Signals and Systems: Introduction

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Signals and Systems: Introduction

1- Introduction to Signals and Systems
2- Continuous-Time Signals
3- Discrete-Time Signals
4- Description of Systems
5- Time-Domain System Analysis
6- Spectral Method
7- Fourier Transform
8- Sampling
9- Discrete Fourier Transform (DFT)
1- Introduction to Signals and Systems
Introduction

-A **signal** is any physical phenomenon which conveys information.

-**Systems** respond to signals and produce new signals.

-**Excitation** signals are applied at system **inputs** and **response** signals are produced at system **outputs**.
Signal types

- **Continuous-Time Continuous-Value Signal**: $x(t)$
- **Continuous-Time Discrete-Value Signal**: $x(t)$
- **Continuous-Time Random Signal**: $x(t)$
- **Discrete-Time Continuous-Value Signal**: $x[n]$
- **Discrete-Time Digital Signal**: $x(t)$
- **Noisy Digital Signal**: $x(t)$
Conversions Between Signal Types
-A communication system has an **information** signal plus **noise** signals.
Voice Signal

Adult Male Voice Saying the Word, "Signal"
(a) Segment of a continuous-time speech signal $x(t)$.
(b) Sequence of samples $x[n] = x(nT)$ obtained from the signal in part (a) with $T=125 \mu s$. 
2- Continuous-Time Signals
All continuous signals that are functions of time are continuous-time, but not all continuous-time signals are continuous functions of time.
**Impulse**

\[
\delta(t) = 0, \quad t \neq 0
\]

\[
\int_{-\infty}^{\infty} \delta(t) \, dt = 1
\]

-The impulse is not a function in the ordinary sense because its value at the time of its occurrence is not defined.

-It is a functional.
-It is represented graphically by a vertical arrow.

-Its strength is either written beside it or is represented by its length.
Properties of the Impulse

Sampling Property

\[ \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = f(0) \]

\[ \int_{-\infty}^{\infty} f(t) \delta(t - t_0) \, dt = f(t_0) \]

-The sampling property “extracts” the value of a function at a point.
Scaling Property

\[ \delta(a(t-t_0)) = \frac{1}{|a|} \delta(t-t_0) \]

-This property illustrates that the impulse is different from ordinary mathematical functions.
The periodic impulse is a sum of infinitely many uniformly-spaced impulses.

\[ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad [n \text{ an integer}] \]
Step

\[ u(t) = \begin{cases} 
1 & , \ t > 0 \\
1/2 & , \ t = 0 \\
0 & , \ t < 0 
\end{cases} \]

Precise Graph

Commonly-Used Graph
Signum

\[ \text{sgn}(t) = \begin{cases} 
1 & , \ t > 0 \\
0 & , \ t = 0 \\
-1 & , \ t < 0
\end{cases} \]

Precise Graph

Commonly-Used Graph
Ramp

\[ \text{ramp}(t) = \begin{cases} 
  t & , t > 0 \\
  0 & , t \leq 0 
\end{cases} = \int_{-\infty}^{t} u(\tau) d\tau = tu(t) \]
Pulse (Rectangle)

\[
\text{rect}(t) = \begin{cases} 
1 & , \quad |t| < 1/2 \\
1/2 & , \quad |t| = 1/2 \\
0 & , \quad |t| > 1/2 
\end{cases} = u(t + 1/2) - u(t - 1/2)
\]
Real Sinusoid and Real Exponential

\[ g(t) = A \cos(2\pi f_0 t + \theta) \]

\[ g(t) = A e^{t/\tau} \]
Complex Exponential

\[ g(t) = A e^{\frac{t}{\tau}} e^{j2\pi ft} \]

\[ e^{j2\pi ft} = \cos(2\pi ft) + j \sin(2\pi ft) \]

\[ e^{-j2\pi ft} = \cos(2\pi ft) - j \sin(2\pi ft) \]

\[ \cos(2\pi ft) = \frac{e^{j2\pi ft} + e^{-j2\pi ft}}{2} \]

\[ \sin(2\pi ft) = \frac{e^{j2\pi ft} - e^{-j2\pi ft}}{2j} \]
Scaling and Shifting

\[ g(t) \rightarrow Ag\left(\frac{t - t_0}{a}\right) \]

Amplitude Scaling

Time Scaling

Time Shifting

Miguel A. Muriel - Signals and Systems: Introduction - 25
\( g(t) \rightarrow A g(bt - t_0) \)
Differentiation

\[ g(t) = \frac{dx(t)}{dt} \]
Integration

\[ g(t) = \int_{-\infty}^{t} x(\tau) \, d\tau \]
Even and Odd Signals

Even Functions
\[ g(t) = g(-t) \]

Odd Functions
\[ g(t) = -g(-t) \]
Even and Odd Parts of Signals

\[ g(t) = g_e(t) + g_o(t) \]

The **even part** of a function is

\[ g_e(t) = \frac{g(t) + g(-t)}{2} \]

The **odd part** of a function is

\[ g_o(t) = \frac{g(t) - g(-t)}{2} \]

- The derivative of an even/odd function is odd/even.
- The integral of an even/odd function is an odd/even function, plus a constant.
Periodic Signals

\[ g(t) = g(t + nT) \]

\( n \) is any integer
\( T \) is a **period** of the function

-The minimum positive value of \( T \) for which \( g(t) = g(t + T) \) is called the **fundamental period** \( T_0 \) of the function.

-The reciprocal of the fundamental period is the **fundamental frequency** \( f_0 = 1/T_0 \).
Examples of periodic functions with fundamental period $T_0$
Signal Energy

The signal energy of a signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$
Signal Power

- Some signals have infinite signal energy.
- This usually occurs because the signal is not time limited.
- In that case it is more convenient to deal with average signal power.

