Global & Local Properties of Lie-Vessiot Systems

Juan J. Morales-Ruiz
U. Politècnica de Catalunya
juan.morales-ruiz@upc.edu

David Blázquez-Sanz
U. Sergio Arboleda
david.blazquez-sanz@usa.edu.co

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Outline

I. Motivation

I.i. Motivation

- A superposition law for a non autonomous differential equation
$$\frac{dx_i}{dt} = F_i(t, x) \quad i = 1, \ldots, n$$
is a set of formulae,
$$\varphi_i(x^{(1)}, \ldots, x^{(r)}, \lambda),$$
expressing the general solution,
$$x(t) = \varphi(x^{(1)}(t), \ldots, x^{(r)}(t), \lambda)$$
as function of a fundamental system of solutions, and n arbitrary constants λ_i.

I.i. Motivation (II)

- The main example is the linear superposition for solutions of a linear system,

$$\dot{x} = A(t).x$$

- n linearly independent solutions give the general solution by linear combinations.
- This is the linear superposition law for linear equations,

$$\varphi\colon \mathbb{C}^n \times \mathbb{C}^n_\lambda \to \mathbb{C}^n$$

$$\varphi(x^{(1)}, \dots, x^{(n)}, \lambda) = \sum_{i=1}^{n} \lambda_i x^{(i)}$$

I.i. Motivation (III)

- There are non-linear equations related to linear system. They also admit superposition Laws. The main example is the Riccati equation.

$$\dot{x} = a(t) + b(t)x + c(t)x^2$$

- The anharmonic ratio of 4 different solutions is a constant of the equation,

$$\frac{d}{dt}\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_4)} = 0.$$

- We can then express the fourth solution as a function of three know solutions and the constant anharmonic ratio λ,

$$x = \frac{x_3(x_1 - x_2) - \lambda x_1(x_3 - x_2)}{(x_1 - x_2) - \lambda(x_3 - x_2)}.$$

I.i. Motivation (IV)

- We can find also genuine non-linear superposition laws.
- There is also a main example, the generalized Weierstrass equation:

$$\dot{x}^2 = f(t)(x^3 - g_2 x - g_3)$$

- Its general solution is of the form,

$$x(t) = \wp\left(\int_0^t f(\tau)d\tau + \lambda\right).$$

- Weierstrass' addition formula for $\wp$ function,

$$\wp(t+u) = -\wp(t) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(t) - \wp'(u)}{\wp(t) - \wp(u)}\right)^2$$

is then a superposition formula for the solutions.

I.ii. Historical Development

- Differential equations admitting superposition laws were introduced By S. Lie in 1885.
- Local conditions for the existence of a superposition law were given by S. Lie and G. Scheffers in 1893. Some other advances in the earlier theory came from E. Vessiot and A. Gulbdberg.
- In the context of mathematical physics several authors carried out the research in 20th and 21th century: Winternitz, Shnider, Shorine, Cariñena, Grabowsky, Marmo, Ramos.
- Superposition laws are also interesting for differential algebraists. The work of K. Nishioka relates superposition Laws with Kolchin's strongly normal extensions of differential fields.
- Contemporary approach to SNE, due to J. Kovacic and R. Churchill is also related with superposition laws.

II. Lie's Superposition Theorem

II.i. Superposition Laws

Let $\vec{X}$ be a non-autonomous vector field in M with time varying in a Rieman surface S, with a fixed meromorphic derivation ∂:

$$\vec{X} = \partial + \sum_{i=1}^{n} f_i(x,t)\frac{\partial}{\partial x_i}.$$

Definition

A superposition law for $\vec{X}$ is an analytic map,

$$\varphi\colon U \times M \to M,$$

where:

- U is an analytic susbet of M^r
- U is union of orbits of $\vec{X}^r$.

Such that, for any local solution $(x^{(1)}(t),\ldots,x^{(r)}(t))$ of $\vec{X}^r$ taking values in U, defined for t varying in $S' \subset S$,

$$x_\lambda(t) = \varphi(x^{(1)}(t),\ldots,x^{(r)}(t),\lambda)$$

II.ii. Lie's Superposition Theorem

- Consider $\vec{X} = \partial + \sum_i f_i(x,t)\frac{\partial}{\partial x_i}$, meromorphic in $S \times M$ and holomorphic in $S^\times \times M$.
- For $t_0 \in S^\times$, define $\vec{X}_{t_0}$ in $\mathfrak{X}(M)$,

$$\vec{X}_{t_0} = \sum f_i(x,t_0)\frac{\partial}{\partial x_i}.$$

- $\vec{X}$ is seen as a map $\vec{X}\colon S^\times \to \mathfrak{X}(M)$

Definition

We call Lie-Vessiot-Guldberg algebra of $\vec{X}$ to the Lie algebra $\mathfrak{g}(\vec{X})$ of vector fields in M spanned by the image of the above defined map.

