William John Macquorn Rankine (1820-1872) was one of the main figures in establishing engineering science in the second half of the 19th Century. His *Manual of Applied Mechanics* (1858) gathers most of his contributions to strength of materials and structural theory. A few additions are to be found in his *Manual of Civil Engineering* (1862). The book is based in his Lectures on Engineering delivered in the Glasgow University, and formed part of his intention of converting engineering science in a university degree [Channell, 1982; Buchanan, 1985]. Both in plan and in content the book shows and enormous rigor and originality. It is difficult to read. As remarked by Timoshenko [Timoshenko, 1953, 198]: "In his work Rankine prefers to treat each problem first in its most general form and only later does he consider various particular cases which may be of some practical interest. Rankine's adoption of this method of writing makes his books difficult to read, and they demand considerable concentration of the reader". Besides, Rankine does not repeat any demonstration or formula, and sometimes the reader must trace back the complete development through four or five previous paragraphs. The method is that of a mathematician. However, the *Manual* had 21 editions (the last in 1921) an exerted a considerable influence both in England and America.

In this article we will concentrate only in one of the more originals contributions of Rankine in the field of structural theory, his Theorems of Transformation of Structures. These theorems have deserved no attention either to his contemporaries or to modern historians of structural theory. It appears that the only exception is Timoshenko (1953, 198-200) who cited the general statement and described briefly its applications to arches. The present author has studied the application of the Theorems to masonry structures [Huerta and Aroca, 1989; Huerta, 1990, 2004, 2007].

Rankine discovered the Theorems during the preparation of his Lectures for his Chair of Engineering in the University of Glasgow. He considered it very important, as he published it in a short note communicated to the Royal Society in 1856 [Rankine, 1856]. He included it, also, in his article "Mechanics (applied)" for the 8th edition of the Encyclopaedia Britannica [Rankine, 1857]. Eventually, the Theorems were incorporated in the Manual of applied mechanics and applied to frames, cables, rib arches and masonry structures. The theorems were also included in his *Manual of civil engineering* (1862), generally in a shortened way, but with some additions.

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1 In the words of one of his biographers: "The extraordinary genius which he displayed... in pure and applied mathematics seems to have owed little or nothing to any external or adventitious aid in the shape of professional instruction: he was a born mathematician". [Mayer, 1973, p.205].
The first formulations of the Theorem

The first formulation of the Theorem which Rankine sent to the Royal Society was restricted to a plane structure: “Let a structure of a given uniform transverse section be stable under a system of forces represented by given lines in the plane of section: Then will any other structure whose transverse section is a projection by parallel lines of that of the first structure upon any other plane, be stable under the system of forces represented by the projections, upon the new plane, of the lines representing the first system of forces” [Rankine, 1856, 60].

As an example, the Theorem is applied to the design of a bridge as a transformation of an existing, “equilibrated”, bridge [Fig.1]. The masonry bridge in Figure 1 has been designed for a horizontal road, and its arch has a certain span CD, height OA and thickness at the crown OK; it is required to design a bridge for a different span and an inclined road, maintaining invariable the vertical distances AB, AO and OK. The solution, by parallel projection is to take any plan passing through the line BK (but not coinciding with the plane of the figure), draw on it the line cod with the given length and inclination and project the original figure tracing parallels to the lines Cc and Dd. Rankine remarks that if R are the reactions at the base of the abutments, the transformed lines r will give the direction and magnitude of the new reactions. Once the right form has been obtained, then, the horizontal foundations can be added.

In the article for the Encyclopaedia Britannica of 1857, the enunciation of the Theorem is more general. First, he defines “parallel projection” in a general way, so that it may be applied to three-dimensional figures. Then, he demonstrates that a system of forces in equilibrium, represented by straight lines, will remain in equilibrium after suffering a parallel projection (his exposition was much extended and improved in the Manual, see below. However, the enunciation of the Theorem is still restricted to what Rankine later would call “blockwork structures”, structures formed by a system of blocks in contact through surface (plane) joints, that is, to masonry structures.

In this kind of structures, a joint may fail either by overturning, when the point of application of the resultant is outside the matter, or by sliding, when the inclination of the resultant is outside of the friction cone of the plane of joint. There are, then, two conditions for stability to be fulfilled: “stability of position”, achieved when the resultant is safely within the joint, and “stability of friction”, obtained when the resultant is within the friction cone.
The Theorem now states: "If a structure of a given figure have stability of position under a system of forces represented by a given system of lines, then will any structure, whose figure is a parallel projection of that of the first structure, have stability of position under a system of forces represented by the corresponding projection of the first system of lines" [Rankine, 1857, 383].