-The average signal power of a signal $x(t)$ is:

$$ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt $$
For a periodic signal $x(t)$ the average signal power is:

$$P_x = \frac{1}{T} \int_{T} |x(t)|^2 \, dt$$

where $T$ is any period of $|x(t)|^2$
-A signal with finite signal energy is called an **energy signal**.

-A signal with infinite signal energy and finite average signal power is called a **power signal**.

-All periodic signals are power signals.
3- Discrete-Time Signals
Sampling and Discrete Time

-Sampling is the acquisition of the values of a continuous-time signal at discrete points in time.

- $x(t)$ is a continuous-time signal, $x[n]$ is a discrete-time signal.

$$x[n] = x(nT_s)$$  \[T_s \text{ is the time between samples}\]

(a) An ideal sampler  (b) An ideal sampler sampling uniformly
Creating a discrete-time signal by sampling a continuous-time signal
Unit Impulse

-The discrete-time unit impulse is a function in the ordinary sense (in contrast with the continuous-time impulse).
-It has a sampling property

\[
\sum_{n=-\infty}^{\infty} A \delta[n - n_0] x[n] = A x[n_0]
\]

-But no scaling property, \( \delta[n] = \delta[an] \) for any non-zero, finite integer \( a \).
Periodic Impulse

\[ \delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN] \]
Unit Sequence

\[ u[n] = \begin{cases} 
1, & n \geq 0 \\
0, & n < 0 
\end{cases} \]
Signum

\[
\text{sgn}[n] = \begin{cases} 
1 & , \quad n > 0 \\
0 & , \quad n = 0 \\
-1 & , \quad n < 0 
\end{cases}
\]
Unit Ramp

\[ ramp[n] = \begin{cases} 
  n & , \quad n \geq 0 \\
  0 & , \quad n < 0 
\end{cases} = nu[n] \]
Exponentials

\[ x[n] = A\alpha^n \quad [A \text{ is a real constant}] \]

**Real** \( \alpha \)

- \( 0 < \alpha < 1 \)
- \( -1 < \alpha < 0 \)
- \( \alpha > 1 \)
- \( \alpha < -1 \)

**Complex** \( \alpha \)

- \( |\alpha| < 1 \)
- \( |\alpha| > 1 \)

Re(\( x[n] \)) \quad \text{Im}(\( x[n] \))
Sinusoids

\[ g[n] = A \cos(2\pi F_0 n + \theta) \]

- \( A \) is a real constant, \( \theta \) is a real phase shift in radians
- \( F_0 \) is a real number, and \( n \) is discrete time

- Unlike a continuous-time sinusoid, a discrete-time sinusoid is not necessarily periodic.
- \( g[n] \) periodic \( \rightarrow F_0 \) must be a ratio of integers (a rational number).
Four discrete-time sinusoids
Two sinusoids whose analytical expressions look different

\[ g_1[n] = A \cos(2\pi F_{01} n + \theta) \quad \text{and} \quad g_2[n] = A \cos(2\pi F_{02} n + \theta) \]

may actually be the same if \( F_{02} = F_{01} + m \), where \( m \) is an integer.

\[ A \cos(2\pi F_{02} n + \theta) = A \cos(2\pi F_{01} n + 2\pi mn + \theta) = A \cos(2\pi F_{01} n + \theta) \]
Two cosines with different $F$’s but the same functional behavior
\[ x[n] = \cos(2\pi F n) \] Dashed line is \( x(t) = \cos(2\pi F t) \)

A discrete-time sinusoid with frequency \( F \) repeats every time \( F \) changes by one
Scaling and Shifting Functions

Let \( g[n] \) be graphically defined by:

\[
 g[n] = 0, \quad |n| > 15
\]
Time shifting \( n \rightarrow n + n_0 \) \( [n_0 \text{ an integer}] \)

\[
\begin{array}{c|cccccccccccc}
 n & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
g[n+3] & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 5 & 0 \\
\end{array}
\]
Time compression $n \rightarrow Kn \quad [K \text{ an integer } > 1]$

Time expansion $n \rightarrow n/K \quad [K \text{ an integer } > 1]$

-For all $n$ such that $n/K$ is an integer, $g[n/K]$ is defined.

-For all $n$ such that $n/K$ is not an integer, $g[n/K]$ is not defined.
Time compression for a discrete-time function
Time expansion for a discrete-time function \((K=2)\)
Differencing

\[ g[n] = x[n] - x[n-1] \]

**Backward Differences**

\[ g[n] = x[n+1] - x[n] \]

**Forward Differences**
Accumulation

\[ g[n] = \sum_{m=-\infty}^{n} h[m] \]
Even and Odd Signals

Even Function

\[ g[n] = g[-n] \]

Odd Function

\[ g[n] = -g[-n] \]

\[ g_e[n] = \frac{g[n] + g[-n]}{2} \]

\[ g_o[n] = \frac{g[n] - g[-n]}{2} \]
Periodic Signals

\[ g[n] = g[n + mN] \]

- The minimum positive value of \( N \) for which \( g[n] = g[n + N] \) is called the fundamental period \( N_0 \) of the function.

- The reciprocal of the fundamental period is the fundamental frequency \( F_0 = 1/N_0 \)
Examples of periodic functions with fundamental period $N_0$
**Signal Energy**

The signal energy of a signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$x[n] = 0$, $n > 31$
Signal Power

- Some signals have infinite signal energy.
- This usually occurs because the signal is not time limited.
- In that case it is more convenient to deal with average signal power.

- The average signal power of a signal $x[n]$ is

$$P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |x[n]|^2$$
-For a periodic signal $x[n]$ the average signal power is:

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2$$

(The notation $\sum_{n=\langle N \rangle}$ means the sum over any set of consecutive $n$'s exactly $N$ in length.)
-A signal with finite signal energy is called an energy signal.

-A signal with infinite signal energy and finite average signal power is called a power signal.

-All periodic signals are power signals.
4- Description of Systems
Systems

-Broadly speaking, a system is anything that responds when stimulated or excited.

