UNIVERSIDAD POLITÉCNICA DE MADRID

ESCUELA TÉCNICA SUPERIOR DE INGENIEROS AERONÁUTICOS

Impact of the Unsteady Aerodynamics of
Oscillating Airfoils on the
Flutter Characteristics of Turbomachines

TESIS DOCTORAL

Almudena Vega Coso

Ingeniera Aeronáutica

Madrid, 2016

Departamento de Mecánica de Fluidos y Propulsión
Aeroespacial

Escuela Técnica Superior de Ingenieros Aeronáuticos

Impact of the Unsteady Aerodynamics of
Oscillating Airfoils on the
Flutter Characteristics of Turbomachines

Autor
Almudena Vega Coso
Ingeniera Aeronáutica

Director
Roque Corral García
Doctor Ingeniero Aeronáutico

Madrid, 2016

Tribunal nombrado por el Sr. Rector Magfco. de la Universidad Politécnica de Madrid, el día de de 20.....

Presidente: Benigno Lázaro

Vocal: Damian Vogt

Vocal: Carlos Martel

Vocal: Mehdi Vahdati

Secretario: Manuel Antonio Burgos

Realizado el acto de defensa y lectura de la Tesis el día 1 de Diciembre de 2016 en la Escuela Técnica Superior de Ingeniería Aeronáutica

Calificación

EL PRESIDENTE

LOS VOCALES

EL SECRETARIO

*You can't connect the dots looking forward; you
can only connect them looking backwards. So
you have to trust that the dots will somehow
connect in your future.*

Steve Jobs

Abstract

This thesis studies the unsteady aerodynamics of oscillating airfoils in the low reduced frequency regime, with special emphasis on its impact on the scaling of the work per cycle curves, using an asymptotic approach and numerical experiments.

The unsteady aerodynamics associated with the vibration of turbine and compressor bladed-discs and stator vanes is nowadays routinely analysed within the design loop of the aeroengine companies, and it has also been the subject of dedicated experiments. The final aim is the derivation of the aerodynamic stability of rotor blades and the quantification of the aerodynamic damping, which is the result of the application of the unsteady pressures on the airfoil displacements. Little attention has been historically paid to the understanding of vibrating airfoil aerodynamics, since this is not a figure of merit in itself for the aeroelastic analyst, and then, little understanding has been gained in recent years about the physics of these type of flows and its impact on the aerodynamic damping. As a consequence, although there are several trends that are well known by the aeroelastic community, such as the stabilisation with the reduced frequency, or the different shape of the work per cycle curves for LPTs and compressors, the physics which is behind this behaviour is not really well understood and these statements lack of a sound theoretical support.

A perturbation analysis of the linearised Navier-Stokes equations for real modes at low reduced frequency is presented and some conclusions are drawn. The first important result is that a new parameter, the unsteady loading of the airfoil, ULP, plays an essential role in the trends of the phase and modulus of the unsteady pressure caused by the vibration of the airfoil. This parameter depends solely on the steady flow-field

on the airfoil surface and the vibration mode-shape. As a consequence, the effect of changing the design operating conditions or the vibration mode onto the work-per-cycle curves (and therefore onto the stability) can be easily predicted and, what is more important, quantified without conducting additional flutter analysis.

For lightly unsteady loaded airfoils, the unsteady pressure and the influence coefficients scale linearly with the reduced frequency whereas the phase departs from $\pi/2$ and changes linearly with the reduced frequency. As a consequence, the work-per-cycle scales linearly with the reduced frequency for any inter-blade phase angle, and it is independent of its sign (the work-per-cycle curves are symmetric with respect inter-blade phase angle zero, even for an asymmetric cascade, which is a surprising effect). For highly unsteady loaded airfoils, the unsteady pressure modulus is fairly constant exhibiting only a small correction with the reduced frequency, while the phase departs from zero and varies linearly with it. In this case only the mean value of the work-per-cycle scales linearly with the reduced frequency. This behavior is independent of the geometry of the airfoil and the modeshape in first order approximation in the reduced frequency. For symmetric cascades the work-per-cycle scales linearly with the reduced frequency irrespectively of whether the airfoil is loaded or not. It is readily concluded that for lightly unsteady loaded airfoils, or symmetric cascades, it is not possible to change the stability by reduced frequency criterion.

These conclusions have been numerically verified on several airfoil geometries. With this aim, simulations using a frequency domain linearised Navier-Stokes solver have been carried out on rows of a low-pressure turbine airfoil section, the NACA65 and NACA0012 sections, and a flat plate (which is commonly considered as the simplest representation of a compressor airfoil, and the tip of the last stages of heavy duty land-based power turbines), to show the correlation between the actual value of the unsteady loading parameter (ULP) and the flutter characteristics for different airfoils, operating conditions and mode-shapes. Both, the traveling-wave and influence coefficient formulations of the problem are used in combination to increase the understanding of the ULP influence in different aspects of the unsteady flow-field. It is concluded that, for a blade oscillating in a prescribed motion at

design conditions, the ULP can quantitatively predict the effect of loading variations due to changes in the incidence, and also in the mode shape. It is also proved that the unsteady loading parameter can be used to compare the flutter characteristics of different airfoils.

The beauty of the ULP derived in the present thesis is that it is able to account for the effect on the stability of the work-per-cycle curves of the mode-shape and the aerodynamic loading distribution (in a quantitative way), and the geometry (in a qualitative way), in the low reduced frequency limit, without performing any unsteady simulation. As a consequence, the parameter is well suited for engineering conceptual studies, since it is able to anticipate the effect of design changes in the aerodynamic damping.

The academic implications of the findings of the present thesis are not negligible either, since: i) it provides a theoretical support for some well-known fundamental concepts related to the stability of oscillating blades, and ii) it introduces new fundamental concepts, such as for example the importance of the symmetry of the cascade, or the unsteady loading parameter, which have been proved being of major relevance in the flutter characteristics of turbomachinery blades.

Resumen

Esta tesis estudia la aerodinámica no estacionaria de perfiles oscilantes, en el régimen de frecuencia reducida baja, con especial énfasis en el impacto que esta tiene en el escalado de las curvas de trabajo por ciclo. Con esta finalidad, se utilizarán métodos asintóticos y experimentos numéricos.

La aerodinámica no estacionaria asociada a la vibración de turbinas y compresores, y vanos de estator se analiza comúnmente de forma rutinaria dentro del bucle de diseño en las compañías de motores de aviación, y ha sido el objeto de muchos experimentos dedicados. El objetivo final de estos esfuerzos es la derivación de la estabilidad aerodinámica de los álabes de rotor y la cuantificación del damping aerodinámico, que es el resultado de la aplicación de las presiones no estacionarias sobre los desplazamientos del álabe.

Históricamente, se ha prestado poca atención a comprender la aerodinámica de perfiles vibrantes, ya que no es una figura de mérito en sí mismo para el analista aeroelástico y, como consecuencia, se ha ganado muy poco conocimiento en los últimos años acerca de la física de este tipo de flujos, y su impacto en el damping aerodinámico. Aunque hay ciertas tendencias que son bien conocidas por la comunidad aeroelástica, como la estabilización con la frecuencia reducida, o la diferente forma de las curvas de trabajo por ciclo de compresores y turbinas de baja presión; la física que está detrás de estos comportamientos no es realmente entendida, y estas observaciones carecen de un soporte teórico.

En la tesis se presentará un análisis de perturbaciones a baja frecuencia reducida de las ecuaciones de Navier-Stokes linealizadas, y se derivarán conclusiones prácticas.

El primer resultado importante es que un nuevo parámetro, la carga aerodinámica no estacionario del álabe (ULP por sus siglas en inglés), juega un papel esencial en la tendencia de la fase y el módulo de la presión no estacionaria generada por la vibración del álabe. Este parámetro depende solo del flujo base estacionario en la superficie del álabe y de la forma modal de la vibración. Como consecuencia directa, el efecto de cambiar las condiciones de operación de diseño en las curvas de trabajo por ciclo (y, por tanto, en la estabilidad) puede ser predicho fácilmente y, lo que es más importante, cuantificado, sin realizar ningún análisis de flameo adicional.

Se concluye que, para perfiles débilmente cargados, la presión no estacionaria y los coeficientes de influencia escalan linealmente con la frecuencia reducida, mientras que su fase parte de $\pi/2$ y cambia linealmente con la frecuencia reducida. Como consecuencia, el trabajo por ciclo escala linealmente con la frecuencia reducida para cualquier valor de inter-blade phase angle, y es independiente de su signo (las curvas de trabajo por ciclo son simétricas con respecto al inter-blade phase angle cero, incluso cuando se considera una cascada asimétrica, lo cual es una conclusión sorprendente). Para perfiles altamente cargados, el módulo de la presión no estacionaria es constante, y solamente presenta una pequeña corrección con la frecuencia reducida, mientras que la fase parte de cero y varía linealmente con este parámetro. En estos casos, solamente el valor medio del trabajo por ciclo escala linealmente con la frecuencia reducida. El comportamiento predicho por la teoría para ambos casos es en primera aproximación, independiente de la geometría del perfil y de la forma modal de vibración. Para cascadas simétricas, el trabajo por ciclo escala linealmente con la frecuencia reducida, independientemente de que el perfil esté cargado o no. La consecuencia directa es que para perfiles débilmente cargados y para cascadas simétricas, no es posible cambiar la estabilidad de la configuración utilizando la frecuencia reducida.

Estas conclusiones se han verificado numéricamente en distintas geometrías de álabes. Se ha realizado una campaña de simulaciones utilizando un solver lineal de las ecuaciones de Navier-Stokes en el dominio de la frecuencia, utilizando los siguientes perfiles: un perfil correspondiente a una turbina de baja presión, los perfiles NACA65 y NACA0012, y placa plana (que se considera representativa de la parte superior de

un álabe de compresor). El objetivo final es probar que se cumplen las tendencias predichas para la amplitud y fase de la presión no estacionaria, y para las curvas de trabajo por ciclo, y mostrar la correlación entre el valor calculado del parámetro de carga no estacionario (ULP) y las características en flameo para distintos perfiles, condiciones de operación y formas modales de vibración.

Se han utilizado las formulaciones de coeficientes de influencia y de onda viajera en combinación para aumentar la comprensión de la influencia del ULP en distintos aspectos del campo fluido no estacionario. Se concluye que, para un álabe oscilando con un movimiento prescrito en condiciones de diseño, el ULP puede predecir de forma cuantitativa el efecto de variar el loading estacionario, debido a cambios en la incidencia, y el efecto de cambiar la forma modal. Se prueba también que el ULP puede ser utilizado también para comparar las características en flameo entre distintos perfiles.

La belleza del parametro de carga no estacionario derivado en la presente tesis es que es capaz de predecir el efecto en la estabilidad de cambios en la forma modal de vibración y en la distribución de carga aerodinámica (de forma cuantitativa), y en la geometría del perfil aerodinámico (de forma cualitativa), en el regimen de frecuencia reducida baja, sin realizar ninguna simulación no-estacionaria. Como consecuencia, el parámetro es muy válido desde el punto de vista ingenieril para estudios aerodinámicos conceptuales, ya que es capaz de predecir el efecto de cambios en el diseño en el damping aerodinámico.

Las implicaciones académicas tampoco son despreciables, ya que: i) aporta un soporte teórico a algunos de los conceptos fundamentales relativos a la estabilidad de perfiles vibrantes existentes hasta la fecha y ii) introduce una serie de conceptos nuevos, como la importancia de la simetría de la cascada, entre otros, a los que no se había prestado atención con anterioridad, y son de gran relevancia en las características de flameo de perfiles de turbomaquinaria.

Acknowledgments

Firstly, I would like to express my gratitude to my supervisor Dr. Roque Corral for giving me the opportunity to perform this thesis, for his technical support throughout my work, and for sharing with me his wide experience in many different aspects, not only the most technical ones. Much of my knowledge on the thesis's subject, and the success of this work stems from his motivating teaching method.

I also gratefully acknowledge ITP S.A for providing me financial support toward this research, and all the necessary logistic conditions to perform my work in a high technology environment.

Special thanks to the people of the Technology and Methods Department of ITP, they have contributed to me to get a vision of the excellence and the rigour in the work, and they have been an example of well-done things all over these years. Last but not least, lot of thanks to the people of the Aerodynamics Department of ITP for their company, support, good discussions, and all the nice coffee breaks and daily meals; they really have boosted my enthusiasm during my stay at ITP.

I would also like to thank my colleagues at the School of Aeronautics, for creating a pleasant working environment during the last year of my thesis, and for transmitting me all their dynamism and positive thoughts.

Special thanks go to my family, it is so difficult to express all my gratitude, for giving me the necessary energy to succeed in this daunting task, with their unconditional trust and infinite love.

Contents

1	Introduction	1
1.1	Context	1
1.2	Motivation	2
1.3	Turbomachinery Aeroelasticity Background	4
1.3.1	Definition	4
1.3.2	Static Aeroelasticity	5
1.3.3	Dynamic Aeroelasticity	6
1.3.4	Forced Response	6
1.3.5	Flutter	7
1.3.6	Non-synchronous Vibrations (NSV)	12
2	Turbomachinery Aeromechanics Fundamentals	13
2.1	Simplified Aeroelastic Model	13
2.1.1	Bladed-Disc Lumped Model	15
2.1.2	Purely structural problem	19
2.1.3	Aerodynamic contribution	21
2.2	Formulation of the Tuned Flutter Problem	24
2.2.1	Structural vibration modes	26
2.2.2	Aerodynamic correction	28

2.2.3	Influence Coefficient Formulation	30
2.3	Aerodynamic Damping Computational Methods	33
2.3.1	Coupled methods	33
2.3.2	Uncoupled methods	36
2.3.2.1	Discussion	36
3	Fundamentals of Unsteady Aerodynamics	39
3.1	Context	40
3.2	Planar Waves in the Euler Equations	42
3.2.1	One-dimensional Flow	42
3.2.2	Vorticity Equation	43
3.2.3	Quasi One-Dimensional Flow	44
3.2.4	Order of Magnitude Estimates	46
3.3	Unsteady Pressure Generation: The Role of Vorticity	50
3.3.1	Kelvin's circulation theorem	50
3.3.2	General Formulation of Acoustic Perturbations	52
3.4	Unsteady Pressure Generation: The Role of Apparent Mass	53
3.4.1	Incompressible flow equations	53
3.4.2	Inviscid Moving Sphere	54
3.5	Unsteady Aerodynamics of Oscillating Flat Plate Cascades	57
3.6	Laminar Boundary with an Oscillating Free Stream	59
3.7	Basic Non-dimensional Analysis of Unsteady Aerodynamics	61
3.7.1	Aerodynamic work-per-Cycle	63
3.7.1.1	Non dimensional aerodynamic work	65

4	Analytical and Numerical Models	69
4.1	Analytical Model	69
4.1.1	Governing Equations	69
4.1.2	Linearisation of the Governing Equations	71
4.1.3	Boundary Conditions	72
4.2	Numerical Model	74
4.2.1	Numerical Formulation	74
4.2.1.1	Spatial Discretisation	75
4.2.1.2	Temporal Discretisation	77
4.2.1.3	Boundary Conditions	78
5	Low Reduced Frequency Limit Analysis	81
5.1	State of the Art	81
5.2	Asymptotic Analysis at Low Reduced Frequency	82
5.2.1	Governing Equations	82
5.2.2	Linearization of the Governing Equations	84
5.2.3	Linearization of the no-penetration boundary condition	85
5.2.4	Asymptotic Analysis of the Linearized Equations	87
5.2.4.1	0th Order Approximation	88
5.2.4.2	1st Order Approximation	88
5.2.4.3	2nd Order Approximation	89
5.2.5	Estimate of the Unsteady Pressure	90
5.2.6	Summary	91
5.3	Calculus of the Unsteady Loading Parameter	92
5.3.1	Integral Unsteady Loading Parameter	92

5.3.2	Interpretation of the Unsteady Loading Parameter	93
5.3.3	Examples	95
5.4	Simplified Analytical Model for the Work-per-cycle	100
5.4.1	Loaded Airfoils	102
5.4.2	Lightly Loaded or Unloaded Airfoils	103
5.4.3	Symmetric Cascades	104
5.5	Review of Theodorsen's Theory	105
5.5.1	Non-circulatory contribution	106
5.5.2	Circulatory contribution	106
5.5.3	Conclusions of Theodorsen theory at $k \ll 1$	107
5.5.3.1	Calculus of the Lift Phase	108
5.6	Conclusions of the Analysis at Low Reduced Frequency	109
6	Numerical Verification of the Theory	113
6.1	Relevance of the Rotational Contribution	118
6.1.1	Asymmetric Flat Plate Linear Cascade	118
6.2	Steady Loading Effects	123
6.2.1	LPT case off-design effect	123
6.2.2	NACA 65 Compressor	127
6.2.2.1	Off-design effects	127
6.2.2.2	Behaviour at $k \sim O(1)$	128
6.2.3	NACA 65 Mach number effects	131
6.3	Mode shape effects	132
6.3.1	LPT Design Case	134
6.3.2	NACA65 Subsonic Case	136

6.4	Geometry Effects	137
6.5	Mach Number Effect	139
6.5.1	Phasing Variation	139
6.5.2	Work-per-cycle Model	141
6.6	Symmetric cascades	147
6.6.1	NACA 0012 Symmetric Case	147
6.6.2	NACA 0012 off-Design Case	149
6.7	Physical Interpretation	151
6.7.1	Mode-shape variation	156
6.7.1.1	Edgewise Mode	156
6.7.1.2	Torsion Mode	161
6.8	Conclusions of the Numerical Validation	164
7	Stabilisation Mechanism of Blade Pairs: Numerical and Experimental Evidence	167
7.1	Context and Problem Description	167
7.2	Experimental Setup	170
7.2.1	Test Facility	171
7.3	Numerical Validation	174
7.3.1	Problem Description	174
7.3.2	Results	177
7.3.2.1	Work-per-Cycle Computation	177
7.3.2.2	Steady State Results	177
7.3.2.3	Single Blade Case	178
7.3.2.4	Welded-Pair Case	180