Theorem (Lie-Scheffers 1893)

A necessary condition for the existence of a superposition law for $\vec{X}$ is $\dim \mathfrak{g}(X) < \infty$.

II.iii. Pretransitive Lie Group Actions

- Consider a complex analytic group G acting in M,

$$G \times M \to M, \quad (\sigma, x) \mapsto \sigma \cdot x.$$

- $\mathcal{R}(G)$ right invariant fields in G.
- $\mathcal{R}(G, M)$ fundamental fields in M.
- We say that G acts pretransitively in M if there is a G-invariant analytic open subset $U \in M^r$ such that the quotient, $U \to U/G$ is a principal bundle.

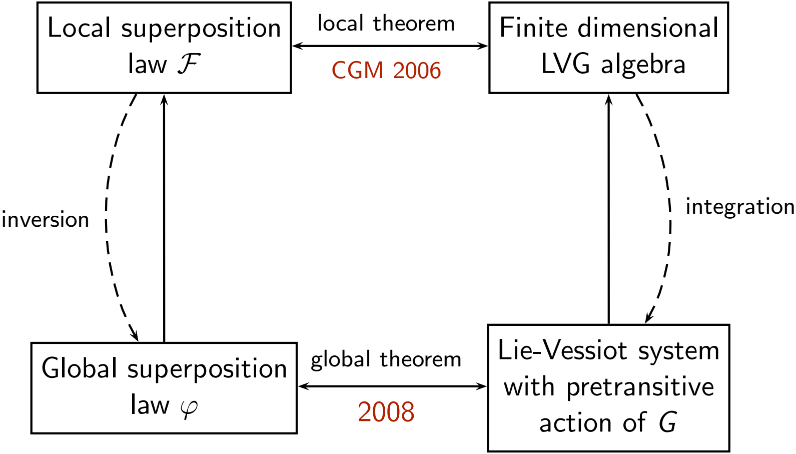
Definition

A non-autonomous vector field $\vec{X}$ is called a Lie-Vessiot system associated to the action of G in M if $\mathfrak{g}(\vec{X}) \subset \mathcal{R}(G, M)$.

Theorem (2008)

A non-autonomous vector field $\vec{X}$ in M admit a superposition law if and only if it is a Lie-Vessiot system associated to a pretransitive action in M.

II.iv. Lie's Superposition Theorem (Scheme)



III. Lie-Vessiot Hierarchy and Galois

III.i. Automorphic System (I)

Definition

We call automorphic system to any Lie-Vessiot system in a Lie group G.

- The superposition law for an automorphic system is the composition law,
$$G \times G \to G.$$
- For small $S' \subset S$ the space of solutions $\mathcal{G}_{\vec{A}}$ is a principal homogeneous space by the right side,
$$\mathcal{G}_{\vec{A}} \times G \to G, \quad (\sigma(t), \tau) \mapsto \sigma(t) \cdot \tau$$
- Example: Let $U(t)$ be a fundamental matrix of solutions of $\dot{x} = A(t).x$. For each non denerate constant matrix $C \in GL(n, \mathbb{C})$, $U(t).C = V(t)$ is another fundamental matrix of solutions.

III.i. Automorphic System (II)

Let M be a faithful G-homogeneous space of finite rank r.

- Let $\vec{X}$ be a Lie-Vessiot system in M with t varying in S. Then, there are functions $f_i(t)$ in $S^\times$ with:

$$\vec{X} = \sum_{i=1}^{s} f_i(t) A_i^M, \quad \mathcal{R}(G, M) = \langle A_1^M, \ldots A_s^M \rangle.$$

- By the isomorphism $\mathcal{R}(G, M) \simeq \mathcal{R}(G)$ we define the associated automorphic system,

$$\vec{A} = \sum_{i=1}^{s} f_i(t) A_i.$$

- Example: $\dot{x} = A(t)x \rightsquigarrow \dot{U} = A(t)U$.

III.ii. Lie-Vessiot Hierarchy

- G-space morphisms,

$$\phi\colon M \to N, \quad \phi(\sigma \cdot x) = \sigma \cdot \phi(x),$$

  carry Lie-Vessiot systems onto Lie-Vessiot systems.
- Let $\vec{A}$ be an automorphic system in G. Throughout the morphism,

$$\mathcal{R}(G) \to \mathcal{R}(G, M),$$

  it induces a Lie-Vessiot system $\vec{A}^M$ in any G-space M.
- If M is a faithful G-homogeneous G-space, then $\vec{A}$ is the automorphic system associated to $\vec{A}^M$.
- Any G-space morphism $M \to N$ carry $\vec{A}^M$ onto $\vec{A}^N$.

