices formed by the eigenvectors of A_1 and A_2 respectively, as well as λ_1 and λ_2 are the eigenvalues corresponding to each of them. Eq. (3.56) show their decomposition.

$$\begin{aligned} A_1 &= M_1\lambda_1M_1^{-1} \\ A_2 &= M_2\lambda_2M_2^{-1} \end{aligned} \quad (3.56)$$

Finally, Eqs. (3.56) are introduced into Eq. (3.55) to obtain Eq. (3.57):

$$M_1^{-1}XM_2 = (\lambda_1 + \lambda_2 + c)^{-1}M_1^{-1}BM_2 \quad (3.57)$$

Eq. (3.55) may be a representation of the Navier-Stokes equations, when the nonlinear term is evaluated with a multi-step algorithm from the previous fractional step. For more details see Appendix B of Theofilis [132], where also the efficiency improvement of the usage of EVD against standard direct methods is highlighted.

3.8 Downwash velocity axes change

Given a counter-rotating vortex pair or group of vortices with total circulation equal to zero, a downward induced velocity will appear, which produces a linear movement of the vortices in direction towards y negative if a x - y plane is considered and the right vortex has net positive circulation. In order to study the problem as steady, a velocity of the same value could be added going in the opposite direction. This is actually the same than performing a rotation of the axes:

$$\theta_d = \arctan(V_d) \quad (3.58)$$

The rotation of an angle θ_d is equivalent to a change of frame of reference. The changes needed to add the descent speed ($V_d > 0$ for descent), are the following:

$$t' = t, \quad x' = x, \quad y' = y + V_d t, \quad z' = z, \quad (3.59)$$

Where the $'$ variables are the computational variables, in which the vortices remain static, and the other are the static frame of reference, where the vortices are going down. These changes have to be converted to changes in the derivatives:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x'}, & \frac{\partial}{\partial y} &= \frac{\partial}{\partial y'}, & \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} \\ \frac{\partial}{\partial t} &= \frac{\partial y'}{\partial t} \frac{\partial}{\partial y'} + \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} = V_d \frac{\partial}{\partial y'} + \frac{\partial}{\partial t'} \end{aligned} \quad (3.60)$$

Therefore, $V_d D_y$ has to be added in the positions xx , yy , zz (diagonal) of the LHS of the eigenvalue problem. Actually, that is the same that adding V_d to the basic flow.

3.9 Integral variables 2D grid

In a similar fashion to the point vortex method, the same integral variables could be calculated for 2D grid simulations. The summations there will be integrals here, which are actually performed by numerical integration. When there is a symmetry, and the total circulation is zero, it is more convenient to calculate these values for half plane. These integrals are defined as follows:

- **Circulation.** The first integral variable to define is the circulation, Γ . It is the surface integral of the vorticity. In some cases this total circulation is zero, and it is more interesting to add separately positive and negative values, which will give more relevant information.

$$\Gamma = \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} \omega(x, y) dx dy \quad (3.61)$$

- **Centroid.** The centroid is the first order moment of the circulation, divided by the total circulation. It is calculated as Eq. (3.62) and as Eq. (3.63) for x and y directions respectively. It is the physical center of the vortex if only one of them is considered, for a system of vortices is the common center of rotation of the whole system. When the total circulation is zero, its value is ∞ and it is interesting to take half of the domain, if it has some symmetry, to have a net nonzero circulation.

$$x_c = \frac{1}{\Gamma} \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} x \omega(x, y) dx dy \quad (3.62)$$

$$y_c = \frac{1}{\Gamma} \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} y \omega(x, y) dx dy \quad (3.63)$$

- **Second order moment.** The second order moment of the circulation is calculated by means of Eq. (3.64) and Eq. (3.65) for x and y directions respectively, and Eq. (3.66) gives its total value.

$$J_x = \frac{1}{\Gamma} \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} (x - x_c)^2 \omega(x, y) dx dy \quad (3.64)$$

$$J_y = \frac{1}{\Gamma} \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} (y - y_c)^2 \omega(x, y) dx dy \quad (3.65)$$

$$J = J_x + J_y \quad (3.66)$$

- **Full vortex radius.** The vortex radius is the root of the second order moment. Is more commonly used because it is a more physical parameter. For a single vortex, it represents its physical radius, both in x and y directions, as well as the total magnitude.

$$a_x = \sqrt{\frac{J_x}{\Gamma}} \quad (3.67)$$

$$a_y = \sqrt{\frac{J_y}{\Gamma}} \quad (3.68)$$

$$a = \sqrt{a_x^2 + a_y^2} \quad (3.69)$$

- **Hamiltonian.** The Hamiltonian is not as commonly used for DNS as it is for vortex methods. Furthermore, as its calculation involves a double integral, it is very computationally costly, and is usually not calculated. Its expression is given for completeness, taken from Batchelor [12].

$$H = -\frac{1}{8\pi} \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} \left\{ \int_{y_{min}}^{y_{max}} \int_{x_{min}}^{x_{max}} \omega(x, y) \omega(x', y') \ln [(x - x')^2 + (y - y')^2] dx' dy' \right\} dx dy \quad (3.70)$$

Chapter 4

Vortex Recognition

In order to understand the flow and the different structures that appear in it, its topology needs to be studied. As part of this analysis, it is needed to recognise vortices, which are quite relevant structures in the flow. Their identification is based on the different vortex definitions exposed in subsection 1.1.1 of Chapter 1. This chapter offers comparisons of results obtained by some of these vortex definitions.

The calculus performed to identify the regions that can be considered vortices is based on a few definitions of flow variables, which are collected here as an exposition of the general methodology. Later, the particular values of the flow variables are substituted inside these equations for each specific case. For that purpose, the velocity gradient tensor ($\nabla \mathbf{U}$), as well as its symmetric and antisymmetric parts, the vorticity ($\mathbf{\Omega}$) and strain ($\mathbf{S}$) tensors are defined here:

$$\nabla \mathbf{U} = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{\partial U_x}{\partial y} & \frac{\partial U_x}{\partial z} \\ \frac{\partial U_y}{\partial x} & \frac{\partial U_y}{\partial y} & \frac{\partial U_y}{\partial z} \\ \frac{\partial U_z}{\partial x} & \frac{\partial U_z}{\partial y} & \frac{\partial U_z}{\partial z} \end{pmatrix} \quad (4.1a)$$

$$\mathbf{S} = \begin{pmatrix} \frac{\partial U_x}{\partial x} & \frac{1}{2} \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) & \frac{\partial U_y}{\partial y} & \frac{1}{2} \left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y} \right) & \frac{\partial U_z}{\partial z} \end{pmatrix} \quad (4.1b)$$

$$\mathbf{\Omega} = \begin{pmatrix} 0 & \frac{1}{2} \left(\frac{\partial U_x}{\partial y} - \frac{\partial U_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \right) \\ -\frac{1}{2} \left(\frac{\partial U_x}{\partial y} - \frac{\partial U_y}{\partial x} \right) & 0 & \frac{1}{2} \left(\frac{\partial U_y}{\partial z} - \frac{\partial U_z}{\partial y} \right) \\ -\frac{1}{2} \left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \right) & -\frac{1}{2} \left(\frac{\partial U_y}{\partial z} - \frac{\partial U_z}{\partial y} \right) & 0 \end{pmatrix} \quad (4.1c)$$

The square modules of these tensors are defined by $|\mathbf{S}|^2 = |S_{ij}S_{ij}|$ and $|\mathbf{\Omega}|^2 = |\Omega_{ij}\Omega_{ij}|$, where the Einstein summation convention is used. And the Q criterion by Dallmann [38] will then define a vortex if $Q = [|\mathbf{\Omega}|^2 - |\mathbf{S}|^2]/2 > 0$.

The parameter $R = \det \nabla \mathbf{U}$ is also defined, in order to follow the Δ definition proposed

by Chong et al. [30], which defined a vortex in the region $\Delta = (Q/3)^3 + (R/2)^2 > 0$.

$$R = \det \nabla \mathbf{U} = \begin{vmatrix} \frac{\partial U_x}{\partial x} & \frac{\partial U_x}{\partial y} & \frac{\partial U_x}{\partial z} \\ \frac{\partial U_y}{\partial x} & \frac{\partial U_y}{\partial y} & \frac{\partial U_y}{\partial z} \\ \frac{\partial U_z}{\partial x} & \frac{\partial U_z}{\partial y} & \frac{\partial U_z}{\partial z} \end{vmatrix} = \frac{\partial U_x}{\partial x} \left(\frac{\partial U_y}{\partial y} \frac{\partial U_z}{\partial z} - \frac{\partial U_y}{\partial z} \frac{\partial U_z}{\partial y} \right) - \frac{\partial U_x}{\partial y} \left(\frac{\partial U_y}{\partial x} \frac{\partial U_z}{\partial z} - \frac{\partial U_y}{\partial z} \frac{\partial U_z}{\partial x} \right) + \frac{\partial U_x}{\partial z} \left(\frac{\partial U_y}{\partial x} \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial y} \frac{\partial U_z}{\partial x} \right) \quad (4.2)$$

In order to obtain a vortex using the λ_2 criterion by Jeong and Hussain [75], the sign of the second eigenvalue of the matrix $\mathbf{M}_{\lambda_2} = \mathbf{S}^2 + \mathbf{\Omega}^2$ must be negative. Therefore, this matrix needs to be evaluated, which, written in index form is: $M_{\lambda_2 ij} = S_{ik}S_{kj} + \Omega_{ik}\Omega_{kj}$.

Finally, the $\mathbf{M}_Z$ criterion by Haller [58, 59] is analysed. For that purpose, the tensor $\mathbf{M} = \partial_t \mathbf{S} + \mathbf{U} \cdot \nabla \mathbf{S} + \mathbf{S}(\nabla \mathbf{U}) + (\nabla \mathbf{U})^T \mathbf{S}$ is defined. Again, it is written in index form to leave it more clear:

$$M_{ij} = \frac{\partial S_{ij}}{\partial t} + U_k \frac{\partial S_{ij}}{\partial x_k} + S_{ik} \frac{\partial U_k}{\partial x_j} + \frac{\partial U_k}{\partial x_i} S_{kj} \quad (4.3)$$

In addition, Z must also be defined. It is a zero strain region such as $Z = \{\xi \mid \langle \xi, \mathbf{S}\xi \rangle = 0\}$. This is the same as calculating ξ such as they follow the equation $\xi_i S_{ij} \xi_j = 0$:

$$S_{xx}\xi_x^2 + S_{yy}\xi_y^2 + S_{zz}\xi_z^2 + (S_{xy} + S_{yx})\xi_x\xi_y + (S_{xz} + S_{zx})\xi_x\xi_z + (S_{yz} + S_{zy})\xi_y\xi_z = 0 \quad (4.4)$$

These ξ are actually the eigenvectors of $\mathbf{S}$. In a two-dimensional case, they correspond to two sets of perpendicular directions, $\xi^\pm$, and the vortex is in the region where $\mathbf{M}_Z$ is indefinite: $\langle \xi^-, \mathbf{M}\xi^- \rangle \langle \xi^+, \mathbf{M}\xi^+ \rangle < 0$.

4.1 Rankine vortex

The Rankine vortex, whose velocity components are given by Eqs. (1.19), is defined by a region of constant vorticity inside its radius a , surrounded by irrotational flow. No matter the vortex definition used, the result should be the same, that there is a vortex for $r < a$. It is then a basic validation for the different vortex definitions.

In order to proceed with the Q criterion, the velocity gradient ($\nabla \mathbf{U}$), the vorticity ($\mathbf{\Omega}$) and strain ($\mathbf{S}$) tensors are evaluated for the Rankine vortex:

$$\nabla \mathbf{U} = \begin{cases} \begin{pmatrix} 0 & -\frac{q}{a^2} & 0 \\ \frac{q}{a^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} \frac{2qxy}{r^4} & \frac{q(y^2-x^2)}{r^4} & 0 \\ \frac{q(y^2-x^2)}{r^4} & -\frac{2qxy}{r^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.5a)$$

$$\mathbf{S} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} \frac{2qxy}{r^4} & \frac{q(y^2-x^2)}{r^4} & 0 \\ \frac{q(y^2-x^2)}{r^4} & -\frac{2qxy}{r^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.5b)$$

$$\mathbf{\Omega} = \begin{cases} \begin{pmatrix} 0 & -\frac{q}{a^2} & 0 \\ \frac{q}{a^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.5c)$$

These are very enlightening results, as it is clearly shown how the strain is zero inside the vortex, and different from zero outside, while the opposite occurs for the vorticity. Then, $|\mathbf{S}|^2 = 0$ for $r \leq a$ and $|\mathbf{S}|^2 = 2q^2/r^4$ for $r > a$, while $|\mathbf{\Omega}|^2 = 2q^2/a^4$ for $r \leq a$ and $|\mathbf{\Omega}|^2 = 0$ for $r > a$. The Q criterion is then:

$$\begin{aligned} Q = q^2/a^4 > 0 &\Rightarrow \text{vortex for } (r \leq a) \\ Q = -q^2/r^4 < 0 &\Rightarrow \text{no vortex for } (r > a) \end{aligned} \quad (4.6)$$

The Δ criterion is now evaluated. For that, R needs to be calculated. Note that, as the flow is 2D, the determinant of $\nabla \mathbf{U}$ will be zero, resulting $\Delta = q^6/(9a^{12})$ for $r \leq a$, $\Delta = -q^6/(9r^{12})$ for $r > a$, and both criteria produces the same result.

The matrix $\mathbf{S}^2 + \mathbf{\Omega}^2$, needed to evaluate the vortex by the λ_2 criterion for the Rankine vortex, is as follows:

$$\mathbf{S}^2 + \mathbf{\Omega}^2 = \begin{cases} \begin{pmatrix} -\frac{q^2}{a^4} & 0 & 0 \\ 0 & -\frac{q^2}{a^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} \frac{q^2}{r^4} & 0 & 0 \\ 0 & \frac{q^2}{r^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.7)$$

The eigenvalues are then:

$$\begin{aligned} &\left. \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= -\frac{q^2}{a^4} \\ \lambda_3 &= -\frac{q^2}{a^4} \end{aligned} \right\} & (r \leq a) \\ &\left. \begin{aligned} \lambda_1 &= \frac{q^2}{r^4} \\ \lambda_2 &= \frac{q^2}{r^4} \\ \lambda_3 &= 0 \end{aligned} \right\} & (r > a) \end{aligned} \quad (4.8)$$

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where it is observed that λ_2 is negative if $r \leq a$ and there is a vortex in this region, while it is positive if $r > a$ and there is no vortex.

Finally, the $\mathbf{M}_Z$ criterion is evaluated. For that, the matrix $\mathbf{M}$ should be defined for the Rankine vortex:

$$\frac{\partial \mathbf{S}}{\partial t} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.9a)$$

$$\mathbf{U} \cdot \nabla \mathbf{S} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} -\frac{2q^2(y^2-x^2)}{r^6} & \frac{4q^2xy}{r^6} & 0 \\ \frac{4q^2xy}{r^6} & \frac{2q^2(y^2-x^2)}{r^6} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.9b)$$

$$\mathbf{S}(\nabla \mathbf{U}) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} \frac{q^2}{r^4} & 0 & 0 \\ 0 & \frac{q^2}{r^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.9c)$$

$$(\nabla \mathbf{U})^T \mathbf{S} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} \frac{q^2}{r^4} & 0 & 0 \\ 0 & \frac{q^2}{r^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.9d)$$

$$\mathbf{M} = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r \leq a) \\ \begin{pmatrix} \frac{4q^2x^2}{r^6} & \frac{4q^2xy}{r^6} & 0 \\ \frac{4q^2xy}{r^6} & \frac{4q^2y^2}{r^6} & 0 \\ 0 & 0 & 0 \end{pmatrix} & (r > a) \end{cases} \quad (4.9e)$$

For the region $r \leq a$, $\mathbf{S} = 0$ and $\xi^\pm$ could be chosen arbitrarily. As in this region, also $\mathbf{M} = 0$, no matter which directions are taken but $\mathbf{M}_Z = 0$. It is in the limit of being indefinite, but it is actually zero. In the outer region, the principal directions of the strain

tensor are $\xi^+ = (x, y)$ and $\xi^- = (y, -x)$, then:

$$\begin{aligned} \langle \xi^-, \mathbf{M} \xi^- \rangle \langle \xi^+, \mathbf{M} \xi^+ \rangle &= \left(\frac{4q^2 x^2 y^2}{r^6} - \frac{4q^2 x^2 y^2}{r^6} - \frac{4q^2 x^2 y^2}{r^6} + \frac{4q^2 x^2 y^2}{r^6} \right) \\ &\quad \left(\frac{4q^2 x^4}{r^6} + \frac{4q^2 x^2 y^2}{r^6} + \frac{4q^2 x^2 y^2}{r^6} + \frac{4q^2 y^4}{r^6} \right) \\ &= (0) \left(\frac{4q^2}{r^2} \right) = 0 \end{aligned} \quad (4.10)$$

Strictly speaking, the $\mathbf{M}_Z$ criterion fails to identify the Rankine vortex as a vortex. The first thought will be to reject this criterion. However, if the aim of the criterion is reviewed, the vortex should be a vortex, no matter the frame of reference. For a Rankine vortex, if the frame of reference is rotated at the same angular velocity of the Rankine vortex, then the vortex disappears.

It is been observed that every criteria, except the $\mathbf{M}_Z$, coincide to locate the vortex for $r \leq a$. Even the more subjective criterion of defining a vortex as a region of concentrated vorticity, will also give the same result, as the vorticity is constant and concentrated inside $r \leq a$, while it is zero in the outer region.

Reached this point, it is clear that for simple two-dimensional flows, it is best to go for the simplest and easier to apply criterion, which will be simply to look to the vorticity field, or use the Q criterion, because all of them coincide, and forget about the complexity and more strict definition of the $\mathbf{M}_Z$ criterion. But nevertheless, this objective definition should not be forgotten, as maybe something that seems clear that is a vortex, as the Rankine vortex, may not be a vortex in an strict sense.

4.2 q -vortex

The velocity components of the q -vortex are given by Eqs. (1.15). This is a vortex model of Gaussian shape both for the vorticity and the axial flow. Before performing the analysis, it is expected to obtain a vortex for $r \approx a$.

The velocity gradient ($\nabla \mathbf{U}$), the vorticity ($\boldsymbol{\Omega}$) and strain ($\mathbf{S}$) tensors, evaluated for the q -vortex are:

$$\nabla \mathbf{U} = \begin{pmatrix} \frac{2qxy}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a} \right)^2} \right\} & \frac{q}{r^2} \left\{ \frac{y^2 - x^2}{r^2} \left[1 - e^{-\left(\frac{r}{a} \right)^2} \right] - \frac{2y^2}{a^2} e^{-\left(\frac{r}{a} \right)^2} \right\} & 0 \\ \frac{q}{r^2} \left\{ \frac{y^2 - x^2}{r^2} \left[1 - e^{-\left(\frac{r}{a} \right)^2} \right] + \frac{2x^2}{a^2} e^{-\left(\frac{r}{a} \right)^2} \right\} & -\frac{2qxy}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a} \right)^2} \right\} & 0 \\ \frac{2W_0 x}{a^2} e^{-\left(\frac{r}{a} \right)^2} & \frac{2W_0 y}{a^2} e^{-\left(\frac{r}{a} \right)^2} & 0 \end{pmatrix} \quad (4.11a)$$

$$\mathbf{S} = \begin{pmatrix} \frac{2qxy}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a} \right)^2} \right\} & \frac{q(y^2 - x^2)}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a} \right)^2} \right\} & \frac{W_0 x}{a^2} e^{-\left(\frac{r}{a} \right)^2} \\ \frac{q(y^2 - x^2)}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a} \right)^2} \right\} & -\frac{2qxy}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a} \right)^2} \right\} & \frac{W_0 y}{a^2} e^{-\left(\frac{r}{a} \right)^2} \\ \frac{W_0 x}{a^2} e^{-\left(\frac{r}{a} \right)^2} & \frac{W_0 y}{a^2} e^{-\left(\frac{r}{a} \right)^2} & 0 \end{pmatrix} \quad (4.11b)$$

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$$\mathbf{\Omega} = \begin{pmatrix} 0 & -\frac{q}{a^2}e^{-\left(\frac{r}{a}\right)^2} & -\frac{W_0x}{a^2}e^{-\left(\frac{r}{a}\right)^2} \\ \frac{q}{a^2}e^{-\left(\frac{r}{a}\right)^2} & 0 & -\frac{W_0y}{a^2}e^{-\left(\frac{r}{a}\right)^2} \\ \frac{W_0x}{a^2}e^{-\left(\frac{r}{a}\right)^2} & \frac{W_0y}{a^2}e^{-\left(\frac{r}{a}\right)^2} & 0 \end{pmatrix} \quad (4.11c)$$

In order to evaluate the Q criterion, first $|\mathbf{S}|^2$ and $|\mathbf{\Omega}|^2$ are evaluated:

$$|\mathbf{S}|^2 = \frac{2q^2}{r^4} \left\{ 1 - \left[1 + \left(\frac{r}{a} \right)^2 \right] e^{-\left(\frac{r}{a}\right)^2} \right\}^2 + \frac{2W_0^2 r^2}{a^4} e^{-2\left(\frac{r}{a}\right)^2} \quad (4.12)$$

$$|\mathbf{\Omega}|^2 = \frac{2q^2}{a^4} e^{-2\left(\frac{r}{a}\right)^2} + \frac{2W_0^2 r^2}{a^4} e^{-2\left(\frac{r}{a}\right)^2} \quad (4.13)$$

The Q criterion is then:

$$Q = -\frac{q^2 e^{-2\left(\frac{r}{a}\right)^2}}{r^4} \left[e^{\left(\frac{r}{a}\right)^2} - 1 \right] \left[e^{\left(\frac{r}{a}\right)^2} - 1 - 2 \left(\frac{r}{a} \right)^2 \right] \quad (4.14)$$

Then, Q is positive if $r < r_Q$, being r_Q the solution of $e^{(r_Q/a)^2} - 1 - (r_Q/a)^2 = 0$, and negative if $r > r_Q$, and then, there is a vortex for $r < r_Q$. This value, r_Q cannot be obtained by analytical means, but the numerical solution is $r_Q \approx 1.1209a$.

For the Δ criterion, R must be calculated. However, again, as the flow is z independent, the determinant of $\nabla \mathbf{U}$ is zero, being:

$$\Delta = - \left[\frac{q^2 e^{-2\left(\frac{r}{a}\right)^2}}{27r^4} \right]^3 \left[e^{\left(\frac{r}{a}\right)^2} - 1 \right]^3 \left[e^{\left(\frac{r}{a}\right)^2} - 1 - 2 \left(\frac{r}{a} \right)^2 \right]^3 \quad (4.15)$$

As $\Delta = Q^3/3$ for this case, Q and Δ have the same sign for the same r and then the criteria produces the same result, there is a vortex for $r < r_Q$.

The λ_2 criterion is based on the eigenvalues of the matrix $\mathbf{S}^2 + \mathbf{\Omega}^2$, which for the q -vortex is:

$$\mathbf{S}^2 = \begin{pmatrix} \frac{q^2 f_1^2\left(\frac{r}{a}\right)}{r^4} + \frac{W_0^2 x^2 e^{-2\left(\frac{r}{a}\right)^2}}{a^4} & \frac{W_0^2 xy e^{-2\left(\frac{r}{a}\right)^2}}{a^4} & \frac{qW_0 y f_1\left(\frac{r}{a}\right) e^{-\left(\frac{r}{a}\right)^2}}{r^2 a^2} \\ \frac{W_0^2 xy e^{-2\left(\frac{r}{a}\right)^2}}{a^4} & \frac{q^2 f_1^2\left(\frac{r}{a}\right)}{r^4} + \frac{W_0^2 y^2 e^{-2\left(\frac{r}{a}\right)^2}}{a^4} & -\frac{qW_0 x f_1\left(\frac{r}{a}\right) e^{-\left(\frac{r}{a}\right)^2}}{r^2 a^2} \\ \frac{qW_0 y f_1\left(\frac{r}{a}\right) e^{-\left(\frac{r}{a}\right)^2}}{r^2 a^2} & -\frac{qW_0 x f_1\left(\frac{r}{a}\right) e^{-\left(\frac{r}{a}\right)^2}}{r^2 a^2} & \frac{W_0^2 r^2 e^{-2\left(\frac{r}{a}\right)^2}}{a^4} \end{pmatrix} \quad (4.16)$$

$$\mathbf{\Omega}^2 = \begin{pmatrix} -q^2 - W_0^2 x^2 & -W_0^2 xy & qW_0 y \\ -W_0^2 xy & -q^2 - W_0^2 y^2 & -qW_0 x \\ qW_0 y & -qW_0 x & -W_0^2 r^2 \end{pmatrix} \frac{e^{-2\left(\frac{r}{a}\right)^2}}{a^4} \quad (4.17)$$

$$\mathbf{S}^2 + \mathbf{\Omega}^2 = \begin{pmatrix} -Q & 0 & \frac{qW_0 y \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] e^{-\left(\frac{r}{a}\right)^2}}{a^2 r^2} \\ 0 & -Q & -\frac{qW_0 x \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] e^{-\left(\frac{r}{a}\right)^2}}{a^2 r^2} \\ \frac{qW_0 y \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] e^{-\left(\frac{r}{a}\right)^2}}{a^2 r^2} & -\frac{qW_0 x \left[1 - e^{-\left(\frac{r}{a}\right)^2} \right] e^{-\left(\frac{r}{a}\right)^2}}{a^2 r^2} & 0 \end{pmatrix} \quad (4.18)$$

where:

$$f_1\left(\frac{r}{a}\right) = 1 - \left[1 + \left(\frac{r}{a}\right)^2\right] e^{-\left(\frac{r}{a}\right)^2} \quad (4.19)$$

$$-Q = \frac{q^2 e^{-2\left(\frac{r}{a}\right)^2}}{r^4} \left[e^{\left(\frac{r}{a}\right)^2} - 1 \right] \left[e^{\left(\frac{r}{a}\right)^2} - 1 - 2 \left(\frac{r}{a}\right)^2 \right] \quad (4.20)$$

The eigenvalues are then:

$$\begin{aligned} \lambda_1 &= -Q + \sqrt{Q^2 + \frac{q^2 W_0^2 \left[1 - e^{-\left(\frac{r}{a}\right)^2}\right]^2 e^{-2\left(\frac{r}{a}\right)^2}}{a^4 r^2}} \\ \lambda_2 &= -Q \\ \lambda_3 &= -Q - \sqrt{Q^2 + \frac{q^2 W_0^2 \left[1 - e^{-\left(\frac{r}{a}\right)^2}\right]^2 e^{-2\left(\frac{r}{a}\right)^2}}{a^4 r^2}} \end{aligned} \quad (4.21)$$

where $\lambda_2(r/a)$ is a function that is negative if $Q > 0$, that is, Q and λ_2 criteria coincide for the q -vortex.

The analytical calculus of the vortex location by the $\mathbf{M}_Z$ criterion, becomes too tedious and it is not performed here. However, it is calculated numerically over a q -vortex defined on a grid. The result obtained is that there is a vortex for $r < r_{M_Z}$, being $r_{M_Z} \approx 2.2$, while the same methodology gives an approximation to r_Q of around 1.1, so the error is around 2% for Q and λ_2 criteria and it is assumable that it will be of the same order for M_Z . The result is not as precise as an analytical one, but will be good enough as an illustration that the region defined as a vortex is much bigger for the M_Z criterion, which a priori cannot be said to be more or less accurate. Figs. (4.1) show the values of the numerically obtained vortex recognition criteria over a q -vortex defined for an equispaced squared grid for $x = [-10 : 10]$, $y = [-10 : 10]$ and 500^2 grid points. Fig. (4.1(a)) show the general values in the interval r from 0 to 5, where the shapes of the Q and λ_2 are clear to be equal with different sign, while Fig. (4.1(b)) shows a scaling of the values to appreciate better their shapes, and finally Fig. (4.1(c)) shows only M_Z and its change of sign.