7.4	Discussion of the Stabilisation Mechanism	181
7.5	Concluding Remarks	183
A	Orthogonality of Aeroelastic Modes	197

List of Figures

1.2.1 NON DIMENSIONAL WORK PER CYCLE CURVE TRENDS AS A FUNCTION OF THE REDUCED FREQUENCY FOR AN LPT (A) AND THE NACA65 CONFIGURATION (B)	4
1.3.1 COLLAR'S TRIANGLE OF FORCES	5
1.3.2 SCHEMATIC OF A CAMPBELL DIAGRAM	7
1.3.3 CLASSICAL COMPRESSOR FLUTTER MAP	8
1.3.4 PHASE LAG BETWEEN BLADE MOTION AND AERODYNAMIC FORCES . . .	9
1.3.5 MACH NUMBER CONTOUR PLOT OF A CHOKED COMPRESSOR AIRFOIL . .	12
2.1.1 SKETCH OF THE NATURAL FREQUENCIES OF A BLADED-DISC AS A FUNCTION OF THE NODAL DIAMETER	14
2.1.2 SIMPLEST LUMPED MODEL OF A BLADED-DISC	16
2.1.3 SCHEMATIC OF A LINEAR VIBRATING CASCADE	18
2.1.4 FREQUENCY CHARACTERISTICS OF THE BLADED-DISC LUMPED MODEL .	21
2.1.5 SKETCH OF THE FREQUENCY CORRECTION (LEFT) AND DAMPING (RIGHT) COMPONENTS OF THE QUASI-STATIONARY AERODYNAMIC FORCES AS A FUNCTION OF THE IBPA	23
2.3.1 LOGARITHMIC DECREMENT AS A FUNCTION OF THE NORMALISED TIME .	35
3.2.1 SKETCH OF A SLOWLY VARYING DUCT	45
3.3.1 GENERATION OF CIRCULATION BY MEANS OF VORTEX SHEDDING	51

3.4.1 CONTROL VOLUME FOR A MOVING SPHERE	55
3.5.1 SKETCH OF THE WHITEHEAD'S CASCADE	58
3.5.2 SUPERRESONANT AND SUBRESONANT REGIONS AS A FUNCTION OF THE IBPA FOR CONSTANT REDUCED FREQUENCY AND MACH NUMBER, LEFT AND RIGHT RESPECTIVELY.	59
3.6.1 VARIATION OF PHASE ANGLE BETWEEN WALL SHEAR AND OSCILLATING EXTERNAL VELOCITY FOR A LAMINAR FLOW OVER A FLAT PLATE ($\Delta V_\infty/V_\infty = 15\%$)	61
3.7.1 MOVING SURFACE OF AN OSCILLATING AIRFOIL	63
3.7.2 NON-DIMENSIONAL WORK-PER-CYCLE AS A FUNCTION OF THE IBPA FOR THE FLAP AND EDGE MODES FOR $k = 0.1$	67
4.2.1 TYPICAL HYBRID-CELL GRID AND ASSOCIATED DUAL MESH	75
4.2.2 CLOSE-UP OF THE GRID ABOUT AN LPT AIRFOIL.	76
4.2.3 SCHEMATIC SHOWING COMPUTATIONAL DOMAIN AND BOUNDARY CONDI- TIONS	79
5.2.1 POSITION OF THE AIRFOIL IN AN ARBITRARY TIME INSTANT	84
5.3.1 NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA FOR THE LPT CASE VIBRATING IN FLAP ($M = 0.74$, $k = 0.1$) FOR A VISCOUS AND AN INVISCID CASE.	94
5.3.2 SNAPSHOT OF THE UNSTEADY PRESSURE CAUSED BY THE VIBRATION OF THE CENTRAL BLADE IN AN LPT AIRFOIL FOR $k = 0.3$	96
5.3.3 DISTRIBUTION OF THE NORMAL GRADIENT OF THE VELOCITY ALONG AN LPT AIRFOIL COMPUTED USING THREE DIFFERENT APPROACHES	96
5.3.4 UNSTEADY LOADING PARAMETER DISTRIBUTION (LOADING CONTRIBU- TION, ULP_L , AND ROTATIONAL CONTRIBUTION, ULP_R) ALONG AN LPT AIRFOIL FOR DIFFERENT MODE-SHAPES.	97

5.3.5	ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD FOR THE LPT AND THE NACA65 CASES	98
5.3.6	UNSTEADY LOADING PARAMETER DISTRIBUTION	99
5.3.7	UNSTEADY LOADING PARAMETER DISTRIBUTION FOR THE NACA65 AIRFOIL. (A): $M_1 = 0.7$. (B): $M_1 = 0.8$	99
5.3.8	IMAGINARY PART OF THE UNSTEADY PRESSURE OF THE NACA65 COMPRESSOR FOR $k = 0.1$, FOR $M_{in} = 0.7$ (A) AND FOR $M_{in} = 0.8$ (B)	101
5.4.1	MEAN VALUE AND 1 st HARMONIC OF THE NON-DIMENSIONAL WORK AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE LPT CASE	103
5.5.1	SKETCH OF THE DISPLACEMENTS OF AN ISOLATED OSCILLATING PLATE .	106
5.6.1	SKETCH OF THE MAIN CONCLUSIONS	110
6.0.1	ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD (A) AND SKETCH OF THE CONFIGURATIONS (B)	116
6.1.1	(A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE FLAT PLATE CASE VIBRATING IN FLAP (FILLED SYMBOLS) AND TORSION AROUND THE L.E. (OPEN SYMBOLS)	119
6.1.2	NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA FOR AN ASYMMETRIC CASCADE OF FLAT PLATES VIBRATING IN FLAP MODE (FILLED SYMBOLS) AND TORSION ABOUT THE PLATE MIDPOINT (OPEN SYMBOLS) ($M = 0.45, \theta = 45^\circ, s/c = 0.5, k = 0.03$)	120
6.1.3	REAL (A) AND IMAGINARY (B) COMPONENTS OF THE UNSTEADY PRESSURE FOR AN ASYMMETRIC CASCADE OF FLAT PLATES VIBRATING IN TORSION ABOUT THE L.E. ($M_{in} = 0.45, \theta = 45^\circ, s/c = 0.5, k = 0.1$) . .	122
6.1.4	(A) REAL AND (B) IMAGINARY COMPONENTS OF THE UNSTEADY PRESSURE FOR AN ASYMMETRIC CASCADE OF FLAT PLATES VIBRATING IN BENDING ($M_{in} = 0.45, \theta = 45^\circ, s/c = 0.5, k = 0.1$)	122

6.2.1	ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD FOR THE LPT CASE FOR THE DESIGN AND OFF DESIGN CONDITIONS . .	124
6.2.2	(A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE OF THE CENTRAL AIRFOIL AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE LPT CASE VIBRATING IN FLAP MODE FOR THE DIFFERENT OPERATING CONDITIONS	125
6.2.3	NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA AND THE STEADY FLOW CONDITIONS FOR THE LOW PRESSURE TURBINE CASE VIBRATING IN FLAP ($M = 0.74$, $k = 0.1$)	126
6.2.4	ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD FOR THE NACA65 COMPRESSOR FOR SUBSONIC DESIGN AND OFF DESIGN CONDITIONS, AND FOR DESIGN TRANSONIC CONDITIONS	128
6.2.5	(A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 CASE VIBRATING IN FLAP MODE FOR THE $i = 0^\circ$ (FILLED SYMBOLS) AND $i = -3^\circ$ (OPEN SYMBOLS) CASES	129
6.2.6	(A) NON-DIMENSIONAL IC MODULUS OF THE 0^{th} AND $\pm 1^{st}$ AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY AND (B) NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE NACA65 COMPRESSOR ($M_{in} = 0.7$)	130
6.2.7	PHASES OF THE 0^{th} AND ± 1 AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 COMPRESSOR ($M_{in} = 0.7$)	130
6.2.8	(A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 CASE VIBRATING IN FLAP MODE. $M_1 = 0.8$: FILLED SYMBOLS AND $M_1 = 0.7$: OPEN SYMBOLS.	132
6.2.9	REAL (LEFT) AND IMAGINARY (RIGHT) PART OF THE UNSTEADY PRESSURE OF THE NACA65 COMPRESSOR FOR $k = 0.1$, FOR $M_{in} = 0.8$ (A) AND FOR $M_{in} = 0.7$ (B)	133

6.3.1	NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA FOR THE LPT CASE VIBRATING IN FLAP, EDGE AND DIFFERENT TORSION MODES ($M =$ $0.74, k = 0.1$)	135
6.3.2	(A) NON-DIMENSIONAL IC MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE LPT CASE VIBRATING IN FLAP (FILLED SYMBOLS) AND EDGE MODE (OPEN SYMBOLS).	135
6.3.3	(A) NON-DIMENSIONAL IC MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 CASE VIBRATING IN FLAP (FILLED SYMBOLS) AND EDGE MODE (OPEN SYMBOLS)	137
6.4.1	MEAN (FILLED SYMBOLS) AND MINIMUM (OPEN SYMBOLS) DAMPING AS A FUNCTION OF THE REDUCED FREQUENCY OF THE LPT AND NACA65 (SUBSONIC AND TRANSONIC) CONFIGURATIONS.	138
6.5.1	PHASE OF THE INFLUENCE COEFFICIENTS OF THE 0TH AND +1ST AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY AND MACH NUMBERS FOR A LPT AIRFOIL.	140
6.5.2	MODULUS OF THE FORCE INFLUENCE COEFFICIENTS AS A FUNCTION OF THE REDUCED FREQUENCY FOR $M_2 = 0.75$	141
6.5.3	RATIO BETWEEN THE MODULUS OF INFLUENCE COEFFICIENT 0 AND +1 AS A FUNCTION OF THE REDUCED FREQUENCY AND MACH NUMBER . . .	142
6.5.4	VARIATION OF THE INFLUENCE COEFFICIENT OF THE CENTRAL BLADE WITH THE SQUARE OF THE MACH NUMBER AND THE REDUCED FREQUENCY	143
6.5.5	NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE MACH NUMBER	145
6.5.6	MEAN VALUE AND VARIATION WITH THE IBPA OF THE NON-DIMENSIONAL WORK AS A FUNCTION OF THE MACH NUMBER	145
6.5.7	ISENTROPIC MACH NUMBER DISTRIBUTION NORMALIZED WITH THE EXIT MACH NUMBER FOR DIFFERENT EXIT MACH NUMBERS.	146

6.5.8 EFFECT OF THE REDUCED FREQUENCY AND THE M ONTO THE WORK- PER-CYCLE CURVES	147
6.6.1 SNAPSHOT OF THE UNSTEADY PRESSURE FOR THE NACA0012 SYMMET- RIC CASE FOR $k = 0.2$	148
6.6.2 (A) NON-DIMENSIONAL IC MODULUS AND (B) PHASE OF THE 0TH AND ± 1 ST AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA 0012 SYMMETRIC (FILLED SYMBOLS) AND ASYMMETRIC (OPEN SYMBOLS) CASES.	149
6.6.3 NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE NACA 0012 SYMMETRIC (A) AND ASYMMETRIC (B) CASES.	150
6.6.4 SNAPSHOT OF THE UNSTEADY PRESSURE FOR THE NACA0012 ASYM- METRIC CASE FOR $k = 0.2$	151
6.7.1 SNAPSHOT OF THE UNSTEADY PRESSURE CAUSE BY THE VIBRATION OF THE CENTRAL BLADE IN A LPT AIRFOIL FOR $k = 0.3$	152
6.7.2 ISENTROPIC MACH NUMBER AND UNSTEADY PRESSURE MODULUS DISTRI- BUTIONS ALONG BLADE SURFACE FOR THE: (A) 0TH AIRFOIL AND (B) +1ST AIRFOIL	153
6.7.3 PHASE DISTRIBUTION ALONG THE 0-TH (LEFT) AND +1-TH (RIGHT) AIRFOILS FOR DIFFERENT REDUCED FREQUENCIES. SOLID LINE: SUCTION SIDE. DASHED LINE: PRESSURE SIDE	155
6.7.4 MACH NUMBER ISO-CONTOURS AND STREAMLINES	156
6.7.5 NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE EDGE MODE	157
6.7.6 RATIO BETWEEN THE MODULUS OF INFLUENCE COEFFICIENT -1,0 AND +1 AS A FUNCTION OF THE REDUCED FREQUENCY AND MACH NUMBER. FYLLER SYMBOLS: AIRFOIL -1, EMPTY SYMBOLS: AIRFOIL 0	158

6.7.7 MEAN VALUE AND 1 st HARMONIC (VARIATION WITH THE IBPA) OF THE NON-DIMENSIONAL WORK-PER-CYCLE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE EDGE MODE	158
6.7.8 PHASES OF THE ZERO th AND 1 st AIRFOILS (FILLED SYMBOLS), AND -1 AIRFOIL (SOLID SYMBOLS) AS A FUNCTION OF k AND M	159
6.7.9 NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE MACH NUMBER FOR THE EDGEWISE MODE.	160
6.7.10 NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE TORSION MODE.	161
6.7.11 NON-DIMENSIONAL MOMENT INFLUENCE COEFFICIENTS OF THE TORSION MODE AT TWO DIFFERENT REDUCED FREQUENCIES AND $M_2 = 0.75$	162
6.7.12 MODULUS OF THE UNSTEADY PRESSURE DISTRIBUTION ALONG THE AIRFOILS OF A LINEAR CASCADE COMPOSED OF NINE AIRFOILS ($M_2 =$ 0.75).	163
6.7.13 PHASES OF THE INFLUENCE COEFFICIENTS OF THE TORSION MODE FOR DIFFERENT MACH NUMBERS.	163
7.1.1 LAYOUT OF DIFFERENT BLADED-DISC ASSEMBLY CONFIGURATIONS . . .	168
7.1.2 1 st FLAP MODE-SHAPE OF A WELDED-IN-PAIR ROTOR BLADE CONFIGUR- ATION	170
7.2.1 SCHEMATIC VIEW OF THE NON ROTATING ANNULAR TEST FACILITY (A) AND TURBINE ROTOR CASCADE INSTRUMENTATION, VIEW FROM DOWNSTREAM SIDE (B)	171
7.2.2 BLADE SUSPENSION SYSTEM (LEFT) AND PROBE TRAVERSE LOCATIONS (RIGHT)	173
7.3.1 BENDING DIRECTION AND TORSION AXIS	175
7.3.2 ISENTROPIC MACH NUMBER DISTRIBUTION FOR DIFFERENT SPAN HEIGHTS	178

7.3.3 SURFACE STREAMLINES AND ISOCONTOURS OF THE UNSTEADY PRESSURE MODULUS (FLOW MOVES FROM LEFT TO RIGHT) OF THE SUCTION (LEFT) AND PRESSURE (RIGHT) SIDES FOR THE BENDING MODE AND IBPA=90°	179
7.3.4 COMPARISON OF EXPERIMENTAL AND ANALYTICAL RESULTS OF THE NON- DIMENSIONAL WORK PER CYCLE AS A FUNCTION OF THE IBPA FOR: (A) BENDING MODE AND (B) TORSION MODE , FOR SINGLE BLADE (∇), AND BLADE-PAIR (Δ) CONFIGURATIONS, AT $k = 0.7$	179
7.3.5 EXPERIMENTAL (FILLED SYMBOLS) AND NUMERICAL (OPAQUE SYMBOLS) UNSTEADY PRESSURE MODULUS DISTRIBUTION ALONG THE BLADE CHORD FOR SINGLE BLADE AND BLADE PAIR PACKAGE CONFIGURATIONS FOR IBPA=-108° AND ISPA=-108, RESPECTIVELY AT $k = 0.7$ FOR A BENDING MOTION	180
7.3.6 UNSTEADY PRESSURE MODULUS OBTAINED FROM THE SIMULATIONS FOR THE TORSION MODE AT $k = 0.7$. IBPA/ISPA=180°. (A) SINGLE BLADE CONFIGURATION AND (B) PAIR CONFIGURATION	181
7.4.1 NON-DIMENSIONAL WORK PER CYCLE AS A FUNCTION OF THE IBPA FOR SINGLE BLADE AND BLADE PAIR PACKAGE CONFIGURATIONS FOR: (A) BENDING AND (B) TORSION MODE, AT $k = 0.1$.	183

List of Tables

5.1	CIRCULANT AND NON-CIRCULANT CONTRIBUTIONS TO THE NON-DIMENSIONAL LIFT LIFT	108
6.1	UNSTEADY LOADING PARAMETER VALUE FOR THE LPT CONFIGURATION FOR DIFFERENT INCIDENCE ANGLES	124
6.2	UNSTEADY LOADING PARAMETER FOR THE NACA65 CONFIGURATION FOR DIFFERENT INCIDENCE ANGLES AT $M_1=0.7$	128
6.3	UNSTEADY LOADING PARAMETER FOR THE NACA65 CONFIGURATION FOR DIFFERENT INLET MACH NUMBER	131
6.4	UNSTEADY LOADING PARAMETER FOR THE LPT CONFIGURATION FOR THE TWO FUNDAMENTAL BENDING DIRECTIONS AND DIFFERENT TORSION CENTRES	134
6.5	UNSTEADY LOADING PARAMETER FOR THE NACA65 CONFIGURATION FOR THE TWO FUNDAMENTAL BENDING DIRECTIONS	137
6.6	UNSTEADY LOADING PARAMETER AND SLOPE OF THE $\phi - k$ CURVES FOR THE LPT AND THE NACA65 CONFIGURATIONS FOR THE TWO FUNDAMENTAL BENDING DIRECTIONS	138
6.7	MACH NUMBER EFFECT IN THE PHASES OF THE INFLUENCE COEFFICIENTS OF THE AIRFOILS. V: VORTEX SHEDDING DOMINATED. A: ACOUSTICALLY INFLUENCED.	166
7.1	UPSTREAM AND DOWNSTREAM TEST CONDITIONS AT 50% SPAN	176

Nomenclature

Latin symbols

a	Sound speed
c	Axial chord
c_p	$\frac{p'}{p_0 \frac{\delta}{c}}$
f	Blade excitation frequency (Hz)
F	Convective fluxes
G, H	Viscous fluxes
h	Amplitude of blade translations
H	Blade height
i	Incidence
$k = \frac{\omega c}{v_c}$	Reduced frequency
k_b	Blade stiffness
k_d	Disc stiffness
K	Stiffness matrix
l	Influence coefficient modulus
L_{aero}	Matrix of aerodynamic forces
L	Influence coefficients matrix
M	Mach number
M	Mass matrix
M_d	Disc mass

m_b	Blade mass
n	Nodal diameter
N	Number of blades
p_{dyn}	Dynamic pressure
p'	Unsteady pressure
Re	Reynolds number
s	Pitch
S	Blade surface
t	Time
T	Period of oscillation
U	Vector of conservative variables
u, v, w	Velocity components in cartesian coordinates
v_c	Characteristic velocity
V_∞	Free stream velocity
$\tilde{\mathbf{W}}$	Non-dimensional velocity vector
$\tilde{\mathbf{W}}_0$	Non-dimensional mean velocity
$\tilde{\mathbf{w}}(\tilde{\mathbf{x}}, \tau)$	Non-dimensional perturbed velocity
W_{cycle}	Work-per-cycle per span unit
x	Chordwise direction

Greek Symbols

α	Torsion amplitude (rad)
$\alpha_{1,2}$	Inlet/outlet flow angles
$\nabla \tilde{\mathbf{W}}_0$	Velocity gradient in the wall normal direction
δ	Vector of the airfoil vibration amplitude
μ	Mass ratio