III.iii. Meromorphic Solutions and Galois Group

Consider a meromorphic automorphic system,

$$\vec{A} = \partial + \sum f_i(t) A_i, \quad f_i(t) \text{meromorphic in } S$$

we re-define,

$$S^{\times} = S \setminus \{\text{poles of } \partial \text{ and } f_i(t)\}.$$

- For a closed analytic subgroup $H \subset G$, the space of cosets G/H is an homogeneous G-space.
- Any homogeneous G-space is isomorphic to a space of cosets G/H for certain H.
- Let $\mathfrak{C}$ be the set of congugacy classes of analytic subgroups of G.
- For $\mathfrak{c} \in \mathfrak{C}$ let $M(\mathfrak{c})$ be its corresponding homogeneous space. $\vec{A}^{M(\mathfrak{c})}$ is a meromorphic Lie-Vessiot system.

III.iii. Meromorphic Solutions and Galois Group (II)

- Define:

$$\mathcal{M}(\vec{A}, M(\mathfrak{c})) = \{\text{meromorphic solutions of } \vec{A}^{M(\mathfrak{c})}\}$$

- And $\mathcal{M}(\vec{A})$ the total set of meromorphic solutions:

$$\mathcal{M}(\vec{A}) = \bigcup_{\mathfrak{c} \in \mathfrak{C}} \mathcal{M}(\vec{A}, M(\mathfrak{c})).$$

Definition

For t_0 in $S^\times$ we define the analytic Galois group,

$$Gal_{t_0}(\vec{A}) = \bigcap_{x(t) \in \mathcal{M}(\vec{A})} H_{x(t_0)}$$

III.iv. Analytic Galois Bundle

Theorem

The Galois group $Gal_t(\vec{A})$ depends meromorphically on t in S.

Definition

We call Galois bundle of $\vec{A}$ to the complex analytic in S and meromorphic in $S^\times$ principal fiber bundle,

$$Gal(\vec{A}) = \bigcup_{t \in S^\times} Gal_t(\vec{A}) \xrightarrow{\pi} S^\times.$$

III.v. Analytic Galois Bundle and Picard Vessiot

- The case $GL(n, \mathbb{C})$ corresponds to systems of linear homogeneous differential equations.
- We should use Picard-Vessiot theory to examine these systems.
- The algebraic differential Galois group is the stabilizer of all meromorphic tensorial invariants of the equation (Tanakian approach).
- By Chevalley's theorem we have a correspondence between tensorial invariants and meromorphic solutions of Lie-Vessiot systems in algebraic homogeneous spaces.

- Theorem. The analytic differential Galois group is a Zarisky dense subgroup of the algebraic differential Galois group provided by Picard-Vessiot theory.
- Toy Example: The following differential equation is an automorphic system in the additive group C:

$$\dot{x} = \frac{1}{t}$$

- Its Galois group in Picard-Vessiot theory is $\mathbb{C}$
- There is an analytic action of $\mathbb{C}$ in $\mathbb{C}^*$, $x \star z = e^x z$. The induced Lie-Vessiot has a meromorphic solutions $z = \lambda t$.
- The analytic Galois group is $2\pi i\mathbb{Z} \subset \mathbb{C}$.

IV. Lie's Reduction Method

IV.i. Gauge Transformations

- Consider $S' \subset S$ and a curve $\sigma(t) \in \mathcal{O}(S', G)$. We define the gauge transformations,

$$L_{\sigma(t)} \colon G \times S' \to G \times S', \quad (\tau, t) \mapsto (\sigma(t) \cdot \tau, t).$$

$$L_{\sigma(t)} \colon M \times S' \to M \times S', \quad (x, t) \mapsto (\sigma(t) \cdot x, t).$$

- They transform automorphic (Lie-Vessiot) systems $\vec{A}$ onto automorphic (Lie-Vessiot) systems.

IV.ii. Lie's theorem on Reduction

For $H \subset G$, we have $\mathcal{R}(H) \subset \mathcal{R}(G)$. Automorphic systems in H are, in particular, automorphic systems in G. Let us consider:

- $\vec{A}$ automorphic in G.
- M homogeneous G-space.
- $\vec{X}$ induced Lie-Vessiot in M.
- $x_0 \in M$, and $H \subset M$ its isotropy group.

Lemma

If $x(t) = x_0$ is a solution of $\vec{X}$, then $\vec{A}$ is an automorphic system in $H \subset G$.