The demonstration follows from the property already mentioned for systems of forces in equilibrium. Rankine stressed the importance of the Theorem: "It is useful in practice, by enabling the engineer easily to deduce the conditions of equilibrium and stability of structures of complex and unsymmetrical figures from those of structures of simple and symmetrical figures". In this way, elliptical arches can be easily designed from circular arches and elliptical domes from hemispherical domes. Besides, it is possible to find the figures of arches fitted to resist the thrust of earth, which is less horizontally than vertically in a certain given ratio, can be deduced by projection from those of arches fitted to resist the, thrust of a liquid, which is of equal intensity, horizontally and vertically.

The stability of oval or elliptic arches has been treated briefly in some previous works. The study of oval or elliptic domes, and of arches subject to variable horizontal thrusts (i.e. soil thrust), were completely new problems.

The Manual of Applied Mechanics

The Theorems of Transformation of Structures are fully explained and applied in Parts I, "Principles of Statics", and II, "Theory of Structures", of his Manual of Applied Mechanics. In fact, as is apparent from the structure of the manual, Rankine considered the Theorems a fundamental tool in structural design and particularize the enunciation for every structural type studied, namely, trusses (which he called frames), cable structures, arches and masonry structures.

Parallel Projections of Statics

The first three chapters of Part I explain with great rigor and generality the laws of equilibrium of forces and couples. After defining briefly in Chapter 1 the fundamental concepts of force, its balance and measurement, Rankine treats in Chapter 2 "The theory of couples and of the balance of parallel forces", before stating and demonstrating the law of composition of forces in Chapter 3 "Balance of inclined forces". This disposition is unusual and was criticized in some contemporary reviews of the Manual, but it serves Rankine to prepare the way to his Chapter IV "On parallel projections of statics", which constitutes the basis of his Theorems of Transformation of Structures.

With respect to couples he explains first their composition, first when they have the same axis (i.e. are in parallel planes), and then the composition of couples of different axis. He then studies in detail the equilibrium and composition of parallel forces, a most important case in structural theory. He defines the concept of "centre of parallel forces", which he will use afterwards for example in obtaining centers of gravity, and the way to obtain its coordinates. For any system of parallel forces, acting on certain points, in the plane or in the space, there is a resultant which passes through this centre, "whatsoever may be the absolute magnitude of those forces, and the angular position of their lines of action" [Rankine, 1858, 32]. Inclined forces are considered in the same abstract and general way. Eventually, he arrived to the conclusion (now common knowledge in any undergraduate course on statics) "that

2 See for example the review in Mechanics Magazine, vol. 1, 1859, pp.118, 134, 234.
every system of forces which is not self-balanced", is equivalent either, to a single force, to a couple, or
to a force, combined with a couple whose axis is parallel to the line of action of the force [Rankine,
1858, 44].

Having made a complete study of the equilibrium and composition of forces in the most general way,
Rankine passes on to discuss how parallel projection could affect any system of forces. Rankine first
define a parallel projection in a geometrical way: "If two figures be so related, that for each point in one there
is a corresponding point in the other, and that to each pair of equal and parallel lines in the one there
corresponds a pair of equal and parallel lines in the other, those figures are said to be parallel projections of
each other" [Rankine, 1858, 45].

Afterwards, he gives the mathematical definition: Let any figure be referred to axes of coordinates,
whether rectangular or oblique; let x, y, z, denote the co-ordinates of any point in it, which may be
denoted by A: let a second figure be constructed from a second set of axes of co-ordinates, either
agreeing with, or differing from, the first set as to rectangularity or obliquity; let \( X = ax, y = by, z = cz \),
be the co-
ordinates in the second figure, of the point A whose co-ordinates are related to the co-ordinates of A, that for each pair of corresponding points,
A, A', in the two figures, the three pairs of corresponding co-ordinates shall bear to each other three
constant ratios, such as then are these two figures parallel projections of each other.

This kind of projective transformation is an affine projection, and affine figures present most interesting
gometrical properties in statics, which Rankine enumerates:

I A parallel projection of a system of three points, lying in one straight line and dividing it in a given
proportion, is also a system of three points, lying in one straight line and dividing it in the same
proportion.

II A parallel projection of a system of parallel lines whose lengths bear given ratios to each other, is
also a system of parallel lines whose lengths bear the same ratios to each other.

III A parallel projection of a closed polygon is a closed polygon.

IV A parallel projection of a parallelogram is a parallelogram.

V A parallel projection of a parallelepiped is a parallelepiped.

VI A parallel projection of a pair of parallel plane surfaces, whose areas are in a given ratio, is also a
pair of parallel plane surfaces, whose areas are in the same ratio.

VII A parallel projection of a pair of volumes having a given ratio, is a pair of volumes having the same
ratio [Rankine, 1858, 45-46].

This geometrical property with previous properties demonstrated for parallel forces (including
couples), centers of parallel forces, and inclined forces acting on one point, permits him to apply
parallel projection to any system of forces, concluding that: 
"... if a balanced system of forces acting through
any system of points be represented by a system of lines, then will any parallel projection of that system of lines
represent a balanced system of forces; and that if any two systems of forces be represented by lines which are
parallel projections of each other, the lines, or sets of lines, representing their resultants, will be corresponding
parallel projections of each other" [Rankine 1858, 47].