$\mu_d = M_d/m_b$	Disc to rotor blade mass ratio
$\kappa = k_{disc}/k_{blade}$	Disc to blade stiffness ratio
ρ_a	Characteristic density of the air
σ	Inter-blade phase angle
Σ	Computational domain boundary
$\tau = \omega t$	Non-dimensional time
$\tau_{\xi\eta}$	Viscous shear stresses
Ω	Control volume
Υ	Unsteady loading parameter
Γ	Vorticity
ξ	x-direction
η	y,z-directions
ω	Oscillation frequency (rad/s)

Abbreviations

<i>ADP</i>	Aerodynamic Design Point
EO	Engine order
HEO	High engine order
h.o.t	Higher-order terms
HPT	High pressure turbine
IBPA	Inter blade phase angle
IC	Influence coefficient
ISPA	Inter sector phase angle
l.e.	Leading edge
LEO	Low engine order
LPT	Low Pressure Turbine

NSV	Non-synchronous vibration
PS	Pressure side
RANS	Reynolds averaged Navier Stokes
SS	Suction side
r.h.s	Rigth hand side
t.e	Trailing edge
TW	Traveling wave
ULP	Unsteady loading parameter
URANS	Unsteady Reynolds averaged Navier Stokes

Subscripts

c	Characteristic
cyc	Per-cycle
in	Inlet
I	Imaginary part of a complex number
is	Isentropic
0	Base flow
out	Outlet
t	Total
R	Real part of a complex number

Superscripts

'	Unsteady value
^	Complex number
~	Non-dimensional value

Chapter 1

Introduction

Start by doing what's necessary;
then do what's possible; and
suddenly you are doing the
impossible

Francis of Assisi

Contents

1.1	Context	1
1.2	Motivation	2
1.3	Turbomachinery Aeroelasticity Background	4

1.1 Context

The unsteady pressure field caused by airfoil vibration plays a major role in the flutter and forced response of turbomachinery blade rows. Flutter is a self excited aeromechanic instability by which energy is extracted from the main stream and transferred to the structure. When the energy dissipated by the structure is smaller than that taken out from the flow, the vibration amplitude grows, leading eventually

to either an instantaneous or a high cycle fatigue (HCF) failure of the structure, depending on the mechanical damping level. On the other hand, blades and vanes undergo periodic excitations due to the relative motion of neighboring blades. The vibration caused by these excitations may also provoke an HCF failure if the resulting vibration amplitude is large enough. It is well known that the largest response of the structure takes place at resonant conditions, when the frequency of the excitation matches one of the natural frequencies of the system. Under these conditions the vibration amplitude is directly proportional to the excitation force and inversely proportional to the overall damping, of which a significant part may be due to the air stream in which the airfoil is immersed. In both phenomena, the unsteady pressure field induced by the airfoil oscillation and its subsequent interaction with the structure are essential in determining the aerodynamic damping and, therefore, the life of the component.

1.2 Motivation

Aerodynamics of oscillating airfoils is equally important for both flutter and forced response. However the range of reduced frequencies, k , i.e., the ratio between the flow-through or residence time of the fluid, $t_r = c/v_c$, and the inverse the angular frequency of the vibration, ω , may be quite different. Whereas flutter typically occurs at the lowest modes of the assembly, forced response is usually of interest at higher modes, since resonant crossings with the first modes are avoided whenever possible by design; an exception of this case being the analysis of crossings at large off-design conditions. As a consequence, the reduced frequency of a flutter analysis is, generally speaking, much lower than that of a forced response study.

Flutter primarily affects high-aspect ratio rotor blades found in the fan, fore high-pressure compressor stages, and aft low-pressure turbine stages, because low natural frequencies and high axial velocities give rise to low reduced frequencies. Aeronautical low-pressure turbines (LPTs) are made of very slender and thin airfoils because their weight and cost have a large impact on the engine (about 20% of the total

weight and 15% of the total cost). Natural frequencies of LPT rotor blades are very low exhibiting very low reduced frequencies as well ($k \sim 0.1 - 0.2$) and, as a consequence, LPT bladed-discs are prone to flutter. Nowadays flutter may become a dominant constraint on the design of modern LPTs precluding the use of more efficient aerodynamic configurations [1, 2]. Compressor airfoils have a much lower aspect ratio than that of the LPTs, and usually the reduced frequency of their lowest modes is of order unity ($k \sim 1.0$). However, it is still interesting to study their behavior in the $k \ll 1$ limit, as a means to increase the overall understanding of vibrating airfoils at low frequency, and in particular the sensitivity to the airfoil shape.

The unsteady aerodynamics associated with the vibration of turbine and compressor bladed-discs and stator vanes is nowadays routinely analyzed within the design loop of the aeroengine companies, and it has been the subject of dedicated experiments in the past [3, 4, 5]. The aim of these simulations is the derivation of the aerodynamic stability of the rotor blades and the quantification of the aerodynamic damping, which is the result of the application of the unsteady pressures on the airfoil displacements. With a few exceptions in the early days of aeromechanics [6, 7], little attention has been paid to the understanding of vibrating airfoil aerodynamics since this is not a figure of merit in itself for the aeroelastic analyst. Designers obtain and use aerodynamic data from efficient numerical tools, but they do not usually examine critically the resulting unsteady flow-field, since they are really just interested in the global aerodynamic damping. As a direct consequence of using aerodynamic codes as black boxes, little understanding has been gained in recent years about the physics of these type of flows and its impact on the aerodynamic damping. Although there are several trends that are well known, such as the stabilization with the reduced frequency (see Fig. 1.2.1), the physics which is behind this behavior is not really well understood and the statement lacks of a sound theoretical support. Another example is the shape of the damping curves as a function of the inter-blade phase angle (IBPA) of LPTs and compressors, that look quite different without a clear rational of which is the underlying reason for this (see Fig. 1.2.1). On the other hand, vibrating cascade experiments are scarce, and performed well apart in

Figure 1.2.1: NON DIMENSIONAL WORK PER CYCLE CURVE TRENDS AS A FUNCTION OF THE REDUCED FREQUENCY FOR AN LPT (A) AND THE NACA65 CONFIGURATION (B)

time in different institutions [8, 9, 10, 11, 12] with different techniques, parameter range, etc., and therefore, are not well suited for conceptual and systematic studies. The main objective of this thesis is hence to study the unsteady aerodynamics of oscillating airfoils in the low reduced frequency regime, with special emphasis on finding a theoretical support which allows us to predict the behaviour of the work-per-cycle curves, understand the physics of the problem and predict the work-per-cycle trends as a function of different physical parameters, even before carrying out the actual numerical simulations. However, although the topic is strongly related with turbomachinery flutter, and many conclusions can be directly drawn from the results, this thesis is not believed to address the whole flutter problem, since relevant mechanical issues such as mistuning and friction are not treated at all in this work.

1.3 Turbomachinery Aeroelasticity Background

1.3.1 Definition

Aeroelastic problems has been around since the early days of turbomachinery design and they have been object of study for many years.

The science that studies physical phenomena related with the interaction between aerodynamic, elastic and inertial forces is called aeroelasticity, and it is of a highly interdisciplinary character. Collar's triangle (see [13]), shown in Fig. 1.3.1, and which

Figure 1.3.1: COLLAR'S TRIANGLE OF FORCES

is formed by putting in each of its vertex the forces contributing to the aeroelastic problem, illustrates this interrelation. From this triangle, it can be shown the different types of problems studied in aeroelasticity as a function of the most relevant forces in that problem. The interactions among the three types of forces, located in the vertexes of the triangle in Figure 1.3.1, give rise to four disciplines: classical vibration, which is concerned with the interaction of inertial and elastic forces, stability and control, which are affected by the interaction of aerodynamic and inertial forces, static aeroelasticity, which describes the interaction of aerodynamic and elastic forces and finally dynamic aeroelasticity, which is the discipline which deals with the interaction of all the three forces. In turbomachinery both static and dynamic aeroelasticity are a major cause of concern since blade rows are subjected to high centrifugal loads and unsteady aerodynamic forces.

1.3.2 Static Aeroelasticity

Static aeroelasticity describes the interaction of aerodynamic and elastic forces. It is therefore a vibration-free phenomenon and refers to the deformation of a structure under steady aerodynamic loads. Turbomachinery blade-rows are subject to large variations of rotational speeds and flow conditions (aircraft engines must endure take off, acceleration, cruise, descent, and landing conditions during their flight

envelope). Flow conditions and rotational speed dictate the gas and centrifugal loads. As the centrifugal loads change, so does the stiffness of the blades, which deform elastically from their initial manufactured (or cold) shapes to their running shapes. This deformation can be significant and its prediction crucial to stability. Static turbomachinery aeroelasticity is related to the study of such deformations. The determination of the final shape of the rotor blade can be of paramount importance for fans operating at transonic conditions, where minor modifications in the final position of the blade can lead to significant variations in its aerodynamics.

1.3.3 Dynamic Aeroelasticity

Dynamic aeroelasticity is the study of problems caused by the interaction of unsteady fluids with blade vibration. The main aeroelastic phenomena of interest are: forced response, flutter and non-synchronous vibrations (NSV). Both, flutter and forced response are a potential source of blade failure in turbomachines and, sometimes, it is difficult to distinguish between them and to know the exact reason for failure.

1.3.4 Forced Response

Forced response is a synchronous problem which occurs when the frequency of an unsteady aerodynamic load, that is the result of the rotor's motion through a non-uniform flow, coincides with one of the bladed-disc natural frequencies. The excitation frequency is a multiple of the so-called engine order, EO , i.e.: $f = EO \Omega$. It can be distinguished between two types of forced response: classical forced response (high engine order), which is caused by the rotation of a rotor through the wake or potential field of an upstream or downstream stator, and low engine order (LEO) forced response caused by the rotation through a long wave-length non-uniformity (much greater than the pitch of the stator vanes) created, for example, by gusts, aerodynamically mistuned assemblies or staggered blades. The characterization of this type of forcing remains still today as a major challenge [14].

Figure 1.3.2: SCHEMATIC OF A CAMPBELL DIAGRAM

Campbell's diagram (Fig. 1.3.2) represents the bladed rotor eigenfrequencies versus the rotational speed and it is an useful and widely used aeroelastic tool to predict potential forced response mechanisms, at the crossings of the excitation sources (EO) with the natural frequencies. In the same figure, the points corresponding to flutter are marked up in blue. The latter phenomenon is described in the following lines.

1.3.5 Flutter

Flutter is a 'sustained oscillation due to the interaction between aerodynamic forces, elastic response and inertia forces' [15], but unlike forced response, flutter is a self excited vibration, and it does not require the presence of any unsteadiness coming from the interaction with upstream or downstream rows. Technically speaking flutter is an aero-mechanic dynamic instability. As flutter occurrences are not bound to engine order lines, and therefore can occur at any rotational speed, the predictive assessment of flutter can not be done recurring to the Campbell's diagram.

The vibration occurs at the natural modes and frequencies of the whole aeroelastic problem, but the aerodynamic forces resulting from the oscillations could be large enough to alter the vibration characteristics of the purely structural problem. The perturbation of the non-dimensional frequency being of the order of the critical damping ratio, if several modes are closely spaced in frequency, the aerodynamics can significantly modify the mode-shapes, and the structure exhibits flutter in a

Figure 1.3.3: CLASSICAL COMPRESSOR FLUTTER MAP

coupled mode (which is different, in mode-shape and frequency, from the original natural modes). The extent to which purely structural natural modes are affected by aerodynamic forces depends on its mass ratio (the ratio of the mass of the rotor blade to the mass of the surrounding fluid). Wings, for example, are light weight structures which typically flutter in a coupled bending-torsion mode whereas traditional turbomachinery blades predominantly flutter in a single mode shape close to their in-vacuo frequency. However, coalescence (coupled bending-torsion) flutter has also been observed on modern low mass ratio, low solidity fans [16], and turbine sectored vanes. [17].

A few major differences separate aeroelastic phenomena on turbomachinery-blades from the ones for an aircraft wing: the construction of the rotor blade, the cyclic symmetry of the structure, and the aerodynamic coupling effects between neighbouring blades. Wing flutter is driven by a ‘mode coalescence’ phenomena whereas turbomachine flutter is mostly driven by the blade-to-blade coupling. Neighboring blades can perform vibrations with a certain inter-blade phase difference, which considerably influences the time-dependent aerodynamic response. Another main difference is that the coupling between aerodynamics and structural dynamics in turbomachinery flutter problems is relatively weak compared to the aeroelasticity of isolated wings.

Flutter is an important subject in the design phase of fans, first compressor stages

Figure 1.3.4: PHASE LAG BETWEEN BLADE MOTION AND AERODYNAMIC FORCES

and aft turbines stages, where the blades are slender. Four main categories of flutter are encountered in turbomachinery, namely: classical flutter, stall flutter, acoustic flutter, and choke flutter. Figure 1.3.3 represents a sketch of a compressor map and the regions where different types of flutter may occur. Flutter in turbines is only of concern at Maximum Take-Off (MTO) and a similar sketch does not exist.

Classical Flutter

Flutter is named classical when the flow remains attached to the blade with no separation along the whole vibration cycle, and the phenomenon is essentially inviscid. Flutter instability is due to the phase between the aerodynamic forces acting on the airfoil and the airfoil displacements. Depending on whether the force is lagging or leading the displacement the flow feeds in or absorbs energy from the blade, and the motion is amplified or damped respectively. This fact is sketched in Fig. 1.3.4. It can be said that flutter is all about phasing, the only difference among the different types is the origin of the phasing. In classical flutter the lag between the blade motion and the pressure is due to vortical and acoustic waves, and the interaction with the neighbouring airfoils. This type of flutter is the one which is closely with the subject of this thesis.

Stall Flutter

Stall flutter is an aeroelastic instability affecting the blades of fans and compressors that occurs near the stall line. Typically, stall flutter occurs at part speed, and at high mean flow incidence angles and/or low reduced frequencies. As the mass flow decreases and the pressure ratio increases (near the stall boundary), the incidence of the blade increases and the flow eventually stalls. For stall flutter, flow separation is the key driver because: i) it creates significant unsteadiness behind the shock and ii) it changes the phase of the unsteady pressure so that it is out-of-phase with the blade motion. Frequently, the mode of blade vibration, in large fans, is the first bending mode, but flutter may occur in torsion as well; though other compressor and low-pressure turbine blades may also suffer from such instabilities. In the main, flutter is observed in one to six forward traveling nodal diameter assembly modes [18]. Usually stall flutter correlations express maximum incidence as a function of the reduced frequency [19], but these correlations heavily rely on a specific type of blade and base flow [20, 21, 22].

Near design speed, a fan operating at supersonic tip Mach numbers can exhibit supersonic stalled flutter at high pressure ratios. It is believed that this type of flutter is caused by movement of a strong passage shock and boundary layer separation [23]. On the other hand, near design speed, a fan operating at supersonic tip Mach numbers at low pressure ratios can exhibit supersonic unstalled flutter. In this case, the phenomenon is caused by the motion of a impinging shock originating from the neighbouring blade and is therefore associated with an inter-blade phase angle [24].

Acoustic Flutter

Acoustic flutter is encountered when acoustic waves, generated by the blade vibration, are reflected back onto the vibrating blade and feed the vibration. For this to happen, the temporal frequency of the acoustic wave must be equal to one of the natural frequencies of the blade, and must correspond to a resonant (or cut-on) mode of the annulus. For example, some acoustic flutter are known to occur due to the

interaction between the fan blades and the engine intake. Acoustic flutter may occur on its own since the wave reflection mechanism is independent of flow separation. However, acoustic flutter is unlikely to be a problem on its own since blade-only damping is usually quite high when there is no flow separation and there will also be some inherent mechanical damping in the system. Nevertheless, acoustic flutter can exacerbate the stall flutter very significantly. It is important to say that the use of blade frequency, say increase the reduced frequency by stiffening the blade, for stability control will not necessarily alleviate the acoustic flutter, since damping is not a monotonically increasing function of blade frequency [25, 18].

This type of flutter can occur even when the slope of the pressure rise characteristic is still negative, since stall is not the driven mechanism, and is referred to as “flutter-bite” sometimes. It happens in low nodal diameter forward cut-on traveling waves, frequently in first flap (1F) mode of blade vibration.

Choke Flutter

Compressors operate in a wide range of pressure ratios and shaft speeds, that give rise to quite different base flow-fields. A particular case is that in which there is a shock wave inside the compressor passage (see Fig. 1.3.5). The shock-wave extends from the trailing-edge of the suction side to the pressure side. In this case, the instability is caused by the shock wave oscillation between neighbouring airfoils caused by the airfoil vibration that modifies the throat area. The instability occurs at large reduced frequencies ($k = 0.3\text{--}0.4$) and the sensitivity to this parameter is low. Choke flutter is less common than other types of compressor flutter and it is usually experienced by middle and rear compressor stages. The numerical prediction of the instability is difficult (see [26]) since this is the result of the compensation of two large quantities: the oscillation of the shock-wave causes large unsteady perturbations in its vicinity, both in the pressure and suction sides. The pressure side is stabilizing while the suction side is destabilizing. The global result depends on the subtraction of both quantities.

Figure 1.3.5: MACH NUMBER CONTOUR PLOT OF A CHOKED COMPRESSOR AIRFOIL

The problem is highly non-linear, since the perturbations caused by the oscillation of the shock-wave are large. In this case, it has been shown that the stability of the system depends on the vibration amplitude. For small amplitudes the system may be unstable, but for large vibration amplitudes the non-linearity may lead to a stable situation [26].

1.3.6 Non-synchronous Vibrations (NSV)

Non-Synchronous Vibration (NSV), is the interaction of an aerodynamic instability (like leading-edge vortex shedding or rotating stall) with natural turbomachinery blade vibrations. NSV can occur in the fan, compressor, and turbine stages of the turbomachine, though it is most often found in compressors. NSV has been shown to be very sensitive to parameters like tip clearance shape and size, and back-pressure [27]. It also occurs at many different pressure ratios and mass flow conditions, and at non-integral multiples of the shaft rotational frequencies, [28]. The blade vibrations due to NSV are generally frequency and phase locked. Hence, NSV occurs at one dominant frequency and one inter-blade phase angle.

Chapter 2

Turbomachinery Aeromechanics Fundamentals

Everything should be made as
simple as possible, but not simpler

Albert Einstein

Contents

2.1	Simplified Aeroelastic Model	13
2.2	Formulation of the Tuned Flutter Problem	24
2.3	Aerodynamic Damping Computational Methods	33

2.1 Simplified Aeroelastic Model

Nowadays, linear bladed-disc dynamics is computed routinely by gas turbine manufacturers using finite element methods. Complex blade shapes and discs are discretised using automatic grid generators, the governing equations are written in discrete form, and the eigenvalue problem of the resulting set of ordinary differential equations is solved, to obtain the natural frequencies and mode-shapes of the assembly.