4.3 Sheared Taylor grid

The sheared Taylor grid is analysed here. It is defined as a two-dimensional array of vortices with shear:

$$\omega = 2 \sin(\pi x) \sin(\pi y) + 2c_s \cos(\pi x) \cos(\pi y) \quad (4.22)$$

When $c_s = 0$ the Taylor grid is recovered, and the greater this parameter is, the more important is the shear. When $c_s = 1$, the grid is completely sheared. The Taylor grid is shown in Fig. (4.2), as well as the results from the Q and M_Z criteria. It is observed that both give the same result. The shaded area is the negative value of the variable, then the

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Figure 4.1: Q , λ_2 and M_Z criteria for the q -vortex in the interval r from 0 to 5.

vortices are in the clear region for the Q criterion and in the shaded region for the M_Z criterion.

Then the M_Z criterion is applied to see where are the vortices for several values of shear, showing that the vortices are in the same place, Fig. (4.3).

4.4 Vortex recognition discussion

The M_Z criterion by Haller [58, 59] is clearly a stronger and more powerful tool to identify vortices, as it is not only Galilean invariant, but also valid for rotating frames of reference. However, for the flows discussed here, which are mostly two dimensional, all the criteria offer good results. In fact, for these flows, the vorticity has a single relevant component, and the plot of that component, may be enough to identify the vortices in these cases.

Figure 4.2: Taylor grid without shear.

Figure 4.3: Taylor grid with shear.

Chapter 5

Vortex Movement

This chapter is based on the methodologies of section 2.1 and section 2.2. Some of the results presented are trivial, and even obtainable analytically and are not completely new to the field, so they could be seen as validations of the methodologies. However, these results are important here, as they form a whole and provide with a good understanding of the different phenomena analysed in the variety of cases studied.

Apart from the validation of the simple cases, the methodologies used in this chapter, are also used to obtain basic flows (steady solutions of the equations), that are later analysed in Chapter 6. Actually, a few results at the end of the current chapter present some flow instabilities, but are kept in here because they are based on IVP as the rest of the results of this chapter.

One of the aims of this chapter is also to expose how the ideas presented in the previous chapters materialise in actual solutions. There is, in addition, a duality between the two different routes used to approach the study of vortex instability problems: vortex methods and classical grid methods. The comparison of these two methods, is the main concern of this thesis, and it acquires special relevance in this and the following chapter, because the actual comparison of results between the two approaches is achieved hereinafter. The similarities of results obtained by 2D DNS and point vortex methods are extraordinary, even for moderate Re numbers in the viscous DNS compared to the inviscid point vortex simulations.

5.1 Single vortex

One single vortex will cause a rotating motion around itself decreasing with the distance. The vortex centroid position (see subsection 3.5.2 and section 3.9) must be keep constant for a single vortex, no matter the model that is used. If the vortex is punctual, there is only one parameter in the problem, which is the circulation defined by Γ , and the vortex will not move.

5.1.1 Single point vortex analytical evolution

In the simple case of a single point vortex, the velocity of any point in the field is proportional to the vortex intensity Γ and inversely proportional to the distance to the point, defined as r :

$$U_{\theta} = \frac{\Gamma}{2\pi r} \tag{5.1}$$

A single point vortex does not impose any influence over itself, so it will be a steady point in space.

5.1.2 Single inviscid Gaussian vortex by point vortices

The idea of this section is to study a Gaussian vorticity distribution, Eqs. (1.16), by means of a set of point vortices. For the purpose of this analysis, the following variables are fixed: intensity of the vortex, $q_0 = 1/2$ and vortex radius, $a_0 = 1$. These together produce a total circulation $\Gamma_0 = \pi$. Given these values, the vortex definition should be unique. However, there is not a single way to distribute the point vortices in the domain to achieve this, in fact, the result is very dependent on the choice selected, which is really a problem of this methodology. Three approaches are taken: First, the point vortices are located by means of a grid of vorticity, for two different number of grid points, 100^2 and 500^2 , and the same domain, x and y from -10 to 10; and later, they are located radially in circles around the Gaussian vortex center. Details of the methodologies are given in subsection 3.5.1, note that all the methodologies analysed have a random component, which will produce deviations of the results, mainly for small number of points. The aspect of the three initial configurations is given for 10, 100, 1000 and 10000 point vortices in Figs. (5.1, 5.2). These figures show how the radial positioning is much more organised, but more artificial, while the grid positioning looks too irregular.

Tables (5.1-5.3) offer initial values of absolute value of radial centroid, error in total circulation, error in ratio between radius in y and x directions $E = a_y/a_x$, and error in vortex radius, for configurations of 10, 20, 50, 100, 200, 500, 1000, 2000, 5000, 10000, 20000 and 50000 point vortices for the same three approaches as in Figs. (5.1, 5.2). The centroid is pretty much well calculated in the radial positioning, due to the randomness that the grid positioning has, because of the grid size, and that is also the reason for having a better centroid in the grid with more points. The circulation is well calculated up to machine precision accuracy in all cases. However, it is noticeable that it is best calculated for a smaller number of vortices, and that is because the only error comes from the round off errors of the computer calculations, which accumulate for higher number of point vortices. The ratio between radius in y and x directions, E , has almost no error for the radial positioning, while the error for the grid positioning is very big for small number of point vortices and moderate for a big number of point vortices, and not too different for the two different grids. The error in the vortex radius, is quite big in all the cases, due to different reasons for the different approaches. In the grid positioning, there are two cumulative errors, first, that already mentioned, random location of the points inside the cell, produces that the bigger the cell is, the bigger the error is, but, in addition to this, the number of points is an integer, which will not match the vorticity of the cell, and therefore, the vorticity is moved from some regions to another. In the radial positioning, the reason is different, the algorithm is not optimised, and does not locate the vortex radius exactly where it should. Therefore, from the numbers seen, in general, radial positioning seems the best choice. However, when dealing with more vortices, it will be seen that the vortex radius is crucial, and therefore, grid positioning will be best in many cases, not to mention other cases where the initial distribution is not analytical.

Till this point, only the initial configuration is analysed. Now, the IVP simulations

Figure 5.1: Initial point vortex location by: *left*: grid location and 100^2 grid points, *center*: grid location and 500^2 grid points, and *right*: radial location. 10 and 100 point vortices. A square of size 2 centred at the origin of coordinates is shown to compare the vortex size. The point size is bigger for the figures with less number of points.

are performed. As the number of points and their circulations are not varied during the execution of the simulations, the total circulation remains constant. The values of the centroid are not plotted, as their error remains of the same order as it was in the initial state for each of the cases, very small for radial positioning and bigger for the grid positioning. The roundness of the vortex is studied by the parameter E . Simulations are performed for 500 and 1000 vortices and the three cases studied before. The selected time step is $\Delta t = 10^{-2}$ and final time of simulation 10. Fig. (5.3) shows the evolution of the parameter E in these cases. It is clear the high performance of the radial method, for very round and non oscillating vortices, while the other cases have some periodic oscillations in which the vortex becomes more elongated in the x and y directions repeatedly. The vortex radius also have some variations, it grows monotonically for the grid cases, and oscillates for the radial cases, but the variations are much smaller than the different initial values, and therefore, a visual comparison is not produced.

In addition, a study of the time step is performed for the radial positioning case, using 1000 point vortices. Two additional time steps, $\Delta t = 10^{-1}$ and $\Delta t = 10^{-3}$, are added to the previous, $\Delta t = 10^{-2}$. First, the comparison of the ratio horizontal and vertical vortex radii, E , Fig. (5.4) shows no difference for the three cases. Later, the comparison of the

Figure 5.2: Initial point vortex location by: *left*: grid location and 100^2 grid points, *center*: grid location and 500^2 grid points, and *right*: radial location. 1000 and 10000 point vortices. A square of size 2 centred at the origin of coordinates is shown to compare the vortex size. The point size is bigger for the figures with less number of points.

vortex radius, Fig. (5.5) shows noticeable differences, although of really small absolute value. The result is understandable, the bigger time step produce larger errors, that provoke that the point vortices do not follow the circular orbits that they should follow, instead, they tend to go straight, and therefore, the vortex radius increases slowly. The plot is clear, $\Delta t = 10^{-1}$ is too big, at least for long time evolution, while $\Delta t = 10^{-2}$ is enough. Due to the small differences, the variable plotted is the difference between the vortex radius and the vortex radius of the initial configuration. Doing so, the value plotted is small and the scale shows the difference.

5.1.3 Single viscous Gaussian vortex by 2D DNS

The Gaussian vortex is now analysed by a 2D DNS. The quality of the initial configuration is not a problem here, as it was for the point vortex simulations, because the initial vorticity distribution is directly imposed on the selected grid. However, despite this advantage for the initial configuration, the method is dependant of a few parameters that should be evaluated in order to validate the simulations and check their convergence.

Table 5.1: Parameters for the initial point vortex distribution using placement from a grid of size 100^2 and boundaries at distance 10 in x and y from the vortex centre.

N	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$E - 1$	$ (a - a_0)/a_0 $
10	2.035×10^{-2}	0	2.996×10^{-1}	6.962×10^{-1}
20	7.038×10^{-3}	0	9.503×10^{-3}	6.146×10^{-1}
50	3.125×10^{-2}	8.481×10^{-16}	8.951×10^{-3}	4.307×10^{-1}
100	2.145×10^{-2}	1.414×10^{-15}	9.205×10^{-3}	1.896×10^{-1}
200	5.328×10^{-3}	3.534×10^{-15}	2.181×10^{-3}	1.183×10^{-1}
500	1.137×10^{-3}	7.068×10^{-15}	8.157×10^{-3}	7.139×10^{-2}
1000	7.932×10^{-3}	1.795×10^{-14}	2.680×10^{-3}	2.312×10^{-2}
2000	2.555×10^{-3}	4.000×10^{-14}	1.221×10^{-3}	3.819×10^{-3}
5000	1.741×10^{-3}	4.820×10^{-14}	2.160×10^{-4}	6.822×10^{-3}
10000	2.616×10^{-4}	1.343×10^{-13}	1.058×10^{-3}	6.956×10^{-3}
20000	6.544×10^{-4}	3.363×10^{-13}	2.579×10^{-5}	1.075×10^{-2}
50000	3.483×10^{-4}	9.040×10^{-13}	6.261×10^{-4}	1.218×10^{-2}

In this case, different mappings could be used. The main idea is to cluster points near to the vortex centre, which is attempted via a tangent mapping (see section B.3), for different ratios of points clustering, compared to an equispaced grid. Table 5.4 is shown for that purpose. Surprisingly, the equispaced grid outperforms the tangent grid for all the clustering ratios. The explanation to the phenomena is that the code performs best when there is not stretching and the cells are all squared, but the exact reason of that is not known with certainty, it could be that the DNS algorithm is degrading its accuracy when the grid is not squared, or it may be that the mapping has some bug somewhere, which is downgrading the performance, being the effect augmented for stretched cells. The variables shown are the centroid distance to the original vortex centre, the error in the circulation compared to the initial circulation, and the difference between the vortex radius and the theoretical vortex radius, given by $a_{th} = a_0 \sqrt{1 + 4\nu t/a_0^2}$, see González et al. [54].

As the equispaced grid has the best performance, it will be selected from now on. The next parameter to evaluate is the domain size. Note that the DNS code implemented, is based on a Fourier discretisation, and therefore, is imposing periodic boundaries. Consequently, as the idea is to analyse an isolated vortex, the effect of the periodic boundaries must be negligible, which means, independent of the simulation domain. Table 5.5 shows very good results for all the domains, which means that even the smallest domain is good enough. Taking into account that the improvement of selecting the biggest domain, which has 100 times more points, is in the best scenario (error in vortex radius) only two times better, the smallest of the domains is selected.

The next parameter to analyse is the grid points density. The domain is fixed, $[-10 : 10]^2$, and the selection of the number of grid points for the simulation will be directly related with the grid spacing: 50^2 points give a $\Delta x = 0.4$, 100^2 give $\Delta x = 0.2$, 200^2 give $\Delta x = 0.1$ and 500^2 give $\Delta x = 0.04$. It is a general rule in Computational Fluid Dynamics (CFD) to keep the ratio between grid spacing and time step constant for validations, as

5. Vortex Movement

Table 5.2: Parameters for the initial point vortex distribution using placement from a grid of size 500^2 and boundaries at distance 10 in x and y from the vortex centre.

N	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$E - 1$	$ (a - a_0)/a_0 $
10	9.487×10^{-3}	5.654×10^{-16}	1.548×10^{-1}	9.459×10^{-1}
20	2.491×10^{-3}	2.827×10^{-16}	1.549×10^{-2}	9.231×10^{-1}
50	5.340×10^{-3}	2.827×10^{-16}	1.031×10^{-2}	8.857×10^{-1}
100	3.065×10^{-3}	1.979×10^{-15}	1.144×10^{-2}	8.406×10^{-1}
200	3.039×10^{-4}	3.817×10^{-15}	5.309×10^{-3}	7.724×10^{-1}
500	9.056×10^{-4}	4.241×10^{-15}	1.538×10^{-3}	6.432×10^{-1}
1000	5.985×10^{-4}	1.937×10^{-14}	1.014×10^{-3}	4.949×10^{-1}
2000	4.666×10^{-4}	4.071×10^{-14}	1.031×10^{-3}	2.862×10^{-1}
5000	6.618×10^{-5}	4.114×10^{-14}	1.111×10^{-4}	1.236×10^{-1}
10000	1.453×10^{-4}	1.308×10^{-13}	2.134×10^{-4}	7.317×10^{-2}
20000	1.714×10^{-4}	3.380×10^{-13}	4.906×10^{-5}	4.181×10^{-2}
50000	8.802×10^{-5}	9.040×10^{-13}	1.317×10^{-4}	1.928×10^{-2}

then the CFL¹ will be also constant. However, here, the CFL is much lower than 1, even for the smaller grid size, so the simulations are all performed for the same Δt . The rest of the parameters are keep equal to the previous simulation, and table 5.6 shows the results. An improvement is always seen for a reduction of the spacing, meaning that the CFL limit is not reached for the time step used. However, the previous improvement is quite small from $\Delta x = 0.2$ and smaller, and therefore, this spacing, which correspond to 100^2 points for the domain selected is chosen.

The last parameter to validate is the time step. Usually this parameter does not influence too much on the results, as long as it is small enough to keep the CFL in the safe margin. Using again a domain $[-10 : 10]^2$, 100^2 grid points, equispaced grid, and $Re = 100$. The results of using different time steps are collected in table 5.7 for final time $t = 1$. The result is as expected, as long as the simulation is working (CFL < 1), every time step gives quite similar results which respect to their accuracy.

All the numerical parameters have been analysed. However, there is also a very important physical parameter, which accounts for the ratio between the inertial and the viscous forces, the Reynolds number Re . In this case, it only accounts for diffusion of the vortex, so the bigger the Re , the longer the time that the vortex will need to diffuse. Table 5.8 collects results for the previously studied parameters and $Re = 10^1, 10^2, 10^3, 10^4$ and 10^5 . In the time interval analysed, the case $Re = 10^1$ experiments some important diffusion, while the case $Re = 10^2$ diffuses very little. The rest of the cases have almost no diffusion in such a short time. Maybe that is the reason to see that both for centroid position and error in circulation, the case $Re = 10^1$ suffers a two orders of magnitude degrading of performance, while the case $Re = 10^2$ suffers a one order of magnitude degradation, both compared to the rest of the cases. Results for the vortex radius compared to the theoretical are similar in all the cases.

¹CFL: Courant-Friedrichs-Lewy number = $\frac{\Delta t U}{\Delta x}$ must be smaller than one for numerical convergence, see Courant et al. [31].

Table 5.3: Parameters for the initial point vortex distribution using radial location of point vortices.

N	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$E - 1$	$ (a - a_0)/a_0 $
10	2.925×10^{-16}	0	2.000×10^{-16}	2.389×10^{-1}
20	1.104×10^{-16}	0	1.110×10^{-16}	2.027×10^{-1}
50	7.727×10^{-17}	8.481×10^{-16}	2.220×10^{-16}	1.371×10^{-1}
100	1.516×10^{-16}	1.414×10^{-15}	2.220×10^{-16}	1.321×10^{-1}
200	1.383×10^{-16}	3.534×10^{-15}	3.331×10^{-16}	1.006×10^{-1}
500	2.531×10^{-16}	7.068×10^{-15}	6.661×10^{-16}	1.228×10^{-1}
1000	4.414×10^{-16}	1.795×10^{-14}	4.000×10^{-16}	1.060×10^{-1}
2000	7.794×10^{-16}	4.000×10^{-14}	4.441×10^{-16}	5.966×10^{-2}
5000	5.796×10^{-16}	4.820×10^{-14}	1.221×10^{-15}	3.588×10^{-2}
10000	2.282×10^{-16}	1.343×10^{-13}	4.000×10^{-16}	3.860×10^{-2}
20000	8.892×10^{-16}	3.363×10^{-13}	9.000×10^{-16}	2.141×10^{-2}
50000	5.098×10^{-16}	9.040×10^{-13}	0	1.802×10^{-2}

Table 5.4: Mapping comparison for the 2D DNS simulation of a single Gaussian vortex at $Re = 100$, domain $[-10 : 10]^2$ and 100^2 number of points. $\Delta t = 0.01$, and final time of the simulation, at which parameters are shown, $t = 1$.

Mapping	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$ (a - a_{th})/a_{th} $
Tangent, half points at 0.5	4.183×10^{-5}	2.946×10^{-6}	5.949×10^{-5}
Tangent, half points at 1	4.297×10^{-6}	6.519×10^{-8}	3.656×10^{-6}
Tangent, half points at 2.5	2.746×10^{-7}	5.555×10^{-10}	1.024×10^{-7}
Tangent, half points at 5	3.762×10^{-8}	2.546×10^{-11}	1.018×10^{-8}
Equispaced	4.715×10^{-11}	6.966×10^{-12}	1.293×10^{-9}

It is interesting that, for this particular problem, Re is not a real parameter. If instead of t , already nondimensionalised by the length $a_0 = 1$ and the velocity obtained from $q = 1/2$, it is used t/Re , all the results for different Re coalesce. In particular, the vortex radius, which is increasing with time, will increase in the same rate for all the cases. Fig. (5.6) shows this behaviour, where the difference comes from the cumulative error of the different cases, because the higher the Re , the higher number of steps to reach the same t/Re at the same Δt .

Figs. (5.7) illustrate the vortex dissipation process. The illustrations have been obtained from the $Re = 10$ simulation, but the aspect of any other Re would have been the same, as long as the same t/Re is selected. In this process the peak of the Gaussian diminishes as the same time as the vortex radius increases.

Figure 5.3: Ratio between horizontal and vertical vortex radii point vortex simulations comparison for different initial configurations of a Gaussian vortex.

Table 5.5: Domain size comparison for the 2D DNS simulation of a single Gaussian vortex at $Re = 100$, equispaced grid of $\Delta x = 0.2$. $\Delta t = 0.01$, and final time of the simulation, at which parameters are shown, $t = 1$.

Domain size	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$ (a - a_{th})/a_{th} $
$[-10 : 10]^2$	4.715×10^{-11}	6.966×10^{-12}	1.293×10^{-9}
$[-20 : 20]^2$	8.214×10^{-11}	5.499×10^{-12}	1.073×10^{-9}
$[-50 : 50]^2$	3.161×10^{-10}	8.858×10^{-12}	7.730×10^{-9}
$[-100 : 100]^2$	2.846×10^{-10}	2.976×10^{-12}	5.739×10^{-10}

Figure 5.4: Ratio between horizontal and vertical vortex radii point vortex simulations comparison for different time steps.

Figure 5.5: Vortex radius point vortex simulations comparison for different time steps.

Table 5.6: Grid spacing comparison for the 2D DNS simulation of a single Gaussian vortex at $Re = 100$, and domain $[-10 : 10]^2$. $\Delta t = 0.01$, and final time of the simulation, at which parameters are shown, $t = 1$.

Δx	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$ (a - a_{th})/a_{th} $
0.4	7.943×10^{-8}	1.501×10^{-11}	2.327×10^{-8}
0.2	4.715×10^{-11}	6.966×10^{-12}	1.293×10^{-9}
0.1	2.071×10^{-11}	2.967×10^{-12}	1.140×10^{-9}
0.04	1.115×10^{-11}	1.551×10^{-12}	1.021×10^{-9}

Table 5.7: Time step comparison for the 2D DNS simulation of a single Gaussian vortex at $Re = 100$, equispaced grid of $\Delta x = 0.2$, and domain $[-10 : 10]^2$. Final time of the simulation, at which parameters are shown, $t = 1$.

Δt	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$ (a - a_{th})/a_{th} $
0.1	4.728×10^{-11}	6.795×10^{-12}	1.293×10^{-9}
0.01	4.715×10^{-11}	6.966×10^{-12}	1.293×10^{-9}
0.001	4.244×10^{-11}	7.961×10^{-12}	1.321×10^{-9}

Table 5.8: Reynolds comparison for the 2D DNS simulation of a single Gaussian vortex at $Re = 100$, equispaced grid of $\Delta x = 0.2$, and domain $[-10 : 10]^2$. $\Delta t = 0.01$, and final time of the simulation, at which parameters are shown, $t = 1$.

Re	$\sqrt{x_c^2 + y_c^2}$	$ (\Gamma - \Gamma_0)/\Gamma_0 $	$ (a - a_{th})/a_{th} $
10^1	5.460×10^{-10}	7.829×10^{-11}	3.146×10^{-9}
10^2	4.715×10^{-11}	6.966×10^{-12}	1.293×10^{-9}
10^3	4.034×10^{-12}	7.633×10^{-13}	1.160×10^{-9}
10^4	1.870×10^{-12}	1.907×10^{-13}	1.148×10^{-9}
10^5	1.996×10^{-12}	1.346×10^{-13}	1.148×10^{-9}

Figure 5.6: Error in vortex radius comparison for different Re numbers. The rest of the parameters are equispaced grid of $\Delta x = 0.2$, domain $[-10 : 10]^2$, $\Delta t = 0.01$, and total time of simulation $t/Re = 1$.

Figure 5.7: Single Gaussian vortex 2D DNS simulation. Vorticity field and vorticity contours of 0.2 between contour lines. $Re = 10$, equispaced grid of $\Delta x = 0.2$, domain $[-10 : 10]^2$ and $\Delta t = 0.01$.

5.2 Pair of counter-rotating vortices

The movement of two vortices of the same strength, and different sign of rotation, is analysed in this section. The circulation around each of them will be defined by Γ , and the distance between them by b , which are the only two parameters of the flow. The two vortices will move downwards (in a direction perpendicular to the line that pass through them) in a straight line with constant velocity if the right vortex circulation is defined positive.

The case of counter-rotative vortices, is very interesting, because it is a good model of a wide variety of real cases. Aircraft wake is only one of them, but there are many others, for example, vortex rings cross section also have this shape.

5.2.1 Pair of counter-rotating vortices analytical evolution

This problem could be solved analytically for two point vortices. The velocity of descent, will be equal to the circulation, divided by 2π times the distance between them:

$$v_d = -\frac{\Gamma}{2\pi b} \quad (5.2)$$

If the absolute value of the circulation of the vortices, Γ , as well as the distance between them, b , are taken to be one, the value of the descent velocity is: $v_d = -1/(2\pi) = -0.159154943091895\dots$

5.2.2 Pair of counter-rotating point vortices evolution

A 2D point vortex code (section 2.2) could be used to calculate the movement of these two points to see if the movement is reproduced. This case, is replicated by the code exactly, just the accumulation of machine resolution error is appreciated. That means that the error in the equations and the scheme is zero. The accumulation of error is seen because the solution get worse for an increase of number of steps, but does not change if different time steps or evolution schemes are taken. The error is always of the order of the machine resolution, times the number of steps used to find the solution. Exact values of these errors are calculated for the case of four vortices, in subsection 5.5.2 and is not worthy to repeat the same here.

5.2.3 Pair of counter-rotating inviscid Gaussian vortices by point vortices

The same algorithm of subsection 5.1.2 is employed here. As the validations were performed there, here only the novelty for the counter-rotating pair is exposed. The two counter-rotating vortices are of equal size and strength of that of subsection 5.1.2: intensity of the vortex, $q_0 = 1/2$ and vortex radius, $a_0 = 1$. Only the sign of this intensity is changed for the left side vortex. Now the vortex distance b is the parameter to consider. The simulations have been performed for 2000 point vortices in total, 1000 point vortices

for each Gaussian vortex. The two initial positioning approaches are also considered, the radial positioning, as well as the grid positioning for sufficiently big grids and $\Delta x = 0.1$.

The counter-rotating pair moves downwards with constant speed inversely proportional to the distance of the centroids, if this distance is much bigger than the vortex radius, as calculated in subsection 5.2.1. That is, this velocity grows as the vortices get closer. However, when the vortices are too close, they tend to neutralise each other and this velocity also diminishes. Table 5.9 shows this effect, as the final y_c location of the right vortex is shown, together with other parameters, all calculated for the right half plane, as the vortex size or the ratio of vortex sizes, for both of the approaches taken.

Table 5.9: Comparison of some values for the point vortex simulation of a counter-rotating Gaussian vortex pair at $t = 25$.

b	Initial Positioning	y_c	E	a
0.25	Radial	-0.7822	1.498	0.8947
0.25	Grid	-0.8689	1.788	0.8219
0.50	Radial	-1.548	1.771	0.9975
0.50	Grid	-1.647	2.118	0.8723
0.90	Radial	-2.530	1.964	1.082
0.90	Grid	-2.679	2.029	0.9066
1.00	Radial	-2.726	2.047	1.095
1.00	Grid	-2.892	1.938	0.9031
1.10	Radial	-2.936	2.147	1.134
1.10	Grid	-3.087	1.897	0.9114
1.50	Radial	-3.449	2.293	1.240
1.50	Grid	-3.687	1.632	0.8985
2.00	Radial	-3.713	2.134	1.252
2.00	Grid	-4.046	1.375	0.8808
5.00	Radial	-2.405	1.215	1.100
5.00	Grid	-2.479	1.098	0.9013

Figs. (5.8) and Figs. (5.9) expose a comparison of the initial and final ($t = 25$) point vortex distributions for $b = 0.25, 1$ and 5 . The proximity of the vortices in the first case, makes the dipole to move very slowly, as the final position is located very close to the last. It is also very clear that the grid positioning is much more compact than the radial positioning.

5.2.4 Pair of counter-rotating viscous Gaussian vortices by 2D DNS

The same configuration is now studied via 2D DNS. The same vortex separation values are analysed: $b=0.25, 0.5, 0.9, 1, 1.1, 1.5, 2$ and 5 . And three different Reynolds numbers: $Re=10^2, 10^3$ and 10^4 . The grid spacing is $\Delta x = 0.2$ and the time step $\Delta t = 0.1$.

The equivalent to the inviscid table 5.9 obtained by point vortices, is calculated here for the three values of Reynolds number in table 5.10. The results show how the influence

Figure 5.8: Counter-rotating Gaussian vortex pair by a point vortex simulation using grid initial positioning. Initial and final snapshots are shown. Positive circulation is shown red, as negative is shown purple.

of the viscosity diminishes the total circulation, which reduces the vortex speed, and the final vortex position moves less than for the inviscid case.