-Engineering system analysis is the application of mathematical methods to the design and analysis of systems.

-Systems have **inputs** and **outputs**.

-Systems accept **excitations** or **input signals** at their inputs and produce **responses** or **output signals** at their outputs.
- Continuous-time systems are usually described by differential equations.

- Discrete-time systems are usually described by difference equations.

- The properties of discrete-time systems have the same meaning as they do in continuous-time systems.
Systems are often represented by block diagrams.

A single-input, single-output system block diagram:

\[ x(t) \rightarrow \mathcal{H} \rightarrow y(t) \]

A multiple-input, multiple-output system block diagram:

\[ x_1(t) \rightarrow \mathcal{H}_1 \rightarrow y_1(t) \]

\[ x_2(t) \rightarrow \mathcal{H}_2 \rightarrow \mathcal{H}_4 \rightarrow y_2(t) \]
Some Block Diagram Symbols

Three common block diagram symbols for an amplifier

Three common block diagram symbols for a summing junction
-The block diagram symbols for a summing junction and an amplifier are the same for discrete-time systems as they are for continuous-time systems.

![Block diagram symbol (continuous-time systems) for an integrator](image)

Block diagram symbol (continuous-time systems) for an **integrator**

![Block diagram symbol (discrete time systems) for a delay](image)

Block diagram symbol (discrete time systems) for a **delay**
An Electrical Circuit Viewed as a System

-An $RC$ lowpass filter is a simple electrical system.
-It is excited by a voltage $v_{in}(t)$ and responds with a voltage $v_{out}(t)$.
-It can be viewed or modeled as a single-input, single-output system.
Homogeneity

-In a **homogeneous** system, multiplying the excitation by any constant (including complex constants), multiplies the response by the same constant.

**Homogeneous System**

\[ x(t) \rightarrow [H] \rightarrow y(t) \]

**Multiplier**

\[ x(t) \rightarrow \times \rightarrow Kx(t) \rightarrow [H] \rightarrow Ky(t) \]
Additivity

-If a system when excited by an arbitrary $x_1(t)$ produces a response $y_1(t)$, and when excited by an arbitrary $x_2(t)$ produces a response $y_2(t)$ and $x_1(t) + x_2(t)$ always produces the zero-state response $y_1(t) + y_2(t)$, the system is **additive**.
Time Invariance

-If an arbitrary input signal $x(t)$ causes a response $y(t)$, and an input signal $x(t - t_0)$ causes a response $y(t - t_0)$, for any arbitrary $t_0$, the system is said to be **time invariant**.

**Time Invariant System**

\[
\begin{align*}
x(t) & \quad \rightarrow \quad \mathcal{H} \quad \rightarrow \quad y(t) \\
x(t) & \quad \rightarrow \quad \text{Delay, } t_0 \quad \rightarrow \quad x(t - t_0) \quad \rightarrow \quad \mathcal{H} \quad \rightarrow \quad y(t - t_0)
\end{align*}
\]
Linearity and LTI Systems

- If a system is both \textit{homogeneous} and \textit{additive} it is \textbf{linear}.

- If a system is both \textit{linear} and \textit{time-invariant} it is called an LTI system.

- The \textbf{eigenfunctions} of an LTI system are complex exponentials.

- The \textbf{eigenvalues} of an LTI system are either real or, if complex, occur in complex conjugate pairs.
-Any LTI system excited by a complex sinusoid respond with another complex sinusoid of the same frequency, but generally with different amplitude and phase (multiplied by a complex constant).

-Using the principle of superposition for LTI systems, if the input signal is an arbitrary function that is a linear combination of complex sinusoids of various frequencies, then the output signal is also a linear combination of complex sinusoids at those same frequencies. This idea is the basis for the methods of Fourier transform analysis.

-All these statements are true of both continuous-time and discrete-time systems.
Causality

-Any system for which the response occurs only during or after the time in which the excitation is applied is called a causal system.
-Strictly speaking, all real physical systems are causal.

Stability

-Any system for which the response is bounded for any arbitrary bounded excitation, is called a bounded-input-bounded-output (BIBO) stable system.
5- Time-Domain System Analysis
Continuous Time
-Once the response to an impulse is known, the response of any LTI system to any arbitrary excitation can be found.
Impulse Response $h(t)$

-For an LTI system, the impulse response $h(t)$ of the system is a complete description of how it responds to any signal.
Response of a linear shift-invariant system to impulses
If a continuous-time LTI system is excited by an arbitrary excitation, the response could be found approximately by approximating the excitation as a sequence of continuous rectangular pulses of width $T_p$. 

![Diagram](image)
\[ x(t) \approx \sum_{n=-\infty}^{\infty} x(nT_p) \text{rect} \left( \frac{t-nT_p}{T_p} \right) \]

\[ x(t) \approx \sum_{n=-\infty}^{\infty} T_p x(nT_p) \frac{1}{T_p} \text{rect} \left( \frac{t-nT_p}{T_p} \right) \]

Unit pulse response

\[ y(t) \approx \sum_{n=-\infty}^{\infty} T_p x(nT_p) h_p(t-nT_p) \]
\[ T_p \rightarrow d\tau \]

\[ x(t) = \sum_{n=-\infty}^{\infty} T_p x \left( \frac{nT_p}{\tau} \right) \frac{1}{T_p} \text{rect} \left( \frac{t-nT_p}{T_p} \right) \]

\[ \int \delta(t-\tau) \]

Sampling property of \( \delta(t) \)

\[ x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \]

\[ y(t) = \sum_{n=-\infty}^{\infty} T_p x \left( \frac{nT_p}{\tau} \right) h_p \left( t-nT_p \right) \]

\[ \int h(t-\tau) \]

Convolution integral

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]
Convolution Integral

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) \, d\tau = x(t) \ast h(t) \]

\[ x(t) \xrightarrow{\text{System}} y(t) = x(t) \ast h(t) \]
Convolution Graphical Example