The property depends on the representation of forces by lines, and Rankine remarks "... that couples are
to be represented by pairs of lines, as pairs of opposite forces, or by areas, and not by single lines along their axes".
Centers of Gravity

Rankine studies the centers of gravity in a chapter dedicated a “Distributed forces”. The treatment is excellent, combining the arithmetic with geometrical properties. Besides the usual statements about symmetry axis he explains the methods of addition and subtraction, now again in every undergraduate course, and a curious method of transposition (where a part of a figure, whose centre is known, is “cut” and “pasted” in another place).

The centre of gravity is the centre of the parallel forces originated by the force of gravity, and Rankine dedicates a paragraph to the “Centre of gravity found by projection”, where he explains the advantages of the method for obtaining easily the centre of gravity of figures, or parts of figures, which are parallel projections of more symmetrical figures. He cites the example of a centre of gravity of a sector of an ellipse [Fig.2].

Of course, the application is not restricted to plane figures, and in Figure appears its application to hemi-spherical lunes and portion of lunes, the parallel projection of a hollow sphere of uniform thickness is a hollow ellipsoid, but with variable thickness. The formulae of the coordinates of the centre of gravity are quite long for the regular sphere\(^3\), but for ellipsoids it was, then, completely unmanageable.

Frames (Trusses)

The first application of parallel projection is to frames which for Rankine are trusses composed for bars untied through hinge joints. His definition is entirely general and allow, for example, curved bars: “Frame is here used to denote a structure composed of bars, rods, links, or cords, attached together or supported by joints of the first class described in the last Article [hinges], the centre of resistance being

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\(^3\) Rankine gives the expression for the complete lune, Figure 3 left, covering an angle of 90° [Rankine, 1858, 68]. The expression for a given angle from the apex can be found in [Lamé and Clapeyron, 1823, 805]. Then, by the method of substraction, the coordinates for any part of a lune can be found.
at the middle of each joint, and the line of resistance, consequently, a polygon whose angles are at the centers of the joints" [Rankine, 1858, 132].

In these definitions the usual assumption of hinging at the joints is explicit. It may be interesting to mention how much time it took to make such an assumption, and, also how late the practical geometrical applications came. Though already in the 18th, some simple applications of the law of parallelogram of forces were made to obtain the forces in some elements of trusses, particularly in centers of arches, [López Manzanares, 1996], it appears that the first to consider hinge joints in trussed roofs was John Robison in his article "Carpentry" for the supplement to the third edition (1801) of the Encyclopaedia Britannica (for convenience we have used the second edition of 1803). Robison is

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4 Centre of resistance: "The point where its line of action traverses the joint is called the centre of resistance of that joint". Line of resistance: "A line of Resistance is a line, straight, angular, or curved, traversing the centres of resistance of the joints of a structure". [Rankine, 1858, 131].

5 John Robison (1739-1805) was professor of natural philosophy in the University of Edinburgh. He played a fundamental role in the development of applied mechanics [Dorn, 1970, 1975]. His contributions to the Encyclopaedia Britannica (third edition and supplement) constituted a fundamental advance in the science of engineering due to his ability to combine the theory with the facts deduced from practical observation. The articles were compiled and published, with notes, by David Brewster [Robison, 1822].
explicit: The joints are supposed perfectly flexible, or to be like compass joints; the pin of which only keeps the pieces together when one or more of the pieces draws or pulls. The carpenter must always suppose them all compass joints when he calculates the thrusts and draughts of the different pieces of his frames [Robison, 1803, 166].

This simple supposition unlocks the problem of obtaining the forces in (statically determinate) frame elements using the law of the parallelogram of forces, and Robison explains its use in simple frames, [Fig.4] (a) and (b), and discusses the influence of geometry on the internal forces (how acute angles increases the forces, etc.). Eventually he makes the first graphical analysis of a simple frame, figure 4 (c). The article has some errors and inconsistencies which were commented by Thomas Young in his additions for its inclusion in the Supplement to the forth, fifth and sixth editions of the Encyclopaedia [Robison and Young, 1824], but it was widely read and freely used by late authors like Tredgold (1820). The next analysis of more complex frames were made in the 1840's and all employed, implicitly, Robison's supposition of hinge-joints 6. The treatment of distributed forces was a source of complication, and it appears that it was Rankine who first proposed to convert the distributed forces on the elements on point loads at the joints. The primary forces are obtained, and the secondary forces of flexure are superposed afterwards [Rankine, 1858, 137].

Therefore, it appears that Rankine was the first to formulate a complete theory of trusses, considering hinges at the joints, concentrating the distributed loads on hinges and differentiating the real structure and the ideal frame form by the lines of resistance. Thus, the “frame” of the two structures represented in [Fig.5 (a)] is the same, though visually are completely different, and in figure 5 (b) Rankine explains the analysis of a hammer-beam truss with curved elements, [Fig.5 (b)]: the dotted lines represent the line of resistance AD of the whole secondary truss ABDC, and AC of the curved element AC. The polygon of forces OGH permits to obtain the forces in every element, as we shall see later.