IV.ii. Lie's theorem on Reduction (II)

- Let $x(t)$ be a particular solution of $\vec{X}$.
- Assume that there exist a curve $\sigma(t)$, such that $\sigma(t) \cdot x_0 = x(t)$.
- The Gauge transformantion $L_{\sigma(t)^{-1}}$ sends $\vec{A}$ to a new automorphic system $\vec{B}$.
- The lemma applies to $\vec{B}$.

The problem is to find a Global $\sigma(t)$!

IV.ii. Lie's theorem on Reduction (III)

For a given meromorphic $x\colon S \to M$ we consider.

$$\mathcal{H} = \{(\sigma, t) \,|\, \sigma \cdot x_0 = x(t)\} \subset G \times S,$$

projection $\pi\colon \mathcal{H} \to S$ is a fiber bundle. Our gauge transformation comes from a global section.

- If S is non-compact and H is connected, Grauert theorem ensures that there is a global meromorphic section.
- We say that H is special is any principal bundle modeled over H has a global meromorphic section.

Theorem

Assume that there is a meromorphic solution $x(t)$ of $\vec{X}$ defined in S. Assume one of the following:

1. H is special
2. S is non-compact and H connected

then there is a meromorphic gauge transformadion in $G \times S$ reducing $\vec{A}$ to H.

IV.iii. Reduction to Galois Group

Theorem

Assume that $Gal_t(\vec{A}) \subseteq H \subseteq G$ and one of the following,

1. H is special group,
2. S is non-compact and H is connected.

Then, there is a meromorphic gauge transformation of $G \times S$ that reduces $\vec{A}$ to an automorphic system in H.

IV.iv. Quadratures in Abelian Groups

- G connected abelian group, the exponential map,

$$\mathcal{R}(G) \to G, \quad A \mapsto \exp(A),$$

  is the universal covering.
- If fact it is a group morphism.
- $\mathcal{R}(G)$ is a vector group, and automorphic systems are integrated by a quadrature.
- By composition we integrate automorphic systems in G as exponential of integrals:

$$\sigma(t) = \exp\left(\sum_{i=1}^{s} \int_{t_0}^{t} f_i(\tau) A_i(\tau) d\tau\right),$$

  where $d\tau$ is the meromorphic 1-form in S dual with ∂.

IV.v. Solvable Groups

Theorem

Assume that $Gal_t(\vec{A}) \subseteq H \subseteq G$, with H a connected solvable group, and one of the following,

1. H is special group
2. S is non-compact Riemann surface

then $\vec{A}$ is integrable by means of quadratures of closed meromorphic 1-forms in S and the exponential map in G.

Proof.

It is done, by Lie's reduction to H, and then by iterarion of Lie's reduction method and quadratures in quotient abelian groups.

IV.vi. Symmetries and Reduction

- Infinitesimal symmetries of $\vec{A}$ define a sheaf of vector fields in $M \times S$.
- Any symmetry reduces to its transversal part,

$$\vec{L} = \sum_i f(\sigma, t) \frac{\partial}{\partial \sigma_i}, \quad [\vec{L}, \vec{A}] = 0.$$

- The sheaf of transversal symmetries is characterized as follows:

$$Lie(\vec{X}) \simeq \mathcal{L}(G) \otimes_{\mathcal{C}} \mathcal{O}^{\vec{A}}_{S \times G},$$

- Where $L(G)$ is the algebra of left-invariant vector fields in G, and $\mathcal{O}^{\vec{A}}_{S \times G}$ is the sheaf of first integrals of $\vec{A}$.

IV.vi. Symmetries and Reduction (II)

- A General symmetries of $\vec{A}$ does not descend to a symmetrie the Lie-Vessiot system $\vec{X}$.
- Descending symmetries are right-invariant symmetries,

$$L = \sum_i f(t)\vec{A}_i, \quad [\mathcal{L}, \vec{A}] = 0, \quad \vec{A}_i \in \mathcal{R}(G).$$

- The right invariant symmetries are determined by a Lie-Vessiot system, the system induced in $\mathcal{R}(G)$ by the adjoint action of G.

$$\partial\vec{L} = [\vec{A}, \vec{L}]$$

- We can apply Lie's reduction by terms of known symmetries.

IV.vi. Symmetries and Reduction (III)

- Let $Right(\vec{A})$ be the space of meromorphic right invariant symmetries of $\vec{A}$.
- For $t \in S^{\times}$, $Right_t(\vec{A})$ is a Lie subalgebra of $\mathcal{R}(G)$.

Theorem

For al t in $S^{\times}$, The centralizer subgroup of $Right_t(\vec{A})$ contains the Galois group $Gal_t(\vec{A})$.

Corollary.

Let $\vec{A}$ be an automorphic system of the symplectig group $Sp(2n, C)$. If $Right(\vec{A})$ contains an abelian algebra of dimension n, then the Galois group of $\vec{A}$ is virtually abelian.