Figs. (5.10) are equivalent to Figs. (5.8) and Figs. (5.9) but calculated into a 2D DNS code with $Re = 10^4$. The matching with the two methodologies is impressive, in particular for the grid positioning case.

5.2.5 Pair of counter-rotating viscous Gaussian vortices by 3D PNS

As the interaction between the two real vortices will deviate their aspect from the original radial shape, PNS is needed to obtain a realistic basic flow for two axially diffusing vortices that are close. The PNS equations run in the z direction, but the coordinate system is modified in order to account for the downwash velocity due to the induced velocities between the vortices. More details of the PNS solution can be found in Paredes et al. [115]. The PNS marching integration is initialised by two Batchelor [11] vortices, given by Eqs. (1.8), which is a self-similar solution assuming that the vortex core radius r behaves as $r \propto \sqrt{z}$ when the axial direction is $z \gg 1$.

The domain is discretised using $N_z = 160$ points for the z -direction, mapped using

Figure 5.9: Counter-rotating Gaussian vortex pair by a point vortex simulation using radial initial positioning. Initial and final snapshots are shown. Positive circulation is shown red, as negative is shown purple.

Eq. (B.19) setting $z_\infty = 40$ and $c_z = 0.90$ to concentrate points in the vortex cores, and $N_y = 181$ points, mapped using Eq. (B.9), $y_\infty = 40$ and $y_h = 8$. A constant marching step $\Delta x = 0.5$ is selected. Compact finite differences of order 10, FD-q10, see inside subsection 3.2.2, are used for the differentiation along the two spatial directions. The boundary conditions imposed by the symmetries are $(\partial U_x / \partial y, U_y, \partial U_z / \partial y) = (0, 0, 0)$ at $y = 0$ and asymptotic boundary condition for the streamwise velocity, $U_z = 1$, and normal velocity $U_z = 0$ are imposed at y_∞ , where it is also set the total pressure level, $p_t = 0$. Periodic boundary conditions are imposed at x_∞ . The result is Fig. (5.11), which will serve as an example of a counter-rotating vortex pair flow obtained by this technique.

The PNS computed results converge to those of the isolated vortex as distance between vortices increases. It is found that, as vortices become closer, the mutual interaction results in an increase of the vortex radii, accompanied by a decrease in the downward induced velocity. It should be noted that the steady nature of the PNS equations prevents the recovery of elliptic instability induced by one vortex on the other. Subsequent stability analysis of the computed base flows using three-dimensional Parabolised Stability Equations (PSE-3D) is shown in subsection 6.2.3 of Chapter 6.

Table 5.10: Comparison of some values for the DNS simulation of a counter-rotating Gaussian vortex pair at $t = 25$.

b	Re	y_c	E	a
0.25	10^2	-0.5604	1.564	1.209
0.25	10^3	-0.7878	1.708	0.9238
0.25	10^4	-0.8234	1.751	0.8918
0.50	10^2	-1.098	1.663	1.243
0.50	10^3	-1.507	2.049	0.9901
0.50	10^4	-1.565	2.137	0.9607
0.90	10^2	-1.863	1.857	1.315
0.90	10^3	-2.459	2.284	1.073
0.90	10^4	-2.539	2.303	1.041
1.00	10^2	-2.029	1.899	1.333
1.00	10^3	-2.656	2.279	1.088
1.00	10^4	-2.738	2.282	1.055
1.10	10^2	-2.184	1.935	1.350
1.10	10^3	-2.834	2.263	1.102
1.10	10^4	-2.918	2.256	1.068
1.50	10^2	-2.685	2.018	1.407
1.50	10^3	-3.370	2.158	1.141
1.50	10^4	-3.456	2.145	1.108
2.00	10^2	-3.043	2.002	1.451
2.00	10^3	-3.661	1.967	1.153
2.00	10^4	-3.731	1.941	1.112
5.00	10^2	-2.417	1.374	1.459
5.00	10^3	-2.459	1.200	1.069
5.00	10^4	-2.463	1.174	1.023

Figure 5.10: Counter-rotating Gaussian vortex pair by a DNS simulation using $Re = 10^4$. Initial and final snapshots are shown. Vorticity field, positive vorticity red and negative vorticity blue. Green no vorticity. Vorticity is scaled to range of $t = 0$ for each simulation. Vorticity contours having 9 contours per vortex.

Figure 5.11: Iso-contours and iso-surfaces of axial vorticity of the non-parallel counter-rotating vortex-pair at $Re = 3000$ for the initial parameters $\kappa_0 = 0.4$ and $\gamma_0 = 0.8$, computed by PNS methodology, Paredes et al. [115].

5.3 Pair of co-rotating vortices

Again, the movement of two vortices of the same strength is analysed, but this time they also have the same sense of rotation. The definition of Γ and b is similar, having now the two vortices the same circulation sign. They will move in circles around the common center for this case, and they will rotate in counter-clockwise direction if their circulation is positive.

5.3.1 Pair of co-rotating vortices analytical evolution

This problem could be solved analytically for two point vortices. The velocity of rotation, the orbit velocity, will be equal to the circulation, divided by 2π times the distance between them:

$$v_o = \frac{\Gamma}{2\pi b} \quad (5.3)$$

As the formula is similar to Eq. (5.2), and if Γ and b , are taken to be one, the value of the orbit velocity is the same: $v_o = 1/(2\pi) = 0.159154943091895 \dots$

5.3.2 Pair of co-rotating point vortices evolution

This time, as the vortices do not follow straight lines, the high accuracy that was found in the previous case is not repeated. Here, as the time step gets higher, the vortex circular movement is approximated by polygonal lines that get more and more separated from the original circumference. Therefore, the higher the time step, the higher the error.

However, the error remains small, for a selection of time step moderate, $\Delta t = 0.01$, using a Runge-Kutta order 2 method, the error after two turns (1591 iterations) is of the order of 10^{-4} .

5.3.3 Pair of co-rotating inviscid Gaussian vortices by point vortices

This case represents the merger of two vortices depending on the ratio radius by vortex distance. Here, a two vortex merge is shown in Figs. (5.12). The point vortex simulation uses a total of 1000 points and radial positioning. As the simulation is very similar to that of subsection 5.2.3 the details are omitted.

5.3.4 Pair of co-rotating viscous Gaussian vortices by 2D DNS

Again, details of the simulation are not needed, and were already commented in subsection 5.2.4. The vortex merger is represented by Figs. (5.13), which are equivalent to Figs. (5.12) but calculated into a 2D DNS code with $Re = 10^3$ and 256^2 points. The matching, as in subsection 5.2.4 between the two methodologies is again impressive.

Figure 5.12: Co-rotating Gaussian vortex pair by a point vortex simulation using radial initial positioning. A sequence of snapshots is shown to represent the vortex merger. Positive circulation is shown red.

(a) $t = 0$.

(b) $t = 10$.

(c) $t = 20$.

(d) $t = 30$.

Figure 5.13: Co-rotating Gaussian vortex pair by a DNS simulation using $Re = 10^3$. A sequence of snapshots is shown to represent the vortex merger. Vorticity field, positive vorticity red and zero vorticity blue. Vorticity is scaled to range of $t = 0$ and vorticity contours are also shown.

5.4 Three merging vortices

This section deals with a particular configuration of vortices that produces a very rapid merger in a finite time for punctual vortices. This configuration is an excellent validation case of codes to vortex merger, see Schneider et al. [121] as an example.

5.4.1 Three merging vortices analytical evolution

The vortex coordinates and circulations for the three point vortices configuration that leads to a rapid merger are shown in table 5.11, from Marchioro and Pulvirenti [102].

Table 5.11: Vortex center positions and circulations of the three vortex configuration that merges at time $t = 3\sqrt{2}\pi$, given in Marchioro and Pulvirenti [102].

Vortex	x	y	Γ
1	-1	0	2
2	1	0	2
3	1	$\sqrt{2}$	-1

5.4.2 Three merging point vortices evolution

The point vortices of table 5.11 are located as initial conditions in a point vortex simulation with $\Delta t = 0.01$ up to final time $t = 13.3$. The result of the point vortices trajectories is shown in Fig. (5.14).

5.4.3 Three merging inviscid Gaussian vortices by point vortices

The configuration of three merging vortices is used to validate the point vortex code, producing at the same time very beautiful pictures. Two simulations have been done for 10000 point vortices for vortex radius, $a = 0.4$, and the two approaches used, radial positioning, Figs. (5.15), and grid positioning, Figs. (5.16). The point vortex simulation uses $\Delta t = 0.01$. The vortices with finite size radius, do not completely merge as the three point vortices, but the results are still very useful, because the procedure of this merger is complicated and serves to examine how good is the simulation to analyse this kind of flows. The results for grid and radial positioning are almost identical, with shows that, for complex flows, there is not a clear candidate between the two to be said to perform best.

5.4.4 Three merging viscous Gaussian vortices by 2D DNS

The same configuration is performed now by a 2D DNS. The simulation uses 160^2 grid points and the same vortex radius and time step than the point vortex simulations, $a = 0.4$

Figure 5.14: Point vortices trajectories for the three point vortices merger. Initial and final positions are highlighted

and $\Delta t = 0.01$. The Reynolds number used is $Re = 1000$, which is a compromise between too diffusive or too hard to compute. Actually, it could be said to be quite diffusive, as the difference between the maximum value of vorticity among the results is clearly seen, but not too much, as the vortex merger could be appreciated. Figs. (5.17) show the results, where the similarity with the point vortex inviscid simulations is revealing.

(a) $t = 0$.

(b) $t = 5$.

(c) $t = 10$.

(d) $t = 15$.

(e) $t = 20$.

(f) $t = 25$.

Figure 5.15: Snapshots of the point vortex evolution of the three vortex merger. Simulations for 1000 vortices. Radial positioning. Positive circulation is shown red, as negative is shown purple.

(a) $t = 0$.

(b) $t = 5$.

(c) $t = 10$.

(d) $t = 15$.

(e) $t = 20$.

(f) $t = 25$.

Figure 5.16: Snapshots of the point vortex evolution of the three vortex merger. Simulations for 1000 vortices. Grid positioning. Positive circulation is shown red, as negative is shown purple.

Figure 5.17: Snapshots of the 2D DNS evolution of the three vortex merger. Simulations for 160^2 grid points and $Re = 1000$. Vorticity field and vorticity contours of 0.2 between contour lines.

5.5 Four aligned vortices

In this section, the configuration of two vortex pairs encountered by Rennich and Lele [116] is studied. This kind of wake, although it is not as common as other configurations, can be also encountered in reality when aircraft have a particular flap deployment configuration, as they show in their paper. This configuration has the peculiarity that it is an equilibrium position in 2D for the point vortices, whose equilibrium will be found to be quite stable. This means that when such a configuration occurs in a 2D plane, the distances between the four vortices remain unaltered and the full field moves together as a rigid solid. The Rennich and Lele configuration, represented in Fig. (5.18) can reproduce the wake left by the wing tips of a plane, plus the effect of the fuselage and a special configuration of flaps.

Figure 5.18: Aircraft wake obtained by Rennich and Lele [116] and extracted from that paper.

Rennich and Lele [116] obtained this configuration for four point vortices analytically. This 2D analysis is repeated here because its stability is analysed by different means along the work. A 2D evolution code is presented to show how the points keep distances with time. The same is done for a set of point vortices that represent a Gaussian distribution in 2D, and later for a 2D DNS code. The fact that this configuration in 2D do not change with time, make it ideal for stability analysis. Also, it is known that this wake produces some 3D instabilities that lead to a rapid wake destruction. Both of this points are the main reasons for the selection of this configuration.

5.5.1 Four aligned vortices analytical evolution

The purpose of this section is to explain how this vortex equilibrium configuration is obtained. What follows in here is basically a summary of the procedure carried out by Rennich and Lele [116].

When Eq. (2.15) is particularised for four aligned vortices in the configuration presented in Fig. (5.18), horizontal velocities are always zero, but the vertical velocities are in general different for the inner and the outer vortices. The only thing that has to be done to achieve this configuration is to make them equal. In order to do that, two variables are introduced,

the dimensionless span ratio, Eq. (5.4), and the dimensionless circulation ratio, Eq. (5.5). Once they have been defined, Eq. (5.6) gives the relationship between them, needed to have the four vortex moving as a rigid solid.

$$\mathcal{X} = b_2/b_1 \quad (5.4)$$

$$\mathcal{G} = \Gamma_2/\Gamma_1 \quad (5.5)$$

$$\mathcal{X}^3 + 3\mathcal{G}\mathcal{X}^2 + 3\mathcal{X} + \mathcal{G} = 0 \quad (5.6)$$

In their paper, Rennich and Lele [116] also relate the solutions of Eq. (5.6) with the range of values for actual aircraft. From this point, $\mathcal{G}$ was chosen to be -0.4 which leads a value of $\mathcal{X}$ of around 0.14. b_2 was chosen 8 and $\Gamma_1 = \pi(0.025b_1)^2$ to be realistic. The parameters used in the paper will be these shown in table 5.12. Table 5.13 represent the positions of the four vortices at time 1000 calculated analytically.

Table 5.12: Vortex center positions and circulations used for the solution of the Rennich-Lele configuration up to double precision accuracy.

Vortex	x	y	Γ
1	28.513378591600937	0.0000000000000000	6.3853872750792977
2	4.0000000000000000	0.0000000000000000	-2.5541549100317193
3	-4.0000000000000000	0.0000000000000000	2.5541549100317193
4	-28.513378591600937	0.0000000000000000	-6.3853872750792977

Table 5.13: Vortex center positions at time 1000 up to double precision accuracy.

Vortex	x	y
1	28.513378591600937	-21.901161463511873
2	4.0000000000000000	-21.901161463511873
3	-4.0000000000000000	-21.901161463511873
4	-28.513378591600937	-21.901161463511873

5.5.2 Four aligned point vortices evolution

Here, information of table 5.12 is introduced into Eq. (2.25). The evolution problem is solved both with an Euler scheme (subsection 3.1.1) and a Runge-Kutta order 4 scheme (subsection 3.1.2). The accuracy of the solution is impressive, only the accumulation of machine resolution error is appreciable. This is, however, a consequence of the simplicity of the problem, because the points follow a rectilinear and constant velocity movement. In other vortical configurations, the points follow circular or random movements, which are represented numerically as polygonal lines whose segment lengths are velocity times step time.

The four point vortex positions in x are solved up to machine accuracy. Only position in y suffers the accumulation of error. This lead to a paradox, because while the time step of the scheme lies inside its stability region, the accuracy of the solution is better as the time step is bigger. The explanation is simple, in order to achieve the same final time, a scheme with shorter time step will need more steps and then, the accumulation of error will be higher. Usually, other errors in the solution are much bigger, and they grow with time step size. Therefore, it is more common to find final solution errors decreasing when time step is also decreased. Table 5.14 compares the error found at time 1000 for the different schemes and time steps, compared with the quantity obtained from the multiplication of the machine resolution with the number of time steps. The errors mentioned here, are calculated as Eq. (5.7) shows, where $x_{i, \text{ideal}}$ are the initial positions and $y_{i, \text{ideal}}$ are calculated as the time times the velocity of descent calculated analytically. In table 5.14, numbers for Euler scheme and Runge-Kutta order 4 are the same, because as the solution is exact, the scheme of resolution is not important, only it is the machine resolution.

$$Error_{4pv} = \sum_{i=1}^4 \frac{\sqrt{(x_i - x_{i, \text{ideal}})^2 + (y_i - y_{i, \text{ideal}})^2}}{4} \quad (5.7)$$

Table 5.14: Errors for the 4 point vortices evolution code.

Temporal step	Euler	RK4	Machine error by step number
0.5	6.2172×10^{-13}	6.2172×10^{-13}	2.0×10^{-13}
0.1	3.3573×10^{-12}	3.3573×10^{-12}	1.0×10^{-12}
0.01	4.6024×10^{-11}	4.6024×10^{-11}	1.0×10^{-11}
0.001	3.5082×10^{-10}	3.5082×10^{-10}	1.0×10^{-10}
0.0001	3.2760×10^{-10}	3.2760×10^{-10}	1.0×10^{-9}
0.00001	2.1367×10^{-08}	2.1367×10^{-08}	1.0×10^{-08}

5.5.3 Four aligned inviscid Gaussian vortices by point vortices

Up to this point. Only it has been spoken about point vortices. But actual vortices will have a vorticity distribution. The shape selected here is Gaussian, because it is one of the more realistic and easier to implement. The vorticity distribution will be represented by Eq. (5.8).

$$\omega(x, y) = \sum_{i=1}^4 \frac{2q_i}{a_i^2} e^{-((x-x_i)^2 + (y-y_i)^2)/a_i^2} \quad (5.8)$$

In Eq. (5.8), two new variables appear, q_i and a_i , because now the vortex has a radius and it is not only a point. This two terms are related to the circulation of the vortex, Γ_i , therefore only one of the parameters is truly new. They follow the relationship Eq. (5.9), and as Γ_i is known now for the four vortices, one more parameter has to be fixed. The way how it is done is fixing the ratio between the radius of the outer vortex and its

distance to the other outer vortex for the outer vortices. The value is chosen to be 0.025 for aeronautical interest. For the inner vortices, a_i is taken one.

$$\Gamma_i = 2\pi q_i \quad (5.9)$$

The parameters used for Gaussian will be these shown in table 5.15, which together with table 5.12 give all the parameters of the four initial Gaussian vortices.

Table 5.15: Gaussian vortex parameters used for the solution of the Rennich-Lele configuration up to double precision accuracy. Table 5.12 shows their positions and circulations.

Vortex	a	$2q/a^2$
1	1.4256689295800469	1.0000000000000000
2	1.0000000000000000	-0.81301275870796674
3	1.0000000000000000	0.81301275870796674
4	1.4256689295800469	-1.0000000000000000

The information given is enough to assign vorticity to points on any grid. This vorticity grid values are valid both for the initiation of 2D DNS, as well as for the input for the algorithm of locate point vortex from a grid. Now, the different initial conditions obtained using this algorithm are discussed. A thin mesh is selected, and the number of point vortices considered are modified. It is observed a great dispersion of results, as the point vortices are positioned randomly in each square of the mesh. Tables (5.16-5.19) illustrate this dispersion, showing values of the coordinates of the centroid, ‘Full Vortex Radius’ and Hamiltonian for the right plane. They represent, for the same number of point vortices, different simulations at the initial time. It should be noted that, as the differences are high, there is no need to take a higher number of decimal points. Table 5.20 shows the exact values for the configuration of four ideal filament vortices. In addition, Figs. (5.19) plots some captions of one of the outer vortices for a selection of various different number of point vortices. Although the values approach to those desired when the number of point vortices is increased, the errors are big for any number of them, and even the differences obtained for the same initialization parameters, produced by the random positioning of the vortices are important. Therefore, this method will not be considered for this case any more. However, it should be mentioned that qualitative results can be good and that this method can be used to represent any vorticity field, not only analytical ones.

Table 5.16: Point vortex parameters at initial time for a number of 100 point vortices using the grid positioning.

Simulation	x centroid	y centroid	Full Vortex Radius	Hamiltonian
1	32.3749	3.4733×10^{-2}	10.4721	6.8789
2	32.3640	-8.2769×10^{-3}	10.4465	6.7100
3	32.3810	2.1088×10^{-2}	10.4821	6.7585
4	32.3999	7.9529×10^{-3}	10.4744	6.7968

5. Vortex Movement

Table 5.17: Point vortex parameters at initial time for a number of 500 point vortices using the grid positioning.

Simulation	x centroid	y centroid	Full Vortex Radius	Hamiltonian
1	43.6738	6.7058×10^{-3}	24.5179	7.6894
2	43.6815	-9.4788×10^{-4}	24.5250	7.6692
3	43.6832	-2.6727×10^{-3}	24.5292	7.7072
4	43.6807	1.9965×10^{-3}	24.5162	7.7051

Table 5.18: Point vortex parameters at initial time for a number of 1000 point vortices using the grid positioning.

Simulation	x centroid	y centroid	Full Vortex Radius	Hamiltonian
1	45.4578	2.0552×10^{-3}	26.4866	7.4287
2	45.4479	-1.6355×10^{-3}	26.4762	7.4347
3	45.4491	-9.3914×10^{-3}	26.4810	7.4364
4	45.4518	-1.3339×10^{-3}	26.4852	7.4359

On the other hand, positioning using radial placement, seem much more efficient. Various algorithms have been implemented, but only one of them is considered here. The algorithm implemented, locates one vortex at the center, 3 vortices on the first circle, 4 on the following and so on. The two last circles have a number of points in such a way that the jump between the last two vortices is minimized. The radius of each circle of vortices is calculated to fit the Gaussian distribution, and the location of the vortices inside the circle has been modified a random angle (keeping the distance between vortices), not to have all the first vortices of each circle aligned. It should be mentioned that for this method, the point vortices belonging to different Gaussian vortices do not have exactly the same circulation. This is because the number of point vortices belonging to each Gaussian vortex is an integer which is used to calculate the circulation of each of the point vortices. As the number of total point vortices increases, the difference will diminish. This could have been overcome by changing the value of the circulation of each point vortex to the total circulation, divided by the number of point vortices, but the representation of the initial field would have been worst. Furthermore, the difference of circulation is usually very small, and therefore, the visualization of the field will be quite accurate if only points are represented.

Table 5.19: Point vortex parameters at initial time for a number of 5000 point vortices using the grid positioning.

Simulation	x centroid	y centroid	Full Vortex Radius	Hamiltonian
1	44.7898	-5.4118×10^{-4}	25.7664	6.9882
2	44.7902	-1.3169×10^{-3}	25.7655	6.9839
3	44.7914	-4.3602×10^{-3}	25.7678	6.9868
4	44.7876	-3.4575×10^{-3}	25.7646	6.9819

Table 5.20: Ideal point vortex parameters for four point vortices.

x centroid	y centroid	Full Vortex Radius
44.855630986001565	0.0000000000000000	25.839369831823152

Tables (5.21-5.24) show the dispersion of results for positioning using radial placement, in a similar fashion that tables (5.16-5.19) did for grid positioning. Again, the values of the coordinates of the centroid, 'Full Vortex Radius' and Hamiltonian for the right plane, are shown for the same number of point vortices and different number of simulations at the initial time. This time, all decimals are shown up to double precision, because the differences are in general, around machine accuracy error. It maybe noted that some of the values increase with the number of vortices, but far from being worst representations, the fact is that when the number the points increases, the number of operations to calculate an integral magnitude also increases, and some errors may accumulate. Figs. (5.20) plots some captions of one of the outer vortices for a selection of various different number of point vortices. These four representations, show that this time, the Gaussian behaviour is represented quite well from small to large number of points, and the circular shape of them is better appreciated. In this case, the improvement with the number of points do not seem very big, and maybe it is not worthy to use a big number of them.

Table 5.21: Point vortex parameters at initial time for a number of 100 point vortices using circular positioning.

Simulation	x centroid	y centroid	Full Vortex Radius
1	44.855630986001628	-7.2445×10^{-17}	25.839369831823177
2	44.855630986001657	1.0867×10^{-17}	25.839369831823177
3	44.855630986001643	-2.8978×10^{-17}	25.839369831823177
4	44.855630986001621	1.0142×10^{-16}	25.839369831823177

Table 5.22: Point vortex parameters at initial time for a number of 500 point vortices using circular positioning.

Simulation	x centroid	y centroid	Full Vortex Radius
1	44.855630986001962	-1.8836×10^{-16}	25.839369831823262
2	44.855630986001948	-9.7801×10^{-17}	25.839369831823262
3	44.855630986001920	1.4489×10^{-16}	25.839369831823262
4	44.855630986001941	-5.0712×10^{-17}	25.839369831823273

Using the initial distribution exposed before, and the same algorithm than in subsection 5.5.2, an evolution problem is solved. This time, only Runge-Kutta order 4 is used. Various time steps are used as well as different initial distributions of 100, 500, 1000 and 5000 point vortices. The results do not show anything worth to be mentioned, the four vortices formed by point vortices go down at constant speed.

(a) 100 point vortices

(b) 500 points vortices

(c) 1000 points vortices

(d) 5000 points vortices

Figure 5.19: Point vortex initial positions for an increasing number of them using the grid positioning. Only one of the outer Gaussian vortices is shown.

5.5.4 Four aligned viscous Gaussian vortices by 2D DNS

In this simulation, the code uses a Fourier discretisation and the leading problem is solved using an eigenvalue decomposition algorithm with a Runge-Kutta order 4 as a temporal evolution scheme.

The code has been validated already in many different situations. However, as it is a new configuration, it is needed to run a set of simulations to perform variation of time step, limit of the boundaries and spacing of the mesh.

As the code uses a Fourier spectral method to calculate the derivative matrices, the boundary conditions are periodic, therefore, the boundaries have to be far from where the vortices stay. The limits of the boundaries, called x limit in the figures, are actually the values of the coordinates at the boundaries. They have been checked as Figs. (5.21) shows. This figure illustrates how closer to 24π (75.4) the results start to diverge, as the vortices

Table 5.23: Point vortex parameters at initial time for a number of 1000 point vortices using circular positioning.

Simulation	x centroid	y centroid	Full Vortex Radius
1	44.855630986002247	2.9567×10^{-16}	25.839369831823348
2	44.855630986002161	5.4606×10^{-16}	25.839369831823351
3	44.855630986002282	9.9613×10^{-17}	25.839369831823340
4	44.855630986002190	7.1540×10^{-16}	25.839369831823340

Table 5.24: Point vortex parameters at initial time for a number of 5000 point vortices using circular positioning.

Simulation	x centroid	y centroid	Full Vortex Radius
1	44.855630986001742	1.4113×10^{-15}	25.839369831823134
2	44.855630986001536	5.1165×10^{-16}	25.839369831823152
3	44.855630986001458	7.2853×10^{-16}	25.839369831823163
4	44.855630986001550	-2.9703×10^{-16}	25.839369831823163

are too close to the boundaries. For values of the boundary farther than 48π (150.8), the convergence is quite good. The variables plotted here are calculated in the right plane, so the centroid x will be the centroid of the two right vortices, as well as the centroid y , that will coincide for half plane and full plane. The total circulation is zero for the full domain, which is the reason for using only half plane. The last magnitude is the momentum in y direction, defined as $a_y = \sqrt{\int \int [\omega(x, y)(y - y_c)^2 dx dy]} / \Gamma$. These results are not good enough and already something can be improved, because the grid is very coarse in these cases for the Re used.

For that, the grid spacing have been modified to check convergence. Figs. (5.22) presents that results. Anything coarser than 0.12π (0.377) gives very bad results, which are not plotted, but also it is seen that for anything thinner than 0.08π (0.251) the convergence is very good.

Time step has been also varied, Figs. (5.23) shows no difference at all, and the convergence is good enough for the time steps studied. Above 0.5, taking 0.6 for example, the code works with almost the same accuracy up to a point when it stop converging, as one might expect.

(a) 100 point vortices

(b) 500 points vortices

(c) 1000 points vortices

(d) 5000 points vortices

Figure 5.20: Point vortex initial positions for an increasing number of them using the circular positioning. Only one of the outer Gaussian vortices is shown.

(a) Centroid x

(b) Centroid y

(c) Circulation

(d) Momentum y direction

Figure 5.21: Parameter comparisons for various mesh limits, all them calculated with a time step of 0.01, a spacing of 0.377 and $Re=1000$.

Figure 5.22: Parameter comparisons for various mesh spacings, all them calculated with a time step of 0.01, a boundary distance of 150.8 and $Re=1000$.

(a) Centroid x

(b) Centroid y

(c) Circulation

(d) Momentum y direction

Figure 5.23: Parameter comparisons for various time steps, all them calculated with a mesh spacing of 0.377, a boundary distance of 150.8 and $Re=1000$.