Figure 2.1.1: SKETCH OF THE NATURAL FREQUENCIES OF A BLADED-DISC AS A FUNCTION OF THE NODAL DIAMETER

Rotor blades are coupled with their neighbours via the structural connections that inevitably exists either through the discs, the casings in the case of stator vanes, or through the shroud rings, that are included to stiffen the assembly. The structural coupling among different rotor blades significantly influences the natural frequencies and mode-shapes of the assembly, and need to be retained in the simulations. Analysis including just a single rotor blade must be regarded as a first step in the simulation process. Perfectly tuned bladed-discs exhibit cyclic symmetry, and then the whole wheel may be splitted in N identical sectors, also containing several airfoils, as it is always the case for sectorized vanes, and impose cyclic boundary conditions in the periodic boundaries to reduce the computational domain. Solving a single sector using cyclic symmetry boundary conditions considerably reduces the computational time required to solve the problem.

Discs, or any continuous structure with cyclic-symmetry vibrate in double nodal diameters. Mode-shapes with more nodal diameters have higher natural frequencies since the mode is stiffer. As the excitation frequency increases the disc may exhibit nodal circles which are points at constant radius with zero displacement.

Bladed-Discs are constructed fixing rotor blades to a disc, either using fir-tree or dove tail attachments, or welding the rotor blades to the disc, or even machining integral

parts with discs and blades, as it is usually the case in small compressors and fans (blisks).

Figure 2.1.1 displays the frequency versus the nodal diameter of a generic bladed-disc. Horizontal lines represents the frequency that would be obtained by clamping the rotor blade at the attachment, that does not depend on the nodal diameter, since the disc is not included in the representation. The parabola-like curve represents the natural frequencies of disc dominated modes, that steeply increase with the nodal diameter.

A very short review of the underlying concepts of bladed-disc dynamics resorting to a simplified lumped model that contains all the key ingredients is presented in the next sub-sections. Aerodynamic forces are also included in the mass-springs model, to show their impact on the solution and the differences with the purely mechanical problem. These are essential to explain the basic concepts and characteristics of the aeroelastic problem.

2.1.1 Bladed-Disc Lumped Model

A tuned bladed-disc formed by a disc and a set of rotor blades attached to it can be represented from a conceptual point of view (see Fig. 2.1.2) as a ring on N identical masses, M_d , joined by a spring of stiffness k_d , that represents the disc, and N identical masses, m_b , that represent the rotor blades, linked to the corresponding disc sector through a spring of stiffness k_b . The aerodynamic forces can be written in terms of the influence coefficients as:

$$f_j^{aero} = \sum_{i=-1}^{i=1} \left(l_i y_{j+i} + l_i^* \frac{dy_{j+i}}{dt} \right) \quad (2.1.1)$$

where l_i is the force influence coefficient of the blade i^{th} acting on the blade 0^{th} . Here only the effect of the immediately neighbouring airfoils has taken into account for the sake of simplicity, but this fact does not qualitatively change the problem. Actually, this is the most typical situation. The influence coefficients will be defined and properly explained later on, at this point they are just introduced in order to

Figure 2.1.2: SIMPLEST LUMPED MODEL OF A BLADED-DISC

formulate the aeroelastic problem.

If only the 0^{th} term of the above expression is included, the force is always in phase with the displacement and there is no damping. In general, a second term, that accounts for the effect of the velocity, l_i^* , is included to better represent the physics of the problem at high frequencies. It is important to highlight that this is a linear model of the aerodynamic forces, since these are always proportional to the amplitude of the displacements of the neighbouring blades. The discussion of the validity of this approach is postponed to the next chapters, so far it is enough to know that l_i and l_i^* are given constant coefficients. The governing equations of the simplified bladed-disc, including the aerodynamic contribution, which in turn constitute the simplest aeroelastic problem, are then:

$$\begin{aligned} M_d \frac{d^2 x_j}{dt^2} &= +k_{blade}(y_j - x_j) + k_{disc}(x_{j+1} - x_j) - k_{disc}(x_j - x_{j-1}) \\ m_b \frac{d^2 y_j}{dt^2} &= -k_{blade}(y_j - x_j) + \sum_{i=-1}^{i=+1} \left(l_i y_{j+i} + l_i^* \frac{dy_{j+i}}{dt} \right) \end{aligned} \quad (2.1.2)$$

That in non-dimensional form becomes:

$$\begin{aligned} \mu_d \frac{d^2 x_j}{d\tau^2} &= +(y_j - x_j) + \kappa(x_{j+1} - 2x_j + x_{j-1}) \\ \frac{d^2 y_j}{d\tau^2} &= -(y_j - x_j) + \sum_{i=-1}^{i=+1} \left(\bar{l}_i y_{j+i} + \bar{l}_i^* \frac{dy_{j+i}}{d\tau} \right) \end{aligned} \quad (2.1.3)$$

where $\mu_d = M_d/m_b$ is the disc to rotor blade mass ratio, $\kappa = k_{disc}/k_{blade}$ is the disc to blade stiffness ratio, $\bar{l}_i = \frac{l_i}{m_b \omega_b^2}$ and $\bar{l}_i^* = \frac{l_i^*}{m_b \omega_b}$ are the non-dimensional influence coefficients, and $\tau = \omega_b t$ the non-dimensional time, where $\omega_b^2 = k_{blade}/m_b$ is the natural frequency of the mass representing the rotor blade clamped to a infinitely stiff disc.

Looking for solutions in travelling-wave form:

$$x_j = \hat{x}e^{i(n\theta_j + \tilde{\omega}\tau)}, \quad y_j = \hat{y}e^{i(n\theta_j + \tilde{\omega}\tau)} \quad (2.1.4)$$

where $\theta_j = j\frac{2\pi}{N}$ and $n = 0, \dots, \pm N/2$ if N is even. The non-dimensional angular frequency, $\tilde{\omega}$, is defined as $\tilde{\omega} = \omega/\omega_b$. Introducing the previous expressions in the governing equations, the following eigenvalue problem is obtained:

$$\begin{pmatrix} 1 - \mu_d \tilde{\omega}^2 + 4\kappa \sin^2 \frac{\sigma}{2} & -1 \\ -1 & 1 - \tilde{\omega}^2 - \hat{f}_{aero} \end{pmatrix} \begin{Bmatrix} \hat{x} \\ \hat{y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2.1.5)$$

where

$$\sigma = 2\pi \frac{n}{N} \quad (2.1.6)$$

represents the temporal phase lag between the motion of an airfoil and its neighbour, which is usually referred to as the inter-blade phase angle (IBPA), and the non-dimensional aerodynamic force coefficient, $\hat{f}_{aero}$, is :

$$\hat{f}_{aero} = \bar{l}_0 + i\tilde{\omega}\bar{l}_0^* + (\Delta\bar{l}^+ + i\tilde{\omega}\Delta\bar{l}^{*+}) \cos \sigma + i(\Delta\bar{l}^- + i\tilde{\omega}\Delta\bar{l}^{*-}) \sin \sigma \quad (2.1.7)$$

where $\Delta\bar{l}^+ = \bar{l}_{+1} + \bar{l}_{-1}$, $\Delta\bar{l}^- = \bar{l}_{+1} - \bar{l}_{-1}$, $\Delta\bar{l}^{*+} = \bar{l}_{+1}^* + \bar{l}_{-1}^*$ and $\Delta\bar{l}^{*-} = \bar{l}_{+1}^* - \bar{l}_{-1}^*$.

Inter-Blade Phase Angle Concept

The IBPA is defined as the temporal shift between the motion of a blade and the motion of its neighbour, and is one of the most fundamental concepts in aeroelasticity of turbomachines. Assuming that all the airfoils from the cascade are identical, separated by the same pitch, vibrating with the same common angular frequency, with a constant phase shift between adjacent blades (i.e.: in travelling-wave form), the IBPA can be written as a function of the number of diameters at which the airfoils have zero displacement, n , (also called nodal diameters) and the number of blades in the current blade-row, N , as $\sigma = 2\pi n/N$.

In the case of a rotor blade, when the vibration is propagating in the same direction as the rotor, it is said that the vibration is travelling forward, and the nodal diameter number is positive. On the other hand, when the vibration mode is rotating in

Figure 2.1.3: SCHEMATIC OF A LINEAR VIBRATING CASCADE

the direction opposite to the rotor, it is said that the vibration mode is travelling backwards, and the nodal diameter number is negative.

Influence Coefficient Concept

Let us consider a linear or annular cascade of N airfoils, such that the one represented in Fig. 2.1.3; if only the central, or the 0^{th} , airfoil is oscillated, the resulting unsteady pressure field in the i^{th} airfoil, p'_i , can be integrated along the airfoil surface, Σ , to obtain the influence coefficient:

$$\mathbf{l}_i = \int_{\Sigma} p'_i \mathbf{n} dA \quad (2.1.8)$$

The influence coefficient, $\mathbf{l}_i$, is a vector although frequently only its projection along a given direction is used. In general the above integral can be performed with the in-phase, out-of-phase, or the modulus of the unsteady pressure. ICs have therefore both, modulus and phase.

2.1.2 Purely structural problem

If $\hat{f}_{aero} = 0$, the characteristic equation reduces to:

$$\mu_d \tilde{\omega}^4 - \left[(\mu_d + 1) + 4\kappa \sin^2 \frac{\sigma}{2} \right] \tilde{\omega}^2 + 4\kappa \sin^2 \frac{\sigma}{2} = 0 \quad (2.1.9)$$

and then, the natural frequencies of the problem depend on three parameters:

$$\tilde{\omega}^2 = \frac{\omega_n^2}{\omega_{blade}^2} = \mathcal{F} \left(\mu_d = \frac{M_{disk}}{m_{blade}}, \kappa = \frac{k_{disc}}{k_{blade}}, \sigma = 2\pi \frac{n}{N} \right) \quad (2.1.10)$$

the disc to rotor blade mass ratio, μ_d , the disc to rotor blade mass stiffness, κ , and the non-dimensional wave number, nodal diameter, or IBPA, σ , depending on the nomenclature the reader is more familiar with.

There are a number of conclusions that may be drawn from the mathematical structure of the problem that have their physical counter part:

1. All the eigenvalues, $\tilde{\omega}^2$, of the model problem are real. This may be readily seen by noting that the characteristic equation 2.1.9 is a second order equation of the form $\tilde{\omega}^4 - 2b\tilde{\omega}^2 + c = 0$, whose solution is $\tilde{\omega}^2 = b \pm \sqrt{b^2 - c}$. Since b and c are always positive, and $b^2 > c$, then $\tilde{\omega}^2$ is always real and greater than zero. This physically means that the system is not damped and it is conservative, since the solution can be expanded in a series of sines and cosines of constant amplitude according to 2.1.4. A more general view of the problem can be obtained by noting that the matrix of the eigenvalue problem is symmetric, and hence all its eigenvalues are real. In general, the matrices resulting from a purely structural eigenvalue problem with cyclic symmetry boundary conditions are Hermitian, and therefore all the eigenvalues are again real.
2. The eigenvalues are double, since Eq. 2.1.4 does not change for positive or negative nodal diameters or wave numbers, n , i.e $\tilde{\omega}^2(\sigma) = \tilde{\omega}^2(-\sigma)$. This means that from a purely structural point of view the system is symmetric and then eigenfunctions with the same eigenvalue can be arbitrary combined between them. This directly implies that the solution of the purely structural problem can be expressed in the form of either traveling waves (TW) or standing waves

(SW). The angular wave-length of the waves is $\lambda_\theta = 2\pi/n$. It can be anticipated that this symmetry of the structural problem is spoiled when the aerodynamic forces are introduced.

3. From a mathematical point of view the natural form of looking at the solution is as pairs of forward and backward travelling waves of wave-length λ and phase velocity $c = \pm\tilde{\omega}/n$

$$x(t) = A_n \cos(n\theta + \tilde{\omega}\tau) + B_n \cos(n\theta - \tilde{\omega}\tau + \phi) \quad (2.1.11)$$

however this solution may be recombined and expressed as two standing-waves, setting $\phi = 0$:

$$x(t) = (A_n + B_n) \cos(n\theta) \cos(\tilde{\omega}\tau) - (A_n - B_n) \sin(n\theta) \sin(\tilde{\omega}\tau) \quad (2.1.12)$$

Frequently rotor blade and disc masses are comparable, $\mu_d \sim 0(1)$. This is true for low-pressure turbines and compressors whereas high-pressure turbines tend to have much heavier discs, due to the combined effect of high shaft rotational speeds and high temperatures. Following the same line, compressor blisks tend to have very light blades. The disc is dynamically very stiff in the high nodal diameter limit if $\kappa \gg \mu_d$; more strictly when, i.e.: $4\frac{\kappa}{\mu_d} \sin^2 \frac{\sigma}{2} = \frac{1}{\epsilon} \gg 1$. The dispersion relation reduces in this case to:

$$\epsilon\tilde{\omega}^4 - \tilde{\omega}^2 + 1 = 0 \quad (2.1.13)$$

and hence $\tilde{\omega}_{Disk}^2 = \frac{1}{\epsilon}$ and $\tilde{\omega}_{Blade}^2 = 1$. This means that the natural frequencies of the rotor blade dominated modes are quite similar to that of the blade alone case (see Fig. 2.1.4). There is always a region, for low enough nodal diameters ($\sigma \sim \sqrt{\mu_d/\kappa}$), in which the flexibility of the disc can be noticed. For very stiff discs this region is fairly small and a good approximation to characterise the damping is to consider mode families of constant frequency. This approach has been systematically followed in this thesis.

Figure 2.1.4: FREQUENCY CHARACTERISTICS OF THE BLADED-DISC LUMPED MODEL

2.1.3 Aerodynamic contribution

When the aerodynamic forces are retained ($\hat{f}_{aero} \neq 0$), the new characteristic equation is:

$$\mu_d \tilde{\omega}^4 - \left[(\mu_d + 1) + 4\kappa \sin^2 \frac{\sigma}{2} - \mu_d \tilde{f}_{aero} \right] \tilde{\omega}^2 + 4\kappa \sin^2 \frac{\sigma}{2} (1 - \tilde{f}_{aero}) - \tilde{f}_{aero} = 0 \quad (2.1.14)$$

It is important to express the aerodynamic forces (Eq. 2.1.1) as a function of a physical scaling and in non-dimensional form, to better understand the functional dependence of the whole problem.

$$\tilde{f}_j^{aero} = \frac{f_j^{aero}}{m_b \omega_b^2} = \sum_{i=-1}^{i=1} \left(\frac{1}{\mu k^2} \hat{l}_i(k) y_{j+i} + \frac{1}{\mu k} \tilde{l}_i^*(k) \frac{dy_{j+i}}{d\tau} \right) \quad (2.1.15)$$

where the mass ratio, μ , is defined as $\mu = \frac{m_b/H}{\rho_a c^2}$, and the reduced frequency as $k = \omega_0 c / v_c$, being H the airfoil span, ρ_a , the characteristic density of the air, c the airfoil chord, and v_c the characteristic velocity. Since $m_b/H \sim \rho_s c t$, where t is the characteristic thickness of the airfoil, and ρ_b the density of the rotor blade, the order of magnitude of the mass ratio for a solid rotor blade is $\mu \sim (\rho_b/\rho_a)(t/c)$. The density ratio can range between 10^3 to 10^4 , whereas the relative thickness is around 10^{-1} , therefore the density ratio for a solid blade is close to 10^3 . Then, even if the reduced frequency is very low, $k \sim 10^{-1}$, $\tilde{f}_j^{aero} \ll 1$ and is only a small, but relevant, correction to the structural part.

Equation 2.1.15 has been derived assuming that the force per unit deflection due to the quasi-static displacement of the airfoils is proportional to the dynamic head and

the airfoil span, $l_i \propto \rho_a U_c^2 H$, and the force per unit velocity is $l_i^* \propto \rho_a U_c c H$. The scaling rational will be specifically addressed in other chapters. It is worth noting that the new aerodynamic based non-dimensional form of the influence coefficients is $\tilde{l}_i = \frac{l_i}{\rho_a v_c^2 H}$ and $\tilde{l}_i^* = \frac{l_i^*}{\rho_a v_c^2 H}$.

In this case the natural frequencies of the problem depend on:

$$\frac{\omega_n^2}{\omega_{blade}^2} = \mathcal{F} \left(\mu_d = \frac{M_{disk}}{m_{blade}}, \kappa = \frac{k_{disc}}{k_{blade}}, \sigma = 2\pi \frac{n}{N}, k = \frac{\omega_b c}{v_c}, \mu \right) \quad (2.1.16)$$

the dependence of the non-dimensional aerodynamic influence coefficients on the reduced frequency, cannot be derived from dimensional considerations at this stage, since it requires a deeper inspection of the aerodynamic problem.

Considering the case in which the rotor blade and disc masses are comparable, $\mu_b \sim 0(1)$, the nodal diameter is high, i.e.: $4 \frac{\kappa}{\mu} \sin^2 \frac{\sigma}{2} \gg 1$, and the aerodynamic correction is small, $\tilde{f}_{aero} \ll 1$, then:

$$\tilde{\omega}^2 \simeq 1 - \tilde{f}_{aero} \quad (2.1.17)$$

Similarly as the purely structural case, there are a number of conclusions that can be drawn from this result:.