6 After Timoshenko it was Whipple (1847) the first “to publish a book containing useful information regarding the analysis of trusses”, though the Russian Jourawski formulated a complete theory of the analysis of trusses in 1844 [Timoshenko, 1953, 185].

7 See for example, [Waddington, 1849], cited by [Yeomans, 1992].
Rankine is also credited as being the first to publish the idea of a “polygon of forces” representing the forces in the elements of the frame, as a separate drawing. He employed already force diagrams in his Glasgow lectures in 1856. In 1857 he published it for the first time and he included it in 1858 in the Manual. Rankine took the idea from the force polygons of funicular polygons and generalized it to trusses [Fig. 6].

A few years later, in 1864, Maxwell proposed another type of force diagrams which were “reciprocal figures” of the truss or frame figure. The idea was probably foreshadowed already by Rankine who a few months before published a “Principle of Equilibrium of Polyhedral Frames” where he already discussed in general way the ideas of reciprocity between framed and diagrams of forces. But no doubt, it is to [Maxwell, 1864b, 1867, 1870] to whom we must credit the application of the properties of reciprocal figures to diagrams of forces with great detail and rigor, and also the study of truss rigidity [Maxwell, 1864a]. Rankine recognized explicitly the superiority of Maxwell’s approach and substituted the corresponding figure both in the Manual of applied mechanics and in the Manual of civil engineering. However, Rankine’s method is correct and permits, sometimes, for simple structures a better understanding. In [Fig. 7] both diagrams of forces have been represented extracted from different edition of the Manual of applied mechanics.

Rankine explained also in the Manual his method of sections, which he considered more convenient in the majority of cases. Indeed, for trusses formed by an upper and lower horizontal bars united by a series of a diagonal bars, the method of sections is much easier and the internal forces can be computed readily. However, for trusses with bars with different inclinations the graphical method of Maxwell is better. Besides, as Maxwell pointed out, the graphical method “affords a security against errors, since if any mistake is made the diagram cannot be completed” [Maxwell, 1867, 402].

8 After [Jenkin, 1869] and [Bow, 1873] already in the 1840’s some engineers and draughtsmen have used diagrams of forces.
9 I am indebted to Professor C.R. Calladine, who first noticed the difference between the first method of Rankine and that of Maxwell, and communicated it to me in a letter [Calladine, 2005].
10 This method of sections has been attributed many times to [A. Ritter, 1863].

Figure 6
Open frames and polygons of forces
[a] [Rankine, 1857]
[b] [Rankine, 1858]

Figure 7
Two graphical representations of the diagram of forces in the Manual of applied mechanics, for the same truss [a];
(b) the original diagram of Rankine in the first edition [Rankine, 1858];
(c) the reciprocal diagram of Maxwell, which substituted figure [b] in all editions after the publication of [Maxwell, 1864b].
Maxwell method obtained great diffusion. [Jenkin, 1869] gave a “popular” account of the method, which he described again in detail in his article “Bridge” for the Encyclopaedia Britannica (1876). Drawing reciprocal diagrams is not easy without a certain practical method which could be applied to any frame; Bow [Bow, 1873] developed such a method which was incorporated to the majority of handbooks of graphical statics after the 1880’s, and published a collection of more than one hundred diagrams of trusses, [Fig.8]. Cremona [Cremona, 1872] completed the theoretical work of Maxwell on reciprocal figures and his book, translated into French, 1885, and English, 1890, contributed greatly to the diffusion of the method, which in some countries is known as “Cremona’s method”. The work of the English authors was mostly ignored in Germany. Culmann in his Graphische Statik (1866) employs diagrams of forces and discusses plane reciprocal figures, but does not mention either Rankine or Maxwell. In the second edition of this work, Culmann (1875) cites Maxwell but undervalue his contribution and gives all the merit to Cremona11.

TRANSFORMATION OF FRAMES

Graphical methods have several advantages; one of them have been already mentioned, that of the visual checking of calculations. Sometimes were, also, more convenient than analytical methods. But, perhaps the most important advantage is the possibility of "visual thinking". A diagram may suggest much more than an algebraic expression. We don't know what was the process of thinking of Rankine, but it is most probable that he discovered his Transformation Theorems thinking on the geometrical properties of the diagrams which he invented and used, first in his teaching in Glasgow and afterwards for his technical manuals.

The first theorem refers to frames (trusses), and follows almost directly of his theorems of parallel projection in statics (see above). The theorem states: "If a frame whose lines of resistance constitute a given figure, be balanced under a system of external forces represented by a given system of lines, then will a frame whose lines of resistance constitute a figure which is a parallel projection of the original figure, be balanced under a system of forces represented by the corresponding parallel projection of the given system of lines; and the lines representing the stresses along the bars of the new frame, will be the corresponding parallel projections of the lines representing the stresses along the bars of the original frame" [Rankine, 1858, 162].