- Let $x(t)$ be this smooth waveform and let it be approximated by a sequence of rectangular pulses.
The approximate excitation is a sum of rectangular pulses
The approximate response is a sum of pulse responses.
Exact and approximate excitation, unit-impulse response, unit-pulse response and exact and approximate system response with $T_p=0.2$ and $0.1$
\( \delta(t) \)

\[ \begin{array}{c}
\text{Amplitude} \\
\text{Time}
\end{array} \]

\( h(t) \)

\[ \begin{array}{c}
\text{Amplitude} \\
\text{Time}
\end{array} \]

\( x(t) \)

\[ \begin{array}{c}
\text{Amplitude} \\
\text{Time}
\end{array} \]

\( \ast \)

\( y(t) \)

\[ \begin{array}{c}
\text{Amplitude} \\
\text{Time}
\end{array} \]
Graphical Illustration of the Convolution Integral

-The convolution integral is defined by

\[ x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

-For illustration purposes let the excitation \( x(t) \) and the impulse response \( h(t) \) be the two functions below.

![Graphical Illustration](image)
Process of convolving
Convolution Integral Properties

Convolution with an shifted impulse

\[ x(t) * A\delta(t - t_0) = Ax(t - t_0) \]

Commutativity

\[ x(t) * y(t) = y(t) * x(t) \]

Associativity

\[ \left[ x(t) * y(t) \right] * z(t) = x(t) * \left[ y(t) * z(t) \right] \]

Distributivity

\[ \left[ x(t) + y(t) \right] * z(t) = x(t) * z(t) + y(t) * z(t) \]
If $y(t) = x(t) * h(t)$

**Differentiation Property**

$$y'(t) = x'(t) * h(t) = x(t) * h'(t)$$

**Area Property**

Area of $y = (\text{Area of } x) \times (\text{Area of } h)$

**Scaling Property**

$$y(at) = |a| x(at) * h(at)$$
The image contains several graphs and mathematical operations. The graphs illustrate the convolution of two signals, $x(t)$ and $h(t)$, resulting in $y(t)$. This is depicted as $y(t) = x(t) * h(t)$. Additionally, the derivative $d/dt$ is shown acting on $x(t)$ to produce $x'(t)$, and the integral $\int$ is shown acting on $h(t)$ to produce $y'(t)$. The diagrams visually represent these operations.
System Interconnections

-If the output signal from a LTI system is the input signal to a second LTI system, the systems are said to be **cascade** connected.

-From the properties of convolution:

```
x(t) → h_1(t) → x(t) * h_1(t) → h_2(t) → y(t) = [x(t) * h_1(t)] * h_2(t)
```

```
x(t) → h_1(t) * h_2(t) → y(t)
```

Cascade Connection
-If two LTI systems are excited by the same signal and their responses are added, they are said to be **parallel** connected.

-From properties of convolution:

\[ y(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * [h_1(t) + h_2(t)] \]
Step Response

- One of the most common signals used to test systems is the step function. The response of an LTI system to a unit step is:

\[ h_{-1}(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau \]

- The response of an LTI system excited by a unit step is the integral of the impulse response.
- As the unit step is the integral of the impulse, the unit-step response is the integral of the unit-impulse response.
- In fact, this relationship holds for any excitation.
- If any excitation is changed to its integral, the response also changes to its integral.
Relations between integrals and derivatives of excitations and responses for an LTI system.
Discrete Time
System Response

-Once the response to a unit impulse is known, the response of any LTI system to any arbitrary excitation can be found.

-Any arbitrary excitation is simply a sequence of amplitude-scaled and time-shifted impulses.

-Therefore the response is simply a sequence of amplitude-scaled and time-shifted impulse responses.
Unit Impulse Response $h[n]$

-For an LTI system, the unit impulse response $h[n]$ of the system is a complete description of how it responds to any signal.
Convolutions Sum

-The response $y[n]$ to an arbitrary excitation $x[n]$ is of the form:

$$y[n] = \cdots x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \cdots$$

-This can be written in a more compact form (Convolution Sum):

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

-Compare with

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
\[ y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] = x[n] \ast h[n] \]

\[ x[n] \quad \text{System} \quad h[n] \quad \Rightarrow \quad y[n] = x[n] \ast h[n] \]
System Response Example

System Excitation

System Impulse Response

System Response
Convolution Sum Properties

Convolution with a shifted unit impulse

\[ x[n] * A\delta[n - n_0] = Ax[n - n_0] \]

Commutativity

\[ x[n] * y[n] = y[n] * x[n] \]

Associativity

\[ (x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n]) \]

Distributivity

\[ (x[n] + y[n]) * z[n] = x[n] * z[n] + y[n] * z[n] \]
If \( y[n] = x[n] \ast h[n] \)

**Differentiation Property**

\[
y[n] - y[n-1] = x[n] \ast (h[n] - h[n-1]) = (x[n] - x[n-1]) \ast h[n]
\]

**Sum Property** (Sum of \(y\) = (Sum of \(x\)) \ast (Sum of \(h\))

\[
\sum_{n=-\infty}^{\infty} y[n] = \left( \sum_{n=-\infty}^{\infty} x[n] \right) \ast \left( \sum_{n=-\infty}^{\infty} h[n] \right)
\]
System Interconnections

-If the output signal from a LTI system is the input signal to a second LTI system, the systems are said to be **serie/cascade** connected.

-From the properties of convolution:

\[
x[n] \rightarrow h_1[n] \rightarrow x[n] \ast h_1[n] \rightarrow h_2[n] \rightarrow y[n] = (x[n] \ast h_1[n]) \ast h_2[n]
\]

\[
x[n] \rightarrow h_1[n] \ast h_2[n] \rightarrow y[n]
\]

Serie/Cascade Connection
-If two LTI systems are excited by the same signal and their responses are added they are said to be parallel connected.