1. If $\tilde{l}_{+1} \neq \tilde{l}_{-1}$ and/or $\tilde{l}_{+1}^* \neq \tilde{l}_{-1}^*$, which effectively means that the contribution of the adjacent airfoils of the cascade to the aerodynamic force of the reference airfoil is not symmetric, the symmetry of the system is broken and $\tilde{\omega}^2(\sigma) \neq \tilde{\omega}^2(-\sigma)$. As a consequence, the eigenmodes of the aeroelastic problems are travelling waves. Contrary to the purely structural problem, standing waves are not in this case a potential solution of the eigenvalue problem.
2. The second consequence is that $\text{Im}(\tilde{\omega}) \neq 0$. The system is not conservative and is either damped or undamped. The asymmetry induced by the aerodynamics can give to an instability of the system, as it can be readily seen by looking at Eq. 2.1.1, that shows that the force terms $\Delta \tilde{l}^-$ and $\Delta \tilde{l}^{*+}$ contribute to the imaginary part of the eigenvalue, through the expression of the aerodynamic

Figure 2.1.5: SKETCH OF THE FREQUENCY CORRECTION (LEFT) AND DAMPING (RIGHT) COMPONENTS OF THE QUASI-STATIC AERODYNAMIC FORCES AS A FUNCTION OF THE IBPA

forces:

$$\tilde{f}_{aero} = \eta(\sigma) + i\xi(\sigma) \quad (2.1.18)$$

where the viscous damping part of the aerodynamic forces (effect of the velocity) has been neglected in order to simplify the expression, but it can be easily seen that the same conclusion is obtained if the velocity terms are included. According to the expression 2.1.18, it can clearly be seen that there are two contributions of different nature to the aerodynamic forces, and then to the aerodynamic work-per-cycle (see Fig. 2.1.5): a non-dimensional frequency shift, η , resulting from the real part of the aerodynamic forces (Fig. 2.1.5a) and the non-dimensional aerodynamic damping, ξ , that comes from the imaginary part of the aerodynamic forces (Fig. 2.1.5 b)

3. If only the contribution of the oscillating blade on itself is considered, which means that $\Delta\tilde{l}^+ = \Delta\tilde{l}^- = \Delta\tilde{l}^{*+} = \Delta\tilde{l}^{*-} = 0$, then the expression 2.1.7 reduces to:

$$\tilde{f}_{aero} = \tilde{l}_0 + i\omega_0\tilde{l}_0^* \quad (2.1.19)$$

where it can be clearly seen that the contribution of the oscillating blade is always of stable nature. In this case, the eigenvalues are:

$$\omega^2 = 1 - \tilde{l}_0 - i\tilde{l}_0^* \quad (2.1.20)$$

2.2 Formulation of the Tuned Flutter Problem

Most analytical, numerical and experimental time-dependent aerodynamic studies on oscillating turbomachine blades have been performed in the so-called "travelling wave mode" (TW), in which it is assumed that all blades vibrate with constant and identical vibration frequencies, amplitudes and inter-blade phase angles. Another way to perform an unsteady aerodynamic study is to consider the influence which one oscillating blade has on its non-vibrating neighbours (the so-called "influence coefficient technique" (IC)). It can be shown (see for example [29]) that both models give identical results within the frame of linearized assumptions. Although from a practical point of view most of the analyses devoted to the computation of aerodynamic work per cycle are performed in TW form injecting the modes extracted from a structural model, from the point of view of physics understanding it is believed that the extraction of the influence coefficients from the simulations and the traversing of the corresponding unsteady fields provides more physical insight, and then they should be used in combination with the more widespread simulations in form of TWs. This approach has been followed throughout this thesis. The formulation presented here follows closely that presented in [30], which is partially reproduced here for the sake of continuity.

The equations of motion for small vibration amplitude of a perfectly tuned bladed disc with N identical sectors, in the absence of any external forcing except for the linear motion dependent aerodynamic forces, take the form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{L}(\mathbf{x})_{\text{aero}}, \quad (2.2.1)$$

where the bladed disc degrees of freedom (after an appropriate FEM discretisation) are stored in $\mathbf{x}(t)$, $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices respectively, both symmetric and positive definite, and $\mathbf{L}_{\text{aero}}$ accounts for the motion dependent aerodynamic forces. Structural damping has been neglected; the only source of damping considered in the formulation above is that coming from the aerodynamic terms, which is typically the dominating one at low reduced frequencies, see e.g. [31].

The vibration modes of the system can be written using complex notation as

$$\mathbf{x} = \mathbf{X}e^{i\omega t} + \text{c.c.}, \quad (2.2.2)$$

where the vibration mode shape $\mathbf{X}$ and the mode oscillation frequency and damping ω (c.c. stands for the complex conjugate) are given by the following eigenvalue problem

$$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{X} = \mathbf{L}_{\text{aero}}(\omega)\mathbf{X}, \quad (2.2.3)$$

which requires the knowledge of the matrix of aerodynamic forces $\mathbf{L}_{\text{aero}}$ as a function of ω . The calculation of $\mathbf{L}_{\text{aero}}$ for a given ω value involves the solution of the linearized Navier-Stokes equations that prescribe the airflow around the blades for all linearly independent blade displacements. Therefore, the computation of the bladed disc vibration characteristics using the fully coupled structural-aerodynamic formulation given by Eq. (2.2.3) is extremely costly and simply not realizable for any realistic configuration of practical interest.

The computational cost of this problem is drastically reduced if one takes into account the fact that the aerodynamic effects produce only small variations of the purely structural vibration frequencies. As it has been previously mentioned, this is true if the mass ratio is large ($\mu \gg 1$). With this idea in mind, we will first rewrite Eq. (2.2.3) using the structural mode basis, and then apply an asymptotic perturbation method to obtain the first order aerodynamic correction of the purely structural vibration characteristics. This is a well known procedure to compute the aerodynamic stability properties of tuned bladed discs (see [1, 32] for a recent application of this approach in two and three dimensional configurations).

2.2.1 Structural vibration modes

If the response vector (2.2.2) is expressed as:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \quad (2.2.4)$$

with the vector X_j containing the displacements of the m degrees of freedom associated with sector j , then the problem that gives the natural oscillations of the tuned bladed disc without aerodynamic effects (Eq. (2.2.3) with $\mathbf{M}_{\text{aero}} = 0$), takes the form:

$$\left(\begin{bmatrix} K & K_c & 0 & \cdots & K_c^T \\ K_c^T & K & K_c & \cdots & 0 \\ & & \ddots & \ddots & \\ K_c & 0 & \cdots & K_c^T & K \end{bmatrix} - \omega^2 \begin{bmatrix} M & M_c & 0 & \cdots & M_c^T \\ M_c^T & M & M_c & \cdots & 0 \\ & & \ddots & \ddots & \\ M_c & 0 & \cdots & M_c^T & M \end{bmatrix} \right) \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (2.2.5)$$

where the $m \times m$ sector stiffness and mass matrices K and M are symmetric, and the coupling between adjacent sectors is represented by the coupling stiffness and mass matrices K_c and M_c . In our simplified model problem the mass coupling matrix was $M_c = 0$ and the number of DOF of the sector is $m = 2$. The cyclic symmetry of the bladed disc (that is composed of a sector repeated N times and arranged periodically) is now evident in the matrices in Eq. (2.2.5), that exhibit a block circulant structure, i.e., each row can be obtained from the previous one after a wrap-around forward block shift. The associated complex eigenvectors:

$$\begin{bmatrix} X_1 \\ \vdots \\ X_j \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} \hat{X}_k e^{i(\frac{2\pi k}{N})1} \\ \vdots \\ \hat{X}_k e^{i(\frac{2\pi k}{N})j} \\ \vdots \\ \hat{X}_k e^{i(\frac{2\pi k}{N})N} \end{bmatrix}, \quad \text{for } k = 1 \dots N, \quad (2.2.6)$$

correspond to travelling waves with k nodal diameters that, taking into account Eq. (2.2.2), rotate around the annulus with constant angular velocity $\frac{\omega}{(2\pi k/N)}$. The complex vector $\hat{X}_k$ of size m contains the mode-shape details and the only eigenvector variation from sector to sector is just a phase increment of amount $\sigma_k = \frac{2\pi k}{N}$ (the so-called “inter-blade phase angle”). The vector $\hat{X}_k$ can be regarded as the k -th mode of the discrete Fourier transform of the N -periodic sequence of the sector degrees of freedom along the complete structure, $X_1, \dots, X_N$; recall that the relation between Fourier modes $\hat{X}_1, \dots, \hat{X}_N$ and sector degrees of freedom $X_1, \dots, X_N$ is given by the direct and inverse discrete Fourier transforms

$$X_j = \sum_{k=1}^N \hat{X}_k e^{i\sigma_k j}, \quad \text{for } j = 1 \dots N, \quad \text{and} \quad (2.2.7)$$

$$\hat{X}_k = \frac{1}{N} \sum_{j=1}^N X_j e^{-i\sigma_j k}, \quad \text{for } k = 1 \dots N. \quad (2.2.8)$$

Note also that it is enough to consider the value of k modulo N ; the change $k \rightarrow k+N$, that corresponds to the addition of a complete turn around the rotor, leaves all the above expressions unchanged.

$$\sigma_{-k} = \sigma_{N-k} = \frac{2\pi(-k)}{N} \simeq 2\pi - \frac{2\pi k}{N} \quad (2.2.9)$$

This effectively means that the structure is symmetric and does not distinguish between forward and backward travelling-waves.

If we now insert expression (2.2.6) into Eq. (2.2.5), N decoupled eigenvalue problems of size m are obtained

$$\begin{aligned} ((K_c - \omega^2 M_c) e^{i\sigma_k} + (K_c - \omega^2 M_c)^T e^{-i\sigma_k} + K - \omega^2 M) \hat{X}_k &= 0, \\ \text{for } k &= 1 \dots N, \end{aligned} \quad (2.2.10)$$

which, for each k , have m real eigenvalues, $\omega_{k1}^2, \dots, \omega_{km}^2 \geq 0$, which give the oscillation frequencies of the structure. The associated eigenvectors, $\hat{X}_{k1}, \dots, \hat{X}_{km}$, are, in general, complex and can be taken to be such that the square matrix whose columns are the eigenvectors for the nodal diameter k ,

$$P_k = [\hat{X}_{k1} | \hat{X}_{k2} | \dots | \hat{X}_{km}] \quad (2.2.11)$$

verifies

$$P_k^H (M_c e^{i\sigma_k} + M_c^T e^{-i\sigma_k} + M) P_k = I \quad \text{and} \quad (2.2.12)$$

$$P_k^H (K_c e^{i\sigma_k} + K_c^T e^{-i\sigma_k} + K) P_k = \Omega_k^2 = \begin{bmatrix} \omega_{k1}^2 & & 0 \\ & \ddots & \\ 0 & & \omega_{km}^2 \end{bmatrix}, \quad (2.2.13)$$

for $k = 1 \dots N$,

which directly means that the eigenvectors are orthonormal with respect to the mass matrix and turn into diagonal form the sector stiffness matrix (the super-index ^H stands for the conjugate transpose matrix). Notice that the cyclic symmetry allows to drastically reduce the computational effort required to obtain the vibration modes of the tuned bladed disk: from $\mathcal{O}(N^3 m^3)$ for solving Eq. (2.2.5) to only $\mathcal{O}(Nm^3)$ for the N eigenvalue problems in Eq. (2.2.6).

The eigenvalue equation Eq. (2.2.10) remains invariant under the changes

$$k \rightarrow N - k, \quad \omega_k^2 \rightarrow \omega_{N-k}^2 \quad \text{and} \quad \hat{X}_k \rightarrow \hat{X}_{N-k}, \quad (2.2.14)$$

and thus, for every travelling wave rotating in one direction, there is also another identical one that rotates in the opposite direction, with the same velocity. This symmetry comes from the fact that the original system (2.2.1) remains unchanged after time is reversed, $t \rightarrow -t$, because it does not exhibit any dissipation at all. Travelling wave modes come thus in counter-propagating pairs, except for those associated with $k = 0$ and $k = \frac{N}{2}$ (present only for $N = \dot{2}$), which have real X_k and can be seen as non propagating standing waves. Property (2.2.14) implies that the plot of the natural frequencies versus the number of nodal diameters, k , is symmetric, and only its first half (inter-blade phase angle from 0 to π) is normally plotted.

2.2.2 Aerodynamic correction

In order to evaluate the aerodynamic correction of the vibration characteristics of the bladed disc it is convenient to first perform in Eq. (2.2.3) the following change of

variables to a purely structural travelling wave basis

$$\mathbf{X} = \mathbf{P}\mathbf{A}, \quad (2.2.15)$$

with

$$\mathbf{P} = \frac{1}{\sqrt{N}} \begin{bmatrix} P_1 e^{i\sigma_1 1} & \dots & P_N e^{i\sigma_N 1} \\ \vdots & & \vdots \\ P_1 e^{i\sigma_1 j} & \dots & P_N e^{i\sigma_N j} \\ \vdots & & \vdots \\ P_1 e^{i\sigma_1 N} & \dots & P_N e^{i\sigma_N N} \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_j \\ \vdots \\ A_N \end{bmatrix}, \quad (2.2.16)$$

and where the vectors $A_j = [A_{j1}, \dots, A_{jm}]^T$ contain the amplitudes of the different TW modes. The eigenvalue problem in Eq. (2.2.3) takes now the form

$$\begin{bmatrix} \Omega_1^2 - \omega^2 I & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Omega_N^2 - \omega^2 I \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix} = \mathbf{P}^H \mathbf{L}_{\text{aero}}(\omega) \mathbf{P} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}, \quad (2.2.17)$$

with diagonal mass and stiffness matrices, see Eq. (2.2.12) and (2.2.13), and block diagonal aerodynamic matrix

$$\mathbf{P}^H \mathbf{L}_{\text{aero}}(\omega) \mathbf{P} = \mathbf{C} = \begin{bmatrix} C_1(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_N(\omega) \end{bmatrix}, \quad (2.2.18)$$

as a result of the fact that the linearized aerodynamic flow problem has also cyclic symmetry and, therefore, the TW modes with different nodal diameters are completely independent.

The fact that pressure TWs with k nodal diameters are orthogonal to TWs with $j \neq k$ nodal diameters, or in physical terms, that the work of pressure TW in the displacement of another TW is null (see Appendix A), together with the hypothesis that the in-sector modes are separated enough in frequency, allows the

full diagonalisation of the aerodynamic matrix in the TW basis. In other words:

$$C_k = \begin{bmatrix} c_{k,1}(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c_{k,m}(\omega) \end{bmatrix} \quad (2.2.19)$$

where $c_{k,j}$ is a complex number in general. This means that TWs are the aeroelastic modes of the tuned system, and the system fully decouples in $N \times m$ eigenvalue problems of a single DOF if the in-sector frequencies are well separated. This fact has deep implications in the formulation of the problem and it is one of the reasons why unsteady aerodynamics is computed using a single passage in the TW basis. It is interesting to note that, whereas standing waves are as well eigenfunctions of the structural problem, this is not the case for the whole aeroelastic problem because the aerodynamic matrix, $\mathbf{L}$, is not symmetric.

The aerodynamic terms are small if $\mu \gg 1$, and give rise to a small correction of the purely structural vibration frequencies. For a given structural frequency $\omega_{\tilde{k}\tilde{j}}$ (corresponding to the $\tilde{j}$ -th TW mode with $\tilde{k}$ nodal diameters), its aerodynamically corrected counterpart ω is given by Eq. (2.2.17), and it can be shown (as discussed in [30]) that the correction to the structural frequencies and mode shapes is small if the structural frequencies are well separated. Modes with different natural frequencies but with the same nodal diameter are orthogonal also.

2.2.3 Influence Coefficient Formulation

It is interesting to further explore the structure of the aerodynamic matrix $\mathbf{L}$. Equation 2.2.3 does not makes any hypothesis about the structure of $\mathbf{L}$, which is known in the literature as influence coefficient matrix. According to Eq. 2.2.3, $\mathbf{L}$ is a $N \times m$ full matrix, but expression 2.2.18 shows that $\mathbf{L}$ can be expressed as a function of only $N \times m$ coefficients.

For a single family of modes, m , at high nodal diameters usually the structural frequency and the mode-shape are constant, if the disc is stiff enough. The problem can then be further simplified to perform conceptual studies assuming that

$\omega_{k,m} = \omega_{0,m}$ and $\hat{X}_{k,m} = \hat{X}_{0,m}$ for all the nodal diameters. In this case, the problem in TW formulation (Eq. 2.2.17) can be written as:

$$\begin{bmatrix} \Omega_1^2 - \omega^2 I & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \Omega_N^2 - \omega^2 I \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix} = \mathbf{C}(\omega) \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix}. \quad (2.2.20)$$

We can note that the block diagonal matrices, Ω_k , and the vectors A_k , of the previous equation become scalars in the case of a single family of modes, and hence $\Omega_k \rightarrow \omega_k$ and $A_k \rightarrow a_k$. Furthermore we can assume that $\omega_k = \omega_0$, to obtain:

$$\begin{bmatrix} \omega_0^2 - \omega^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} c_1(\omega) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c_N(\omega) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}, \quad (2.2.21)$$

We can now simplify the matrix of change of variables, $\mathbf{P}$, as $\mathbf{P} = E \otimes \hat{X}_m$, where $E_{k,l} = e^{ik\sigma_l}$, since all the square matrices, P_k , are identical, or in other words, the eigenmodes does not change with the nodal diameter, and contain only the mode of interest, and therefore become a column matrix of size m . Matrix E has size $N \times N$, whereas matrix $\mathbf{P}$ is made up of column vectors elements of size m . Matrix $\mathbf{P}$ has therefore size $Nm \times m$. Analogously $\mathbf{P}^H = E^H \otimes \hat{X}_m^H$ and $E_{k,l}^H = e^{il\sigma_k}$. We can express again Eq. 2.2.21 in the physical DOFs, $\mathbf{X}$, but since matrix $\mathbf{P}$ contains only a single eigenvector, only N of the $N \times m$ equations are linearly independent. The new system of equations expressed in the modal coordinates of each individual blade, $\mathbf{q}$, is:

$$m \begin{bmatrix} \omega_0^2 - \omega^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_0^2 - \omega^2 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} = L \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix}, \quad (2.2.22)$$

The aerodynamic coefficient matrix, L , is full, and couples the initially uncoupled mechanical system (q_k are the blade coordinates). Although matrix L is full, it has a very special form and only contains N independent terms, due to the cyclic symmetry structure of the problem and the matrix E . Actually all the elements of

the sub-diagonals are identical,

$$[L] = \begin{bmatrix} L_0 & L_{-1} & & L_{-N/2} \\ L_1 & L_0 & \ddots & \\ & \ddots & \ddots & L_{-1} \\ L_{N/2} & & L_1 & L_0 \end{bmatrix} \quad (2.2.23)$$

it may also be shown that $L_p = \frac{1}{N} \sum_{m=1}^N e^{i\frac{2\pi p}{N}m} c_m$, i.e.: L_p is the p^{th} Fourier coefficient of the force coefficients in TW form, c_p . The aerodynamic coefficients as they appear in matrix L give tremendous insight into the unsteady aerodynamic interactions in a cascade. Each term in the aerodynamic influence matrix $[L]$ has a unique physical significance. The term in the first row and the second column, for example, designates the force acting on the first blade due to the motion of the second blade. Each term on the diagonal represents the force felt by a blade due to its own motion. There are some experimental facilities, for example the one at KTH, [33], which have been specifically designed to run following this approach: the central airfoil of the cascade is forced to oscillate with a prescribed vibration mode-shape, while the other airfoils of the cascade are at rest. The unsteady pressures generated in each blade are then measured, and the solution is reconstructed in TW form.

Generally speaking, the contribution to the unsteady pressure of the 0-th airfoil of the airfoils vibrating far away from it is small, and therefore the off-diagonal terms located apart from the main diagonal become smaller and smaller. This effectively means that the mean value of the work-per-cycle, is due solely to the effect of the airfoil motion in itself, represented by L_0 . It has been shown that this is the only term in the influence coefficient matrix which can provide a net stabilizing influencing on the rotor, [34] and [35]. The last assertion has been mathematically demonstrated in 2.1.3.