5.6 Shear layers

The point vortex methods with periodic conditions are an excellent tool to analyse shear layers. In particular, viscosity stratified flows with shear, can be modelled by this technique, with is done in the following subsection.

5.6.1 Stratified flow

Stratified flows are analysed here by vortex methods. The idea is to model viscosity jumps at a given spatial location as vorticity jumps, because the shear flow slope registers a sharp jump at these points, see Fig. (5.24). In an unstratified flow, the velocity in the shear region is $u = Uy/h$, (without loss of generality, the velocity difference between upper and lower layers will be U and the layer thickness h) which gives a constant stress value, $\tau = \mu \partial u / \partial y = \mu U / h$ in the shear layer. When there is stratification, that is, there is a viscosity change inside the shear layer, the velocities have to be $u_1 = 2\mu_2 Uy / [(\mu_1 + \mu_2)h]$ and $u_2 = 2\mu_1 Uy / [(\mu_1 + \mu_2)h] + (\mu_2 - \mu_1)U / (\mu_1 + \mu_2)$ in the two regions, because in the viscosity jump, the stress must be equal: $\tau_1 = \tau_2 = 2\mu_1\mu_2 U / [(\mu_1 + \mu_2)h]$. Note that the viscosity jump is supposed in the middle of the shear region to simplify the results. Earth outer core flow is the most relevant example of this type of flow, as extreme viscosity jumps appear in that flow between layers of different materials, see Forte and Peltier [48]. As these jumps are several orders of magnitude in viscosity, they are extremely difficult to study by standard DNS and, therefore, vortex methods are a powerful tool to analyse them.

Figure 5.24: Comparison of the shear layer for unstratified and stratified flow.

The key question addressed in the present section is whether instabilities in viscosity-stratified flows are inviscid in origin, and based on wave interaction, as suggested by the earlier work of Sahu and Govindarajan [120]. Although some linear results support this hypothesis, it is of interest to examine whether nonlinear instability can also be predicted.

Vortex methods, although not often used in this context, are a mature methodology, which already more than three decades ago provided with solutions of relevant fluid mechanics problems, see Leonard [92]. The methods have now been extended by introducing a wide range of modifications that allow to calculate complete 3D viscous flows, i.e. Winckelmans et al. [144] and Chatelain et al. [28].

Meiburg and others [9, 46, 56, 106] have studied stratified fluid flows where the

Reynolds number is zero but the Peclet number is infinity. Such flows are seen in chemical industrial applications; but Earth flows have very large Reynolds number. Here we study a flow where both the Peclet and Reynolds numbers are infinity, although our work has the potential to be extended to finite Peclet numbers.

The numerical method is applied on an array of several layers of vortices. In this particular case, eight layers in total are taken, with half of them given a value of circulation different of those of the other layers. An infinite number of vortices per row are given (due to the periodicity), but the actual number of vortices per row employed in the code varies from a few to a few hundreds. An schematic of the vortex array is shown in Fig. (5.25).

Figure 5.25: Vortex array scheme.

After performing simulations of the above array of vortices, with ratio of vorticities one half, Figs. (5.26-5.28) are obtained, where it is seen that the flow begins to destabilize around a non-dimensional time of $t = 3.3$. Between times $t = 6.6$ and $t = 10$ it is observed how the wave packets move to the right in the lower layers, and to the left in the upper, with amplitudes growing in time. The speed of these spots seems to grow with the distance to the discontinuity. The most noticeable feature of the dynamics is that the upper-half disturbances show a much smaller growth than the lower half, i.e., disturbance growth is greater in the region of higher vorticity. This is consistent with intuitive expectations. At time $t = 150$ some vortices start to develop in the lower rows, and at time $t = 200$ the vortices of these rows fill all the lower half, and the upper rows too show some undulations. This behaviour is augmented for time $t = 250$, where the three lower vortices are more clear. The bigger of the three lower vortices begins to deform the upper layer clearly at $t = 300$, forming something like a Kelvin-Helmholtz wave. Time $t = 350$ shows mainly one big vortex as the result of this interaction between layers, which has assimilated the other vortices. The whole set of layers is progressively undulating from $t = 350$ to $t = 450$, where all the set of point vortices are forming a single vortex.

All the above results have been obtained with an in house code based on Eq. (3.45). This is not very efficient and introduces important errors if a large number of vortices is not used. Therefore, simulations based on Eq. (3.46) have also been performed².

Simulations have been done for 2, 4, 8, 16, 32, 64 and 128 rows, 50, 100 and 200 vortices per period. Half lower layers have a circulation Γ , while half upper layers have

²Code provided by Suryanarayanan et al. [127] which uses optimised algorithms

Figure 5.26: 8 rows of 100 point vortices, half vorticity for the upper 4 rows (colored). (a) Initial condition, $t = 0$. (b) $t = 3.3$. (c) $t = 6.6$. (d) $t = 10$.

Figure 5.27: 8 rows of 100 point vortices, half vorticity for the upper 4 rows (colored). (a) $t = 150$. (b) $t = 200$. (c) $t = 250$. (d) $t = 300$.

circulations Γ , $\Gamma/2$, $\Gamma/4$, $\Gamma/10$, $\Gamma/100$ or 0.

The thickness of the layers is first studied. Actually, as the flow is periodic, a change of the thickness is similar to a change of the maximum periodicity length allowed for instabilities to grow. The results are similar to these that could be found for the Kelvin Helmholtz instabilities of the unstratified shear layer, see Fig. (5.30). For very thick layers, the shorter instability length of the Kelvin Helmholtz instability is longer than the periodic domain, and therefore, these instabilities do not develop, see Fig. (5.31).

Now, the viscosity ratio effect, modelled here as a circulation ratio, is analysed. For low viscosity ratios, as in Fig. (5.30) the effect is similar to that of having the average viscosity (circulation), but more dominated by layer with less viscosity (more circulation). If the ratio is only 10, Fig. (5.32), it is seen how the more viscous region (less vortical) is basically dragged from the less viscous (more vortical). But when this ratio is taking to be 100, Fig. (5.33), the effect of the particles with less circulation is almost negligible and they act more as tracer particles.

Figure 5.28: 8 rows of 100 point vortices, half vorticity for the upper 4 rows (colored). (a) $t = 350$. (b) $t = 400$. (c) $t = 450$. (d) $t = 500$.

Figure 5.29: Point vortex preliminary results showing 3 periodic domains, $t = 500$.

Figure 5.30: Kelvin-Helmholtz like instabilities for thin layers. Figures show three periods and different times where the lower wavelength instabilities of each case are shown.

(a) 64 rows of vortices.

(b) 128 rows of vortices.

Figure 5.31: Instabilities saturation for thick layers. Figures show three periods and long times where it can be seen that instabilities are not growing.

Figure 5.32: Viscosity ratio 1:10. Figures show three periods and different times where the lower wavelength instabilities of each case are shown.

In order to understand better the instabilities, the evolution of the RMS (Root Mean Square or quadratic mean) of the stream function along the x axis (interface at initial time) is defined, Eq. (5.10). This is a single magnitude for all the domain which allows the understanding of the phenomena. Changes on the slope are due to the pairing of vortices, and then straight lines are due to the development of the vortices. It is observed in Fig. (??) how the slope of the ratio 1:1 is half of that of the ratio 0:1, while intermediate cases are in between.

$$\psi_{RMS}(t) = \sqrt{\sum_{i=1}^N [\psi^2(x_i, t) + (\psi_{mean})^2(t)] / N} \quad \psi_{mean}(t) = \sum_{i=1}^N \psi(x_i, t) / N \quad (5.10)$$

To summarise the results, Kelvin Helmholtz instabilities are recovered when long wavelengths are allowed to grow. For the stratified case, the higher viscosity ratio, the smaller the effective thickness, the instability is seen earlier, the wavelength is smaller, the mixing seems to be less and nonlinear evolution is different.

(a) 8 rows of vortices.

(b) 16 rows of vortices.

(c) 32 rows of vortices.

Figure 5.33: Viscosity ratio 1:100. Figures show three periods and different times where the lower wavelength instabilities of each case are shown.

Figure 5.34: $\psi_{RMS}(t)$ for various viscosity ratios.

5. Vortex Movement

Figure 5.35: ψ'_{RMS} along the interface as a function of time steps.

Figure 5.36: ψ'_{RMS} along the interface as a function of time steps.

5.7 Realistic vortex wake, elliptic wing

The flow behind an elliptic wing is analysed by point vortices. A slice is extracted from a 3D DNS simulation at $Re_c = 1750$ of an elliptic wing of low aspect ratio $AR = 1.714$, wing span ($b = 1.714$) to root chord ($c_r = 1$) ratio, formed by a succession of Eppler E-387 aerofoils, at a distance of two root chords from the wing trailing edge. A general view of the wing wake roll up by point vortices is shown in Fig. (5.41), extracted from He et al. [63].

Simulations have been performed for 2000, 5000 and 25000 vortices using symmetry conditions from time $t = 0$ to $t = 18$. Note that the wing velocity is one, and as far as there is similarity between distance and time, these times could be converted to distances of 2 to the trailing edge, the place where the snapshot was taken, to a distance of 20 to the trailing edge for the time 18. Due to small errors in the initial vortex positioning, the integral values are not the same for the three simulations considered. Table 5.25 shows these differences. Then, Fig. (5.37) shows the rolling process, as this parameter is very small at the beginning, but tends to one as the vortical sheet is rolled up. There are significant variations between the simulations, but the general behaviour is equivalent. Fig. (5.38) and Figs. (5.39, 5.40) show a general view for the roll up process in the three point vortex simulations.

Table 5.25: Comparison of some values for the initial point vortex array for the wake behind an elliptical wing at distance two chords from the trailing edge.

N	Γ	x_c	y_c	E	a
2000	0.1178	0.6851	-8.267×10^{-2}	0.5179	0.1621
5000	0.1178	0.6852	-8.371×10^{-2}	0.5192	0.1628
25000	0.1178	0.6809	-8.352×10^{-2}	0.5148	0.1643

Figure 5.37: Ratio between horizontal and vertical vortex radii point vortex simulations comparison in the wake of an elliptic wing.

Finally, it should be mentioned that the wake will not only develop inviscid instabilities in the Oxy plane due to the inflectional velocity profile in the x direction. This type of instabilities will also appear on Oyz planes too, as an inflectional velocity profile is also to be found along the z direction at some point between the axis of symmetry and the vortex core. This feature is captured by the point-vortex simulations as has already been seen in the insert Fig. (5.41). However, complete description of all linear instability mechanisms beyond the leading perturbations identified by global methods is beyond the scope of the present work.

(a) $t = 0$, 2000 points.

(b) $t = 0$, 5000 points.

(c) $t = 4.5$, 2000 points.

(d) $t = 4.5$, 5000 points.

(e) $t = 9$, 2000 points.

(f) $t = 9$, 5000 points.

(g) $t = 13.5$, 2000 points.

(h) $t = 13.5$, 5000 points.

(i) $t = 18$, 2000 points.

(j) $t = 18$, 5000 points.

Figure 5.38: Snapshots of the point vortex evolution of the wake behind an elliptic wing. Simulations for 2000 and 5000 vortices. Positive circulation is shown red, as negative is shown purple.

(a) $t = 0$, 25000 points.

(b) $t = 4.5$, 25000 points.

(c) $t = 9$, 25000 points.

Figure 5.39: Snapshots of the point vortex evolution of the wake behind an elliptic wing. Simulation for 25000 vortices. Positive circulation is shown red, as negative is shown purple. $t = 0, 4.5$ and 9.

(a) $t = 13.5$, 25000 points.

(b) $t = 18$, 25000 points.

Figure 5.40: Snapshots of the point vortex evolution of the wake behind an elliptic wing. Simulation for 25000 vortices. Positive circulation is shown red, as negative is shown purple. $t = 13.5$ and 18.

Figure 5.41: Three-dimensional perspective of the flow obtained by inviscid point-vortex methods over the elliptic wing at $AoA = 5^\circ$. Vortex core lines being those obtained in the 3D DNS. Inviscid instability developing in the wake at $x = 40$ (*right insert*).

5.8 Vortex movement discussion

This chapter exposes clearly the duality of DNS (Navier-Stokes equations) and point vortex (Biot-Savart equations) as methodologies for the solution of IVP for vortical flows. They both produce results with reasonable agreement when the flow is sufficiently well calculated and Re is big enough for the comparison.

The point vortex methods, are incredibly fast when the number of point vortices is small, producing results with reasonable quality. However, as the number of points is increased to big values, lets say around 10000, the calculus in single core could take a few hours, and because the computational time grows as N^2 , the calculus for bigger number of points reaches prohibitively times if the number of points is increased. Therefore, for the cases analysed, point vortices offer good results in very short times, but may not offer good quality results, even for very long simulations.

In the other side, DNS simulations need a minimum number of points that is not small in order to obtain physical results, so they do not offer this low cost possibility that the point vortices offer. They number of points increase as N^2 because is a 2D methodology, but also, due to the CFL, the number of time steps should also be increased when the number of points is increased. Therefore, doubling the linear number of points N will at least increase the computational time by N^3 , of course depending on the algorithm. However, despite the number of operations needed and its dependence with the number of grid points, the DNS for laminar flow will produce very robust results, as far as there are enough grid points to resolve the smaller scales of the flow, but not more. Therefore, when affordable, the quality of results achieved by DNS outperforms the equivalent point vortex simulation significantly.

In addition to all that, there is one more issue to take into account. Most of the flows become unstable for high Reynolds numbers, which produces a lot of issues when dealing with DNS codes and not wanting to resolve turbulence. For such cases, the point vortex methods, as they are inviscid, they resolve the limit $Re \rightarrow \infty$ case very easily, when the DNS simply cannot do it.

There have been shown comparisons of the two methodologies for simple, but interesting cases opening the possibility of obtaining basic flows for the subsequent instability analysis. In addition, the stratified flow and the elliptic wing, are cases where the point vortex simulation offer directly very interesting linear and non-linear instability results.

Chapter 6

Vortex Instabilities

The main results of the thesis are shown in this chapter. As well as for Chapter 5, a classification is done from the simpler to the more complex. The two different approaches to study vortex instability problems, vortex methods and classical grid methods, are the main concern in this thesis, and they acquire special relevance in this chapter, because the actual comparison of results between them is achieved here.

6.1 Single vortex instabilities

First, the instability of an isolated vortex is just performed as a validation. Later it can be compared with the case of two vortices. As seen in subsection 1.1.2, there are a variety of vortex models, which can be taken as basic flows. One of the most analysed due to its simplicity and closeness to actual vortices is the q -vortex approximation of the classic Batchelor [11] vortex. For many of such vortex models, both the axial and the azimuthal spatial directions can be taken as homogeneous, and be resolved by local analysis, see Lessen et al. [94], Mayer and Powell [105]. As expected, when BiGlobal and local analysis are applied to the same flow, they deliver identical instability results [19, 112], although in BiGlobal analysis the assumption of azimuthal homogeneity is not invoked and all different azimuthal eigenmode results of local analysis are obtained in a single BiGlobal computation. However, when the axial homogeneity assumption is relaxed, the physics of vortex instability changes not only quantitatively but also qualitatively.

6.1.1 Single parallel vortex instabilities by local and BiGlobal

The q -vortex has been selected for the validations of a single vortex for $W_0 = 0$, $Re = 100$ and $q = 0.475$, being the Reynolds number based on the jet velocity. The results shown in table 6.1 are BiGlobal results by Paredes [112], compared to those local of Mayer and Powell [105] and employing matrix-formation and the FD-q method at variable orders.

Just as examples, a few results of BiGlobal of a single vortex are shown in Fig. (6.1), where results for a variety of azimuthal wavenumbers m are obtained at once, while each would have required a separate local analysis. Fig. (6.2) shows a mode of $m = 8$ directly extracted from the BiGlobal, where the reconstruction of the lobes is not needed.

Spatial discretisation	N^2	$\omega_r \times 10^{-2}$	$\omega_i \times 10^{-3}$
CGL	20^2	2.833	9.64
	30^2	2.8352	9.617
	40^2	2.835189	9.616908
	50^2	2.8351897	9.6169098
	60^2	2.83518966	9.61690985
	70^2	2.83518966	9.61690985
FD-q8	40^2	2.835	9.62
	60^2	2.8352	9.617
	80^2	2.835183	9.6167
	100^2	2.8351898	9.616912
FD-q10	40^2	2.8353	9.617
	60^2	2.8351899	9.61698
	80^2	2.8351895	9.616913
	100^2	2.83518965	9.61690986
MP			9.61

Table 6.1: Leading eigenvalue of a q -vortex at $W_0 = 0$, $Re = 100$ and $q = 0.475$ [105, 112]

Figure 6.1: Amplified modes of the q -vortex, obtained by spatial BiGlobal analysis [112] and ordered according to their respective m -value as if they have been obtained by local LST.

Figure 6.2: Spatial structure of the BiGlobal amplitude function mode, assimilable to a $m = 8$ mode of local LST [112].

6.2 Pair of counter-rotating vortices instabilities

Local theory is not applicable to vortex systems analysis, because such flows do not have axial symmetry and have two inhomogeneous directions. For both, viscous BiGlobal and inviscid point vortex analysis, the axial direction should be homogeneous, but there are other tools as PSE-3D, which allow a slow variation of the flow in this axial direction.

The counter-rotating vortex pair flow could develop a variety of instabilities. Among the most relevant are the long wave or Crow [35] instability and the short wave, elliptic or also called Widnall [141] instability, both of them included into the category of cooperative instabilities, because they are based on the interaction between vortices. The instabilities that are not cooperative could be found by the analysis of a single vortex, being the most relevant of those the core instabilities.

6.2.1 Pair of counter-rotating point vortices inviscid instability [35]

In this section, first the two counter-rotating point vortices are analysed assuming that they are not perturbed in the direction perpendicular to the plane. Later, this assumption is eliminated and the 3D case is analysed, but the 2D result should be recovered when the wavelength is taken to be zero, $k = 0$, in the 3D case.

First, the 2D eigenvalue problem is obtained substituting into Eqs. (2.74) the following: $N = 2$, $p = 1, 2$, $X_1 = -b/2$, $X_2 = b/2$, $Y_{1,2} = 0$, $\Gamma_1 = -\Gamma$ and $\Gamma_2 = \Gamma$, where b is the distance between the vortices and Γ is the circulation of the positive vortex, which makes the downwash between the vortices to move downwards. Note that the choice of these parameters derives from the existence of an equilibrium configuration in which $\bar{\mathbf{U}} = -(\Gamma/2\pi b)\mathbf{e}_y$ is the downwash speed of the vortex pair.

The equations are written in non-dimensional form using the distance between the vortices as the characteristic length b , as well as the circulation of the right vortex as the characteristic circulation Γ . From that, the characteristic time scale is derived, $2\pi b^2/\Gamma$, which makes the non-dimensional downwash velocity unity. The values of the new non-dimensional variables are: $\tilde{\omega} = 2\pi\omega b^2/\Gamma$, $\tilde{x}_i = \hat{x}_i/b$ and $\tilde{y}_i = \hat{y}_i/b$. The system of equations can be written in the form of an eigenvalue problem, $-i\tilde{\omega}\tilde{\mathbf{x}} = \mathbf{A}\tilde{\mathbf{x}}$:

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} \quad (6.1)$$

Solving Eq. (6.1) analytically, only one quadruple eigenvalue $-i\tilde{\omega} = 0$ is obtained. The physical explanation of this trivial results is that if any perturbation is introduced on the initial positions, it will be neutrally stable and the system will equilibrate in the new position.

Now, the two counter-rotating vortex filaments with homogeneous perturbations along the axial direction are analysed. They are one of the most interesting configurations, since

they provide realistic results (Crow's [35] instability) by means of a very simple model of aircraft wake.

As it was demonstrated in the analysis of the 2D case, this configuration is in equilibrium in a moving frame of reference, with speed $\bar{\mathbf{U}} = -(\Gamma/2\pi b)\mathbf{e}_y$, and can be used as a steady base flow in the analysis. The parameters are the same as in the 2D analysis ($N = 2$, $p = 1, 2$, $X_1 = -b/2$, $X_2 = b/2$, $Y_{1,2} = 0$, $\Gamma_1 = -\Gamma$ and $\Gamma_2 = \Gamma$), and in addition a cutoff parameter $d_1 = d_2 = d$ is considered. All values are substituted into Eqs. (2.80) and non-dimensionalised, using again the same time ($2\pi b^2/\Gamma$) and distance (b) scales. Two new parameters appear in the equations, d and $1/k$, which non-dimensionalised give: $\tilde{d} = d/b$ and $\tilde{k} = kb$. The equations are finally written in matrix form using the same nomenclature as before and introducing $\tau(\theta) = \cos \theta - 1 + \theta \sin \theta - \theta^2 Ci(\theta)$ and Bessel functions relations¹:

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 + \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} & 0 & -\tilde{k}^2 K_0(\tilde{k}) - \tilde{k} K_1(\tilde{k}) \\ 1 - \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} & 0 & -\tilde{k} K_1(\tilde{k}) & 0 \\ 0 & \tilde{k}^2 K_0(\tilde{k}) + \tilde{k} K_1(\tilde{k}) & 0 & -1 - \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} \\ \tilde{k} K_1(\tilde{k}) & 0 & -1 + \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} \quad (6.2)$$

The eigenvalue problem that defines the case of analysis, presented in Eq. (6.2), was first obtained by Crow [35]. By contrast to the 2D case, the equation depends on two parameters, $\tilde{k}$ and $\tilde{d}$, that have to be defined ahead of the analysis. Once the values of both parameters are set, the solution can be obtained, but caution should be exercised that not every choice of the pair of values will be physically meaningful. Note also that $\tau(\tilde{k}\tilde{d})/\tilde{d}^2$ diverges when $\tilde{d} \rightarrow 0$, as the cutoff parameter was needed for the integrals not to diverge. That does not contradict the 2D case, because in that approach, $\tilde{k}$ goes to zero faster than the cutoff distance.

As the cutoff distance d is related to the vortex core size a , following the approach from Crow [35] for the case of the Rankine vortex as an averaged value, $d/a \approx 1/3$. Additionally, the vortex cores are not supposed to interact with each other, and therefore, $b \gg 2a$. These two constraints lead to $\tilde{d} \ll 0.2$.

Furthermore, the length of the oscillation cannot be smaller than the cutoff distance, because it has been removed from the integral. Therefore, an additional restriction should be imposed: $1/\tilde{k} > \tilde{d}$.

The stability of the counter-rotating vortices is governed by Eq. (6.2). The eigenvalue problem is solved for six selections of $\tilde{d}$: 10^{-6} , 0.001, 0.1, 0.2, 0.3 and 0.4, which cover most of the range in which the methodology is applicable. The results are shown for the first three values in the range $0 < \tilde{k} < 2$, as the eigenvalues are zero for higher values inside the region of validity of the method, see Figs. (6.3(a)-6.3(c)). For the last three choices of $\tilde{d}$, the range $0 < \tilde{k} < 6$ is examined, although some of the values shown are in the area out of bounds, which is shown shadowed in the figures. In fact, these three values of cutoff, Figs. (6.3(d)-6.3(f)), are outside of the region of validity for $\tilde{d}$, but they have been reproduced because they are similar to figure 9 of Crow [35], and include the shadowing of the non eligible area. There are two symmetric modes that come closer as $\tilde{d}$

¹ $e^{(\alpha-1)\pi i} K_{\alpha-1}(x) - e^{(\alpha+1)\pi i} K_{\alpha+1}(x) = \frac{2\alpha}{x} e^{\alpha\pi i} K_{\alpha}(x) \Rightarrow K_0(x) - K_2(x) = -\frac{2}{x} K_1(x)$

increases, becoming a single mode at $\tilde{d} \approx 0.25$. An antisymmetric mode also exists, which approaches $\tilde{k} = 0$ for larger values of $\tilde{d}$. Although this is the leading mode, its maximum is always outside of the range of validity of the method. For the area of realizable results, a single symmetric mode, known as Crow mode, grows from $\tilde{k} = 0$, having a maximum for a value of $\tilde{k}$ dependent of $\tilde{d}$, but always in the segment of $0 < \tilde{k} < 1$ (in the range of values considered). The maximum amplification rate of the mode is always in the range 0.5 to 1, varying little with changes of the cutoff parameter.

Figure 6.3: Amplification rate $\tilde{\omega}_i$ as a function of the wavenumber $\tilde{k}$ for $\tilde{d}$: (a) 10^{-6} , (b) 0.001, (c) 0.1, (d) 0.2, (e) 0.3 and (f) 0.4. (d) to (f) same as Crow [35] Fig. 9, highlighting the area out of bounds for high wavenumbers.

The four eigenvalues are always in pairs and are either real or imaginary. The couples have the same value with positive and negative sign. That is, when there is a growing mode, there exists a decaying mode, and the neutral eigenvalues appear always in pairs of positive and negative frequency.

The shape of the modes is introduced with the help of Figs. (6.4-6.6). Fig. (6.5) is a typical example, where the only growing mode is the symmetric Crow mode, Fig. (6.5(a)). On the other hand, Fig. (6.6) represents a case with two growing modes, symmetric Crow modes Fig. (6.5(a)) and antisymmetric non realizable Fig. (6.6(c)), outside of the valid region for the cutoff parameter, $\tilde{k} = 3$ and $\tilde{d} = 0.4$. Note here that the limits that have been taken both for $\tilde{k}$ and $\tilde{d}$ are diffuse, and the antisymmetric growing mode might be realistic in some cases. Its growth rate is also high, so it might compete with the symmetric growing mode if some particular wavelengths are excited. The symmetric growing mode is present in both Figs. (6.5(a), 6.6(a)). In the regions that either the symmetric or antisymmetric mode are neutrally stable, two modes with frequency appear, that neither grow nor decay, but move at a finite frequency in time both upstream and downstream. The neutral modes are represented in Figs. (6.4).

It is also worth mentioning that the amplification rate and frequency are zero for the limit of wavenumber approaching to zero, as seen in Fig. (6.3), which is in agreement with

Figure 6.4: The four modes for $\tilde{k} = 3$ and $\tilde{d} = 0.2$. Four neutral eigenvalues.

Figure 6.5: The four modes for $\tilde{k} = 1.5$ and $\tilde{d} = 0.3$. Two neutral eigenvalues, one growing and one decaying.

the two-dimensional analysis.

6.2.2 Pair of counter-rotating parallel viscous Gaussian vortices instability BiGlobal

The BiGlobal analysis consists basically in solving the eigenvalue problem defined in Eqs. (2.50). However, this solution is not straightforward and some considerations have to be done before. The numerical considerations are dealt with in detail in Chapter 3. In addition, some boundary conditions have to be imposed to the problem, which is a delicate aspect that will be analysed in depth, and also a basic flow and a wavenumber have to be selected, aspect, the first of them, that will be commented next.

Basic Flow

The basic flow is a field for velocities (U_x, U_y, U_z) and its derivatives, which must be a stationary solution of the Navier-Stokes equations. Sometimes it could be given analytically see section 1.1, or it could also be obtained as a solution of a 2D DNS simulation performed until the final result does not vary in time, steady solution.

Furthermore, these basic flows are not exact solutions of the Navier-Stokes equations, they are approximations. This will create some spurious results when calculating, but their importance can be minimised. One of the issues is the descent speed of the vortices,

Figure 6.6: The four modes for $\tilde{k} = 3$ and $\tilde{d} = 0.4$. Two growing eigenvalues and two decaying.

which is not taken into account by default, where a translation of the frame of reference is required in order to have a real basic flow (steady). This term is of order unity, so it should be added to the basic flow. Other important point is that the steady vortex models are inviscid solutions, but, as the real problem is viscous, the vortices will diffuse, and the basic flow will not be steady then. Luckily, if Re is high enough, this issue will be negligible. Finally, these basic flows are analytic solutions for a single vortex, but not for the pair, so the closer that the two vortices are, the further that the basic flow will be from being an actual solution of the flow equations. This can be overcome by means of 2D DNS. The analytical and not steady solution of the flow equation, can be introduced in the DNS code, and evolved some time steps until the solution is steady. In that case, Re should be high for the vortices not to diffuse too much, and disappear before the stationary solution is found. This choice has been taken by many as González et al. [53] among others. In particular, the superposition of two q -vortices is selected for the stability analysis carried out here. The equations were already exposed inside subsection 1.1.3, in particular, given by Eqs. (1.31).