-From properties of convolution:

\[
x[n] \rightarrow h_1[n] \rightarrow x[n] * h_1[n] \\
x[n] \rightarrow h_2[n] \rightarrow x[n] * h_2[n] \\
\]

\[
y[n] = x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \{h_1[n] + h_2[n]\}
\]

Parallel Connection
6- Spectral Method
Introduction

Response of a linear shift-invariant system to impulses
Response of a linear shift-invariant system to a harmonic function
Representing a Signal

-The **Spectral (Fourier) method** represents a signal as a linear combination of **complex exponentials (Harmonic functions)**.

-If an excitation can be expressed as a sum of complex exponentials, the response of an LTI system can be expressed as the sum of responses to complex sinusoids too.

-This is the fundament of the Spectral Method.
\[ x(t) = A_1 e^{j2\pi t/T_1} + A_2 e^{j2\pi t/T_2} + A_3 e^{j2\pi t/T_3} \rightarrow h(t) \rightarrow y(t) \]
\[
\cos(2\pi f, t) = \frac{e^{j2\pi f, t}}{2} + \frac{e^{-j2\pi f, t}}{2} \\
\sin(2\pi f, t) = \frac{j e^{-j2\pi f, t}}{2} - \frac{j e^{j2\pi f, t}}{2}
\]
\[ e^{j2\pi f_o t} = \cos(2\pi f_o t) + jsin(2\pi f_o t) \]
Inner Product

-The inner product of two functions, is **the integral of the product of one function and the complex conjugate of the other function** over an interval.

-Inner Product of $x_1(t)$ and $x_2(t)$ on the interval $t_0 < t < t_0 + T$

\[
\left( x_1(t), x_2(t) \right) = \int_{t_0}^{t_0+T} x_1(t)x_2^*(t)dt
\]
Orthogonality

- **Orthogonal** means that the **inner product** of the two functions of time, on some time interval, **is zero**.

-Two functions \( x_1(t) \) and \( x_2(t) \) are orthogonal on the interval \( t_0 < t < t_0 + T \), if:

\[
\langle x_1(t), x_2(t) \rangle = \int_{t_0}^{t_0+T} x_1(t)x_2^*(t)dt = 0
\]
Orthogonality of complex exponentials

\[ x_1(t) = e^{j\omega_1 t} = e^{j2\pi f_1 t} \]
\[ x_2(t) = e^{j\omega_2 t} = e^{j2\pi f_2 t} \]

\[ \int_{-\infty}^{\infty} e^{-j\omega t} dt = \delta(\omega) \rightarrow \begin{cases} 
( e^{j\omega_1 t}, e^{j\omega_2 t} ) \neq 0 & [\omega_1 = \omega_2] \\
( e^{j\omega_1 t}, e^{j\omega_2 t} ) = 0 & [\omega_1 \neq \omega_2] 
\end{cases} \]

\[ e^{j\omega t} \rightarrow \text{Orthonormal (Orthonormal and Normalized) Basis} \]
7- Fourier Transform
Definition

- Fourier Basis → \( e^{j\omega t} \)

- The Fourier Transform of \( x(t) \) is the projection of this function onto the Fourier basis.
- The Fourier Transform of \( x(t) \) is defined as:

\[
\mathcal{F}(x(t)) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt
\]

\( x(t) \Leftrightarrow X(\omega) \)
-It follows that the Inverse Fourier Transform of $X(\omega)$ is:

$$F^{-1}(X(\omega)) = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

$$X(\omega) \leftrightarrow x(t)$$
-The definitions based on frequency terms are:

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \]

\[ x(t) = \int_{-\infty}^{\infty} X(f) e^{j\omega t} df \]
Time Domain \[\rightarrow\] FT \[\rightarrow\] Frequency Domain
Examples

![Signal and Frequency Response Diagram](image)

- $x(t)$
- $|X(f)|$
- Lowpass
Highpass
Example: extracting frequency-domain information in the signal
Example: blood pressure waveform (sampled at 200 points/s)
FT Pairs

\[ x(t) \xrightarrow{\mathcal{F}} \delta(t) \]

\[ \delta(t) \xrightarrow{\mathcal{F}} 1 \]

\[ 1 \xrightarrow{\mathcal{F}} \delta(f) \]

\[ \delta(f) \xrightarrow{\mathcal{F}} 1 \]

\[ 1 \xrightarrow{\mathcal{F}} 2\pi \delta(\omega) \]

\[ \delta(\omega) \xrightarrow{\mathcal{F}} f \]

\[ f \xrightarrow{\mathcal{F}} \delta(f) \]

\[ \delta(f) \xrightarrow{\mathcal{F}} 1 \]

\[ 1 \xrightarrow{\mathcal{F}} 2\pi \delta(\omega) \]
\[ x(t) \xrightarrow{\mathcal{F}} \delta_{T_0}(f) \xrightarrow{f_0} f_0 \delta_{f_0}(f) \]

\[ f_0 = \frac{1}{T_0} \]

\[ x(f) \]

\[ \delta_{T_0}(t) \xrightarrow{\mathcal{F}} \omega_0 \delta_{\omega_0}(\omega) \]

\[ \omega_0 = \frac{2\pi}{T_0} \]
\[ e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f - f_0) \]

\[ e^{j\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi \delta(\omega - \omega_0) \]
\[ x(t) \]

\[ \cos(2\pi f_0 t) \xrightarrow{\mathcal{F}} \frac{1}{2} \left[ \delta(f - f_0) + \delta(f + f_0) \right] \]

\[ x(t) \]

\[ \cos(\omega_0 t) \xrightarrow{\mathcal{F}} \pi \left[ \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right] \]
\[ \sin(2\pi f_0 t) \xrightarrow{\mathcal{F}} j \frac{1}{2} [\delta(f + f_0) - \delta(f - f_0)] \]