The advantage is that "it enables the conditions of equilibrium of any unsymmetrical frame which happens to be a parallel projection of a symmetrical frame (for example, a sloping lattice girder), to be deduced from the conditions of equilibrium of the symmetrical frame, a process which is often much more easy and simple than that of finding the conditions of equilibrium of the unsymmetrical frame directly".

One must be aware when the point loads at the joints are the resultant of distributed forces, and, in this case, apply some corrections.

The best way to understand the potential of the theorem and the kind of corrections needed is by way of an example. In [Fig.9] (a) a Warren truss and its diagram of forces has been drawn (taken from Jenkin 1876). Let as suppose that the point loads originated in a distributed load per unit of length,
applied in the upper bar. Then the total load will be $W_a$, proportional to the length $AK$. Then, it is easy to obtain the scale of forces in the transformed frames:

$$W_b = (1.35) \ W_a$$
$$W_c = \frac{W_a}{\sin \alpha}$$
$$W_d = (0.80) \ W_a$$

Then, it is easy to measure directly the internal forces in the corresponding diagram. There is, however, a short-cut. In both the drawing of the truss and the force diagram, the bar $IJ$ and its corresponding force has been marked. Of course, the element bar is represented by a line of a certain length and the force by another line, both lines being parallel. A parallel projection will increase or decrease these lengths in the same ratio. It follows, that it is not even necessary to draw the force diagram for the cases (b), (c) and (d). If we call $(F_a, l_a)$ the force and length of the bar $I$ in the orginal truss, and $(F_a, l_a'), (F_a, l_a''), (F_a, l_a''')$, the respective forces and lengths in the transformed trusses, then:

$$F_b = F_a \left( \frac{W_b}{W_a} \right) \left( \frac{l_b}{l_a} \right)$$
$$F_c = F_a \left( \frac{W_c}{W_a} \right) \left( \frac{l_c}{l_a} \right)$$
$$F_d = F_a \left( \frac{W_d}{W_a} \right) \left( \frac{l_d}{l_a} \right)$$

The internal force in any bar is proportional to the variation of the total load and of the length of the bar. This evident corollary was not cited by Rankine. For plane trusses, it may not be so useful, but in the case of spatial trusses, where the force diagrams are difficult to draw, it can be employed with great advantage. Let us consider the Schwedler dome in [Fig.10a], taken from [Föppl, 1900]. For symmetrical loads, it is very easy to obtain the forces in the meridian or hoop bars (diagonal bars would have zero forces). Let us suppose a distributed load per unit of horizontal surface (snow load). Then, the total load $W_a$ acting on the dome would be proportional to the covered surface. In the case (b), then, $W_a = W_b$, and in the case (c) $W_c = k \ W_a$, where $k$ is the stretching factor. Now we can apply the same expressions as before, and the problem reduces to calculate the new lengths of the bars, a simple trigonometric problem.

The limit to the application of the transformation theorems lies in that the new loads must be transformed original loads, represented by lines. This is so in the two examples above. However, if in the case of the dome of [Fig.10], the load would have been the weight of the faces of the polyhedron,
then, the variation of the loads would follow another law, not linear, and we could not apply the transformation theorem.

The case of inclined or unsymmetrical loads, which will make work some or all the diagonals, is subject to the same restrictions. If the loads are true point loads or can be reduced to point loads which vary linearly as the original straights representing the forces, then, the theorem can be applied. This is the case of the two examples reproduced in [Fig.11].

Rankine continued to think about the transformation theorems after the publications of the manuals of applied mechanics and civil engineering. His ambitious program of publication of engineering handbooks precluded the in depth revision and completion of the first editions of the manuals. Successive editions were almost identical, with a few corrections and additions in the text, and some short appendixes at the end. In the years 1863 and 1864, Rankine published two short notes on transformation of structures. The first, with the title “On the application of barycentric perspective to the transformation of structures” contain affirmations about the maintenance of the static properties (rigidity, determinacy, indeterminacy) after the transformation [Tarnai, 2001], and suggested the application of Sylvester’s barycentric perspective as a way to transform structures, not only by parallel projection, but in a completely general way\textsuperscript{12}.