Boundary Conditions

The boundary conditions for the variables $\hat{u}$, $\hat{v}$ and $\hat{w}$ are very easy to impose in the case that the two vortices are taken inside the domain and the boundaries are kept far away enough. In that case, Dirichlet conditions are imposed on all the boundaries. However, the symmetry of the problem allows to calculate the solution as the sum of symmetric and antisymmetric solutions of only one half. In that case, two different sets of conditions have to be imposed. Symmetry conditions mean Neumann boundary conditions, while antisymmetry mean Dirichlet boundary conditions.

Symmetric case: $\hat{v}$ and $\hat{w}$ are symmetric, while $\hat{u}$ is antisymmetric:

$$\begin{aligned}
 \hat{u}(x, y) &= -\hat{u}(-x, y) \\
 \hat{v}(x, y) &= \hat{v}(-x, y) \\
 \hat{w}(x, y) &= \hat{w}(-x, y)
 \end{aligned}
 \tag{6.3}$$

Antisymmetric case: $\hat{u}$ is symmetric, while $\hat{v}$ and $\hat{w}$ are antisymmetric:

$$\begin{aligned}\hat{u}(x, y) &= \hat{u}(-x, y) \\ \hat{v}(x, y) &= -\hat{v}(-x, y) \\ \hat{w}(x, y) &= -\hat{w}(-x, y)\end{aligned}\tag{6.4}$$

Comparison Full boundary and Half Boundary

First of all, a validation case is performed, where it is done a convergence study for the boundary conditions. It will be a case for low Re and q of order unity, which leads to some instabilities. The exact values of the problem are the following:

$$Re = 100, \quad q = 0.475, \quad W_\infty = 0.0, \quad W_0 = -1.0, \quad b = 4.0, \quad a = 1.0, \quad \beta = 0.418$$

Convergence of the solution is shown in table 6.2 for the first two unstable eigenvalues. Good convergence is seen for both eigenvalues, but the symmetric boundary seems to have better behaviour. Fig. (6.7) highlights that the union of the symmetric and antisymmetric modes is the same result as the full solution. Similarities can be found with figure 4b of Hein and Theofilis [66]. These spectra have been calculated with high order finite differences of order 8. The Krylov space was only of 100 points, and that is the reason because the number of eigenvalues is not very big and the continuous part of it is not very well converged, although it is not needed either.

Table 6.2: Table of convergence for the two more unstable modes.

Mesh	N_x	N_y	Mode 1 ω_i	Mode 1 ω_r	Mode 2 ω_i	Mode 2 ω_r
Symmetric	150	300	0.103125945	0.042288781	-	-
Antisymmetric	150	300	-	-	0.083025955	0.057975379
Full	200	200	0.103125949	0.042288787	0.083026013	0.057969180
Symmetric	100	200	0.103125945	0.042288785	-	-
Antisymmetric	100	200	-	-	0.083026064	0.057975206
Full	150	150	0.103125943	0.042288810	0.083026005	0.057969173
Symmetric	75	150	0.103125958	0.042288818	-	-
Antisymmetric	75	150	-	-	0.083025996	0.057963435
Full	100	100	0.103126030	0.042289024	0.083026145	0.057969327
Symmetric	50	100	0.103126312	0.042289742	-	-
Antisymmetric	50	100	-	-	0.083026838	0.057975630
Full	50	50	0.103142154	0.042328588	0.083062988	0.058011763

The modes are complex and therefore, as the grid is not the same for the case of full domain and the case of half domain, the eigenvectors are moved one phase and then they do not look the same for half and full domain. However, the reconstruction in space and time will give the same Figs. (6.8, 6.9), which represent the value of the imaginary part of the modes, only for the full domain simulations. There, the kind of boundary condition in the center is appreciated, where it can be seen that all the variables, but $\hat{u}$ follow the symmetry or antisymmetry of the mode, while $\hat{u}$ does the opposite.

Figure 6.7: Spectrum obtained for both domains.

Effect of adding the descent speed

The results presented before are calculated without adding the descent speed of the vortices to the basic flow and for an analytic and imposed base flow. In this paragraph it is shown how the descent speed, $V_d = q/b = 0.11875$, even if small, displaces the spectrum. For that purpose, Fig. (6.10) is shown, where convergence have been analysed again, as well as the symmetric and antisymmetric meshes are used to show whether the eigenvalues are symmetric or antisymmetric. The shape of the first two modes (not shown here) is very similar to those without descent speed added (again symmetric and antisymmetric respectively), and therefore these two modes correspond to the ones with descent speed equal zero.

Viscous-Inviscid comparisons

Viscous linear stability theory is considered and results of its predictions are compared with those delivered by the inviscid analysis. There are several works that have applied viscous modal analysis to vortical flows among which only a few addressed the issue of direct comparisons between inviscid, vortex-filament-based, and viscous linear stability analysis; the present work intends to provide such a comparison.

Two sets of parameters have been examined, $(\tilde{d} = 0.1, \tilde{a} = 0.1145)$ and $(\tilde{d} = 0.2, \tilde{a} = 0.2290)$, where $\tilde{a} = a/b$ is the non-dimensional vortex core radius, not defined earlier, but using again b for the non-dimensionalisation purpose in this case and $\sim$ for non-dimensional values. The correspondence between $\tilde{d}$ and $\tilde{a}$ given by the Widnall [141] formulas: $\tilde{a} = \tilde{d}/(0.6420e^{1/4-A+C})$ and Eqs. (1.23), particularised A and C for the Lamb-Oseen vortex model, Eqs. (1.25). In the results shown in what follows, $Re = 1000$ was

Figure 6.8: Imaginary part of the four components of the first symmetric mode for the values specified above.

found to be a good compromise between obtaining Reynolds-independent amplification rates while maintaining the cost of solution of Eq. (2.50) tractable. Results are shown in Fig. (6.11), where it can be observed that the matching between the results delivered by the two approaches is much better for the first choice of parameters. The explanation for this behaviour is that the second set of parameters corresponds to vortices that are closer together than those corresponding to the first choice of parameters. At the relatively low Reynolds number value chosen, viscous diffusion makes the idealised inviscid model questionable in that case. Benton and Bons [13] found a similar behaviour in their analysis, the comparison between the two approaches being relatively good when the vortices are far apart, but viscous analysis results departing from those of the inviscid vortex filament theory as the vortices come closer. By contrast, Brion et al. [17] obtained excellent agreement between results delivered by the two approaches even when the two vortices are rather close. However, they used a DNS calculated basic flow, and most probably a better adjustment for the vortex radius, both for the viscous and the inviscid calculations. Despite of that, the results still prove both methodologies to produce reliable results when the vortices are not very close. The limit of validity of these good matching is between $\tilde{d} = 0.1$ ($\tilde{a} = 0.1145$) and $\tilde{d} = 0.2$ ($\tilde{a} = 0.2290$).

Fig. (6.12) presents a superposition of the instability analysis results delivered by the two methodologies at a value of the wavenumber parameter $\tilde{k} = 0.8$, which corresponds to the maximally amplified perturbation in both analyses. The superposition of the base

Figure 6.9: Imaginary part of the four components of the first antisymmetric mode for the values specified above.

flow vorticity and that of the leading eigenmode at a linearly small amplitude parameter, $\varepsilon = 0.1$ is plotted. Lines represent the vortex filament result, while the viscous eigenmode is shown with a finite vortex radius. The direction x of the homogeneous perturbation is scaled to show the vortices in a more compact way, as the wavenumber is very small. From a numerical point of view, introduction of viscosity raises the 3 sec computing time needed for the recovery of the inviscid result to nearly 6 minutes per wavenumber analysis, while the memory needed for the viscous instability work is orders of magnitude larger than that used in the inviscid analysis.

In order to check the dependence with the vortex model, two cases have been selected, where the viscous BiGlobal is performed both for the Lamb-Oseen (LO) and the Rankine (R) vortex, as well as the inviscid filaments methodology is also compared. Table 6.3 shows this comparison for two vortex sizes, $\tilde{d} = 0.1$ ($\tilde{a} = 0.1145$ for LO and $\tilde{a} = 0.1558$ for R) and $\tilde{k} = 0.8$, as well as $\tilde{d} = 0.2$ ($\tilde{a} = 0.2290$ for LO and $\tilde{a} = 0.3115$ for R) and $\tilde{k} = 1.0$, both near to the maximum growth rate of the mode for each vortex size. The Reynolds number is kept the same for the viscous simulations ($Re = 1000$). Both viscous vortex models show degradation of agreement for the vortices with greater cores with respect to the inviscid case. For the two selections of vortex size, the agreement between Rankine and inviscid is better than between Lamb-Oseen and inviscid, but always of the same order.

Figure 6.10: Spectrum to show the effect of the descent speed, in this case, $V_d = 0.11875$.

$\tilde{d}$	$\tilde{k}$	$\tilde{\omega}_i$ Inviscid	$\tilde{\omega}_i$ Viscous LO	$\tilde{\omega}_i$ Viscous R	Relative with Inviscid, LO/R
0.1	0.8	0.8145...	0.7969...	0.8049...	$2.17 \times 10^{-02} / 1.18 \times 10^{-02}$
0.2	1.0	0.7883...	0.6782...	0.7036...	$1.40 \times 10^{-01} / 1.07 \times 10^{-01}$

Table 6.3: Comparison of amplification rates $\tilde{\omega}_i$ for inviscid and viscous ($Re = 1000$) methodologies, showing the vortex model dependence for the two counter-rotating vortex system selecting two different vortex sizes: $\tilde{d} = 0.1$ ($\tilde{a} = 0.1145$ for LO and $\tilde{a} = 0.1558$ for R), $\tilde{k} = 0.8$ and $\tilde{d} = 0.2$ ($\tilde{a} = 0.2290$ for LO and $\tilde{a} = 0.3115$ for R), $\tilde{k} = 1.0$, always for the Crow mode (symmetric).

Spatial BiGlobal

The spatial BiGlobal analysis is used to obtain the most spatial amplified eigenmode for the initial integration station. Two eigenmodes are found with equal growth rate, one symmetric and another antisymmetric. Fig. (6.13) shows the shape functions of the most spatially amplified modes, corresponding to the $d = 6$ case. The spatial structure is similar to the eigenfunction recovered for the isolated vortex flow analysis, with five lobes per vortex, $m = 5$. Same shape functions is observed for $d = 4$ and $d = 8$ with same parameters. Results of this analysis are later used as initialisation of the PSE-3D.

Figure 6.11: Comparison between filament (lines) and BiGlobal theory at $Re = 1000$ (points). Amplification rate $\tilde{\omega}_i$ as a function of the wavenumber $\tilde{k}$ for $\tilde{d} = 0.1$ and $\tilde{a} = 0.1145$ (solid line and circles) and $\tilde{d} = 0.2$ and $\tilde{a} = 0.2290$ (dashed line and squares), from Tendero et al. [131].

Figure 6.12: Basic flow plus perturbation for the leading mode at $\tilde{k} = 0.8$ comparison between the filament positions (filament instability) at $\tilde{d} = 0.1$ (lines) and vorticity isosurfaces (BiGlobal) for $Re = 1000$ and $\tilde{a} = 0.1145$ (surfaces). Perpendicular direction not scaled [131].

Figure 6.13: The most unstable eigenmodes of the counter-rotating Batchelor vortex Eqs. (2.50) with $d = 6$, $\gamma_0 = 0.8$ and $\kappa_0 = 0.4$ at $Re = 3000$ and $\omega = 2.55$, (left) symmetric and (right) antisymmetric eigenmodes plotted with iso-contours $\hat{u}/|u| = (-0.9 : 0.1 : 0.9)$ of normalised real part of axial velocity amplitude function (dashed lines correspond to negative values).

6.2.3 Pair of counter-rotating non-parallel viscous Gaussian vortices instability PSE-3D

At this point, the study of the stability of the counter-rotating vortex pair with axial diffusion can be performed. Examples of the results to expect are as that of Fig. (6.14), for a core instability in a realistic counter-rotating vortex pair flow field. The result is just shown as illustration of the capabilities of the methodology, as the thesis is more focussed in the axially homogeneous flows, and the duality of point vortex and grid methods. It is worth mentioning that PSE-3D is the only analysis methodology capable of the analysis of non-parallel vortex systems, regardless the prohibitively expensive TriGlobal instability analysis.

Paredes et al. [115] solved the three-dimensional Parabolised Navier Stokes equations by fully resolving a plane on which flow depends strongly and marching the equations of motion along the third spatial direction, convergence aided by adjusting the marching spatial direction to take into account the flow downwash. Fig. (5.11) shows the basic state initialised by two counter-rotating (axially-dependent) Batchelor vortices and obtained by PNS at $Re = 3000$. Full description of the parameters used can be found in the original reference. Next results are corresponding to fixed $\omega = 2.55$. The computational domain

in the x - y plane is truncated at $(x_\infty, y_\infty) = (35, 35)$. It is discretised using $N_x \times N_y = 181 \times 121$ FD-q10 nodes mapped by Eq. (B.19) setting $c_x = 0.93$ and $c_y = 0.85$. The result is Fig. (6.14), which shows the iso-surfaces of axial vorticity flowfield for the initial most unstable perturbations (plotting the symmetric eigenmode). The inhomogeneous nature of the underlying flow is evident in the non-axisymmetric shape of the amplitude functions of perturbations.

Of interest in the present context is that the residuals that needed to be damped before a steady PNS solution was obtained were qualitatively identical with the eigenmodes shown in the work of González et al. [54] at a comparable Reynolds number. However, unlike the two-dimensional, axially homogeneous DNS used to obtain those results, here the base flow solution permits axial flow diffusion and mutual vortex interaction. Good agreement has been demonstrated with independently obtained experimental results at a given set of parameters. When the distance between the vortices is increased, the local vortex radii approach the theoretical value known from the isolated vortex model. However, when the vortices come progressively closer, strong deviations from the theoretical result are observed. The interested reader is referred to the original reference for more details.

Figure 6.14: Vorticity of the superposition of the base flow shown in Fig. (5.11) and the leading linear perturbation computed by PSE-3D [112]

6.3 Pair of co-rotating vortices instabilities

The stability analysis of the co-rotating vortex pair is performed next. Despite the typical vortex wake is not formed by co-rotating vortex pairs, pairs of co-rotating vortices could be part of it. It was seen in section 5.3 that the co-rotating vortices tend to merge if they are close enough. Therefore, only the case of distant co-rotating vortices is analysed here by means of point vortices.

6.3.1 Pair of co-rotating point vortices instabilities [76]

First, the two-dimensional case without perturbations in the axial direction is analysed, and later, the vortex filaments are allowed to have sinusoidal perturbations.

The only difference with the case of counter-rotating vortices is the change of sign in the circulation of one vortex. When that is done, it is found that $\bar{\mathbf{U}} = \pm(\Gamma/2\pi b)\mathbf{e}_y$, for the vortex 2 and 1 respectively, which are not the same, and therefore, the configuration is not in equilibrium in that reference frame. In order to overcome this difficulty, Jiménez [76] chose a rotating reference system, where z is a common coordinate for both vortices, x is the radial coordinate, pointing outward, and y is the azimuthal coordinate following the direction of the movement of each vortex. Eqs. (2.74) are re-written changing the sign of $\hat{x}_2$ and $\hat{y}_2$ and taking for the vortex 1, $X_1 = 0$ and $X_2 = -b$ and for the vortex 2, $X_1 = -b$ and $X_2 = 0$, all the rest remaining equal. Using that system, $\bar{\mathbf{U}} = (\Gamma/2\pi b)\mathbf{e}_y$ becomes the same for the two vortices. Furthermore, as the reference system is rotating, a new term appears, as $[\mathbf{dx}'_p/\text{dt}]_{\text{Inertial}} = [\mathbf{dx}'_p/\text{dt}]_{\text{rotating}} + \boldsymbol{\Omega} \times \mathbf{x}'_p$, where $\boldsymbol{\Omega} = \Gamma/(\pi b^2)\mathbf{e}_x$ is the angular velocity. The non-dimensionalization could have been done choosing πb as distance and Γ as circulation. Time would then be non-dimensionalized by $2\pi^2 b^2/\Gamma$, which is the time needed to propagate a distance πb , that is one turn. However, that will lead to a single multiplicative constant π in the equations. For this reason the same non-dimensionalization is used here as for the case of the counter-rotating vortices. The values of the new variables are again: $\tilde{\omega} = 2\pi\omega b^2/\Gamma$, $\tilde{x}_i = \hat{x}_i/b$ and $\tilde{y}_i = \hat{y}_i/b$. The system of equations is written in the form of an eigenvalue problem:

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} \quad (6.5)$$

Solution of Eq. (6.5) now results in one double eigenvalue $-i\tilde{\omega} = 0$ and two complex conjugates $-i\tilde{\omega} = \pm 2i$. In mathematical terms the zero real part of the eigenvalue again implies neutral stability, which manifests itself in physical terms as a situation in which two co-rotating vortices moved from their original positions, continue forming a displaced system of two vortices rotating around the centroid.

Now, the configuration of two co-rotating vortices with homogeneous perturbations in the axial coordinate already analysed by Jiménez [76] is revisited here. The existence of an equilibrium solution was addressed in the two-dimensional analysis, while stability can be analysed by solution of Eqs. (2.80). The pertinent values now are: $N = 2$, $p = 1$,

2, $Y_{1,2} = 0$, $\Gamma_1 = \Gamma_2 = \Gamma$ and $d_1 = d_2 = d$, while $X_1 = 0$ and $X_2 = -b$ for vortex 1 and $X_1 = -b$ and $X_2 = 0$ for vortex 2. The sign of $\hat{x}_2$ and $\hat{y}_2$ must be changed and a new term has to be introduced, due to the rotation of the system of reference. The same non-dimensionalization scales are used, namely $2\pi b^2/\Gamma$ for time and b for distances. The equations are written as an EVP using the same nomenclature as before:

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 + \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} & 0 & -\tilde{k}^2 K_0(\tilde{k}) - \tilde{k} K_1(\tilde{k}) \\ -3 - \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} & 0 & -\tilde{k} K_1(\tilde{k}) & 0 \\ 0 & -\tilde{k}^2 K_0(\tilde{k}) - \tilde{k} K_1(\tilde{k}) & 0 & 1 + \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} \\ -\tilde{k} K_1(\tilde{k}) & 0 & -3 - \frac{\tau(\tilde{k}\tilde{d})}{2\tilde{d}^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \end{pmatrix} \quad (6.6)$$

This equation is the same as the one found by Jiménez [76]. The values of the two parameters $\tilde{k}$ and $\tilde{d}$, need to be chosen in advance of numerical solution of Eq. (6.6). The same constraints considered in the previous section are also applicable to this section: $\tilde{d}$ should be smaller than around 0.2 (because the vortex cores should be far away) and $1/\tilde{k} > \tilde{d}$ (because the cutoff distance do not consider higher frequency oscillations).

The instability analysis of the co-rotating vortices is performed through the study of the eigenvalues of Eq. (6.6). This configuration is stable for the range of parameters considered, $\tilde{d} \leq 0.2$ and $1/\tilde{k} > \tilde{d}$. There are two unstable modes, with their respective stable modes, symmetric and antisymmetric for a range of wavenumbers $\tilde{k}$ slightly higher than the range $1/\tilde{k} > \tilde{d}$, but because they are out of the range of validity of the method, their existence cannot be justified. These two modes appear almost for the same range of wavenumbers: the symmetric is the first to appear and the more unstable of the two, having a maximum value of $\tilde{\omega}_i$ slightly larger than unity and varying slightly for cutoff parameter $\tilde{d}$ variations.

In the rest of the wavenumber range (the valid one), there are always four neutral modes in pairs, symmetric and antisymmetric. The higher frequency is always for the antisymmetric, whose value is two in non-dimensional values for $\tilde{k} = 0$ and any $\tilde{d}$, which corresponds to the 2D case. The value of the frequency of this mode grows as the wavenumber increases, reaching a maximum just before the end of validity of the approach and depending on the cutoff parameter. The symmetric mode has a similar behavior, having a smaller frequency than the antisymmetric mode. In the limit of $\tilde{k} \rightarrow 0$, $\tilde{\omega}_i \rightarrow 0$ and $\tilde{\omega}_r \rightarrow 0$ or ± 2 , which coincides with the 2D case.

6.4 Four aligned vortices instabilities

This section addresses the instability analysis of the configuration of four aligned vortices, already presented in section 5.5. Therefore, more details are not added here, as they have been already collected there.

6.4.1 Four aligned point vortices instabilities [43, 116]

Configurations of two aligned vortex pairs with a common center of symmetry are analysed in this section. The values for the parameters appearing in Eqs. (2.74) that define these configurations are: $N = 4$, $X_1 = -X_4$, $X_2 = -X_3$, $Y_1 = Y_2 = Y_3 = Y_4 = 0$, $\Gamma_1 = -\Gamma_4$ and $\Gamma_2 = -\Gamma_3$. This set of values does not necessarily lead to an equilibrium configuration, and different situations can occur. The configuration may diverge, it may have a periodic movement or it may be stationary, that is, the vortices will move together keeping the same distances among them. Donaldson and Bilanin [41] first studied these different situations, and Fabre et al. [44] showed the different regions in a more clear way. Only the equilibrium configuration is studied here, which is obtained when $\bar{\mathbf{U}}$ is imposed to be the same for all four vortices. In fact, due to the symmetry of the problem, this condition only needs to be imposed to the inner and outer vortices on one side. Eq. (2.72a) is used for that purpose, which leads to the celebrated cubic equation first derived by Rennich and Lele [116], (see also Jacquin et al. [74]), Eq. (5.6), where $\mathcal{X} = X_3/X_4$ and $\mathcal{G} = \Gamma_3/\Gamma_4$. The implication is that any ratio between circulations, $\mathcal{G}$, will lead to a ratio between inner and outer vortex positions. If the equation is written as $\mathcal{G}(\mathcal{X}) = -\mathcal{X}(\mathcal{X}^2 + 3)/(3\mathcal{X}^2 + 1)$, the following properties are deduced: $\mathcal{G}(-\mathcal{X}) = -\mathcal{G}(\mathcal{X})$ and $\mathcal{G}(1/\mathcal{X}) = 1/\mathcal{G}(\mathcal{X})$. Therefore, as the solution is odd, negative values of $\mathcal{X}$ do not offer additional results to these obtained by the positive values, and, due to the second property, results for $|\mathcal{X}| > 1$ are already considered when $1/|\mathcal{X}| < 1$ is analysed. Therefore, only the range $0 < \mathcal{X} < 1$ and $-1 < \mathcal{G} < 0$ is considered because it includes all the possible solutions. Fig. (6.15) shows the solution of Eq. (5.6) for that range.

Figure 6.15: Solution of the equation of equilibrium for four point vortices.

With the understanding that $\mathcal{X}$ is a function of $\mathcal{G}$, the parameters appearing in Eqs. (2.74) take the values: $N = 4$, $X_1 = -b_1/2$, $X_2 = -b_2/2 = -\mathcal{X}b_1/2$, $X_3 = b_2/2 = \mathcal{X}b_1/2$, $X_4 = b_1/2$, $Y_1 = Y_2 = Y_3 = Y_4 = 0$, $\Gamma_1 = -\Gamma_0$, $\Gamma_2 = -\mathcal{G}\Gamma_0$, $\Gamma_3 = \mathcal{G}\Gamma_0$, $\Gamma_4 = \Gamma_0$, where b_1 and b_2 are the distances between the outer and the inner vortices respectively. Substituting these parameters into Eqs. (2.74) and non-dimensionalising all variables, an

EVP $-\mathrm{i}\tilde{\omega}\tilde{\mathbf{x}} = \mathbf{A}\tilde{\mathbf{x}}$ results:

$$-\mathrm{i}\tilde{\omega} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \\ \tilde{x}_3 \\ \tilde{y}_3 \\ \tilde{x}_4 \\ \tilde{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & G_5 & 0 & \mathcal{G}G_2 & 0 & -\mathcal{G}G_1 & 0 & -G_3 \\ G_5 & 0 & \mathcal{G}G_2 & 0 & -\mathcal{G}G_1 & 0 & -G_3 & 0 \\ 0 & G_2 & 0 & G_6 & 0 & -\mathcal{G}G_4 & 0 & -G_1 \\ G_2 & 0 & G_6 & 0 & -\mathcal{G}G_4 & 0 & -G_1 & 0 \\ 0 & G_1 & 0 & \mathcal{G}G_4 & 0 & -G_6 & 0 & -G_2 \\ G_1 & 0 & \mathcal{G}G_4 & 0 & -G_6 & 0 & -G_2 & 0 \\ 0 & G_3 & 0 & \mathcal{G}G_1 & 0 & -\mathcal{G}G_2 & 0 & -G_5 \\ G_3 & 0 & \mathcal{G}G_1 & 0 & -\mathcal{G}G_2 & 0 & -G_5 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \\ \tilde{x}_3 \\ \tilde{y}_3 \\ \tilde{x}_4 \\ \tilde{y}_4 \end{pmatrix} \quad (6.7)$$

The eigenvalues $\tilde{\omega}$ of the matrix $\mathbf{A}$ provide the stability of the system. The distances are non-dimensionalised by the distance between the centroids of each side pair of vortices b and the circulation by the sum of the two different circulations Γ , as done by Crouch [32]. The characteristic time is then $2\pi b^2/\Gamma$. $\Gamma = (1 + \mathcal{G})\Gamma_0$, $b = (1 + \mathcal{X}\mathcal{G})b_1/(1 + \mathcal{G})$, $\tilde{\omega} = 2\pi\omega b^2/\Gamma$, $\tilde{x}_i = \hat{x}_i/b$ and $\tilde{y}_i = \hat{y}_i/b$ are the values that have been used in order to obtain G_i of Eqs. (6.7), which are given in Eqs. (6.8).

$$\begin{aligned} G_1 &= \frac{4(1 + \mathcal{X}\mathcal{G})^2}{(1 + \mathcal{G})^3(1 + \mathcal{X})^2} & G_2 &= \frac{4(1 + \mathcal{X}\mathcal{G})^2}{(1 + \mathcal{G})^3(1 - \mathcal{X})^2} \\ G_3 &= \frac{(1 + \mathcal{X}\mathcal{G})^2}{(1 + \mathcal{G})^3} & G_4 &= \frac{(1 + \mathcal{X}\mathcal{G})^2}{(1 + \mathcal{G})^3\mathcal{X}^2} \\ G_5 &= \mathcal{G}(G_1 - G_2) + G_3 & G_6 &= G_1 - G_2 + \mathcal{G}G_4 \end{aligned} \quad (6.8)$$

The eigenvalues $\tilde{\omega}$ in Eq. (6.7) are calculated as a function of the circulation ratio $\mathcal{G}$. Fig. (6.16) shows the variation, where $\tilde{\omega}_i \rightarrow \infty$ as $\mathcal{G} \rightarrow -1$, which means that in that limit this configuration is very unstable: the vortices are very near to each other and any small perturbation of the position will change their relative positions by a very large amount. This limit is actually also prone to larger errors in the computation of the eigenvalues.