Typically only the adjacent airfoils contribute significantly to the aerodynamic damping. Fig. 2.1.3 displays the unsteady pressure field generated by the vibration of the central blade while the rest are at rest. It may be appreciated that only the 0-th and +1th have a significant pressure perturbation while the rest of the blades

remain unperturbed. From a mathematical point of view this means that $L_0 \neq 0$ and $L_1 \neq 0$ while the rest of the terms of matrix L are negligible in first approximation.

2.3 Aerodynamic Damping Computational Methods

2.3.1 Coupled methods

Fully coupled methods try to mimic as close as possible the physics of the fluid-structure problem making use of the less possible number of approximations. Ideally fully coupled methods should simultaneously march in time the RANS equations for the fluid and the elasticity equations for the structure, exchanging the solvers the aerodynamic forces (static pressure and viscous stresses) and the structure displacements at the interface. RANS equations had to be solved in moving grids, which are continuously recomputed every time step to accommodate the structure displacements. [36, 37, 38, 39, 40, 41]

Fully coupled methods are computationally intensive for the simulation of time periodic or quasi-periodic turbomachinery problems, not only because of the system of equations has to be marched in time for many (at least ten) fundamental periods to reach a periodic solution, but because the whole wheel has to be included in the simulation, both in the fluid and the structure. As a consequence, even today, the resulting problem is hardly manageable in a design environment. Typically turbine or compressor rotors are made up of about 100 airfoils, if every passage were discretised with a one million node grid, which is usually considered as a marginal resolution for a purely aerodynamic simulation, the mesh of the whole annulus would have about 100 million points and should be marched in time for 10-100 fundamental periods.

Several methods exist to reduce the computational time of fully coupled methods with a negligible impact on the solution accuracy.

1. **USE OF A SUBSET OF NOMINAL MODES.** In this approach, the structural natural frequencies, ω_r , and mode-shapes, $\phi_r(\mathbf{x})$, of the structure are computed

taking into account the cyclic symmetry of the problem in a pre-processing step and stored. The structure is discretised in a single sector and the mode-shapes and natural frequencies computed imposing cyclic symmetry boundary conditions, which are the counter part of the phase-lagged boundary conditions in unsteady aerodynamic simulations. The displacements of the whole structure are expanded in the basis of the structural modes, and the time evolution of the modal deflections, η_r , is computed as

$$m_r \frac{d^2 \eta_r}{dt^2} + \omega_r^2 \eta_r = \int_{\Sigma} \phi_r(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \, d\sigma \quad (2.3.1)$$

where the right hand side represents the modal force, which is obtained, projecting the aerodynamics forces on the airfoil surface, $\mathbf{f} = -(-p\bar{\bar{I}} + \bar{\bar{\tau}}_v) \cdot \mathbf{n}$, onto the mode-shapes, which are usually normalised to deliver a modal mass, $m_r = 1$. The structural mode-shapes are interpolated from the FE mesh onto the aerodynamic mesh, since both grids are unlikely to be coincident. The cost of evaluating the time response of the structure becomes negligible, since only a few modes of the order of hundreds at most, need to be marched in time. Mode-shapes are usually expressed in travelling-wave form [38] although for simplified problems sometimes the individual blade formulation is preferred [41]. Initial boundary conditions are selected in such a way that all the modes of the structure are perturbed.

Damping is obtained tracking in time the modal amplitudes and computing the logarithmic decrement, $\delta = \ln(x_{n+1}/x_n)$, where x_n and x_{n+1} are two consecutive maxima in the amplitude of the modal deflection (see Fig. 2.3.1). It can be shown that in the free vibration of a linear second order system:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (2.3.2)$$

the logarithmic decrement is

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad (2.3.3)$$

where $\xi = c/2\pi m\omega_0$, is the so-called critical damping ratio, and $\omega_0 = \sqrt{k/m}$, is the purely structural natural angular frequency. Engineering systems are

Figure 2.3.1: LOGARITHMIC DECREMENT AS A FUNCTION OF THE NORMALISED TIME

frequently lightly damped and $\xi \ll 1$. Within this approximation $\delta \simeq 2\pi\xi$. The non-dimensional eigenvalues of the system are:

$$\tilde{\omega} = \frac{\omega}{\omega_0} = i\xi \pm \sqrt{1 - \xi^2} \quad (2.3.4)$$

and therefore δ is a good approximation to the imaginary part of the eigenvalue, that is, the non-dimensional damping.

2. PRE-COMPUTED AERODYNAMICS. A second approach to reduce the cost of the fully coupled methods is to pre-compute the aerodynamics. The standard hypothesis is that the solution of the problem is harmonic, compute the aerodynamics in travelling-wave form and translated or not into influence coefficient form. The hypothesis that the flow is linear and that the unsteady perturbations are proportional to the vibration amplitude is usually used. Eventually the correction of the aerodynamic with the oscillating frequency could be included but this approach is not usually followed. The resulting problem can be cast in the form of an eigenvalue problem to determine the stability [17] of the aeroelastic problem, or as an initial value problem which is marched in time to find the final non-linear saturated solution [41].

2.3.2 Uncoupled methods

When the mass ratio of the system is high ($\mu \gg 1$), the aeroelastic modes and natural frequencies of the bladed-disc are very similar to that of the in-vacuum structure. The free response of the system consists in a slowly-varying envelope of the airfoil amplitudes or of the bladed-disc modal coordinates. Under these conditions the most efficient procedure is to prescribe the harmonic motion and frequency of the mode. In this case, the computation of the aerodynamic damping reduces to the solution of a purely unsteady aerodynamic problem where the displacements on the surface of all the airfoils are given. This approach is simple and can be computationally very efficient if used in combination with frequency domain solvers, either linear or non-linear [42, 43, 44].

In this approach, airfoils are forced to oscillate at a specified mode, with constant amplitude, frequency and IBPA [45]. Since the motion is given, the stability of the system is quantified by the work performed within one cycle by the fluid in the structure. With the standard criterion of signs, if the work is positive, it means that the aerodynamic contribution stabilizes the motion, whereas if the work is negative, the system is unstable and in a free vibration case the oscillation would be amplified. The work exerted by the unsteady flow-field onto the blades is calculated as the result of the application of the unsteady aerodynamic pressure onto the blade displacements.

2.3.2.1 Discussion

It is interesting to use the analytical solution of free-vibration of the 2^{nd} order system represented by Eq. 2.3.2. The energy dissipated per cycle by the viscous damping can be computed by evaluating the difference of the total energy (kinetic and potential) of the system between two consecutive peaks of the response, x_n and x_{n+1} , (see Fig. 2.3.1), i.e.: $W_{cycle} = E_n - E_{n+1}$, where $E_n = \frac{1}{2}\dot{x}_n^2 + \frac{1}{2}x_n^2$. The non-dimensional W_{cycle} is:

$$\hat{W}_{cycle} = \frac{W_{cycle}}{m\omega_0^2 x_n^2} = \frac{1}{2} \left(1 - e^{-\frac{4\pi\xi}{\sqrt{1-\xi^2}}} \right) \quad (2.3.5)$$

where obviously, since the problem is linear there is not a characteristic vibration amplitude. The non-dimensional W_{cycle} is directly related with the logarithmic decrement as $W_{cycle} = \frac{1}{2}(1 - e^{-\delta})$. If we further assume that $\xi \ll 1$, then $\hat{W}_{cycle} \simeq 2\pi\xi$.

If we assume, like in the uncoupled approach, that the motion of the system is harmonic, with the angular frequency of the purely structural problem, ω_0 , the W_{cycle} of the viscous damping would be computed as:

$$\hat{W}_{cycle} = \frac{1}{m\omega_0^2 x_0^2} \int_0^{T=2\pi/\omega_0} c\dot{x}^2 dt = 2\pi\xi \quad (2.3.6)$$

where the motion is given by $x = x_0 \sin \omega_0 t$, and obviously the rest of the forces have a null contribution to the W_{cycle} , since they are conservative.

This effectively means that, if the aerodynamics is linear, as the viscous damping is, the difference in computing the W_{cycle} , using a fully coupled method (Eq. 2.3.5) and an uncoupled approach (Eq. 2.3.6) is an error of second order in ξ , if $\xi \ll 1$,

$$\tilde{W}_{cycle}^{uncoupled} = \tilde{W}_{cycle}^{coupled} + 0(\xi^2) \quad (2.3.7)$$

The coupled method is able to account for the small shift in frequency due to the aerodynamics; the actual non-dimensional vibration frequency of the 2^{nd} order system is $\tilde{\omega} = \sqrt{1 - \xi^2}$, which is the origin of the difference. Summarising, if $\xi \ll 1$ (or $\mu \gg 1$), the use of an uncoupled approach is well justified. The comparison between coupled and uncoupled approaches is sometimes misleading, since it is mixed up with the fluid model. Fully coupled methods are usually used in conjunction with the URANS equations, whereas uncoupled methods are based on the use of linearised frequency domain solvers. The linearity or not of the unsteady aerodynamics with respect the vibration amplitude is a conceptually different issue from the coupling method, and it will be discussed later on, but in practice has been difficult to separate in the past. A proper comparison would require the use of uncoupled non-linear methods, either in the time-domain, or in the frequency domain, with the recent development of non-linear harmonic balance methods.

Apart from flutter stability, coupled methods are well suited to the study of limit-

cycle oscillations, where the vibration amplitude is saturated by nonlinearities, either in the fluid, as it is the case sometimes if shock-waves are present, like in choke flutter, or in the structure, where dry friction can play an important role. Time dependent fully coupled methods are also attractive to deal with separated flows, and in general aerodynamically driven instabilities, such as rotating stall and buffeting. In all these cases the assumption that all the blades vibrate with constant amplitude and IBPA does not usually hold, and the vibration frequency is not known a priori either.

Up to now only systems with a single active mode has been considered. Uncoupled methods work in a mode-by-mode basis and therefore there are not suitable to account for mode interaction. Mode interaction can occur both, in the linear (like in flutter of vane packets [17]), and non-linear (friction saturation [41]) regimes, but in any case the modes need to be close enough in terms of frequency to interact among them. This type of analysis requires the use of fully coupled methods, even if the aerodynamics is linear.

Chapter 3

Fundamentals of Unsteady Aerodynamics

It is during our darkest moments
that we must focus to see the light

Aristotle Onassis

Contents

3.1	Context	40
3.2	Planar Waves in the Euler Equations	42
3.3	Unsteady Pressure Generation: The Role of Vorticity .	50
3.4	Unsteady Pressure Generation: The Role of Apparent Mass	53
3.5	Unsteady Aerodynamics of Oscillating Flat Plate Cas- cades	57
3.6	Laminar Boundary with an Oscillating Free Stream . .	59
3.7	Basic Non-dimensional Analysis of Unsteady Aerody- namics	61

3.1 Context

Unsteady aerodynamics associated to oscillating airfoils is complex, and little can be said with a large degree of generality. Computational methods based on the solution of the linearized Navier-Stokes equations are widely used in the gas turbine industry, and even fully non-linear unsteady methods are used for some specific problems. The accuracy of computational methods is acceptable for most of the cases, especially within the industry, where codes have been tuned for specific flows, however the physical understanding of these flows is reduced.

Unsteady pressure distributions have little to do with steady pressure fields and the variability is even larger. To begin with the base mean flow, it has a strong influence on the perturbations. Steady aerodynamics of low-pressure turbines, which is a high subsonic Mach number flow subject to a large turning, is completely different of compressor aerodynamics that may even change dramatically from operating point to operating point. Compressor flows have low turning but may include shock waves at different positions.

When talking about unsteady fields, new parameters such as the reduced frequency, the phasing between the vibration of neighbouring airfoils or the vibration mode-shapes themselves appear, making the variability of the problem very large.

As a consequence of the complexity of the unsteady aerodynamic associated to vibrating airfoils, and despite the advances in computational fluid dynamics (CFD) and experimental methods, that have enabled flutter and forced response studies, an analytical approach for understanding the basic mechanisms, and for prediction and design is still forthcoming.

Most of the theoretical work in unsteady aerodynamics was performed during the 1920s and 1930s. One of the most important and significant of them, which is owned by Theodorsen, [6], will be briefly reviewed in this thesis, and the main conclusions applicable to vibrating airfoils will be highlighted with the final aim of showing the state-of-the-art of the knowledge about the unsteady aerodynamic behaviour of oscillating airfoils.

The work of Wagner, [46], was the starting point of Theodorsen and some other studies focused on understanding the unsteady flow field originated by an oscillating airfoil. The main finding of Wagner was that the oscillatory motion generates a vortex wake, which in turn affects the entire nature of the forces experienced by the airfoil.

Theodorsen's [6] theory, is based on potential flow and the Kutta condition. He developed a solution for the unsteady loads (lift and moment) on a flat plate undergoing harmonic oscillations in both, torsion and translation, and he applied the results to an analysis of the wing-flutter problem. Garrick, [47], extended Theodorsen's theory and calculated the drag/propulsion forces.

In the turbomachinery field, the unsteady phenomena are very complex and they are a major concern in current designs [48], [49], [50], [51]. Some examples of important unsteady phenomena are: the effects of aerodynamically induced and self-excited blade vibrations on aeroelastic response and stability, the effects of blade row interactions and turbulence on efficiency and noise generation, the effects of inlet flow non-uniformities on compressor and engine aerodynamic stability, and the nature and flow structure of general post-stall engine transients.

Several studies were carried out in the past with the aim of understanding the unsteady flow phenomena and the development of theoretical procedures for predicting such phenomena [20], [52], [51], [53]. The theory derived in the present thesis follows an approach that combines the approaches of other authors, such as [6, 7, 42], since it is a linear theory, for airfoils oscillating at small amplitudes in high-subsonic flow, and the conclusions are derived from an asymptotic analysis of the linearised equations. The difference now is that the theory presented includes real airfoil profiles and viscosity effects, and that the practical conclusions extracted here apply to a wider range of operating flow conditions (even transonic conditions, with the presence of shock waves), and is valid for any vibration mode-shape.

3.2 Planar Waves in the Euler Equations

The understanding of the unsteady compressible flow about an oscillating airfoil is a daunting task. The objective of this section is to briefly remind key elements of the one- and quasi-one-dimensional theory of compressible flows, which constitute the building blocks of the understanding of more complex flows. Special attention is paid to the realistic case in which the unsteady flow perturbations are small with respect to the base flow.

3.2.1 One-dimensional Flow

Let us consider an inviscid flow in a constant area duct, without heat addition, subject to planar, or one-dimensional perturbations, at the inlet or the outlet. The one-dimensional governing equations can be expressed as:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} &= 0 \\ \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} &= 0 \\ \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} &= 0 \end{aligned} \tag{3.2.1}$$

which constitute, respectively, the mass, momentum and entropy equations. In order to close the problem an expression for the entropy as a function of the state variables is needed, $s = s(p, \rho)$. For a perfect ideal gas $s = c_v \ln(p/\rho^\gamma)$. The entropy equation can then be written as:

$$\left(\frac{\partial p}{\partial t} - a^2 \frac{\partial \rho}{\partial t} \right) + u \left(\frac{\partial p}{\partial x} - a^2 \frac{\partial \rho}{\partial x} \right) = 0 \tag{3.2.2}$$

where $a = \sqrt{\gamma p / \rho}$ is the speed of sound. This system of equations can be written as:

$$\frac{\partial \mathbf{V}}{\partial t} + A(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial x} = 0 \tag{3.2.3}$$

where $\mathbf{V}^T = \{\rho, u, p\}$. The problem can be diagonalized to obtain the characteristic form of the equations. The entropy equation was already in diagonal form since entropy is convected with fluid velocity and remains unchanged (Eq. 3.2.2) whereas

the mass and momentum equations are recombined to yield:

$$\left(\frac{\partial p}{\partial t} \pm \rho a \frac{\partial u}{\partial t} \right) + (u \pm a) \left(\frac{\partial p}{\partial x} \pm \rho a \frac{\partial u}{\partial x} \right) = 0, \quad (3.2.4)$$

which represents the upstream and downstream propagation of acoustic waves, in the case of a subsonic flow. The eigenvalues of the problem are: $\lambda_{1,2} = u \pm a$ and $\lambda_3 = u$.

Equations 3.2.2 and 3.2.4 can be linearised about a mean uniform flow. If we assume that the flow can be decomposed as:

$$u = u_0 + u'; \rho = \rho_0 + \rho'; p = p_0 + p' \quad (3.2.5)$$

where u_0, ρ_0, u_0 are constants, and the perturbations are small, i.e.: $p' \ll p_0, \rho' \ll \rho_0$, and $u' \ll u_0$. The systems of equations 3.2.3 in diagonal form becomes:

$$\frac{\partial \mathbf{C}}{\partial t} + \Lambda_0 \frac{\partial \mathbf{C}}{\partial x} = 0 \quad (3.2.6)$$

where

$$\Lambda_0 = \begin{bmatrix} u_0 + a_0 & 0 & 0 \\ 0 & u_0 - a_0 & 0 \\ 0 & 0 & u_0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} p' + \rho_0 a_0 u' \\ p' - \rho_0 a_0 u' \\ p' - a_0^2 \rho' \end{bmatrix} \quad (3.2.7)$$

which represents a system of three independent linear waves propagating the characteristic variables at three different speeds, $u_0 \pm a_0$ and u_0 . Acoustic perturbations propagate upstream and downstream (if $M_0 < 1$) with velocities $u_0 \pm a_0$, conserving $p' \pm \rho_0 a_0 u'$, while entropy is convected downstream with velocity u_0 . Note that if the entropy is constant upstream, as it is the case of a 2D airfoil oscillating in an inviscid flow, then $s = s_0 \forall t$, and will not be possible to see entropy waves propagating through the fluid.