\textsuperscript{12} “The theorems discovered by Mr. Sylvester now afford the means of greatly extending the art of designing structures by transformation from structures of more simple figures; for they obviously give at once the solution of the question given the figure of a structure which is balanced and stable under a load distributed in a given way; given also any perspective or homalographic projection of that figure; to find how the load must be distributed on the transformed structure, in order that it also may be balanced and stable” [Rankine, 1863, p.388]. Rankine is referring to two papers of the English mathematician J. J. Sylvester published in 1863 [Sylveste r, 1863a, 1863b]. Rankine concluded his article with the statement: “This is not the first instance in which theorems of pure science have proved to be capable of practical applications unexpected, perhaps, by their discoverers”. We have no notice of any work on Rankine’s suggestion.
TRANSFORMATION OF CORDS AND CHAINS

After frames, Rankine studies the equilibrium of chords and chains, first under any loads and, then under vertical parallel loads and, eventually, under vertical uniform parallel loads. To Rankine is evident that a cord or chain is an inverted open frame (see figure 6 (b), above). The treatment is clear and systematic. For vertical, parallel loads, the diagram of forces takes the form of [Fig.12a], and he deduces the differential equation of equilibrium, which imposes the tangency of the internal force to the form of the chain. Uniform load the chain adopts the form of a parabola. The usual case is with vertical ties, [Fig.12b], but Rankine considers also the case of inclined parallel ties, [Fig.12c].

Then, Rankine attacks the more general case: a vertical load defined by two curves of intrados (the chain) and extrados (the curve which marks the intensity of the vertical load at each point). Robison mechanical for a bridge, [Fig.13a], explains perfectly the concept. Rankine solves the problem for a horizontal extrados, [Fig.13b], and obtains the mathematical expression (this problem has been solved by many authors within the English “equilibration theory” of ca. 1800, for example by Hutton and later by Young, see [Huerta, 2005]). Rankine realizes that for a certain depth at the keystone, the equation becomes that of the common catenary, i. e., the form adopted by a chord subject to its own weight.

As a loaded chain is a particular case of an open frame, the Theorem of transformation of frames applies also to chains and chords: “The principle of Transformation by Parallel Projection is applicable to continuously loaded cords as well as to polygonal frames: it being always borne in mind, that in order that forces may be correctly transformed by parallel projection, their magnitudes must be represented by the lengths of straight lines parallel to their directions” [Rankine, 1858, 180].
In the case of hanging bridges with parallel ties or of stayed bridges, the application of the Theorem is so straightforward that it requires almost no explanation, the method being the same as that explained already for trusses, [Fig.14]

It has been already mentioned, that the common catenary, besides the form of equilibrium of a chain, is also the form of a chain loaded with a horizontal extrados passing at a distance of the parameter of the catenary from the vertex. Rankine, then, proposes to transform the common catenary to obtain the figure of equilibrium for a chain, with a certain relation span to height, and, thickness to height, [Fig.15]. The procedure is, first, to stretch or contract the common catenary in the vertical direction to obtain the desired relationship thickness to height, and, afterwards, to stretch or contract horizontally, to reach the relationship height to span. The method has been illustrated in [Fig.16]. This procedure is useful in bridge design.

**LINEAR ARCHES**

Linear arches, or equilibrated ribs, are inverted chains, and Rankine is well aware of the static equivalence between chains and arches: "All the propositions and equations [...] respecting cords or chains, are applicable to linear arches, substituting only a thrust for a pull, as the stress along the line of resistance". Of course a linear arch is unstable: "Linear arches do not actually exist; but the propositions respecting them

13 The examples of figure 14 are exercises proposed by the author to the students of the last course of structural design in the 1990's
are applicable to the lines of resistance of real arches and arched ribs, in those cases in which the direction of the thrust at each joint is that of a tangent to the line of resistance, or curve connecting the centers of pressure at the joints” [Rankine 1858, 182].

Rankine is interested in linear arches as a tool to study the equilibrium of arches subject to inclined loads, produced for example by the thrust of a fluid of or of the soil. This matter has been discussed in some detail by [Timoshenko, 1953, 198-200] and we will outline only the main aspects.

After discussing, the equilibrium of circular and elliptical ribs for normal pressure, Rankine passes on to discuss the case of a rib in equilibrium with a hydrostatic pressure; Rankine called these linear arches “hydrostatic arches”. In a circular ring of radius r, subject to a normal pressure p, the compression T is: 

\[ T = p \cdot r \]

Then, in any equilibrated rib subject to normal pressures, it is evident that, “the thrust at any normally pressed point of a rib is the product of the radius of curvature by the intensity of the pressure” [Rankine, 1862, 208]. Therefore, is the pressure is p, proportional to the depth, and the radius of curvature \( \rho \), then in the general case \( T = p \cdot \rho \). Then, it follows, that for the rib to be in equilibrium “the radius of curvature must be inversely as the pressure”, and, “it is further evident, that if the pressure be normal at every point of the rib, the thrust must be constant at every point; for it can vary only by the application of a tangential pressure to the arch; and the radius of curvature must be inversely as the pressure”, [Fig.17a].

Then, if \( x_0 \) and \( \rho_0 \), the depth and the radius of curvature at the vertex, are given, “the property of having the radius of curvature inversely proportional to the vertical ordinate from a given horizontal axis enables the curve to be drawn approximately, by the junction of a number of short circular arcs”, [Fig.17b].