Figure 6.16: Amplification rate $\tilde{\omega}_i$ as function of the circulation ratio $\mathcal{G}$ for the four vortices in equilibrium configuration.

The configuration becomes less unstable when $\mathcal{G} \rightarrow 0$. This limit ($\mathcal{G} = \mathcal{X} = 0$) is actually the same as that of a two-vortex configuration. However, although the eigenvalues corresponding to a two-vortex system in which $\mathcal{G} = 0$ are $\tilde{\omega} = 0$, this is not the case when the two-vortex system is approached as a limit-case of the four-vortex configuration.

The explanation of this contradiction, is that the modes associated with the non-zero eigenvalues only affect the inner and weaker vortices. In the limit, they are infinitely weak and do not really exist, and therefore, the mode itself does not really exist for $\mathcal{G} \rightarrow 0$. The value of the amplification rate in the case $\mathcal{G} = \mathcal{X} = 0$ is increasingly challenging to obtain numerically because the problem becomes singular. It can be solved analytically by means of Taylor series, and two double eigenvalues $\tilde{\omega}_i = \pm 4\sqrt{6}$ (9.79795897...) appear, in addition to the quadruple zero eigenvalue. As $\tilde{\omega}_r = 0$, all modes are stationary, so only the amplification rate ($\tilde{\omega}_i$) is shown in Fig. (6.16).

The configuration formed by a system of four vortices in equilibrium, as proposed by Rennich and Lele [116], is addressed next. In addition to what was presented before, homogeneous perturbations in the third direction are permitted. In order to analyse stability, Eqs. (2.80) are written for $N = 4$, while the remaining parameters have the same values as in the 2D case: $X_1 = -b_1/2$, $X_2 = -b_2/2 = -\mathcal{X}b_1/2$, $X_3 = b_2/2 = \mathcal{X}b_1/2$, $X_4 = b_1/2$, $Y_1 = Y_2 = Y_3 = Y_4 = 0$, $\Gamma_1 = -\Gamma_0$, $\Gamma_2 = -\mathcal{G}\Gamma_0$, $\Gamma_3 = \mathcal{G}\Gamma_0$, $\Gamma_4 = \Gamma_0$. The relation Eq. (5.6) between $\mathcal{G}$ and $\mathcal{X}$ is again fulfilled. The same non-dimensionalization as in the 2D case is used: b is the length scale and $2\pi b^2/\Gamma$ the time scale, being $\Gamma = (1 + \mathcal{G})\Gamma_0$, $b = (1 + \mathcal{X}\mathcal{G})b_1/(1 + \mathcal{G})$, $\tilde{\omega} = 2\pi\omega b^2/\Gamma$, $\tilde{x}_i = \hat{x}_i/b$ and $\tilde{y}_i = \hat{y}_i/b$, and, as before, only the range $-1 < \mathcal{G} < 0$ is considered. In addition, new parameters appear because of the three dimensional perturbations: the wavenumber, k , the inner vortex cutoff, d_I , and the outer vortex cutoff, d_O . Their respective non-dimensional values are: $\tilde{k} = bk$, $\tilde{d}_I = d_I/b$ and $\tilde{d}_O = d_O/b$. Finally, the eigenvalue problem $-\tilde{\omega}\tilde{\mathbf{x}} = \mathbf{A}\tilde{\mathbf{x}}$ can be written, where $\tilde{\mathbf{x}}$ and $\mathbf{A}$ will be defined shortly. It depends on $\mathcal{G}$, $\mathcal{X}$, $\tilde{k}$, $\tilde{d}_O$ and $\tilde{d}_I$, which are constant. Here, the same G_i , given in Eqs. (6.8) are used, as well as $\tau(\theta) = \cos\theta - 1 + \theta\sin\theta - \theta^2 Ci(\theta)$, $K_x(\tilde{k}, \theta) = \tilde{k}^2 K_0(\tilde{k}/\theta) + \tilde{k}\theta K_1(\tilde{k}/\theta)$ and $K_y(\tilde{k}, \theta) = \tilde{k}\theta K_1(\tilde{k}/\theta)$. For simplicity in the notation and to highlight the different terms, the eigenvalue problem will be written as a function of Ξ_O , Ξ_I , K_{xi} and K_{yi} , defined as:

$$\begin{aligned}\Xi_O(\mathcal{G}, \tilde{k}, \tilde{d}_O) &= \frac{\tau(\tilde{k}\tilde{d}_O)}{2(1 + \mathcal{G})\tilde{d}_O^2} \\ \Xi_I(\mathcal{G}, \tilde{k}, \tilde{d}_I) &= \frac{\mathcal{G}\tau(\tilde{k}\tilde{d}_I)}{2(1 + \mathcal{G})\tilde{d}_I^2} \\ K_{xi}(\mathcal{X}, \mathcal{G}, \tilde{k}) &= \frac{1}{(1 + \mathcal{G})} K_x\left(\tilde{k}, \sqrt{(1 + \mathcal{G})G_i}\right) \\ K_{yi}(\mathcal{X}, \mathcal{G}, \tilde{k}) &= \frac{1}{(1 + \mathcal{G})} K_y\left(\tilde{k}, \sqrt{(1 + \mathcal{G})G_i}\right)\end{aligned}\tag{6.9}$$

With the aid of these quantities, $\mathbf{A}$ becomes:

$$\mathbf{A} = \begin{pmatrix} 0 & G_5 + \Xi_O & 0 & \mathcal{G}K_{x2} & 0 & -\mathcal{G}K_{x1} & 0 & -K_{x3} \\ G_5 - \Xi_O & 0 & \mathcal{G}K_{y2} & 0 & -\mathcal{G}K_{y1} & 0 & -K_{y3} & 0 \\ 0 & K_{x2} & 0 & G_6 + \Xi_I & 0 & -\mathcal{G}K_{x4} & 0 & -K_{x1} \\ K_{y2} & 0 & G_6 - \Xi_I & 0 & -\mathcal{G}K_{y4} & 0 & -K_{y1} & 0 \\ 0 & K_{x1} & 0 & \mathcal{G}K_{x4} & 0 & -G_6 - \Xi_I & 0 & -K_{x2} \\ K_{y1} & 0 & \mathcal{G}K_{y4} & 0 & -G_6 + \Xi_I & 0 & -K_{y2} & 0 \\ 0 & K_{x3} & 0 & \mathcal{G}K_{x1} & 0 & -\mathcal{G}K_{x2} & 0 & -G_5 - \Xi_O \\ K_{y3} & 0 & \mathcal{G}K_{y1} & 0 & -\mathcal{G}K_{y2} & 0 & -G_5 + \Xi_O & 0 \end{pmatrix}\tag{6.10}$$

As done by Crouch [32], a new parameter $\tilde{d}$ is defined, which will be equal to the outer vortices non-dimensional cutoff, $\tilde{d} = \tilde{d}_O$. The inner vortices are assumed to follow the relation $\tilde{d}_I = \tilde{d} |\mathcal{G}|^{1/2}$, which is a consequence of having fixed the vorticity peak and distribution. This choice implies that the only differences in circulation are due to the core size. Of course this is only one possible choice. Others, such as Fabre and Jacquin [43] did not follow this approach, and explicitly took $\tilde{d}_I = 0.5\tilde{d}_O$. However, the assumption of Crouch [32], besides reducing the number of parameters in the problem, is realistic in terms of representation of actual vortex systems and will be used here.

The problem now, has three parameters, $\mathcal{G}$, $\tilde{d}$ and $\tilde{k}$, and will be posed in the following ranges: $-1 < \mathcal{G} < 0$, and $\tilde{d}$ and $\tilde{k}$ given in Figs. (6.17(a), 6.17(b)) respectively. The reasons for the choice of range of $\mathcal{G}$ were explained later. The value of $\tilde{d}$ is a function of the vortex core, and, for the model to be valid, the cores of different vortices should not interact. Two kind of interactions could happen, between outer and inner vortex or between the two inner vortices. For $\mathcal{X} \approx 1$, the first interaction is more probable, while the opposite is the case when $\mathcal{X} \approx 0$. The first case, where the difference between the two centroids should be bigger than the sum of the two vortex cores, that is, for a ratio around one third between cutoff and core size: $(1 - \mathcal{X})(1 + \mathcal{G})/[2(1 + \mathcal{X}\mathcal{G})] > 3\tilde{d}(1 + |\mathcal{G}|^{1/2})$. On the other hand, when the distance between the inner vortices is bigger than the inner vortex diameter, $\mathcal{X}(1 + \mathcal{G})/(1 + \mathcal{X}\mathcal{G}) > 6\tilde{d} |\mathcal{G}|^{1/2}$ is considered, again taking the ratio between cutoff and core size to be around one third. The wavenumber $\tilde{k}$ should be given in such a way that the oscillations of the vortex filament have a wavelength bigger than the vortex core, hence, $1/\tilde{k} > \tilde{d}$. In the limits $\mathcal{G} = -1$ and $\mathcal{G} = 0$, $\mathcal{X} = 1$ and $\mathcal{X} = 0$ respectively, and $\tilde{d}$ should be zero as it is shown in Fig. (6.17(a)). That leads to any possible $\tilde{k}$ from Fig. (6.17(b)) and no restriction is applied to the oscillations wavelength.

Figure 6.17: Valid values as a function of $\mathcal{G}$, shadow region.

The results of the Rennich and Lele [116] configuration are analysed now from a three-dimensional point of view. In addition, as a cross-validation of the results presented earlier for two vortices, calculations are performed for $\mathcal{G} = -10^{-5}$ and the results are compared with those of the two counter-rotating vortices configuration. If the figures of the two vortices, Figs. (6.3), are compared with figures for the four vortex approach, four modes are indistinguishable, and other four modes appear, some of them much more unstable, but

affecting the inner vortices, which are very weak in that limit. As an example, the values of $\tilde{\omega}_i$ for $\tilde{k} = 3.5$ are compared in both cases for the two Crow eigenvalues in table 6.4, in which $\tilde{d} = 0.3$ is chosen. This value is well outside the validity region of the method, but the objective here is to cross-verify that the two configurations deliver analogous results.

Mode	$\tilde{\omega}_i$ 2 Vortices	$\tilde{\omega}_i$ 4 Vortices	Relative Error
Antisymmetric	1.092960238456726	1.092994599627738	3.14×10^{-05}
Symmetric	0.8669134974479675	0.8669361404957115	2.61×10^{-05}

Table 6.4: Comparison of amplification rates $\tilde{\omega}_i$ for 2 and 4 vortices taking $\mathcal{G} = -10^{-5}$ in the 4 vortex problem, $\tilde{k} = 3.5$ and $\tilde{d} = 0.3$.

Results of the 3D approach should also converge to those of the 2D approach when the wavenumber is taken very small. This has been checked for $\tilde{k} = 10^{-3}$, where the values for 2D ($\tilde{k} = 0$) are recovered. Table 6.5 compares the value of the non-zero eigenvalue both for the 2D problem and in the limit of the 3D when both $\mathcal{G}$ and $\tilde{k}$ are close to zero for $\tilde{d} = 0.3$, as other values of the cutoff will show similar agreement. Here, the 4 values (two positive and two negative) obtained numerically are not equal, therefore, only the digits that are common are presented here.

$\tilde{\omega}_i$ 2D Analytic	$\tilde{\omega}_i$ 3D $\tilde{k} = 10^{-3}$	Relative Error
9.797958971132712	9.7982...	2.46×10^{-05}

Table 6.5: Comparison of amplification rates $\tilde{\omega}_i$ for 4 vortices, both 2D and 3D when both problems approach as $\tilde{k}$ tends to zero. $\mathcal{G} = 0$ in the 2D problem and $\mathcal{G} = -10^{-5}$ and $\tilde{d} = 0.3$ in the 3D problem.

Attention is now turned to the 4-vortex system, in which finite values of $\mathcal{G}$ are considered. The pair of parameters $\tilde{d} = 0.02$ and $\mathcal{G} = -0.4$ is considered, which is in the upper limit for the vortex core taken from Fig. (6.17(a)), where, as seen in Fig. (6.17(b)), $\tilde{k}$ can take values from 0 to approximately 40. Fig. (6.18) shows the amplification rate, where only the range of wavenumbers from 0 to 20 is shown, as all the eigenvalues are neutral above this limit. In that range, there exist three unstable modes. The highest of them is symmetric, solid line in Fig. (6.18), shown in Fig. (6.19(a)), where it can be seen that is a mode affecting mainly the inner vortices. The second in amplification (dashed line in Fig. (6.18), Fig. (6.19(b))) is antisymmetric, has similar growth rate as the symmetric mode and also affects more the inner vortices. In the limit $\tilde{k} \rightarrow 0$ these two modes merge together. The values of the amplification rates of these modes are rather high, which highlights how unstable the configuration is. The third mode in amplification, is the Crow mode for the entire vortex system, dotted line in Fig. (6.18). Its amplification is one order of magnitude lower, and its region of instability is much smaller than the modes discussed previously, but it is significant in terms of aircraft wake destruction, as it also affect the outer vortices. These three modes have their respective decaying counterparts, while two neutral modes also exist in that region, all shown in Fig. (6.19).

Finally, results corresponding to $\tilde{d} = 0.01$ are discussed. In this case, Fig. (6.17(b))

Figure 6.18: Amplification rate $\tilde{\omega}_i$ as a function of the wavenumber $\tilde{k}$ for $\tilde{d} = 0.02$ and $\mathcal{G} = -0.4$. Solid line: symmetric mode for inner vortices Fig. (6.19(a)). Dashed line: antisymmetric mode for inner vortices Fig. (6.19(b)). Dotted line: the Crow mode for the entire vortex system Fig. (6.19(c)).

shows that nearly the entire range of $\mathcal{G}$ is available. The amplification rate is calculated as function of the wavenumber for various values of the circulation ratio. The qualitative shape of the results is similar to Fig. (6.18), so they are not plotted again. However, they have important quantitative differences with the results shown in Fig. (6.18). The amplification of the three unstable modes grows to very large values as $\mathcal{G} \rightarrow -1$. The non-dimensional maximum value of amplification rate of the most unstable mode is around 100 for small values of $\mathcal{G}$, reaching 150 for intermediate values of the circulation ratio and growing very rapidly as $\mathcal{G} \rightarrow -1$. The region of unstable wavenumbers also increases in general with higher values of the absolute value of $\mathcal{G}$. Only one qualitative change occurs for $\mathcal{G} = -0.7$ and below, which is that the second mode has also a maximum.

Fabre and Jacquin [43] performed analogous analysis of the four-vortex system, but they related directly the results with the vortex radius, and did not use the cutoff assumption. Their approach, although is more realistic, particularly for high wavenumbers where the cutoff assumption is not valid, makes the methodology dependent of the choice of vortex model (in their case the Rankine vortex model) and it has been the reason not to use it here, where the approach taken is independent of the vortex model. It is worth to highlight here that the choice of cutoff done in the present work, allows to compare the filament results with results using any vortex model, using the formulas of Widnall [141] to convert cutoff distance into vortex radius, known the vortex shape. They used the following values, in the notation of this work, $\tilde{d}_O = 0.0409$, $\tilde{d}_I = 0.0204$ and $\mathcal{G} = -0.4$, and they did not use the Crouch [32] assumption ($\tilde{d}_I = \tilde{d}_O |\mathcal{G}|^{1/2}$). In order to compare the results of Fabre and Jacquin [43] with those of the present methodology, here kb_1 needs to be multiplied by 1.57 to obtain $\tilde{k}$ and $(2\pi b_1/\Gamma)\sigma_r$ by 4.11 to obtain $\tilde{\omega}_i$. Doing so, the results of the two approaches are indistinguishable up to high values of wavenumber, where their solution is predicted to be more unstable than that presented herein.

6.4.2 Four aligned parallel viscous Gaussian vortices instabilities by BiGlobal

The case is solved for the set of parameters previously addressed, $\tilde{d} = 0.02$, $\mathcal{G} = -0.4$. The corresponding value for the vortex radius is then $\tilde{a} = 0.0229$ in the viscous analysis.

Figure 6.19: The eight modes for $\tilde{k} = 1$, $\tilde{d} = 0.02$ and $\mathcal{G} = -0.4$. Three growing, three decaying and two neutral eigenvalues.

At $Re = 1000$ only the Crow mode of the outer vortices corresponds to the result of the inviscid theory. At $Re = 10000$ the remaining modes known from the inviscid analysis also appear in the viscous eigenspectrum. At high Reynolds number values the viscous instability analysis delivers an increasingly large number of spurious eigenmodes and also the characteristics of the leading perturbations are qualitatively different between the two types of analysis: from the three modes recovered, the leading mode (symmetric and only affecting the inner vortices) is found to have a nonzero frequency in the viscous analysis, while the inviscid theory predicts this mode to be stationary. A number of parameters, whose selection is guided by the results of Fig. (6.18), have been chosen in order to perform the viscous global linear instability work. The choices made are shown as discrete points in Fig. (6.20), alongside the full lines of inviscid results, reproduced from Fig. (6.18). In all cases shown, very good agreement between the results of the two types of analysis is obtained. Although the value $\tilde{d} = 0.02$ might seem small, from Fig. (6.17(a)), it can be seen that it is actually in the limit of validity of the approach. To show this fact, a value slightly higher of vortex core, $\tilde{a} = 0.025$, equivalent to $\tilde{d} = 0.0218$ is calculated. Table 6.6 compares results obtained from the filament inviscid instability and BiGlobal viscous instability for the particular case $\tilde{k} = 1.8$ and antisymmetric mode, given $\tilde{d} = 0.02$ ($\tilde{a} = 0.0229$) and $\tilde{d} = 0.0218$ ($\tilde{a} = 0.025$) both for $\mathcal{G} = -0.4$. It is observed how a very little change of vortex size leads to an increase of the disagreement between the two approaches from less than 1% for $\tilde{a} = 0.0229$ to almost 10% for $\tilde{a} = 0.025$.

$\tilde{a}$	$\tilde{\omega}_i$ Inviscid	$\tilde{\omega}_i$ Viscous	Relative Difference
0.0229	36.189...	36.423...	6.45×10^{-03}
0.025	36.623...	33.047...	9.76×10^{-02}

Table 6.6: Comparison of amplification rates $\tilde{\omega}_i$ for inviscid and viscous ($Re = 10000$) methodologies for the four vortex system selecting $\mathcal{G} = -0.4$ and $\tilde{k} = 1.8$ and two choices of $\tilde{d} = 0.02, 0.0218$ ($\tilde{a} = 0.0229, 0.025$), for the antisymmetric mode.

Figure 6.20: Comparison between inviscid (lines) and viscous linear theory results for the amplification rate $\tilde{\omega}_i$ as a function of wavenumber $\tilde{k}$ for $Re = 10000$ (points). $\tilde{d} = 0.02$ and $\tilde{a} = 0.0229$. Solid line and circles: symmetric mode inner vortices. Dashed line and triangles: antisymmetric mode inner vortices. Dotted line and circle: symmetric Crow mode.

Figure 6.21: Basic flow plus $\varepsilon = 0.02$ perturbation for the second symmetric mode at $\tilde{k} = 1.2$ comparison between the filament positions obtained from the inviscid vortex analysis at $\tilde{d} = 0.02$ (lines) and the vorticity isosurfaces obtained from the BiGlobal analysis for $Re = 10000$ and $\tilde{a} = 0.0229$ (surfaces). The perpendicular direction has not been scaled.

6.5 Vortex arrays instabilities

In the previous examples, wakes are considered in which the axis of the vortices is the same as the direction of motion of the aircraft whose wake is modelled. By contrast, in a two dimensional profile or bluff body, the wake can be seen as an infinite arrangement of aligned vortices downstream the body and its stability may be analysed following the seminal works of von Kármán [138, 139]. In all cases studied in this section, an additional step has to be made due to the infinite number of point vortices considered. A sinusoidal perturbation of periodicity φ times the separation between consecutive vortices is imposed. This assumption allows to reproduce a wide number of different modes of perturbations corresponding to finite values of a single parameter, φ . This parameter is taken to be continuous, $0 \leq \varphi < 2\pi$, since higher values mean that the periodicity length is smaller than the distance between points. The meaning of parameter φ is that it fixes the number of point vortices needed to repeat periodicity as $N = 2\pi/\varphi$, and therefore, although it could be taken from the continuous, it is usually better to select the number of points N and calculate the corresponding value of the parameter φ . Doing that, for each value of the parameter φ , the number of degrees of freedom is equal to twice the number of different points considered.

Figure 6.22: Vortex location scheme for various configurations of infinite number of vortices.

6.5.1 Equidistant aligned vortices rotating in the same sense

This is the most simple configuration of this kind, shown in Fig. (6.22(a)), and is composed of an infinite set of equidistant aligned vortices of the same strength and direction of rotation. This configuration is an idealization of a vortex sheet, that appears in the interface of two regions of different speeds.

In all cases, the number of different points to be considered in the analysis first needs to be identified. All the points are equal here, so only one point vortex needs to be considered, named point 0.

Second, the condition needs to be derived for the system to remain in equilibrium. For that, Eq. (2.72a) is written for a single point $p = 0$. $X_q = ql$ and $Y_q = 0$, in particular, $X_0 = 0$ and $Y_0 = 0$. l is the distance between vortices, $-\infty < q < \infty$ and $\Gamma_q = \Gamma$ for every q . Remember that both p and q are integers as they define different point vortices. In this case $\bar{\mathbf{U}} = 0$ and the configuration is in equilibrium.

The stability analysis is performed using Eqs. (2.74) for the point $p = 0$. Next, a sinusoidal perturbation of the form $(\hat{x}_q, \hat{y}_q)^T = (\hat{x}_0, \hat{y}_0)^T e^{iq\varphi}$ is imposed, which results in:

$$-i\omega\hat{x}_0 = -\frac{\Gamma\hat{y}_0}{\pi l^2} \sum_{q=1}^{\infty} \frac{1 - \cos(q\varphi)}{q^2} \quad (6.11a)$$

$$-i\omega\hat{y}_0 = -\frac{\Gamma\hat{x}_0}{\pi l^2} \sum_{q=1}^{\infty} \frac{1 - \cos(q\varphi)}{q^2} \quad (6.11b)$$

From now on, the positive real function of the parameter φ , $f_1(\varphi)$ is considered¹. If the distance is non-dimensionalised by l and ω by $\Gamma/\pi l^2$, the problem can be written in matrix form as an EVP, for the non-dimensional perturbations to the position vector $(\tilde{x}_0, \tilde{y}_0) = (\hat{x}_0, \hat{y}_0)/l$ and the eigenvalues $\tilde{\omega} = \pi l^2 \omega / \Gamma$.

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix} = \begin{pmatrix} 0 & -f_1(\varphi) \\ -f_1(\varphi) & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix} \quad (6.12)$$

In this case, the eigenvalues are $-i\tilde{\omega} = \pm f_1(\varphi)$. Therefore, there exist one positive and one negative real values and the configuration is unstable due to the positive one. The eigenvector associated with the positive eigenvalue is $(\tilde{x}_0, \tilde{y}_0) = (1, -1)$, which means that the vortices will move in directions forming an angle of 45 degrees with the original vortex line. Furthermore, the maximum value of $f_1(\varphi = \pi) = \pi^2/4$, resulting in $\tilde{\omega}_{i,max} = \pi^2/4 \approx 2.4674$, which means that the higher growth appears for periods of two vortices. The most unstable mode is a pair of vortices moving in the directions $(\tilde{x}_0, \tilde{y}_0) = (1, -1)$ and $(\tilde{x}_0, \tilde{y}_0) = (-1, 1)$ alternatively, which is actually the Kelvin-Helmholtz instability.

6.5.2 Equidistant aligned vortices rotating in alternating senses

This configuration is seen in Fig. (6.22(b)). The full process is similar to the previous case, only that the sign of the vortices has to be changed alternately. Again the vortices are in equilibrium as $\bar{\mathbf{U}} = 0$ and due to the alternation of the sign of the vortices, a new function $f_2(\varphi)$ appears, which is a negative real function of the parameter φ .

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix} = \begin{pmatrix} 0 & -f_2(\varphi) \\ -f_2(\varphi) & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix} \quad (6.13)$$

The eigenvalues of the problem are $-i\tilde{\omega} = \pm |f_2(\varphi)|$, and this configuration is also unstable due to the positive value. $f_2(\varphi) \leq 0$ with a very sharp maximum absolute value at $\varphi = \pi$. The positive growing eigenvalue is $\tilde{\omega}_i = -f_2(\varphi)$ and the corresponding eigenvector $(\tilde{x}_0, \tilde{y}_0) = (1, 1)$. $f_2(\pi) = -\pi^2/4$ and therefore, $\tilde{\omega}_{i,max} = \pi^2/4 \approx 2.4674$, the same as in the previous case.

6.5.3 Two rows of equidistant aligned vortices rotating in the same sense, one on top of the other

The configuration analysed here is shown in Fig. (6.22(c)), and consists of two lines of vortices, all rotating in the same sense, where the vortices coincide one on top of the other. In contrast to the previous cases, there exist two different kind of points, one corresponding to the upper row, denoted as 0^+ , and one to the lower, denoted as 0^- .

As before, the first concern is to ensure that the configuration is in equilibrium. That is inquired by solving Eq. (2.72a) for the two points, $p = 0^+, 0^-$. The respective locations are $X_q = ql$, $Y_{q^+} = e$, $Y_{q^-} = 0$, where e is the distance between the rows and l is the

¹This and subsequent formulas are collected in Appendix A

distance between points. For this configuration, there are two sums of q , one for the upper row, and another for the lower row. The circulations are the same, $\Gamma_q = \Gamma$, for all q .

The result is that $\bar{U}_y = 0$, but $\bar{U}_x = \pm \Gamma f_3(l/e)/(2\pi e)$ (negative for the upper row and positive for the lower), where $f_3(l/e)$ is a real positive function of the ratio between horizontal and vertical spacing, higher than one and tending to infinity as $l/e \rightarrow 0$. $f_3(l/e)$ is written in series form in Appendix A, but it has an analytic form: $f_3(l/e) = \pi/(l/e) \text{Coth}[\pi/(l/e)]$. Therefore, $\bar{U}_x$ is different for the two rows, the upper moves to the left, while the lower does it to the right and the system is not in equilibrium. In other words, the stability question cannot be addressed for this configuration, since the system is not in equilibrium.

6.5.4 Two rows of equidistant aligned vortices rotating in opposite senses, one on top of the other [139]

This configuration, shown in Fig. (6.22(d)), is analogous with that of the previous case, but now, the point vortices of the upper and lower rows rotate in opposite senses. Without loss of generality, the lower row is taken to have positive circulation and negative the upper. The values of the problem parameters are: $p = 0^+, 0^-$, $X_q = ql$, $Y_{q^+} = e$, $Y_{q^-} = 0$, $\Gamma_{q^+} = -\Gamma$ and $\Gamma_{q^-} = \Gamma$.

In contrast with the previous case, this time a reference system in which the two rows are in equilibrium does exist. The system is that moving with the vortices to the left with the velocity $\bar{U}_x = -\Gamma f_3(l/e)/(2\pi e)$, while $\bar{U}_y = 0$, as expected. Therefore, the stability analysis can be performed using Eqs. (2.74) for two points, one from the upper row ($p = 0^+$) and other from the lower row ($p = 0^-$). A sinusoidal perturbation of the form $(\hat{x}_{q^\pm}, \hat{y}_{q^\pm})^T = (\hat{x}_{0^\pm}, \hat{y}_{0^\pm})^T e^{q i \varphi}$ is again imposed.