\[ \sin(\omega_0 t) \xrightarrow{\mathcal{F}} j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \]
\[ x(t) \xrightarrow{\mathcal{F}} \text{sinc}(f) \]
\[ \text{rect}(t) \xrightarrow{\mathcal{F}} \text{sinc}(\omega/2\pi) \]
\[ |X(f)| \text{ and } |X(j\omega)| \]
\[ \Delta X(f) \text{ and } \Delta X(j\omega) \]

\[ x(t) \xrightarrow{\mathcal{F}} \text{rect}(f) \]
\[ \text{sinc}(t) \xrightarrow{\mathcal{F}} \text{rect}(\omega/2\pi) \]
\[ |X(f)| \text{ and } |X(j\omega)| \]
\[ \Delta X(f) \text{ and } \Delta X(j\omega) \]
Sinc Function

\[
sinc(t) = \frac{\sin(\pi t)}{\pi t}
\]
\[ x(t) \xleftarrow{\mathcal{F}} \text{sinc}^2(f) \]

\[ \text{tri}(t) \xleftarrow{\mathcal{F}} \text{sinc}^2(\omega/2\pi) \]

\[ x(t) \xleftarrow{\mathcal{F}} \text{sinc}^2(f) \]

\[ \text{sinc}^2(t) \xleftarrow{\mathcal{F}} \text{tri}(f) \]

\[ \text{sinc}^2(t) \xleftarrow{\mathcal{F}} \text{tri}(\omega/2\pi) \]
\[ e^{-\frac{a^2}{\omega^2 + a^2}}, \text{ Re}(a) > 0 \]
\[ e^{-\frac{a^2}{(2\pi f)^2 + a^2}}, \text{ Re}(a) > 0 \]

\[ e^{-\pi f^2} \leftrightarrow e^{-\frac{\pi}{2} f^2} \]

\[ e^{-\pi \omega^2} \leftrightarrow e^{-\frac{\omega^2}{4\pi}} \]

\[ |X(j\omega)| \text{ and } |X(f)| \]

\[ \Delta X(j\omega) \text{ and } \Delta X(f) \]

\[ \Delta X(f) \text{ and } \Delta X(j\omega) \]
\[ x(t) \]

\[ u(t) \xrightarrow{f} (1/2)\delta(f) + 1/j2\pi f \]

\[ X(f) \]

\[ X(f) \]

\[ x(t) \]

\[ u(t) \xrightarrow{f} \pi\delta(\omega) + 1/j\omega \]

\[ X(j\omega) \]

\[ X(j\omega) \]
FT Properties

Lineality

$$\alpha x(t) + \beta y(t) \xrightarrow{F} \alpha X(f) + \beta Y(f)$$

$$\alpha x(t) + \beta y(t) \xrightarrow{F} \alpha X(j\omega) + \beta Y(j\omega)$$
Domains Duality

\[ X(t) \xrightarrow{F} x(-f) \] and \[ X(-t) \xrightarrow{F} x(f) \]

\[ X(jt) \xrightarrow{F} 2\pi x(-\omega) \] and \[ X(-jt) \xrightarrow{F} 2\pi x(\omega) \]
Time Shifting

\[ x(t-t_0) \xrightarrow{F} X(f)e^{-j2\pi f t_0} \]
\[ x(t-t_0) \xrightarrow{F} X(j\omega)e^{-j\omega t_0} \]
Frequency Shifting

\[ x(t)e^{+j2\pi f_0 t} \overset{F}{\longleftrightarrow} X(f - f_0) \]

\[ x(t)e^{+j\omega_0 t} \overset{F}{\longleftrightarrow} X(j(\omega - \omega_0)) \]
The “Uncertainty” Principle

- The time and frequency scaling properties indicate that if a signal is expanded in one domain it is compressed in the other domain.
- This is called the “uncertainty principle” of Fourier analysis.

\[ e^{-\pi t^2} \xrightarrow{F} e^{-\pi f^2} \]

\[ e^{-\pi (t/2)^2} \xrightarrow{F} 2e^{-\pi (2f)^2} \]
**Time Scaling** (Uncertainty Principle)

\[ x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right) \]

\[ x(at) \overset{F}{\leftrightarrow} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) \]
**Frequency Scaling** (Uncertainty Principle)

\[
\frac{1}{|a|} x \left( \frac{t}{a} \right) \xrightarrow{F} X(af)
\]

\[
\frac{1}{|a|} x \left( \frac{t}{a} \right) \xrightarrow{F} X(ja\omega)
\]
Time Differentiation

\[
\frac{d}{dt}(x(t)) \xrightarrow{\mathcal{F}} j2\pi fX(f)
\]

\[
\frac{d}{dt}(x(t)) \xrightarrow{\mathcal{F}} j\omega X(j\omega)
\]
Time Integration

\[
\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow_F \frac{X(f)}{j2\pi f} + \frac{1}{2} X(0) \delta(f)
\]

\[
\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow_F \frac{X(j\omega)}{j\omega} + \pi X(0) \delta(\omega)
\]
Total - Area Integrals

\[ X(0) = \int_{-\infty}^{\infty} x(t) \, dt \]

\[ x(0) = \int_{-\infty}^{\infty} X(f) \, df \]
Parseval’s Theorem

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \]

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 df \]
Multiplication Convolution Duality

\[ x(t) * y(t) \xrightarrow{F} X(f)Y(f) \]
\[ x(t) * y(t) \xrightarrow{F} X(j\omega)Y(j\omega) \]

\[ x(t)y(t) \xrightarrow{F} X(f) * Y(f) \]
\[ x(t)y(t) \xrightarrow{F} \frac{1}{2\pi} X(j\omega) * Y(j\omega) \]
Time Domain \((t)\)  \quad \text{Frequency Domain} \ (f)\)

\[ y = x * h \quad \text{↔} \quad Y = XH \]
\[ y = x \cdot h \quad \text{↔} \quad Y = X \cdot H \]
\[ y(t) = x(t) \ast h(t) \]

\[ Y(f) = X(f) \times H(f) \]
Modulation

\[ x(t)\cos(2\pi f_0 t) \xrightarrow{F} \frac{1}{2} \left[ X(f - f_0) + X(f + f_0) \right] \]

\[ x(t)\cos(\omega_0 t) \xrightarrow{F} \frac{1}{2} \left[ X(j(\omega - \omega_0)) + X(j(\omega + \omega_0)) \right] \]
\[ \delta(t) \rightarrow \text{System} \rightarrow h(t) \]

\[ 1 \rightarrow H(f) \]
\[ y(t) = x(t) * h(t) \]

\[ Y(f) = X(f) H(f) \]
System Interconnections

- If the output signal from a LTI system is the input signal to a second LTI system, the systems are said to be **serie/cascade** connected.