3.2.2 Vorticity Equation

A complete vision of the unsteady aerodynamics basics is not possible without resorting to the concept of vorticity, $\vec{\omega} = \nabla \times \mathbf{v}$. Although vorticity is a two- or three-dimensional concept, it is included here because of its relevance. The vorticity

equation is obtained taking the curl of the momentum equation:

$$\rho \frac{D(\vec{\omega}/\rho)}{Dt} = \vec{\omega} \cdot \nabla \mathbf{v} - \nabla \left(\frac{1}{\rho} \right) \times \nabla p + \nabla \times \left(\frac{1}{\rho} \right) \nabla \cdot \tau' + \nabla \times \mathbf{f}_m \quad (3.2.8)$$

The second term of the right hand side (r.h.s.) is zero if $p = p(\rho)$, being the case of homeotropic flows ($s = s_0$) a particular case. The third term of the r.h.s. is zero if the flow is inviscid, since in this case the viscous stresses, τ' , are null. The fourth term of r.h.s. is zero if the body forces derive of a potential, U , i.e.: $\mathbf{f}_m = -\nabla U$. Even in the case of the flow about a two-dimensional section corresponding to a rotor blade rotating with constant angular velocity, $\vec{\Omega}$, this term vanishes since centrifugal forces are conservative and the Coriolis forces are perpendicular to the section and do not contribute to the non-null component of the vorticity, ω_z . Therefore, Eq. 3.2.8 for and inviscid homeotropic flow reduces to:

$$\rho \frac{D(\vec{\omega}/\rho)}{Dt} = \vec{\omega} \cdot \nabla \mathbf{v} \quad (3.2.9)$$

Moreover if the flow is two-dimensional, $\vec{\omega} \cdot \nabla \mathbf{v} = 0$, and then the vorticity equation dictates the convection of ω_z/ρ . If we assume that the perturbations are small, $\omega_z = \omega_{z0} + \omega'_z$, and that the base flow is uniform and irrotational, $\omega_{z0} = 0$, the linearised vorticity equation becomes:

$$\frac{D\omega'_z}{Dt} = 0 \quad (3.2.10)$$

which states that the vorticity perturbations are convected downstream in an inviscid tow-dimensional flow. It is important to notice that this is true even if the Mach number of the base flow is of order unity, $M_0 \sim 0(1)$.

3.2.3 Quasi One-Dimensional Flow

All the key concepts outlined up to now are of great help to understand the aerodynamics of oscillating airfoils, but they are still far away from a representative model, even for conceptual studies. In this sub-section we address the problem of the compressible isentropic unsteady flow moving through a slowly varying, both in time and space, duct (see Fig. 3.2.1). The low reduced frequency limit, and the effect of

Figure 3.2.1: SKETCH OF A SLOWLY VARYING DUCT

the varying area, $A(x, t)$, will be discussed.

If it is assumed the case in which the characteristic length in the axial direction of the nozzle, $l_x \sim L$, is much larger than the corresponding in the orthogonal direction, $l_y \sim o(A^{1/2})$, then the axial velocity, u , is much larger than the transversal velocity, v , $v \ll u$, and then the motion is quasi one-dimensional. It can be shown that the governing equations of the problem are slight variant of that of the one-dimensional problem. The mass conservation in this case is:

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho u A)}{\partial x} = 0 \quad (3.2.11)$$

The momentum and entropy equations remain the same within this approximation. The new system of equations can be linearized about a mean flow decomposing the flow as:

$$u = u_0 + u'; \quad \rho = \rho_0 + \rho'; \quad p = p_0 + p' \quad (3.2.12)$$

where $u_0(x)$, $\rho_0(x)$, $p_0(x)$ represent the mean flow and in general depend of the position, but we will assume that are constant. The basic hypothesis is that the area variations are small, $\Delta A_x/A_0 \ll 1$. The linearised mass and momentum equations using the hypothesis of isentropy ($p - p_0 = (\rho - \rho_0)a_0^2$) become:

$$\frac{\partial p'}{\partial t} + \rho_0 a_0^2 \left[\tilde{A}_t + u_0 \tilde{A}_x + \frac{\partial u'}{\partial x} \right] + u_0 \frac{\partial p'}{\partial x} = 0 \quad (3.2.13)$$

$$\frac{\partial u'}{\partial t} + u_0 \frac{\partial u'}{\partial x} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \quad (3.2.14)$$

where $\tilde{A}_t = \frac{1}{A_0} \frac{\partial A}{\partial t}$ and $\tilde{A}_x = \frac{1}{A_0} \frac{\partial A}{\partial x}$. The previous system of equations can be

diagonalised, yielding:

$$\frac{\partial}{\partial t} (p' \pm \rho_0 a_0 u) + (u_0 \pm a_0) \frac{\partial}{\partial x} (p' \pm \rho_0 a_0 u) = -\rho_0 a_0^2 (u_0 \tilde{A}_x + \tilde{A}_t) \quad (3.2.15)$$

Previous equation is identical to that obtain for one-dimensional flow (Eq. 3.2.4), with the exception of the r.h.s. forcing term. This means that area variations, eventually vibrations, give rise to acoustic perturbations, propagating upstream and downstream, in the case of a subsonic flow, but not to entropy perturbations.

3.2.4 Order of Magnitude Estimates

The objective of this section is to use the simple model for the motion of a compressible isentropic flow in a duct of area A , that changes as: $A(x, t) = A_0 + A'(x) \sin \omega t$, to estimate which is the order of magnitude of the different terms of the governing equations, to help to understand later on the mechanisms involved in the pressure generation of an oscillating airfoil, and its magnitude. The momentum equation and the estimate of the order of magnitude of its different terms can be written as:

$$\underbrace{\frac{\partial u}{\partial t}} + \underbrace{u \frac{\partial u}{\partial x}} + \underbrace{\frac{1}{\rho} \frac{\partial p}{\partial x}} = 0$$

$$\begin{array}{ccc} \frac{U_c}{t_c} & \frac{U_c^2}{L} & \frac{\Delta p_c}{\rho_c L} \\ (1) \downarrow & (2) \downarrow & (3) \downarrow \\ \frac{t_r}{t_c} & O(1) & \frac{\Delta p_c}{\rho_c U_c^2} \end{array} \quad (3.2.16)$$

where $U_c, t_c, L, \Delta p_c$ and ρ_c are respectively the characteristics velocity, time, length, pressure variation and density of the problem. In our problem, for instance if we assume that the duct area changes with angular velocity, ω , then $t_c \sim \omega^{-1}$. The residence, or through-flow time, t_r , is defined as $t_r = L/U_c$. The first row under the momentum equation contains the estimate of the order of magnitude, the second row is a label to shorten the reference to that term, whereas the last row contains the relative order of magnitude with respect the term (2), that now is of order unity.

Analogously the continuity equation reads:

$$\begin{array}{ccccccccc}
 \underbrace{\frac{\partial p}{\partial t}} & + & \underbrace{u \frac{\partial p}{\partial x}} & + & \underbrace{\rho a^2 \frac{\partial u}{\partial x}} & = & - & \underbrace{\rho a^2 \frac{A_t}{A_0}} & - & \underbrace{\rho a^2 u \frac{A_x}{A_0}} \\
 \frac{\Delta p_c}{t_c} & & \frac{U_c \Delta p_c}{L} & & \frac{\rho_c a_c^2 U_c}{L} & & & \frac{\rho_c a_c^2 \delta \tilde{A}}{t_c} & & \frac{\rho_c a_c^2 U_c \tilde{A}}{t_c} \\
 (4) \downarrow & & (5) \downarrow & & (6) \downarrow & & & (7) \downarrow & & (8) \downarrow \\
 \frac{\Delta p_c}{\rho_c a_c^2} \frac{t_r}{t_c} & & \frac{\Delta p_c}{\rho_c a_c^2} & & O(1) & & & \delta \tilde{A} \frac{t_r}{t_c} & & \delta \tilde{A}
 \end{array} \tag{3.2.17}$$

We will be discussing now the relative importance of the different terms as a function of the characteristic time, or in other words the oscillation frequency, ω . A full discussion of the problem is very involved, we will focus here in the low reduced frequency limit, $k = t_r/t_c = \omega L/U_c \ll 1$.

Low reduced frequency limit ($t_r \ll t_c$)

If the residence time is much smaller than the characteristic time, i.e.: $t_r \ll t_c \Rightarrow k \ll 1$, then:

- In the momentum equation, inertia terms, (1), are much smaller than the convective terms, (2), and therefore pressure and convective terms are of the same order, (2) $\sim$ (3). We can then estimate the order of magnitude of the pressure variations since, (2) $\sim$ (3) $\Rightarrow \frac{\Delta p_c}{\rho_c U_c^2} \sim O(1)$, or in other words, the pressure variations are of the order of the dynamic head.
- In the continuity equation, inertia terms, (4), are also much smaller than the convective terms (5) and (6). The order of magnitude of (5) with respect (6) is M_c^2 . If $M_c^2 \ll 1$, then (5) $\ll$ (6) and we recover the incompressible limit.
- The unsteady forcing term, (7), is also much smaller than the steady one, (8), and can be neglected in first approximation.

The governing equations reduce in first approximation to that of the steady flow, the time plays the role of a parameter, and the problem is said to be quasi-stationary:

$$\begin{aligned} u \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} &= -\rho a^2 u \frac{A_x}{A_0} \\ u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= 0 \end{aligned} \quad (3.2.18)$$

In the current analysis, we consider the case in which the area variations are small with respect the characteristic area, $\delta A/A_0 \ll 1$. The forcing term in the r.h.s. of the continuity equation is small, and causes a small perturbation in the pressure, p'_0 , and velocity, u'_0 , fields. To estimate the order of magnitude of the perturbations associated to the area variation, we assume, to ease the presentation but without any loss of generality, that the spatial variations of pressure and velocity of the base flow are null, technically speaking that $\Delta p_{x0} \ll p'_0$ and $\Delta u_{x0} \ll u'_0$. In this case the mass and momentum equations for the perturbations with respect the base flow, and the order of magnitude of the different terms are:

$$\begin{aligned} \underbrace{u_0 \frac{\partial p'_0}{\partial x}}_{\downarrow} + \underbrace{\rho_0 a_0^2 \frac{\partial u'_0}{\partial x}}_{\downarrow} &= - \underbrace{\rho_0 a_0^2 u_0 \frac{A_x}{A_0}}_{\downarrow} \\ u_0 \frac{p'_0}{L} \quad \quad \quad \rho_0 a_0^2 \frac{u'_0}{L} \quad \quad \quad \rho_0 a_0^2 u_0 \frac{\delta \tilde{A}_x}{L} \end{aligned} \quad (3.2.19)$$

$$\begin{aligned} \underbrace{u_0 \frac{\partial u'_0}{\partial x}}_{\downarrow} + \underbrace{\frac{1}{\rho_0} \frac{\partial p'_0}{\partial x}}_{\downarrow} &= 0 \\ u_0 \frac{u'_0}{L} \quad \quad \quad \frac{1}{\rho_0} \frac{p'_0}{L} \end{aligned} \quad (3.2.20)$$

The conclusion is that, in low reduced frequency limit, the order of magnitude of the quasi-steady perturbations are: $u'_0/u_0 \sim \delta \tilde{A}_x$ and $p'_0/\rho_0 a_0^2 \sim \delta \tilde{A}_x$, where the non-dimensional spatial variation of the area, $\delta \tilde{A}_x(t)$, is a function of time. The result is Mach number independent, provided that $M_0 \sim 0(1)$, and the perturbations are in phase with the displacements.

The second order correction to the perturbations, p'_1 and u'_1 , are obtained retaining the previously neglected unsteady terms:

$$\begin{aligned}
 & \underbrace{\frac{\partial p'_0}{\partial t}}_{\rho_c u_0^2 \frac{\delta \tilde{A}}{t_c}} + \underbrace{u_0 \frac{\partial p'_1}{\partial x}}_{u_0 \frac{\Delta p'_{1c}}{L}} + \underbrace{\rho_0 a_0^2 \frac{\partial u'_1}{\partial x}}_{\rho_0 a_0^2 \frac{\Delta u'_{1c}}{L}} = \underbrace{-\rho_0 a_0^2 \frac{A_t}{A_0}}_{\rho_0 a_0^2 \frac{\delta \tilde{A}}{t_c}} \\
 & (1) \downarrow \quad (2) \downarrow \quad (3) \downarrow \quad (4) \downarrow \\
 & \delta \tilde{A} \frac{u_0^2}{a_0^2} \frac{t_r}{t_c} \quad \frac{\Delta p'_{1c}}{\rho_c a_0^2} \quad \frac{\Delta u'_{1c}}{u_0} \quad \delta \tilde{A} \frac{t_r}{t_c}
 \end{aligned} \tag{3.2.21}$$

$$\begin{aligned}
 & \underbrace{\frac{\partial u'_0}{\partial t}}_{u_0 \frac{\delta \tilde{A}}{t_c}} + \underbrace{u_0 \frac{\partial u'_1}{\partial x}}_{u_0 \frac{\Delta u'_{1c}}{L}} + \underbrace{\frac{1}{\rho_0} \frac{\partial p'_1}{\partial x}}_{\frac{1}{\rho_0} \frac{\Delta p'_{1c}}{L}} = 0 \\
 & (5) \downarrow \quad (6) \downarrow \quad (7) \downarrow \\
 & \delta \tilde{A} \frac{t_r}{t_c} \quad \frac{\Delta u'_{1c}}{u_0} \quad \frac{\Delta p'_{1c}}{\rho_0 u_0^2}
 \end{aligned} \tag{3.2.22}$$

On the view of the two previous equations two particular cases could be distinguished:

1. $M_0 \sim O(1)$. In this case $u_0 \sim a_0$, and if (2) $\sim$ (3) (or (6) $\sim$ (7)), the order of magnitude of the correction to the pressure perturbation is $\Delta p'_{1c} \sim \rho_0 u_0 u'_{1c}$, then if make (2) $\sim$ (4), we can conclude that: $\frac{\Delta p'_{1c}}{\Delta p'_{0c}} \sim \frac{u'_{1c}}{u'_{0c}} \sim \frac{t_r}{t_c} \sim k$. Moreover, it can be seen that this correction is out of phase with respect the displacement, since it is generated by the A_t term, and then the phase of the solution, ϕ , scales as $\tan \phi \sim k$.
2. $M_0 \ll O(1)$. In this case $u_0 \ll a_0$: for this case, based on the idea that (5) $\sim$ (7), it is obtained that $\Delta p'_{1c} \sim \rho_0 u_0^2 \delta \tilde{A} \frac{t_r}{t_c}$, and since (3) $\sim$ (4) it is obtained $\frac{u'_{1c}}{u_0} \sim \delta \tilde{A} \frac{t_r}{t_c}$. The conclusions are the same than for the $M_0 \sim O(1)$. The only difference being that (1) $\sim$ (2) $\ll$ (3) and the flow is nearly incompressible.

3.3 Unsteady Pressure Generation: The Role of Vorticity

3.3.1 Kelvin's circulation theorem

The Kelvin's circulation theorem states that, for an inviscid and incompressible fluid of constant density in motion in presence of conservative body forces $\mathbf{f} = -\nabla U$ per unit mass, the circulation when considering a closed fluid circuit (the same fluid particles as time proceeds) is:

$$\Gamma = \int_{C(t)} \mathbf{u} dx \quad (3.3.1)$$

and it is independent of time. It is important to note that:

- Incompressibility is not essential for the theorem to fulfill.
- The theorem does not require the fluid region to be simply connected, i.e., it does not require the dyed circuit C to be spannable by a surface S lying wholly in the fluid.

The generation of lift on an airfoil

The shedding of a starting vortex is essential to the generation of lift of an airfoil. Kelvin's theorem can be used to understand why this is so. Kelvin's circulation theorem states that the time rate of change of circulation around a closed curve consisting of the same particles is zero.

Figure 3.3.1 helps to explain the lift generation mechanism on an oscillating airfoil. At time t after the airfoil started moving, the viscous forces and vorticity will be confined to: i) a thin boundary layer on the airfoil, ii) a thin wake and iii) the rolled-up "core" of the starting vortex. If it is sketched in a line in space at time t such that the curve $aecda$ encloses the airfoil but not the wake or the starting vortex, then the circulation around $aecda$ must be equal and opposite to that round $abcea$, since at the initial time instant, the total circulation in the circuit $abcd$ is zero.

Figure 3.3.1: GENERATION OF CIRCULATION BY MEANS OF VORTEX SHEDDING

As the airfoil starts to move, positive vorticity is shed in the form of a starting vortex. By Stokes theorem,

$$\int_S \omega \mathbf{n} dS = \int_C \mathbf{u} dx \quad (3.3.2)$$

where S represents the airfoil surface and C the closed circuit $abcea$, this vorticity gives a positive circulation around $abcea$ (see red arrow 1 in Fig. 3.3.1), which in turn, by Kelvin's circulation theorem, leads to a negative circulation around $aecda$ (see red arrow 2 in Fig. 3.3.1). The vortex shedding continues until the circulation around the airfoil is sufficient to make the main, irrotational flow smooth at the trailing edge, at which stage no further net vorticity is shed into the wake from the boundary layers on the upper and lower surfaces of the airfoil. Thereafter, the airfoil retains its final "Kutta-Joukowski" value of the circulation.

According with the Kutta Joukowski lift theorem, if it is considered a flow which is uniform at infinity with speed U_0 in the x-direction, if the circulation generated around a body is Γ , then the force on the body will be $F = (0, -\rho U_0 \Gamma)$. The next section explains from a mathematical point of view the role of vorticity in the lift generation on an oscillating airfoil.

3.3.2 General Formulation of Acoustic Perturbations

Howe's [54] showed that combining the continuity and momentum equations, and if dissipation is neglected and the flow is homeotropic ($s = s_0$), the following equation for the pressure can be obtained:

$$\left(\frac{D}{Dt} \left(\frac{1}{a^2} \frac{D}{Dt} \right) - \frac{1}{\rho} \nabla \cdot (\rho \nabla) \right) H = \frac{1}{\rho} \nabla \cdot (\rho \omega \wedge \mathbf{v}) \quad (3.3.3)$$

where $H = C_p T + \frac{1}{2} v^2$ is the total enthalpy and ω represents the shed vorticity. For an uniform base flow, $\rho = \rho_0$, $a = a_0$, $\mathbf{v} = \mathbf{v}_0$, the pressure equation can be linearised about the mean flow to yield:

$$\left(\frac{1}{a_0^2} \frac{D^2}{Dt^2} - \nabla^2 \right) \left(\frac{p'}{\rho_0} + v_0 u' \right) = \nabla \cdot (\omega' \wedge \mathbf{v}_0) \quad (3.3.4)$$

which is the equivalent to the classical 2nd order wave equation with a source term. In this context, the sub-index “0” represent the mean flow and the super-index “'” the perturbed variables. The previous expressions clearly shows that when there is a vorticity in the flow (for example, due to the change in the apparent angle of attack for an oscillating airfoil), which is convected downstream with the flow velocity, u , this vorticity is the responsible of the generation of an additional pressure over the airfoil, since it creates a perturbation $\frac{p'}{\rho_0} + v_0 u'$, which is governed by Eq. 3.3.4. This is the ultimate reason for the propagation of pressure perturbations with the flow velocity, as it will be shown later on. This equation can be used in combination with the Stokes theorem to estimate the order of magnitude of the unsteady pressure created by the shed vorticity. The contribution of the unsteady terms in Eq. 3.3.4 can be assumed to be much smaller, or at most of the same order, than the convective terms (i.e.: $\frac{1}{a_0^2} \frac{D^2}{Dt^2} \lesssim \nabla^2$), if the reduced frequency based in the speed of sound $\omega c/a_0 \lesssim 1$. In this case :

$$\nabla^2 \left(\frac{p'}{\rho_0} \right) \sim \frac{\Delta p'_c}{\rho_0 L_c^2}, \text{ and } \nabla \cdot (\omega' \wedge \mathbf{v}_0) \sim \frac{\omega'_c U_c}{L_c} \quad (3.3.5)$$

need to be of the same order. The Stokes theorem provides the order of magnitude of the shed vorticity:

$$\Gamma = \oint \mathbf{u} d\mathbf{x} = \int_{\Sigma} \omega dA \quad (3.3.6)$$

where it can be derived that $\Gamma \sim \omega'_c L_c^2$. Since the force generated by the vorticity onto the airfoils is $F'_c \sim \rho U_c \Gamma'$ then, the order of magnitude for the unsteady pressure is:

$$\frac{\Delta p'_c}{\rho_0 U_c^2} \sim \frac{\omega'_c U_c}{U_c} \sim \frac{F'_c}{\rho_0 U_c^2 L_c} \sim C'_L \quad (3.3.7)$$

This effectively means that the non-dimensional unsteady pressure associated to the vortex shedding, is proportional to the variation of the lift coefficient.