The real functions $f_4(l/e)$ (positive and lower than 1), $f_5(\varphi, l/e)$ and $f_6(\varphi, l/e)$ (both positive or negative for different values of φ and l/e , and their absolute value always lower than unity) are defined in order to simplify the presentation. The problem is also non-dimensionalised, and the notation is further simplified. As originally done by von Kármán [139], new functions are introduced: $A_1(\varphi, l/e)$ and $C_1(\varphi, l/e)$ real and $B_1(\varphi, l/e)$ imaginary. In the notation introduced, the eigenvalue problem can now be stated:

$$-i\tilde{\omega} \begin{pmatrix} \tilde{x}_{0^+} \\ \tilde{y}_{0^+} \\ \tilde{x}_{0^-} \\ \tilde{y}_{0^-} \end{pmatrix} = \begin{pmatrix} 0 & A_1 & B_1 & -C_1 \\ A_1 & 0 & -C_1 & -B_1 \\ B_1 & C_1 & 0 & -A_1 \\ C_1 & -B_1 & -A_1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_{0^+} \\ \tilde{y}_{0^+} \\ \tilde{x}_{0^-} \\ \tilde{y}_{0^-} \end{pmatrix}, \quad (6.14)$$

as first formulated by von Kármán [139]

The eigenvalues for this case have the form $-i\tilde{\omega} = \pm(B_1 \pm \sqrt{A_1^2 - C_1^2})$. B_1 has only imaginary part, so the real part, if present, would come from the term inside the square root. Therefore, if $|A_1| > |C_1|$, there will be two eigenvalues with negative real part and two with positive real part ($\tilde{\omega}_i = \pm\sqrt{A_1^2 - C_1^2}$, $\tilde{\omega}_r = \pm B_1/i$), and the configuration will be unstable due to them. On the other hand, if $|A_1| \leq |C_1|$ the four eigenvalues will only have imaginary part ($\tilde{\omega}_i = 0$, $\tilde{\omega}_r = \pm(B_1/i \pm \sqrt{C_1^2 - A_1^2})$) and the configuration will be neutrally stable. However, A_1 is maximum for $\varphi = \pi$, and for that value, $A_1^2 - C_1^2 = \pi^4/16 > 0$

independent of l/e . That means that the configurations is always unstable to perturbations to vortex pairs for every choice of l/e . The value of the growth rate for the most amplified mode is then $\tilde{\omega}_{i,max} = \pi^2/4$, and, additionally, $B_1 = 0$ for $\varphi = \pi$ and the mode is double and non-traveling ($\tilde{\omega}_r = 0$).

6.5.5 Two rows of equidistant aligned vortices rotating in the same sense, displaced half distance

The configuration of two rows of equidistant aligned vortices rotating in the same sense, one on top of the other is now displaced a distance that is half of the spacing between vortices of the same row in a staggered pattern shown in Fig. (6.22(e)).

As before, there exist two different kind of points, the upper row (0^+) and the lower row (0^-). The point locations are $X_{q^+} = q^+l$, $X_{q^-} = (q^- - \text{sign}(q^-)/2)l$, $Y_{q^+} = e$ and $Y_{q^-} = 0$. The circulations are all the same, $\Gamma_q = \Gamma$ for all q .

The result is the same as obtained in case 3 of this chapter, although function $f_3(l/e)/2$ is changed in this case by $f_7(l/e)$. This is a positive real function of a qualitative shape similar to $1/x$, being unbounded for $l/e \rightarrow 0$ and 0 when $l/e \rightarrow \infty$. The series form of $f_7(l/e)$ is collected in Appendix A, but in this case again, an analytic form exists: $f_7(l/e) = i(\psi[1/2 - i/(l/e)] - \psi[1/2 + i/(l/e)])/(2l/e)$, being $\psi(x)$ is the digamma function [2]. That is, $\bar{U}_y = 0$ and $\bar{U}_x = \pm \Gamma f_7(l/e)/(\pi e)$, and again, the upper row moves to the left and the lower to the right. As in the earlier case 3, this configuration is not in equilibrium and no stability analysis may be performed.

6.5.6 Two rows of equidistant aligned vortices rotating in opposite senses, displaced half distance, [139]

The last configuration of infinite rows analysed is shown in Fig. (6.22(f)). It is analogous with that in the previous case, but the point vortices of the upper and lower rows rotate in opposite senses. The particular parameters of the problem are: $p = 0^+, 0^-$, $X_{q^+} = q^+l$, $X_{q^-} = (q^- - \text{sign}(q^-)/2)l$, $Y_{q^+} = e$, $Y_{q^-} = 0$, $\Gamma_{q^+} = -\Gamma$ and $\Gamma_{q^-} = \Gamma$. This configuration led to the celebrated work of von Kármán [139], which describes the topology of a wake bearing his name, and provides the best point-vortex approximation of the wake of a bluff body.

The procedure to follow is the same as in the earlier cases, the only difference being the displacement of the vortices. Again, $\bar{U}_y = 0$, and the system moves with constant horizontal speed $\bar{U}_x = -\Gamma f_7(l/e)/(\pi e)$, which can be computed analytically.

In addition, new real functions are defined: $f_8(l/e)$, always positive and small, $f_9(\varphi, l/e)$, taking positive or negative values and $f_{10}(\varphi, l/e)$, also taking positive or negative values. These functions are used to define $A_2(\varphi, l/e)$, real, $B_2(\varphi, l/e)$ imaginary, and $C_2(\varphi, l/e)$, real. Using these new functions, the problem follows the same equations as Eqs. (6.14), where subindex 1 is changed by 2. Therefore, the relations between A_1 , B_1 and C_1 , can be directly applied here to A_2 , B_2 and C_2 .

This time the eigenvalues are $-i\tilde{\omega} = \pm(B_2 \pm \sqrt{A_2^2 - C_2^2})$ and therefore, if $|A_2| > |C_2|$,

the configuration will be unstable and if $|A_2| \leq |C_2|$ the configuration will be neutrally stable. Selecting once more $\varphi = \pi$, ($N = 2$), it is found that $C_2 = 0$, and $|A_2| > |C_2|$, unless $A_2 = 0$, which occurs for $l/e \approx 3.564$, solution of $\cosh(\pi e/l) = \sqrt{2}$. This selection gives neutral stability, and every other choice of the parameter l/e produces growth, whose rate is function of the same parameter, $\tilde{\omega}_{i,max} = \pi^2/4 |1 - 2 \cosh^2(\pi e/l)|$. The modes will have frequency as a general basis ($\tilde{\omega}_r \neq 0$).

6.6 Vortex instabilities discussion

The main outcomes of the thesis have been presented in this chapter, being all the previous a very long introduction to it. However, the core of the work is performed before this chapter, as it only consist on the actual execution of the algorithms and ideas already presented.

Said so, as the actual outcomes of the work have been exposed here and will be discussed. Firstly, the duality between grid and gridless methodologies, already presented in Chapter 5, rises a higher state in this chapter. Note that the BiGlobal executions could be more or less cheap computationally when a lot of efforts are done to increase the computational efficiency, among which use of sparse algebra, Arnoldi algorithms for the calculus of eigenvalues, parallelisation, use of compact finite differences, and many more, and then, even though, the convergence of results is delicate and the eigenvalues have to be extracted manually in a quite artistic manner. However, the stability vortex methods, offer very clear results, which could even be obtained analytically in many cases, and are inviscid, which is in many situations a better approach than taking a very low Reynolds number. But of course, not all are advantages, the vortex problem is very simplified, so is its solution, usually explaining only the general inviscid instability mechanisms, with very little detail of vortex shape, axial flow, and many more parameters that may be relevant for the flow.

Talking now about the actual vortical wake problems analysed, the two counter-rotating pair is maybe the most straightforward. Dealing with the axially homogeneous vortices, both BiGlobal and vortex methods offer results with reasonably good agreement, giving the vortex methods a very general plot of the cooperative inviscid instabilities, as well as BiGlobal offers the possibility of studying the viscous problem, core instabilities, axial flow and many more. The star instability of this configuration is that of Crow [35], which is captured by both methodologies, although many more instabilities could be analysed by BiGlobal, being them usually less amplified, but that they could be triggered in order to achieve the earlier wake destruction.

The four vortex wake, also analysed by both methodologies, is much more unstable than the vortex pair. However, the high growth instabilities, affect mainly to the inner vortices, so the most probable scenario is that, after this instability has destroyed the inner vortices, the vortex pair wake will still remain.

Other scenarios are also analysed generalising the von Kármán [139] idea, were periodic wakes and shear flows are analysed. The flows are known from classical fluid mechanics and nothing new is found from this side, only the methodology is proven to be of great

use for the analysis of this type of flows.

Finally, the capability of the combination of PNS and PSE-3D methodologies for the analysis of axially diffusing vortices is exposed, but not explored in depth, leaving that for further studies.

Chapter 7

Conclusions

This thesis has analysed the usage of both classical and vortex methods for the stability analysis of vortical fluid flows. The novelty of this combination for stability is the most relevant quality of the work, and the reason for the big effort that has been given to the theoretical part.

There have been shown a few cases that illustrate only one of the approaches, as the shear flows in the case of vortex methods and the vortices with axial diffusion in the case of grid methods. But more important, there have been found also cases where the comparison has been possible, the analysis of the instabilities of a pair of parallel vortices as in Crow [35] and four aligned vortices as in Fabre and Jacquin [43], Rennich and Lele [116]. For these two examples, it has been found that the agreement of the two approaches is excellent, even for moderate Re and intermediate vortex distances. The advantage of these cases goes to the vortex methods, which are much less demanding of resources. However, for distances of the order of the radius or Re below 1000, the vortex methods fail and the BiGlobal methodology is needed. Even more, if the vortices are not parallel and have variation along their axis, then only the PSE-3D methodology provides with accurate results.

The thesis starts with a deep literature review on the studied topics, presenting the aircraft wake problem in the Introduction, as well as dealing with the vortex definition, vortex models and general vortex instabilities and methodologies to solve the problem in Chapter 1. Then, the core of the thesis is presented in Chapter 2, where all the theoretical concepts behind the methodologies analysed are reviewed and explained in detail. It follows Chapter 3, with a recompilation of all the numerical considerations needed to solve the different problems efficiently. Later, Chapter 4 explains the details of the different vortex recognition criteria, which are of great relevance for complex flows, in particular to turbulent flows, but that finally are not required, as the same time that are too complicated for the vortex systems analysed in this thesis. The results are finally divided into two chapters, Chapter 5 deals with IPV, and focuses more in the evolution of the flow and the obtaining of basic flows, while Chapter 6 focuses on EVP, more suited to analyse the vortex systems instabilities. Now, the conclusions establish the more relevant achievements and outcomes from the thesis. They have been highlighted in the previous paragraphs as a mode of introduction, but after the structure of the thesis is exposed, more detailed remarks focused on the different chapters are given.

The literature survey have left clear that the topic, vortex instabilities, is a very open topic, with a very long history of research, which does not seem close to its end. And in particular, the application to the destruction and control of aircraft and wind turbine vortical wakes, does not look near to be fully understood soon either. Therefore, from

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this end, the main objective, or the hypothesis if it wants to be stated in this way, is to improve the understanding of the physical mechanisms involving instabilities of these flows, but not intending to achieve a unique or optimal solution to the problem, because that would be too ambitious. It must be added that from the outcomes of the work, the objective is certainly accomplished.

But the research undertaken is not only focused on the improvement of the physical knowledge of the fluid behaviour, the purpose is wider, the improvement of the knowledge of the available tools for this analysis is one of the major aims of this thesis. In this line, this goal is also accomplished, as the different methodologies are reviewed both from their theoretical and applied views, and their advantages and disadvantages are highlighted.

Dealing with IVP, the two methodologies analysed for two dimensional flows, are not as different in computational resources needed and quality of results. As a general comment, if low resources are available, then the DNS will not work, but the point vortex method will provide with a solution, which could be qualitatively good. For not very challenging cases, both methodologies work well, but if the flow needs to be analysed in detail, usually, DNS will be more precise. However, more aspects should be taken into account. Vortex methods are good when vorticity is concentrated, so there are big irrotational flow regions. And they calculate the flow in the inviscid limit. On the other hand, DNS is better if the flow is more homogeneous, and has the possibility of analysing the flow viscosity.

These differences between grid and vortex methods are augmented for the case of EVP. Then, the point vortices are very simple and fast, but cannot account for vortex core size, axial flow, viscosity and many more, while their viscous counterpart, the BiGlobal methodology, is more demanding of resources (still affordable on a 2016 standard computer in a few minutes), and can account for these and many other flow characteristics. However, its convergence is sometimes complicated and requires some manual work.

Additionally, following the same fashion than Tendero et al. [131], the inviscid instability analysis methodology based on point vortices is presented in a unified manner to solve a variety of vortical systems [35, 43, 76, 138, 139]. Effort is paid to identify and expose the assumptions underlying this theoretical approach in terms of the permissible parameters in the analysis. Subsequently, the more complete (and computationally orders-of-magnitude more expensive) viscous linear instability theory is also employed to analyse some of the most interesting configurations. Results of the two approaches are generally found to be in agreement with each other, as long as the vortices are not very close, so the filament approach is valid, even when Reynolds number is maintained in moderate values for the viscous analysis. Therefore, the main limitation of the inviscid approach is the impossibility of analysing vortices whose size is comparable to their distance, as well as very large wavenumbers, which imply very short wavelengths, although the last of them can be overcome by different strategies. In addition, viscosity cannot be introduced, but on the contrary, this methods can analyse inviscid configurations, while the viscous methods tent to fail more when the viscosity is low. The viscous methods can analyse closer vortices and larger wavenumbers, but on the other side, they are orders of magnitude more demanding of computational resources, which is the main advantage of the inviscid methods, which some times can be solved even analytically.

The present work includes several novel features, embedded among other previous

results. They are listed below, including both the less and more important ones. First, the generalization of von Kármán [138, 139] work to other similar configurations. The two vortex configurations proposed by Crow [35] and Jiménez [76] are only valid for some range of parameters, due to the cutoff approach. This fact is brought in this thesis, showing which regions are valid and why, in order to avoid unrealistic results. The same is done for the Rennich and Lele [116] four vortex configuration, also studied by Fabre and Jacquin [43], who limit themselves to one particular configuration, while this time the circulation ratio and the cutoff length are studied in a wide range. Their way to employ the cutoff is different to that employed in this article. They developed a more elegant relation between cutoff and vortex radius, that allows them to study short wavelengths, but they had to chose the same vortex model one for all. On the contrary, the approach followed here permits to convert the filament results based on cutoff length to any vortex model. The previous are some of the new outcomes of the work, regarding the vortex filaments approach, however, the comparisons with viscous analysis are also new. In the two counter-rotating vortices case, some previous works exists: Benton and Bons [13] and Brion et al. [17], but they address the problem from a different approach and do not specify as many details of the comparison, because they pursue different aims. For the four vortex configuration, it is the first known case where a quantitative comparison is given between the two analogies, and therefore, this is one of the most valuable results obtained in the present work. The last outcome of that being a very good agreement between both analogies in this context, which will help the community to apply filament vortex stability analysis as a faster substitute for global stability in more situations. Once the novelties of the present research are exposed clearly, it is time to highlight that the main goal of it is the presentation of a comprehensive evaluation of the method, collecting a wide range of studies under a single methodology, which can be use for many more configurations.

Apart from the earlier, a general outcome of the thesis, particularly from the relevance for the aeronautical community, is the fact that this simple and reliable approach, could be used as a general methodology to solve a really wide variety of vortical configurations that can represent the aircraft wake. Being more specific, this methodology has permitted to achieve results as the Kelvin-Helmholtz instability, the von Kármán vortex street, the Crow instability, and the inner vortex instability for systems of four vortices, all these analysed here. It is generally observed that the vortex core shape is irrelevant for all these if the vortices are far apart, and it becomes important (as it should be) when the size of the vortices becomes comparable to their separation.

Although it escapes from the scope of this investigation, it is provided a little insight into the idea of how the present results could help to destroy the aircraft wake, which is the final goal towards all this and other investigations are directed. An starting point for control strategies to destroy the vortex wake could be to select the highest amplification rate wavenumber for a particular configuration. Done so, this mode could be excited, but as the modes found here do not have frequency, the excitation has to be done in space, rather than in time. Still, given that the aircraft is moving at a particular speed, an appropriate excitation in time will provide a spatial excitation, which should match the maximum amplification rate wavelength. Doing so, specifically if the shape of the perturbation matches the shape of the mode, it could be excited earlier, and therefore the wake would be destroyed faster. However, the four vortex case show a particular case where the most unstable mode, only affect the inner vortices, so the excitation of this

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mode, will only destroy these vortices, remaining still a two vortex wake. Nevertheless, the destruction process involves many other non-linear mechanisms, which could most probably also be excited to achieve this in a more effective manner. The interaction between the actual actuator for the wake destabilization and the aircraft performance should be evaluated to avoid undesired behaviours. However, as the lift of the plane is a function of the vortices circulation, as long as it is not modified, the effect on the aircraft performance should be little.

From a physical point of view, a shortcoming of both approaches is the treatment of the axial flow direction as homogeneous, neglecting a viscous diffusion that is a function of this spatial coordinate. This issue is partially treated here, and will be treated in future work.

In order to conclude, it should be mentioned that although two well explored fields are examined, the combination of them is a much more unexplored field that might lead to important contributions to vortex stability. Furthermore, stability by vortex methods may be an alternative to produce results at a much lower computational cost.

Appendix A

Formulas used in section 6.5

A list of formulas used in section 6.5 is provided in this appendix:

$$\begin{aligned}
f_1(\varphi) &= \sum_{q=1}^{\infty} \frac{1 - \cos(q\varphi)}{q^2} \\
f_2(\varphi) &= \sum_{q=1}^{\infty} (-1)^q \frac{1 - \cos(q\varphi)}{q^2} \\
f_3(l/e) &= 1 + 2 \sum_{q=1}^{\infty} \frac{1}{(ql/e)^2 + 1} \\
f_4(l/e) &= \sum_{q=1}^{\infty} \frac{(ql/e)^2 - 1}{[(ql/e)^2 + 1]^2} \\
f_5(\varphi, l/e) &= \sum_{q=1}^{\infty} \frac{(ql/e)^2 - 1}{[(ql/e)^2 + 1]^2} \cos(q\varphi) \\
f_6(\varphi, l/e) &= \sum_{q=1}^{\infty} \frac{q \sin(q\varphi)}{[(ql/e)^2 + 1]^2} \\
f_7(l/e) &= \sum_{q=1}^{\infty} \frac{1}{[(q - 1/2)l/e]^2 + 1} \\
f_8(l/e) &= \sum_{q=1}^{\infty} \frac{[(q - 1/2)l/e]^2 - 1}{[((q - 1/2)l/e)^2 + 1]^2} \\
f_9(\varphi, l/e) &= \sum_{q=1}^{\infty} \frac{[(q - 1/2)l/e]^2 - 1}{[((q - 1/2)l/e)^2 + 1]^2} \cos[(q - 1/2)\varphi] \\
f_{10}(\varphi, l/e) &= \sum_{q=1}^{\infty} \frac{(q - 1/2) \sin[(q - 1/2)\varphi]}{[((q - 1/2)l/e)^2 + 1]^2} \\
A_1(\varphi, l/e) &= f_1(\varphi) + (l/e)^2 [1/2 - f_4(l/e)] \\
B_1(\varphi, l/e) &= 2i(l/e)^3 f_6(\varphi, l/e) \\
C_1(\varphi, l/e) &= (l/e)^2 [1/2 - f_5(\varphi, l/e)] \\
A_2(\varphi, l/e) &= f_1(\varphi) - (l/e)^2 f_8(l/e) \\
B_2(\varphi, l/e) &= 2i(l/e)^3 f_{10}(\varphi, l/e) \\
C_2(\varphi, l/e) &= -(l/e)^2 f_9(\varphi, l/e)
\end{aligned} \tag{A.1}$$

Appendix B

Collection of Mappings

B.1 Linear

B.1.1 Linear v1

$$\eta = \eta_0 + \eta_{end}x \quad (B.1)$$

Table B.1: Various values of $\eta(x)$ for linear v1 mapping.

x	η
-1	$\eta_0 - \eta_{end}$
-0.5	$\eta_0 - 0.5\eta_{end}$
0	η_0
0.5	$\eta_0 + 0.5\eta_{end}$
1	$\eta_0 + \eta_{end}$

$$x = \frac{\eta - \eta_0}{\eta_{end}} \quad (B.2)$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{\eta_{end}} \quad (B.3)$$

$$\frac{\partial^2 x}{\partial \eta^2} = 0 \quad (B.4)$$

B.1.2 Linear v2

$$\eta = \frac{\eta_R - \eta_L}{2}(x + 1) + \eta_L \quad (B.5)$$

$$x = 2\frac{\eta - \eta_L}{\eta_R - \eta_L} - 1 \quad (B.6)$$

$$\frac{\partial x}{\partial \eta} = \frac{2}{\eta_R - \eta_L} \quad (B.7)$$

$$\frac{\partial^2 x}{\partial \eta^2} = 0 \quad (B.8)$$

Table B.2: Various values of $\eta(x)$ for linear v2 mapping.

x	η
-1	η_L
-0.5	$\frac{\eta_R + 3\eta_L}{4}$
0	$\frac{\eta_R + \eta_L}{2}$
0.5	$\frac{3\eta_R + \eta_L}{4}$
1	η_R

B.2 Boundary layer

$$\eta = \frac{\eta_\infty \eta_m (1 - x)}{(\eta_\infty - 2\eta_m)(1 + x) + 2\eta_m} \quad (\text{B.9})$$

$$l = \frac{\eta_\infty \eta_m}{\eta_\infty - 2\eta_m} \quad (\text{B.10})$$

$$s = \frac{2l}{\eta_\infty} = \frac{2\eta_m}{\eta_\infty - 2\eta_m} \quad (\text{B.11})$$

$$\eta = l \frac{1 - x}{1 + x + s} \quad (\text{B.12})$$

Table B.3: Various values of $\eta(x)$ for boundary layer mapping.

x	η
-1	η_∞
-0.5	$\frac{3\eta_\infty \eta_m}{\eta_\infty + 2\eta_m} \approx 3\eta_m$
0	η_m
0.5	$\frac{\eta_\infty \eta_m}{3\eta_\infty - 2\eta_m} \approx \frac{\eta_m}{3}$
1	0

$$x = \frac{\eta_m - \eta}{\eta_m + \eta \left(1 - \frac{2\eta_m}{\eta_\infty}\right)} \quad (\text{B.13})$$

$$x = \frac{l - \eta(1 + s)}{\eta + l} \quad (\text{B.14})$$

$$\frac{\partial x}{\partial \eta} = - \frac{2\eta_m \left(1 - \frac{\eta_m}{\eta_\infty}\right)}{\left[\eta_m + \eta \left(1 - \frac{2\eta_m}{\eta_\infty}\right)\right]^2} \quad (\text{B.15})$$

$$\frac{\partial x}{\partial \eta} = - \frac{(2 + s)l}{(\eta + l)^2} \quad (\text{B.16})$$

$$\frac{\partial^2 x}{\partial \eta^2} = \frac{4\eta_m \left(1 - \frac{\eta_m}{\eta_\infty}\right) \left(1 - \frac{2\eta_m}{\eta_\infty}\right)}{\left[\eta_m + \eta \left(1 - \frac{2\eta_m}{\eta_\infty}\right)\right]^3} \quad (\text{B.17})$$

$$\frac{\partial^2 x}{\partial \eta^2} = \frac{2(2+s)l}{(\eta+l)^3} \quad (\text{B.18})$$

B.3 Tangent

B.3.1 Tangent v1

$$\eta = \eta_0 + \frac{\tan\left(\frac{c\pi}{2}x\right)}{\tan\left(\frac{c\pi}{2}\right)}\eta_\infty \quad (\text{B.19})$$

$$s = \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \quad (\text{B.20})$$

$$\eta = \eta_0 + \frac{\tan\left(\frac{c\pi}{2}x\right)}{s} \quad (\text{B.21})$$

$$\eta_m = \eta_0 + \frac{\tan\left(\frac{c\pi}{4}\right)}{\tan\left(\frac{c\pi}{2}\right)}\eta_\infty \quad (\text{B.22})$$

$$\tan\left(\frac{c\pi}{4}\right) = \sqrt{1 - 2\frac{\eta_m - \eta_0}{\eta_\infty}} \quad \eta_m - \eta_0 \leq \frac{\eta_\infty}{2} \quad (\text{B.23})$$

$$c = \frac{4}{\pi} \arctan\left(\sqrt{1 - 2\frac{\eta_m - \eta_0}{\eta_\infty}}\right) \quad (\text{B.24})$$

Table B.4: Various values of $\eta(x)$ for tangent v1 mapping.

x	η
-1	$\eta_0 - \eta_\infty$
-0.5	$2\eta_0 - \eta_m$
0	η_0
0.5	η_m
1	$\eta_0 + \eta_\infty$

B.3.2 Tangent v2

$$\eta_m + \eta_0 = \eta_0 + \frac{\tan\left(\frac{c\pi}{4}\right)}{\tan\left(\frac{c\pi}{2}\right)}\eta_\infty \quad (\text{B.25})$$

$$\tan\left(\frac{c\pi}{4}\right) = \sqrt{1 - 2\frac{\eta_m}{\eta_\infty}} \quad \eta_m - \eta_0 \leq \frac{\eta_\infty}{2} \quad (\text{B.26})$$

$$c = \frac{4}{\pi} \arctan\left(\sqrt{1 - 2\frac{\eta_m}{\eta_\infty}}\right) \quad (\text{B.27})$$

Table B.5: Various values of $\eta(x)$ for tangent v2 mapping.

x	η
-1	$\eta_0 - \eta_\infty$
-0.5	$\eta_0 - \eta_m$
0	η_0
0.5	$\eta_0 + \eta_m$
1	$\eta_0 + \eta_\infty$

B.3.3 Tangent v3

$$\eta(\pm 0.5) = \eta_0 + \frac{\tan\left(\frac{\pm c\pi}{4}\right)}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty \quad (\text{B.28})$$

$$\eta(\pm 0.5) = \eta_0 \pm \left[1 - \frac{1}{1 + \cos\left(\frac{c\pi}{2}\right)} \right] \eta_\infty \quad (\text{B.29})$$

$$\eta(\pm 0.5) = \eta_0 \pm \frac{\eta_m - \eta_0}{\eta_\infty - \eta_0} \eta_\infty \quad (\text{B.30})$$

$$c = \frac{2}{\pi} \arccos\left(\frac{\eta_m - \eta_0}{\eta_\infty - \eta_m}\right) \quad (\text{B.31})$$

Table B.6: Various values of $\eta(x)$ for tangent v3 mapping.

x	η
-1	$\eta_0 - \eta_\infty$
-0.5	$\eta_0 - \frac{\eta_m - \eta_0}{\eta_\infty - \eta_0} \eta_\infty \approx 2\eta_0 - \eta_m$
0	η_0
0.5	$\eta_0 + \frac{\eta_m - \eta_0}{\eta_\infty - \eta_0} \eta_\infty \approx \eta_m$
1	$\eta_0 + \eta_\infty$

$$x = \frac{2}{c\pi} \arctan\left[\frac{\eta - \eta_0}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right)\right] \quad (\text{B.32})$$

$$x = \frac{2}{c\pi} \arctan[(\eta - \eta_0)s] \quad (\text{B.33})$$

$$\frac{\partial x}{\partial \eta} = \frac{2}{c\pi} \frac{\frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}}{1 + \left[\frac{\eta - \eta_0}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right)\right]^2} \quad (\text{B.34})$$

$$\frac{\partial x}{\partial \eta} = \frac{2}{c\pi} \frac{s}{1 + [(\eta - \eta_0)s]^2} \quad (\text{B.35})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{4}{c\pi} \frac{(\eta - \eta_0) \left[\frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}\right]^3}{\left\{1 + \left[\frac{\eta - \eta_0}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right)\right]^2\right\}^2} \quad (\text{B.36})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{4}{c\pi} \frac{(\eta - \eta_0)s^3}{\{1 + [(\eta - \eta_0)s]^2\}^2} \quad (\text{B.37})$$