- In the frequency domain, the **serie/cascade connection** multiplies the frequency responses instead of convolving the impulse responses.

\[
x(t) \rightarrow h(t) \rightarrow y(t) = h(t) * x(t) \quad X(f) \rightarrow H(f) \rightarrow Y(f) = H(f)X(f)
\]

\[
X(f) \rightarrow H_1(f) \rightarrow X(f)H_1(f) \rightarrow H_2(f) \rightarrow Y(f) = X(f)H_1(f)H_2(f)
\]

\[
X(f) \rightarrow H_1(f)H_2(f) \rightarrow Y(f)
\]

Serie/Cascade Connection
-If two LTI systems are excited by the same signal and their responses are added they are said to be **parallel** connected.

\[
Y(f) = X(f)(H_1(f) + H_2(f))
\]

**Parallel Connection**
8- Sampling
Introduction

-The fundamental consideration in sampling theory is how fast to sample a signal to be able to reconstruct the signal from the samples.

![Signal to be Sampled](image)

**High Sampling Rate**

**Medium Sampling Rate**

**Low Sampling Rate**
Aliasing

\[ |X(f)| \]

\[ f_s = \frac{1}{T_s} \]

\[ |X_\delta(f)| \]

\[ f_s = \frac{1}{T_s} \]
Nyquist rate

- If a continuous-time signal is sampled for all time at a rate $f_s$ that is more than twice the bandlimit $f_m$ of the signal, the original continuous-time signal can be recovered exactly from the samples.

$$f_s \geq 2f_m$$

- The frequency $2f_m$ is called the Nyquist rate.
- A signal sampled at a rate less than the Nyquist rate is undersampled.
- A signal sampled at a rate greater than the Nyquist rate is oversampled.
Discrete-time Signal Formed by Sampling the Continuous-Time Signal

\[ x_s[n] \]

| \[|X_s(F)|\] |
|---|
| 8.3039 |

<table>
<thead>
<tr>
<th>[\Delta X_s(F)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\pi]</td>
</tr>
</tbody>
</table>

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Interpolation

- Interpolation process for an ideal lowpass filter
- The corner frequency set to half the sampling rate
-Sinc-function interpolation is theoretically perfect but it can never be done in practice because it requires samples from the signal for all time.
-Real interpolation must make causal compromises.
-The simplest realizable interpolation technique is what a DAC does.
9- Discrete Fourier Transform (DFT)
Introduction

- A signal can be represented as a linear combination of harmonic complex exponentials.

\[ x[n] = A_1 e^{j2\pi n/N_1} + A_2 e^{j2\pi n/N_2} + A_3 e^{j2\pi n/N_3} \]
Orthonormal (Orthogonal and Normalized Basis)

$\mathbf{e}_{\frac{j(2\pi nk)}{N}}$

$0 \leq n \leq N - 1$

$(t) \rightarrow [n]$

$0 \leq k \leq N - 1$

$(f) \rightarrow [k]$
Definition

-The most common definition of the Discrete Fourier Transform (DFT) of $x[n]$ is:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} \quad x[n] \leftrightarrow \text{DFT} \quad X[k]$$

$x[n] \rightarrow N$ real values

$X[k] \rightarrow N$ complex values $\rightarrow 2N$ real values
-It follows that the Inverse Discrete Fourier Transform (IDFT) of $X[k]$ is:

$$
x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] e^{j \left(\frac{2\pi}{N}nk\right)}
$$

$$
X[k] \xrightleftharpoons{\text{DFT}^{-1}} \frac{N}{N} \rightarrow x[n]
$$
DFT Matrix

\[ W_N = e^{-j \frac{2\pi}{N}} \]

\[ X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk} \]

0 → \( n \) → \( N - 1 \)

\[
\begin{bmatrix}
X[0] \\
X[1] \\
X[2] \\
\vdots \\
X[N-1]
\end{bmatrix}
= 
\begin{bmatrix}
W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\
W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\
W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)}
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[1] \\
x[2] \\
\vdots \\
x[N-1]
\end{bmatrix}
\]

0 → \( k \) → \( N - 1 \)
$N_w = 5, N_0 = 22$

$|X[k]|$

$N_w = 5, N_0 = 44$

$|X[k]|$

$N_w = 5, N_0 = 88$

$|X[k]|$
\[ N_w = 5, \quad N_0 = 22 \]
\[ |X[k]| \]

\[ \cdots \frac{k}{N_0} \]

\[ \cdots k \]

\[ \cdots \]

\[ \cdots \]

\[ N_w = 5, \quad N_0 = 44 \]
\[ |X[k]| \]

\[ \cdots \frac{k}{N_0} \]

\[ \cdots k \]

\[ \cdots \]

\[ \cdots \]

\[ N_w = 5, \quad N_0 = 88 \]
\[ |X[k]| \]

\[ \cdots \frac{k}{N_0} \]

\[ \cdots k \]

\[ \cdots \]

\[ \cdots \]
2-point DFT

DFT of vector \( (a,b) \)

\[
N = 2 \quad \rightarrow \quad W_2 = e^{-j\pi} = -1
\]

\[
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
\end{pmatrix}
= 
\begin{pmatrix}
a + b \\
a - b \\
\end{pmatrix}
\]