Rankine cites Yvon Villarceau (1854) as the only engineer to have treated the problem of the equilibrium of hydrostatic arches. The mathematical function of the curve of intrados of the hydrostatic arch is quite complex and the integration of the differential equation of equilibrium involves the use of elliptic functions [Rankine, 1858, 193]. However, Rankine in barely five pages [Rankine, 1858, 191-195] deduces the mathematical formulae all the relevant parameters of the hydrostatic arch.

Rankine, then, passes to his true objective: to obtain the form of a linear arch in equilibrium with the thrust of the soil: the “geostatic arch”. He shows that the geostatic arches can be obtained as a parallel projection of hydrostatic arches: “reducing the horizontal dimensions of arch ABC [Fig.17a] ... to some
constant ratio without changing the vertical dimensions, he obtains a linear arch for the case where the intensity of the horizontal pressure acting on the arch is only a certain fraction of the vertical pressure as it should be for earth pressure”. Rankine gives the mathematical formulae of conversion.

Rankine uses geostatic linear arches for the analysis of real masonry bridge arches. There is no space here for a detailed explanation, but the approach combines statements of equilibrium and geometry, and it is therefore completely correct [Heyman, 1995]. A bridge arch is stable if it is possible to draw a geostatical arch in equilibrium with the soil which forms the backing within the middle third of the arch ring. Rankine, then, proposed the most complete theory of masonry arches; however, we have no evidence that it has been ever used.

TRANSFORMATION OF BLOCKWORK AND MASONRY STRUCTURES

A masonry structure can be imagined as a series of blocks in dry contact. The essential property of this kind of structure is that the internal forces must be within the masonry, i.e., that the locus of the resultants at every joint, the “line of thrust” must be contained everywhere within the masonry.

The concept of line of thrust was formulated first by Thomas Young [Huerta, 2005], though it is usually attributed to Moseley, who called it “line of resistance”. Rankine first discuss the concept and the different ways to obtain the line of resistance, either graphically or by his “method of sections” [Rankine, 1858, 230]. He, then, establishes an “analogy of Blockwork and Framework”: the polygon formed by the intersection of the resultants at the different joints is analogous to a polygonal frame, and has the same properties. Drawing this polygon of resultants it is possible to obtain the position of the “centre of pressure” at each joint, and also to check the inclination of the resultant respect to the plane. “Then if each centre of pressure falls within the proper limits of position, and the direction of each resultant pressure within the proper limits of obliquity; […], the structure will be balanced” [Rankine, 1858, 232].

The transformations applied to frames are applicable also to blockwork structures, and Rankine states his Theorem of Blockwork Structures: If a structure composed of Blocks have stability of position when acted on by forces represented by a given system of lines, then will a structure whose figure is a parallel projection of the original structure have stability of position when acted on by forces represented by the
Figure 79
Use of the "slicing technique" to study the safety of masonry domes
(a) Domes of revolution; 
(b) Oval domes

corresponding parallel projection of the original system of lines; also, the centers of pressure and the
lines representing the resultant pressures at the joints of the new structure will be the corresponding
projections of the centers of pressure and the lines representing the resultant pressures at the joints of
the original structure.

The relative weights of the blocks which compose the structure remain unaltered, and if these weights
are represented by lines, and are equilibrium, a transformation by parallel projection will give a system
of forces represented by lines, also in equilibrium (see above, parallel projection on statics).

Therefore, if a masonry structure in a state of safe equilibrium, represented by a line of thrust inside the
masonry, a parallel (affine) projection of the structure will also be in a safe state of equilibrium and the new
line of thrust will be the transformation of the original, the relative distances from the border at each joint
remaining the same. The theorem has been illustrated in figure 18 with reference to a simple masonry arch.

The "stability of position", the line of thrust inside the masonry, is maintained. However, the "stability
of friction" is an independent problem, and it may occur that the resultant is outside the cone of friction in
one or several of the transformed joints. However, for arches the form of the line of thrust is quite
unaffected by the family of joints considered [Huerta, 2004, 57] and Rankine recommends simply
altering the direction of these joints: Should the pressure at any joint in the transformed structure prove
to be too oblique, frictional stability can in most cases be secured, without appreciably affecting
the stability of position, by altering the angular position of the joint, without shifting its centre of figure, until
its plane lies sufficiently near to a normal to the pressure as originally determined [Rankine, 1858, 233].

The theorem can be applied to any masonry structure. Its application to domes is particularly simple.
A dome can be imagined as composed by a series of arches obtained slicing the dome by meridian
planes. Every two "orange slices" form an arch; if it is possible to draw a line of thrust within this arch,
then we have found a possible equilibrium state in compression and the dome is safe, it will not
collapse, [Fig.19a] [Heyman, 1995]. The dome may have an oculus as in it a compression ring forms:
the dome build a "keystone" when a ring is closed, and therefore masonry domes can be built without
centering. The safety condition is, then, a geometrical condition. Domes of similar forms and different
sizes have the same degree of safety: the drawing of [Fig.19a] has no scale.