B.4 Displaced tangent A

B.4.1 Displaced tangent A v1

$$\eta = \eta_\infty \frac{\eta_0 + \eta_m + \frac{\tan\left\{\frac{c\pi}{4}[(1+x_m)x - x_m + 1]\right\}}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.38})$$

$$x_m = \frac{2}{c\pi} \arctan \left[(\eta_m + \eta_0) \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right] \quad (\text{B.39})$$

$$s = \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \quad (\text{B.40})$$

$$x_m = \frac{2}{c\pi} \arctan [(\eta_m + \eta_0)s] \quad (\text{B.41})$$

$$\eta = \eta_\infty \frac{\eta_0 + \eta_m + \frac{\tan\left\{\frac{c\pi}{4}[(1+x_m)x - x_m + 1]\right\}}{s}}{\eta_\infty + \eta_m} \quad (\text{B.42})$$

$$\eta(0) = \eta_\infty \frac{\eta_0 + \eta_m + \frac{\tan\left[\frac{c\pi}{4}(1-x_m)\right]}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.43})$$

$$\eta(0) = \eta_\infty \frac{\eta_0 + \eta_m + \frac{1 \pm \sqrt{1 + \left[(\eta_m + \eta_0) \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}\right]^2} - \frac{\eta_m + \eta_0}{\eta_\infty} \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right]}{(\eta_m + \eta_0) \frac{\tan^2\left(\frac{c\pi}{2}\right)}{\eta_\infty} + \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right] \left\{1 \pm \sqrt{1 + \left[(\eta_m + \eta_0) \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}\right]^2}\right\}} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.44})$$

Table B.7: Various values of $\eta(x)$ for displaced tangent A v1 mapping.

x	η
-1	0
0	$\eta_\infty \frac{\eta_0 + \eta_m + \frac{\tan\left[\frac{c\pi}{4}(1-x_m)\right]}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty}{\eta_\infty + \eta_m}$
1	$\eta_\infty \frac{\eta_0 + \eta_m + \eta_\infty}{\eta_\infty + \eta_m} \approx \eta_\infty$

B.4.2 Displaced tangent A v2

$$\eta(0) = \eta_\infty \frac{\eta_0 + \eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2} \tan^2\left(\frac{c\pi}{2}\right) - \frac{\eta_m + \eta_0}{\eta_\infty} \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right]}{\frac{\eta_m + \eta_0}{\eta_\infty} \tan^2\left(\frac{c\pi}{2}\right) + \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right] \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2} \tan^2\left(\frac{c\pi}{2}\right)\right\}} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.45})$$

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$$\eta(0) = \eta_\infty \frac{\eta_0 + \eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]} - \frac{\eta_m + \eta_0}{\eta_\infty} \left[1 \pm \sqrt{1 + \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]}\right]}{\frac{\eta_m + \eta_0}{\eta_\infty} \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right] + \left[1 \pm \sqrt{1 + \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]}\right] \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]}\right\}}}{\eta_\infty + \eta_m} \eta_\infty \quad (\text{B.46})$$

$$c = \frac{2}{\pi} \arccos \left(\frac{\eta_m - \eta_0}{\eta_\infty - \eta_m} \right) \quad (\text{B.47})$$

$$\eta(0) = \eta_\infty \frac{\eta_0 + \eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m - \eta_0}\right)^2 - 1\right]} - \frac{\eta_m + \eta_0}{\eta_\infty} \left\{1 \pm \sqrt{1 + \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m - \eta_0}\right)^2 - 1\right]}\right\}}{\frac{\eta_m + \eta_0}{\eta_\infty} \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m - \eta_0}\right)^2 - 1\right] + \left[1 \pm \sqrt{1 + \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m - \eta_0}\right)^2 - 1\right]}\right] \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m - \eta_0}\right)^2 - 1\right]}\right\}}}{\eta_\infty + \eta_m} \eta_\infty \quad (\text{B.48})$$

$$\eta(0) \approx \eta_0 + \eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_m - \eta_0}\right)^2} \mp \frac{\eta_m + \eta_0}{\eta_m - \eta_0}}{\frac{\eta_m + \eta_0}{\eta_m - \eta_0} \pm \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_m - \eta_0}\right)^2}\right\}} (\eta_m - \eta_0) \quad (\text{B.49})$$

Taking different signs, it can take values close to $2\eta_0$, $2\eta_m$.

Table B.8: Various values of $\eta(x)$ for displaced tangent A v2 mapping.

x	η
-1	0
0	$\approx 2\eta_0, \approx 2\eta_m, \dots$
1	$\eta_\infty \frac{\eta_0 + \eta_m + \eta_\infty}{\eta_\infty + \eta_m} \approx \eta_\infty$

B.4.3 Displaced tangent A v3

$$\eta(0) \approx \eta_0 + \eta_m \quad (\text{B.50})$$

Then:

$$\frac{\tan \left[\frac{c\pi}{4} (1 - x_m) \right]}{\tan \left(\frac{c\pi}{2} \right)} \eta_\infty = 0 \quad (\text{B.51})$$

$$\frac{1 \pm \sqrt{1 + \left[(\eta_m + \eta_0) \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right]^2} - \frac{\eta_m + \eta_0}{\eta_\infty} \left[1 \pm \sqrt{1 + \tan^2 \left(\frac{c\pi}{2} \right)} \right]}{(\eta_m + \eta_0) \frac{\tan^2\left(\frac{c\pi}{2}\right)}{\eta_\infty} + \left[1 \pm \sqrt{1 + \tan^2 \left(\frac{c\pi}{2} \right)} \right] \left\{ 1 \pm \sqrt{1 + \left[(\eta_m + \eta_0) \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right]^2} \right\}} \eta_\infty = 0 \quad (\text{B.52})$$

The result above goes to ∞ when η_∞ goes to ∞ , so the tangent must go to zero even faster. Still there is an indetermination zero divided by zero.

$$\lim_{x \rightarrow 0} \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 x^2} - \frac{\eta_m + \eta_0}{\eta_\infty} \left(1 \pm \sqrt{1 + x^2}\right)}{\frac{\eta_m + \eta_0}{\eta_\infty} x^2 + \left(1 \pm \sqrt{1 + x^2}\right) \left[1 \pm \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 x^2}\right]} \eta_\infty \quad (\text{B.53})$$

The “+” solution is $(\eta_\infty - \eta_m - \eta_0)/2$.

$$\begin{aligned} \lim_{x \rightarrow 0} & \left(\frac{1 - \sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 0^2} + \left[-\frac{\left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 0}{\sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 0^2}} \right] x}{\frac{\eta_m + \eta_0}{\eta_\infty} x^2 + o(x^2)} \eta_\infty \right. \\ & + \frac{\left\{ -\frac{\left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2}{\sqrt{1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 0^2}} + \frac{\left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^4 0^2}{\left[1 + \left(\frac{\eta_m + \eta_0}{\eta_\infty}\right)^2 0^2\right]^{\frac{3}{2}}} \right\} \frac{x^2}{2} + o(x^2)}{\frac{\eta_m + \eta_0}{\eta_\infty} x^2 + o(x^2)} \eta_\infty \\ & \left. - \frac{\frac{\eta_m + \eta_0}{\eta_\infty} \left\{ 1 - \sqrt{1 + 0^2} + \left(-\frac{0}{\sqrt{1 + 0^2}}\right) x + \left[-\frac{1}{\sqrt{1 + 0^2}} + \frac{0^2}{(1 + 0^2)^{\frac{3}{2}}}\right] \frac{x^2}{2} \right\} + o(x^2)}{\frac{\eta_m + \eta_0}{\eta_\infty} x^2 + o(x^2)} \eta_\infty \right) \end{aligned} \quad (\text{B.54})$$

$$\lim_{x \rightarrow 0} \left[\frac{\frac{\eta_m + \eta_0}{\eta_\infty} \left(1 - \frac{\eta_m + \eta_0}{\eta_\infty}\right) \frac{x^2}{2} + o(x^2)}{\frac{\eta_m + \eta_0}{\eta_\infty} x^2 + o(x^2)} \eta_\infty \right] = \frac{\eta_\infty - \eta_m - \eta_0}{2} \quad (\text{B.55})$$

The solution for “−” is the same.

Therefore, $c = 0, 1, \dots$ does not give the desired result, limit equal to zero. However, values of c close to 1, but lower provide with reasonable results.

Table B.9: Various values of $\eta(x)$ for displaced tangent A v3 mapping.

x	η
-1	0
0	$\approx \eta_\infty \frac{\eta_0 + \eta_m + \frac{\eta_\infty - \eta_m - \eta_0}{2}}{\eta_\infty + \eta_m} \approx \frac{\eta_\infty}{2}$
1	$\eta_\infty \frac{\eta_0 + \eta_m + \eta_\infty}{\eta_\infty + \eta_m} \approx \eta_\infty$

$$x = \frac{x_m - 1 + \frac{4}{c\pi} \arctan \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m}{\eta_\infty} \tan \left(\frac{c\pi}{2} \right) \right]}{1 + x_m} \quad (\text{B.56})$$

$$x = \frac{x_m - 1 + \frac{4}{c\pi} \arctan \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m \right) s \right]}{1 + x_m} \quad (\text{B.57})$$

$$\frac{\partial x}{\partial \eta} = \frac{4}{c\pi(1+x_m)} \frac{\frac{\eta_\infty + \eta_m}{\eta_\infty^2} \tan\left(\frac{c\pi}{2}\right)}{1 + \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right) \right]^2} \quad (\text{B.58})$$

$$\frac{\partial x}{\partial \eta} = \frac{4}{c\pi(1+x_m)} \frac{\frac{\eta_\infty + \eta_m}{\eta_\infty} s}{1 + \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m \right) s \right]^2} \quad (\text{B.59})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{8}{c\pi(1+x_m)} \frac{\left[\frac{\eta_\infty + \eta_m}{\eta_\infty^2} \tan\left(\frac{c\pi}{2}\right) \right]^2 \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right) \right]}{\left\{ 1 + \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right) \right]^2 \right\}^2} \quad (\text{B.60})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{8}{c\pi(1+x_m)} \frac{\left[\frac{\eta_\infty + \eta_m}{\eta_\infty} s \right]^2 \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m \right) s \right]}{\left\{ 1 + \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_0 - \eta_m \right) s \right]^2 \right\}^2} \quad (\text{B.61})$$

B.5 Displaced tangent B

B.5.1 Displaced tangent B v1

$$\eta = \eta_\infty \frac{\eta_m + \frac{\tan\left\{\frac{c\pi}{4}[(1+x_m)x - x_m + 1]\right\}}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.62})$$

$$x_m = \frac{2}{c\pi} \arctan \left[\eta_m \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right] \quad (\text{B.63})$$

$$s = \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \quad (\text{B.64})$$

$$x_m = \frac{2}{c\pi} \arctan(\eta_m s) \quad (\text{B.65})$$

$$\eta = \eta_\infty \frac{\eta_m + \frac{\tan\left\{\frac{c\pi}{4}[(1+x_m)x - x_m + 1]\right\}}{s} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.66})$$

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{\tan\left[\frac{c\pi}{4}(1-x_m)\right]}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.67})$$

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 \pm \sqrt{1 + \left[\eta_m \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right]^2} - \frac{\eta_m}{\eta_\infty} \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)} \right]}{\eta_m \frac{\tan^2\left(\frac{c\pi}{2}\right)}{\eta_\infty} + \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)} \right] \left\{ 1 \pm \sqrt{1 + \left[\eta_m \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right]^2} \right\}} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.68})$$

Table B.10: Various values of $\eta(x)$ for displaced tangent B v1 mapping.

x	η
-1	0
0	$\eta_m + \frac{\tan\left[\frac{c\pi}{4}(1-x_m)\right]}{\tan\left(\frac{c\pi}{2}\right)}\eta_\infty$
1	η_∞

B.5.2 Displaced tangent B v2

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 \tan^2\left(\frac{c\pi}{2}\right)} - \frac{\eta_m}{\eta_\infty} \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right]}{\frac{\eta_m}{\eta_\infty} \tan^2\left(\frac{c\pi}{2}\right) + \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right] \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 \tan^2\left(\frac{c\pi}{2}\right)}\right\}} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.69})$$

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]} - \frac{\eta_m}{\eta_\infty} \left[1 \pm \sqrt{1 + \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]}\right]}{\frac{\eta_m}{\eta_\infty} \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right] + \left[1 \pm \sqrt{1 + \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]}\right] \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 \left[\frac{1}{\cos^2\left(\frac{c\pi}{2}\right)} - 1\right]}\right\}} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.70})$$

$$c = \frac{2}{\pi} \arccos\left(\frac{\eta_m}{\eta_\infty - \eta_m}\right) \quad (\text{B.71})$$

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m}\right)^2 - 1\right]} - \frac{\eta_m}{\eta_\infty} \left\{1 \pm \sqrt{1 + \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m}\right)^2 - 1\right]}\right\}}{\frac{\eta_m}{\eta_\infty} \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m}\right)^2 - 1\right] + \left[1 \pm \sqrt{1 + \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m}\right)^2 - 1\right]}\right] \left\{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 \left[\left(\frac{\eta_\infty - \eta_m}{\eta_m}\right)^2 - 1\right]}\right\}} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.72})$$

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 \pm \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)} - \frac{\eta_m}{\eta_\infty} \pm \frac{\eta_m}{\eta_\infty} \mp 1}{\frac{\eta_\infty - 2\eta_m}{\eta_m} + \left(1 \mp 1 \pm \frac{\eta_\infty}{\eta_m}\right) \left[1 \pm \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)}\right]} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.73})$$

Solution “+”:

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 + \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)} - \frac{\eta_m}{\eta_\infty} + \frac{\eta_m}{\eta_\infty} - 1}{\frac{\eta_\infty - 2\eta_m}{\eta_m} + \left(1 - 1 + \frac{\eta_\infty}{\eta_m}\right) \left[1 + \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)}\right]} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.74})$$

$$\eta(0) \approx \frac{2 + 2\sqrt{2}}{2 + \sqrt{2}} \eta_m \quad (\text{B.75})$$

Solution “-”:

$$\eta(0) = \eta_\infty \frac{\eta_m + \frac{1 - \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)} - \frac{\eta_m}{\eta_\infty} - \frac{\eta_m}{\eta_\infty} + 1}{\frac{\eta_\infty - 2\eta_m}{\eta_m} + \left(1 + 1 - \frac{\eta_\infty}{\eta_m}\right) \left[1 - \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)}\right]} \eta_\infty}{\eta_\infty + \eta_m} \quad (\text{B.76})$$

$$\eta(0) \approx \sqrt{2}\eta_m \quad (\text{B.77})$$

So the value is around η_m .

Table B.11: Various values of $\eta(x)$ for displaced tangent B v2 mapping.

x	η
-1	0
0	$\approx \eta_m$
1	η_∞

B.5.3 Displaced tangent B v3

$$\eta(0) \approx \eta_m \quad (\text{B.78})$$

Then:

$$\frac{\tan\left[\frac{c\pi}{4}(1-x_m)\right]}{\tan\left(\frac{c\pi}{2}\right)}\eta_\infty = 0 \quad (\text{B.79})$$

$$\frac{1 \pm \sqrt{1 + \left[\eta_m \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}\right]^2} - \frac{\eta_m}{\eta_\infty} \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right]}{\eta_m \frac{\tan^2\left(\frac{c\pi}{2}\right)}{\eta_\infty} + \left[1 \pm \sqrt{1 + \tan^2\left(\frac{c\pi}{2}\right)}\right] \left\{1 \pm \sqrt{1 + \left[\eta_m \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}\right]^2}\right\}} \eta_\infty = 0 \quad (\text{B.80})$$

The result above goes to ∞ when η_∞ goes to ∞ , so the tangent must go to zero even faster. Still there is an indetermination zero divided by zero.

$$\lim_{x \rightarrow 0} \frac{1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 x^2} - \frac{\eta_m}{\eta_\infty} \left(1 \pm \sqrt{1 + x^2}\right)}{\frac{\eta_m}{\eta_\infty} x^2 + \left(1 \pm \sqrt{1 + x^2}\right) \left[1 \pm \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 x^2}\right]} \eta_\infty \quad (\text{B.81})$$

The “+” solution is $(\eta_\infty - \eta_m)/2$.

$$\lim_{x \rightarrow 0} \left(\frac{1 - \sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 0^2} + \left[-\frac{\left(\frac{\eta_m}{\eta_\infty}\right)^2 0}{\sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 0^2}} \right] x + \left\{ -\frac{\left(\frac{\eta_m}{\eta_\infty}\right)^2}{\sqrt{1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 0^2}} + \frac{\left(\frac{\eta_m}{\eta_\infty}\right)^4 0^2}{\left[1 + \left(\frac{\eta_m}{\eta_\infty}\right)^2 0^2\right]^{\frac{3}{2}}} \right\} \frac{x^2}{2} + o(x^2)}{\frac{\eta_m}{\eta_\infty} x^2 + o(x^2)} \eta_\infty - \frac{\frac{\eta_m}{\eta_\infty} \left\{ 1 - \sqrt{1 + 0^2} + \left(-\frac{0}{\sqrt{1+0^2}}\right) x + \left[-\frac{1}{\sqrt{1+0^2}} + \frac{0^2}{(1+0^2)^{\frac{3}{2}}} \right] \frac{x^2}{2} \right\} + o(x^2)}{\frac{\eta_m}{\eta_\infty} x^2 + o(x^2)} \eta_\infty \right) \quad (\text{B.82})$$

$$\lim_{x \rightarrow 0} \left[\frac{\frac{\eta_m}{\eta_\infty} \left(1 - \frac{\eta_m}{\eta_\infty}\right) \frac{x^2}{2} + o(x^2)}{\frac{\eta_m}{\eta_\infty} x^2 + o(x^2)} \eta_\infty \right] = \frac{\eta_\infty - \eta_m}{2} \quad (\text{B.83})$$

The solution for “−” is the same.

Therefore, $c = 0, 1, \dots$ does not give the desired result, limit equal to zero. However, values of c close to 1, but lower provide with reasonable results.

Table B.12: Various values of $\eta(x)$ for displaced tangent B v3 mapping.

x	η
-1	0
0	$\approx \eta_\infty \frac{\eta_m + \frac{\eta_\infty - \eta_m}{2}}{\eta_\infty + \eta_m} \approx \frac{\eta_\infty}{2}$
1	$\eta_\infty \frac{\eta_m + \eta_\infty}{\eta_\infty + \eta_m} \approx \eta_\infty$

$$x = \frac{x_m - 1 + \frac{4}{c\pi} \arctan \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m}{\eta_\infty} \tan \left(\frac{c\pi}{2} \right) \right]}{1 + x_m} \quad (\text{B.84})$$

$$x = \frac{x_m - 1 + \frac{4}{c\pi} \arctan \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m \right) s \right]}{1 + x_m} \quad (\text{B.85})$$

$$\frac{\partial x}{\partial \eta} = \frac{4}{c\pi(1 + x_m)} \frac{\frac{\eta_\infty + \eta_m}{\eta_\infty^2} \tan \left(\frac{c\pi}{2} \right)}{1 + \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m}{\eta_\infty} \tan \left(\frac{c\pi}{2} \right) \right]^2} \quad (\text{B.86})$$

$$\frac{\partial x}{\partial \eta} = \frac{4}{c\pi(1 + x_m)} \frac{\frac{\eta_\infty + \eta_m}{\eta_\infty} s}{1 + \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m \right) s \right]^2} \quad (\text{B.87})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{8}{c\pi(1 + x_m)} \frac{\left[\frac{\eta_\infty + \eta_m}{\eta_\infty^2} \tan \left(\frac{c\pi}{2} \right) \right]^2 \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m}{\eta_\infty} \tan \left(\frac{c\pi}{2} \right) \right]}{\left\{ 1 + \left[\frac{\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m}{\eta_\infty} \tan \left(\frac{c\pi}{2} \right) \right]^2 \right\}^2} \quad (\text{B.88})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{8}{c\pi(1 + x_m)} \frac{\left[\frac{\eta_\infty + \eta_m}{\eta_\infty} s \right]^2 \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m \right) s \right]}{\left\{ 1 + \left[\left(\eta \frac{\eta_\infty + \eta_m}{\eta_\infty} - \eta_m \right) s \right]^2 \right\}^2} \quad (\text{B.89})$$

B.6 Displaced tangent C (for boundary layer)

$$\eta = \frac{\tan \left[\frac{c\pi(x+1)}{4} \right]}{\tan \left(\frac{c\pi}{2} \right)} \eta_\infty \quad (\text{B.90})$$

$$s = \frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \quad (\text{B.91})$$

$$\eta = \frac{\tan\left[\frac{c\pi(x+1)}{4}\right]}{s} \quad (\text{B.92})$$

$$\eta_m = \frac{\tan\left(\frac{c\pi}{4}\right)}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty \quad (\text{B.93})$$

$$\tan\left(\frac{c\pi}{4}\right) = \sqrt{1 - 2\frac{\eta_m}{\eta_\infty}} \quad \eta_m \leq \frac{\eta_\infty}{2} \quad (\text{B.94})$$

$$c = \frac{4}{\pi} \arctan\left(\sqrt{1 - 2\frac{\eta_m}{\eta_\infty}}\right) \quad (\text{B.95})$$

$$\eta(-0.5) = \frac{\tan\left(\frac{c\pi}{8}\right)}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty \quad (\text{B.96})$$

$$\eta(-0.5) = -\frac{1 \pm \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)}}{1 - 2\frac{\eta_m}{\eta_\infty}} \eta_m \quad (\text{B.97})$$

$\eta(-0.5) > 0$ then minus sign must be chosen:

$$\eta(-0.5) = \frac{\sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)} - 1}{1 - 2\frac{\eta_m}{\eta_\infty}} \eta_m \quad (\text{B.98})$$

$$\eta(0.5) = \frac{\tan\left(\frac{3c\pi}{8}\right)}{\tan\left(\frac{c\pi}{2}\right)} \eta_\infty \quad (\text{B.99})$$

$$\eta(0.5) = \frac{\left[\pm\sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)} + 2\frac{\eta_m}{\eta_\infty}\right]}{\left(1 - 2\frac{\eta_m}{\eta_\infty}\right)\left[2 \pm \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)}\right]} \eta_m \quad (\text{B.100})$$

$\eta(0.5) > \eta_m$ then plus sign must be chosen in first $\pm$ and minus sign in second $\pm$:

$$\eta(0.5) = \frac{\left[\sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)} + 2\frac{\eta_m}{\eta_\infty}\right]}{\left(1 - 2\frac{\eta_m}{\eta_\infty}\right)\left[2 - \sqrt{2\left(1 - \frac{\eta_m}{\eta_\infty}\right)}\right]} \eta_m \quad (\text{B.101})$$

$$x = \frac{4}{c\pi} \arctan\left[\frac{\eta}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right)\right] - 1 \quad (\text{B.102})$$

$$x = \frac{4}{c\pi} \arctan(\eta s) - 1 \quad (\text{B.103})$$

$$\frac{\partial x}{\partial \eta} = \frac{4}{c\pi} \frac{\frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty}}{1 + \left[\frac{\eta}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right)\right]^2} \quad (\text{B.104})$$

Table B.13: Various values of $\eta(x)$ for displaced tangent C mapping.

x	η
-1	0
-0.5	$\frac{\sqrt{2\left(1+\frac{\eta_m}{\eta_\infty}\right)}-1}{1-\frac{2\eta_m}{\eta_\infty}}\eta_m \approx (\sqrt{2}-1)\eta_m$
0	η_m
0.5	$\frac{\sqrt{2\left(1-\frac{\eta_m}{\eta_\infty}\right)}+\frac{2\eta_m}{\eta_\infty}}{\left(2-\sqrt{2\left(1-\frac{\eta_m}{\eta_\infty}\right)}\right)\left(1-\frac{2\eta_m}{\eta_\infty}\right)}\eta_m \approx \frac{\sqrt{2}}{2-\sqrt{2}}\eta_m$
1	η_∞

$$\frac{\partial x}{\partial \eta} = \frac{4}{c\pi} \frac{s}{1 + (\eta s)^2} \quad (\text{B.105})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{8}{c\pi} \frac{\eta \left[\frac{\tan\left(\frac{c\pi}{2}\right)}{\eta_\infty} \right]^3}{\left\{ 1 + \left[\frac{\eta}{\eta_\infty} \tan\left(\frac{c\pi}{2}\right) \right]^2 \right\}^2} \quad (\text{B.106})$$

$$\frac{\partial^2 x}{\partial \eta^2} = -\frac{8}{c\pi} \frac{\eta s^3}{[1 + (\eta s)^2]^2} \quad (\text{B.107})$$

B.7 Double flat tangent

$$\eta = \frac{\eta_1 + \eta_2}{2} + \frac{\tan\left(\frac{c\pi x}{2}\right)}{\tan\left(\frac{c\pi}{2}\right)} \left(\eta_\infty - \frac{\eta_2 - \eta_1}{2} \right) + \frac{\arctan\left(\frac{x}{1-c}\right)}{\arctan\left(\frac{1}{1-c}\right)} \left(\frac{\eta_2 - \eta_1}{2} \right) \quad (\text{B.108})$$

The constant c must be close to 1, but lower, the closer c is to 1, the more number of points are located close to η_1 and η_2 .

Table B.14: Various values of $\eta(x)$ for double flat tangent mapping.

x	η
-1	$\frac{\eta_1 + \eta_2}{2} - \eta_\infty \approx -\eta_\infty$
-0.5	$\approx \eta_1$
0	$\frac{\eta_1 + \eta_2}{2}$
0.5	$\approx \eta_2$
1	$\frac{\eta_1 + \eta_2}{2} + \eta_\infty \approx \eta_\infty$

A function $x(\eta)$ is not achievable analytically, so the values of x must be used instead

of the values of η

$$\frac{\partial x}{\partial \eta} = \frac{1}{\frac{\partial \eta}{\partial x}} = \frac{1}{\frac{c\pi}{2} \frac{\tan^2(\frac{c\pi x}{2})+1}{\tan(\frac{c\pi}{2})} \left(\eta_\infty + \frac{\eta_1 - \eta_2}{2} \right) + \frac{(\eta_2 - \eta_1)}{2(1-c) \left[1 + \left(\frac{x}{1-c} \right)^2 \right] \arctan\left(\frac{1}{1-c}\right)}} \quad (\text{B.109})$$

$$\frac{\partial^2 x}{\partial \eta^2} = \frac{\partial \frac{\partial x}{\partial \eta}}{\partial \eta} = - \frac{\left[\left(\frac{c\pi}{2} \right)^2 \tan\left(\frac{c\pi x}{2}\right) \frac{\tan^2(\frac{c\pi x}{2})+1}{\tan(\frac{c\pi}{2})} (2\eta_\infty + \eta_1 - \eta_2) - \frac{(\eta_2 - \eta_1)x}{(1-c)^3 \left[1 + \left(\frac{x}{1-c} \right)^2 \right]^2 \arctan\left(\frac{1}{1-c}\right)} \right]}{\left[\frac{c\pi}{2} \frac{\tan^2(\frac{c\pi x}{2})+1}{\tan(\frac{c\pi}{2})} \left(\eta_\infty + \frac{\eta_1 - \eta_2}{2} \right) + \frac{(\eta_2 - \eta_1)}{2(1-c) \left[1 + \left(\frac{x}{1-c} \right)^2 \right] \arctan\left(\frac{1}{1-c}\right)} \right]^3} \quad (\text{B.110})$$

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