IDFT

\[
\frac{1}{2}
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
\end{pmatrix}
\begin{pmatrix}
a + b \\
a - b \\
\end{pmatrix}
= 
\begin{pmatrix}
a \\
b \\
\end{pmatrix}
\]
4-points DFT

DFT of vector \((a,b,c,d)\)

\[ N = 4 \rightarrow W_4 = e^{-j\frac{\pi}{2}} = -j \]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -j & -1 & j \\
1 & -1 & 1 & -1 \\
1 & j & -1 & -j \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
d \\
\end{pmatrix}
=
\begin{pmatrix}
a+b+c+d \\
(a-c)-j(b-d) \\
(a+c)-(b+d) \\
(a-c)+j(b-d) \\
\end{pmatrix}
\]
IDFT

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & j & -1 & -j \\
\frac{1}{4} & 1 & -1 & 1 \\
1 & -j & -1 & j
\end{bmatrix}
\begin{bmatrix}
a + b + c + d \\
(a - c) - j(b - d) \\
(a + c) - (b + d) \\
(a - c) + j(b - d)
\end{bmatrix} =
\begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix}
\]
DFT Properties

Linearity
\[\alpha x[n] + \beta y[n] \xrightarrow{\text{DFT}} \alpha X[k] + \beta Y[k]\]

Time Shifting
\[x[n-n_0] \xrightarrow{\text{DFT}} X[k] e^{-j2\pi kn_0/N}\]

Frequency Shifting
\[x[n] e^{j2\pi k_0 n/N} \xrightarrow{\text{DFT}} X[k - k_0]\]

Time Reversal
\[x[-n] = x[N-n] \xrightarrow{\text{DFT}} X[-k] = X[N-k]\]

Conjugation
\[x^*[n] \xrightarrow{\text{DFT}} X^*[-k] = X^*[N-k]\]

Time Reversal
\[x^*[-n] = x^*[N-n] \xrightarrow{\text{DFT}} X^*[k]\]

Time Scaling
\[z[n] = \begin{cases} x[n/m] & , \ n/m \text{ an integer} \\ 0 & , \text{ otherwise} \end{cases} \quad N \rightarrow mN, \quad Z[k] = (1/m)X[k]\]
Change of Period

\[ N \rightarrow qN, \text{ } q \text{ a positive integer} \]

\[ X_q[k] = \begin{cases} X[k/q], & k/q \text{ an integer} \\ 0, & \text{otherwise} \end{cases} \]

Multiplication - Convolution Duality

\[ x[n]y[n] \overset{\text{DFT}}{\longrightarrow} (1/N)Y[k]*X[k] \]

\[ x[n]*y[n] \overset{\text{DFT}}{\longrightarrow} Y[k]X[k] \]

where \[ x[n]*y[n] = \sum_{m=\langle N\rangle} x[m]y[n-m] \]

Parseval's Theorem

\[ \sum_{n=\langle N\rangle} |x[n]|^2 = \frac{1}{N} \sum_{k=\langle N\rangle} |X[k]|^2 \]
DFT Pairs

\[ e^{j2\pi n/N} \xleftrightarrow{\text{DFT}} mN\delta_{mN}[k-m] \]

\[ \cos(2\pi qn/N) \xleftrightarrow{\text{DFT}} (mN/2)(\delta_{mN}[k-mq]+\delta_{mN}[k+mq]) \]

\[ \sin(2\pi qn/N) \xleftrightarrow{\text{DFT}} (jmN/2)(\delta_{mN}[k+mq]-\delta_{mN}[k-mq]) \]

\[ \delta_N[n] \xleftrightarrow{\text{DFT}} m\delta_{mN}[k] \]

\[ 1 \xleftrightarrow{\text{DFT}} N\delta_N[k] \]

\[ (u[n-n_0]-u[n-n_1])\delta_N[n] \xleftrightarrow{\text{DFT}} \frac{e^{-j\pi k(n_1+n_0)/N}}{e^{-j\pi k/N}}(n_1-n_0)drcl(k/N,n_1-n_0) \]

\[ \text{tri}(n/N_w)\delta_N[n] \xleftrightarrow{\text{DFT}} N_wdrcl^2(k/N,N_w), \text{ } N_w \text{ an integer} \]

\[ \text{sin c}(n/w)\delta_N[n] \xleftrightarrow{\text{DFT}} \text{wrect}(wk/N)\delta_N[k] \]
Fast Fourier Transform (FFT)

-The DFT requires $N^2$ complex multiplies and $N(N-1)$ complex additions.

-Algorithms that exploit computational savings are collectively called Fast Fourier Transforms.

-They take advantage of the symmetry and periodicity of the complex exponential.
\[ W_N = e^{-j\left(\frac{2\pi}{N}\right)} \quad X[k] = \sum_{n=0}^{N-1} x[n]W_N^{nk} \]

Symmetry → \[ W_N^{[N-n]k} = W_N^{-nk} = (W_N^{nk})^* \]

Periodicity → \[ W_N^{nk} = W_N^{[n+N]k} = W_N^{n[k+N]} \]

\(N\) length DFT → \(N^2\) multiplications

\[ 2\left(\frac{N}{2}\right) \text{ lengths DFT} → 2 \left(\frac{N}{2}\right)^2 = \frac{N^2}{2} \text{ multiplications} \]
\[ N = 2 \quad \rightarrow \quad W_2 = e^{-j\pi} = -1 \]

\[ X[k] = \sum_{n=0}^{1} x[n]W_2^{nk} \]

\[
\begin{bmatrix}
X[0] \\
X[1]
\end{bmatrix} =
\begin{bmatrix}
W_2^0 & W_2^0 \\
W_2^{-1} & W_2^{-1}
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[1]
\end{bmatrix} =
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
x[0] \\
x[1]
\end{bmatrix}
\]

2-Point DFT

A 2-point DFT
An $N$-point DFT computed using two $N/2$-point DFT's.

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