The same "slicing technique" can be applied to oval domes. Now the elementary arches are different,
but equal the two opposed which can build a safe arch. If it is possible to find a line of thrust within
each pair of elementary arches, the dome will be safe, [Fig.19b].

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As an example of application, in [Fig.20a] it is represented the static analysis of the model of a simple dome proposed by Fontana (1694). In [Fig.20b) it is represented another dome, obtained contracting the height and broad of the original dome. The line of thrust represents the equilibrium of the slice arch having the major axis. The stability remains unaltered, and it may be that the confidence of the architects in the stability of oval domes has its origin in the intuition of this principle.

**Influence. The process of rediscovery**

Rankine's theorems of transformation of structures exerted no influence. Despite the enormous diffusion of his *Manuals of Applied Mechanics and Civil Engineering*, it appears that nobody realized the potential of the theorems. Even in a series of books intended to facilitate the understanding of Rankine's manuals, this part was omitted.

The theorems were rediscovered, in part, by several authors, who apparently arrived independently to their conclusions. It may be curious to make a brief review of these partial rediscoveries.

The first to give a partial statement concerning elliptical arches was [Woodbury, 1858]. He realizes that when contracting vertically an arch, the horizontal thrust remained unaltered and the line of thrust was the contraction of the original line, [Fig.21].
Henry T. Eddy (1878) was also aware of the possibility of using parallel projection to ease the calculation of elliptical arches. In his book on graphical statics, applied the method to the analysis of Brunel's bridge of Maidenhead, [Fig.22]. The book by Eddy is full of interest and was translated into German in 1880. With reference to domes, he invented a graphical method to obtain the internal forces in a “spherical dome of metal” [Eddy, 1878, 53-56], i.e., a dome build with a (bilateral) material resisting tensions, in contraposition with masonry, a (unilateral) material which resists only compressions, [Fig.23a]. In the upper part of the dome there are compressive forces and below an angle of circa 52°, tensions appear. If the dome is of metal a membrane state is possible everywhere; if of masonry, the forces deviate from the middle surface in the zone of tensions and the dome needs some thickness to accommodate the thrusts, [Fig.23b]. (For a more detailed explanation see [Huerta, 2003]). [Föppl, 1900] employed both the device of contraction for elliptic arches and the method of dome analysis, but without giving credit to Eddy.

Josef Gross wrote a dissertation, probably by Föppl suggestion (he was Korreferent of the dissertation), on the affine transformation (parallel projection) of domes [Gross, 1913]. Gross does not cite Rankine
or Eddy. The last omission is surprising, as Eddy's book has been translated into German, as we have said. He studied affine transformation of arches, cross vaults and domes, mainly with graphical methods, [Fig. 24].

The last and most important author in this process of rediscovery is Franz Dischinger. Dischinger, with Bauersfeld, are the founders of the modern shell theory in the 1920's [Specht, 1987]. A practicing engineer, Dischinger combined a sound mathematical knowledge with a clear understanding of the objectives of structural theory. Already in his first book on shells [Dischinger, 1928], Dischinger studied in depth the application of affine transformation to calculate, first elliptical shells, and, then, of shells of any form. He maintained an active interest in the topic, and applied it to some actual structures as the great octagonal domes of the market of Leipzig [Dischinger and Rüsüch, 1929]. Eventually, he published a long, definitive, article on the matter in 1936 [Dischinger, 1936].

The interest of Dischinger was not in stability, as with masonry structures, but in strength, in obtaining the state of stress within the shells. In making an affine transformation of a certain shell, the thickness varies, and it is necessary to make some adjustments to obtain the stresses in the actual shell of constant thickness. There is no space here to handle Dischinger's contribution, but is full of ingenuity and interest. Before the age of computers, Dischinger afforded a tool to analyze shells of any form. Though Flügge incorporated the problem in his book on shells [Flügge, 1934], it appears that it was not used in practice very often. An interesting exception is the membrane analysis of the oval dome of the Abbey of Neresheim, by Balthasar Neumann, made by [Ullrich, 1974], who followed Dischinger's approach [Fig. 25].
Conclusions

Rankine's theorems of transformation of structures have been unjustly ignored, first by his contemporaries, and, second, by contemporary engineers and architects and historians of construction and engineering. They use the approach of equilibrium, validated by modern limit analysis. The theorems afford a useful tool for design, even in our computer age, and proportion a deep insight on the relations among structural design and geometry. They are also relevant to mathematicians and geometers.


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GEOMETRÍA
Y PROPORCIÓN
EN LAS
ESTRUCTURAS

ENSAYOS EN HONOR DE RICARDO AROCA

ESSAYS IN RICARDO AROCA'S HONOUR

GEOMETRY
AND PROPORTION
IN STRUCTURAL
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Cayo Crispio Salustio

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