3.4 Unsteady Pressure Generation: The Role of Apparent Mass

The objective of this section is to qualitatively describe the effect on the unsteady aerodynamics of the so-called apparent or virtual mass. To better present this idea, and to ease the formulation, the fluid is supposed to be inviscid and incompressible. Although this is a strong idealization with respect a real case, yet several and useful results can be obtained. The purpose of this sub-section is to discuss this effect in the context of oscillating airfoils.

3.4.1 Incompressible flow equations

The governing equations of the motion for an inviscid and incompressible flow reduce to:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= 0 \end{aligned} \quad (3.4.1)$$

Applying the curl operator to the momentum equation, and taking into account that:

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla \mathbf{u}^2 + \mathbf{u} \wedge \omega \quad (3.4.2)$$

where $\omega = \nabla \wedge \mathbf{u}$ is the vorticity, the following expression is obtained for the momentum equation:

$$\frac{\partial \omega}{\partial t} + \nabla \wedge (\omega \wedge \mathbf{u}) = 0 \quad (3.4.3)$$

The previous equation can be written as:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = (\omega \cdot \nabla) \mathbf{u} \quad (3.4.4)$$

It can be extracted from the last expression that for an ideal, steady, two-dimensional flow, it is $(\mathbf{u} \cdot \nabla) \omega = 0$, and consequently, the vorticity is constant along a streamline, which in fact means that the vorticity propagates with the convection velocity $\mathbf{u}$. It is important to note that the vorticity does not exist for a one-dimensional problem, since it is by definition a two-dimensional concept.

For an irrotational flow ($\omega = 0$) in any connected region, the velocity potential, ϕ , is defined as:

$$\mathbf{u} = \nabla \phi \quad (3.4.5)$$

The continuity and momentum equations can be written in terms of the potential velocity as:

$$\nabla^2 \phi = 0 \quad (3.4.6)$$

$$\rho \frac{\partial}{\partial t} \nabla \phi + \frac{\rho}{2} \nabla |\mathbf{u}|^2 = -\nabla p \quad (3.4.7)$$

The momentum equation can be written as:

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} \right) = 0 \quad (3.4.8)$$

which leads to the Bernoulli equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{u}|^2 + \frac{p}{\rho} = F(t) \quad (3.4.9)$$

Equations 3.4.6 and 3.4.9, together with the appropriate boundary conditions, define the unsteady motion of an incompressible irrotational flow.

3.4.2 Inviscid Moving Sphere

Let us consider a sphere moving with velocity $U(t)$, and a control volume, Ω , surrounding it (see Fig. 3.4.1), the total kinetic energy of the flow surrounding the

Figure 3.4.1: CONTROL VOLUME FOR A MOVING SPHERE

sphere is:

$$KE = \frac{1}{2}\rho \int_{\Omega} u^2 dV = \frac{1}{2}\rho \int_{\Omega} \nabla\phi \nabla\phi dV \quad (3.4.10)$$

which can also be written in terms of the velocity potential taking into account the continuity equation 3.4.6 as:

$$KE = \frac{1}{2}\rho \int_{\Omega} (\nabla\phi \nabla\phi + \phi \nabla^2\phi) dV = \frac{1}{2}\rho \int_{\Omega} \nabla(\phi \nabla\phi) dV = \frac{1}{2}\rho \int_{\Sigma_c} \phi \nabla\phi \cdot \mathbf{n} d\sigma \quad (3.4.11)$$

where the last expression has been obtained by using the Gauss theorem, and Σ_c represent the surfaces delimiting the control volume Ω . Taking into account that the potential field of a sphere is: $\phi = -\frac{UR^3}{2r^2} \cos\theta$, then it is $\nabla\phi \cdot \mathbf{n} = -\phi_r = \frac{UR^3}{r^3} \cos\theta$ on the sphere surface, whereas $\nabla\phi \cdot \mathbf{n} \rightarrow 0$ at infinity, and then, it can be shown that the total energy associated to the sphere motion is:

$$KE = \frac{1}{2}\rho \int_{\Sigma_c} \phi \nabla\phi \cdot \mathbf{n} d\sigma = \frac{1}{2}\rho \int_{\Sigma_c} \frac{U^2 R}{2} \cos^2\theta d\sigma = \frac{\pi\rho U^2 R^3}{3} \quad (3.4.12)$$

The work required to change the kinetic energy is:

$$U F_m = \frac{d(KE)}{dt} = \frac{2\pi\rho UR^3}{3} \frac{dU}{dt} \Rightarrow F_m = \frac{2}{3}\rho\pi R^3 \frac{dU}{dt} = \frac{M_{sphere}}{2} \frac{dU}{dt} \quad (3.4.13)$$

the apparent mass is then equal to one half of the mass displaced by the sphere.

The aerodynamic force could also be computed from the pressure field by means of the integral $\mathbf{F} = \int_{\Sigma} p \mathbf{n} ds$. The pressure is obtained from the Bernoulli equation

3.4.9:

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} = \frac{p_\infty}{\rho_\infty} \quad (3.4.14)$$

where it has been introduced that when $r \rightarrow \infty \Rightarrow \phi_t \rightarrow 0$ and $\nabla\phi \rightarrow 0$. Taking into account that:

$$\phi_t = -\frac{R^3}{2r^2} \frac{dU}{dt} \cos\theta \text{ and } v^2 = (\nabla\phi)^2 = \phi_r^2 + \left(\frac{\phi_\theta}{r}\right)^2 = \left(\frac{UR^3}{r^3}\right)^2 \cos^2\theta + \left(\frac{UR^3}{2r^3}\right)^2 \sin^2\theta \quad (3.4.15)$$

the aerodynamic force can be written as:

$$\frac{F_x}{\rho} = - \int_{\Sigma_{c,sphere}} \left(\phi_t + \frac{1}{2}(\nabla\phi)^2 \right) \cos\theta d\sigma \quad (3.4.16)$$

It can be shown that:

$$\int_{\Sigma_{c,sphere}} \phi_t \cos\theta d\sigma = \frac{\pi R^3}{3} \frac{dU}{dt} \int_0^\pi d(\cos^3\theta) \text{ and } \int_{\Sigma_{c,sphere}} \frac{1}{2}(\nabla\phi)^2 \cos\theta d\sigma = 0 \quad (3.4.17)$$

and then, the following expression for the aerodynamic force is obtained:

$$F_x = \frac{d}{dt} \int_{\Omega} \rho \mathbf{v} d\Omega = \rho \int_{\Sigma_c} \frac{\partial\phi}{\partial t} \mathbf{n} d\sigma = \frac{2}{3} \rho \pi R^3 \frac{dU}{dt}. \quad (3.4.18)$$

Saffman [55] express the aerodynamic force due to the flow about a moving body without circulation (the *virtual mass* contribution) as $\mathbf{F} = \rho \frac{d}{dt} \mathbf{I}_b$, where $\mathbf{I}_b = \int_{\Sigma} \phi \mathbf{n} d\sigma$, which, as it can be readily seen, is nothing else that the contribution of the inertia of the fluid.

These forces are always in-phase with the displacement, out-of-phase with the velocity, and in the case of an harmonic motion, the net contribution to the work-per-cycle of the inertia terms is zero. The order of magnitude of the inertia forces, F_c , of the harmonic motion ($\delta = \delta_{max} \sin(\omega t)$) of a body in an inviscid flow is $F_c \sim \rho R^3 \delta_{max} \omega^2$, which in non-dimensional terms becomes:

$$\frac{F_c}{\rho U_\infty^2 R^2} \sim \frac{\delta_{max}}{R} k^2 \quad (3.4.19)$$

where U_∞ is the free-stream velocity and $k = \omega R / U_\infty$ the reduced frequency. It can be anticipated that inertia contribution to the unsteady pressure can be neglected if $k \ll 1$, and that, what is more, its contribution to the work-per-cycle is zero.

3.5 Unsteady Aerodynamics of Oscillating Flat Plate Cascades

Whitehead [56, 57] developed a method of calculation for linearised two-dimensional subsonic unsteady flow through a cascade of flat plates operating at zero incidence. The cascade, schematically represented in Fig. 3.5.1, is defined by a blade spacing, s , a stagger angle θ , and an inlet Mach number, M . His method is suitable for calculation of forced and self-excited vibration of the blades, and for the generation, transmission, and reflection of acoustic waves by a cascade. The assumptions of his analysis were:

- The flow is two-dimensional and isentropic. The bending modes of the blades are represented by a translation motion, and the torsional modes by a rotation about a known axis
- The blades are not stalled, so that the flow always follows the blade surface
- The blades are treated as flat plates and then, the effect of the camber and the thickness are not taken into account.
- The blades operate at zero mean incidence.
- The amplitude of the perturbations is small, and then:
 1. The wakes of the blades, which are vortex sheets whose strength varies sinusoidally with distance from the trailing edge, can be assumed to be straight
 2. The governing equations are linear, and then any two solutions for the perturbation can be superposed to obtain a third solution.
- All the blades oscillate with the same angular frequency, ω , and with a constant phase change between each blade and its neighbour, σ .

Figure 3.5.1: SKETCH OF THE WHITEHEAD'S CASCADE

- No viscosity effects are retained, and the Kutta-Joukowski condition at the trailing edges of the blades is satisfied.

He found two kinds of acoustic resonances, one corresponding of an annular duct without blades, and the other in the passages between the blades. He obtained conditions for an acoustic wave to propagate, based on the angle between the wave-fronts (lines of constant pressure) and the cascade direction:

$$\sin \zeta = \frac{\Omega - 2\pi n}{\eta s} \quad (3.5.1)$$

where $\Omega = \sigma - \frac{\omega s}{a} \frac{1}{\beta} M \sin \theta$, $\beta = \frac{1}{\sqrt{1-M^2}}$, a is the speed of sound, n is an integer, and $\eta = \frac{\omega}{a\beta}$. Note that in Fig. 3.5.1 the propagation perturbations are represented by pairs of arrows departing from both, the leading edge and the trailing edge of the plate.

For a given cascade geometry, reduced frequency $k = \omega c/U$, and Mach number, M , the range of σ within the perturbations do not decay and propagate at constant

Figure 3.5.2: SUPERRESONANT AND SUBRESONANT REGIONS AS A FUNCTION OF THE IBPA FOR CONSTANT REDUCED FREQUENCY AND MACH NUMBER, LEFT AND RIGHT RESPECTIVELY.

amplitude is given by:

$$\frac{s}{c}k \frac{M}{1-M^2} \left(M \sin \theta - \sqrt{1 - M^2 \cos^2 \theta} \right) < \sigma < \frac{s}{c}k \frac{M}{1-M^2} \left(M \sin \theta + \sqrt{1 - M^2 \cos^2 \theta} \right). \quad (3.5.2)$$

This region is called superresonant (see Fig. 3.5.2). Outside of this range of IBPAs the perturbations decay and the solution is said to be subresonant. The region of superresonant IBPA grows with the reduced frequency and the Mach number. The separation between the propagating and non-propagating solutions are named resonances. Solvers usually experience problem at these regions since high responses at the inlet perturbations are observed. At the point of appearance of the acoustic waves $\zeta = \pm 90^\circ$, so that the direction of energy propagation is along the cascade. This is a resonance condition, and also corresponds to the cut-off condition discussed by Tyler and Soffrin [58].

3.6 Laminar Boundary with an Oscillating Free Stream

The problem of understanding the unsteady behavior of a boundary layer subject to fluctuations in the external flow is very important and has been the object of

several studies [59, 60, 61, 62]. Here, the solution obtained by Lighthill [59] is briefly reviewed. Lighthill obtained an analytical solution for the laminar boundary layer in two-dimensional flow about a flat plate, when the velocity of the oncoming flow relative to the body oscillates in magnitude but not in direction. He studied the limiting cases of high and low oscillation frequencies, ω .

Lighthill obtained expressions for the reduced skin friction coefficient and the phase angle between the external velocity and the wall shear for the cases in which $k = \omega x/V_\infty \ll 1$ and $k \gg 1$, where x is the distance from the leading edge and V_∞ the free stream velocity. He obtained that there is a phase shift between the force and the movement, due to the viscous terms. The main conclusions he obtained are the following:

- The maximum skin friction at any point anticipate the maximum of the stream velocity, because the pressure gradient needed to speed up the main stream locally produces a given percentage increase in the slow flow near the wall sooner than it can do so in the main stream itself.
- For each point on the body surface there is a critical reduced frequency, $k_0 = 0.6$, such that:
 - If $k > k_0$: the oscillations are to a close approximation ordinary “shear waves” unaffected by the mean flow; the phase advance in the skin friction is then 45°
 - If $k < k_0$: the oscillations are closely approximated as the sum of parts proportional to the instantaneous velocity and acceleration of the oncoming stream; the phase advance, ϕ , in the skin friction is then $\tan \phi = k/k_0$.

Figure 3.6.1, extracted from [62], shows the computed phase angle (between the wall shear stress and external velocity) variation with reduced frequency $\omega x/u_0$ for a laminar flat-plate flow. It is concluded that for low reduced frequencies ($k \ll 1$)

Figure 3.6.1: VARIATION OF PHASE ANGLE BETWEEN WALL SHEAR AND OSCILLATING EXTERNAL VELOCITY FOR A LAMINAR FLOW OVER A FLAT PLATE ($\Delta V_\infty/V_\infty = 15\%$)

the phase between the shear stress and the free stream velocity is small ($\phi \ll 1$) and varies linearly with the reduced frequency ($\phi \propto l$)

3.7 Basic Non-dimensional Analysis of Unsteady Aerodynamics

The motion of a blade profile produces an unsteady pressure distribution that in turn will result in an unsteady force on the profile. The order of magnitude of the modulus of unsteady pressure of an airfoil vibrating with an amplitude δ_{max} , and an angular frequency ω , may be easily estimated by using the momentum equation for an inviscid flow:

$$\rho \frac{Dv}{Dt} = -\nabla p \quad (3.7.1)$$

Following the classical approach [53], if it is assumed that the vibration amplitude is small, the unsteady pressure, $p(\mathbf{x}, t)$, can be considered as a mean or time-averaged value, $p_0(\mathbf{x})$, and a time-dependent perturbation value, $p'(\mathbf{x}, t)$, that is considered to

be harmonic. According with this, the unsteady pressure can be expressed as:

$$p(\mathbf{x}, t) = p_0(\mathbf{x}) + p'(\mathbf{x}, t) \quad (3.7.2)$$

The flow may be decomposed in a similar way as:

$$v(\mathbf{x}, t) = v_0(\mathbf{x}) + v'(\mathbf{x}, t) \quad (3.7.3)$$

where the sub-index 0 denotes the mean flow, and the prime the unsteady perturbed variables. Introducing the expressions 3.7.2 and 3.7.3 in Eq. 3.7.1, the order of magnitude of the unsteady perturbation, Δp_c , may be estimated as:

$$\Delta p_c \sim \rho_c U_c \omega \delta_{max} \max(1, k) \quad (3.7.4)$$

were ρ_∞ and U_∞ are the density and the velocity of the free stream, respectively. Consequently, the order of magnitude of the unsteady force in the airfoil is then:

$$F^a \sim \rho_c U_c \omega \delta_{max} S = F_c \quad (3.7.5)$$

where S is the airfoil surface. As it was shown by Crawley, [63], if it is considered a cascade of vibrating airfoils, the unsteady forced in the i^{th} -airfoil may be expressed as:

$$F_i^a = F_c \tilde{F}(k, \sigma, t) \quad (3.7.6)$$

The most distinctive feature of the unsteady aerodynamics of vibrating airfoils in turbomachinery is that the unsteady force in the i^{th} airfoil depends on the motion of all the airfoils, q_j :

$$F_i^a = F_c \tilde{F}_j(k, q_1, q_2, \dots, q_j, \dots, q_N, t) \quad (3.7.7)$$

In Chapter 1, it was explained that in the particular case that the airfoils motion is harmonic in time, and that all the airfoils move with a common inter-blade phase angle, $\sigma_n = \frac{2\pi n}{N}$, the displacement of the airfoil i^{th} is $q_i = \hat{q}_n e^{j(i\sigma_n + \omega t)}$, and then the force in the i^{th} airfoil may be expressed as a summation of the contribution of the different temporal and spatial harmonics:

$$F_i^a = Re(F_c \sum_{\omega} \sum_n \hat{F}_{n,\omega}(\mathbf{q}) e^{j(i\sigma_n + \omega t)}) \quad (3.7.8)$$

Figure 3.7.1: MOVING SURFACE OF AN OSCILLATING AIRFOIL

As it has been previously mentioned, under the assumption that the unsteady perturbations are much smaller than the mean flow, $p'(\mathbf{x}, t) \ll p(\mathbf{x})$, the unsteady aerodynamics is linear. According with this, a necessary condition is then that:

$$\frac{\Delta p_c}{p_c} \sim \frac{\rho_\infty U_\infty \omega \delta_{max}}{p_c} \sim M^2 k \frac{\delta_{max}}{c} \quad (3.7.9)$$

where c is the blade chord. It can be easily seen that the condition for having linear flow is the blade displacements to be small enough, $\delta_{max}/c \ll 1$.

In the present case, it has been considered that the solution is harmonic in time, i.e.: composed in first approximation of a single harmonic, corresponding with the vibration frequency. The contribution of higher harmonics is negligible since they are the results of non-linear interactions and then, the force in the airfoil i may be expressed as:

$$F_i^a = Re(F_c \sum_n \hat{F}_n(\mathbf{q}) e^{j(i\sigma_n + \omega t)}) \quad (3.7.10)$$

The unsteady pressure is also proportional to the vibration amplitude, and therefore it is interesting to scale the blade force with δ_{max} :

$$F_i^a = Re(F_c / \delta_{max} \sum_n \hat{f}_n(\omega) \hat{q}_n e^{j(i\sigma_n + \omega t)}) \quad (3.7.11)$$

The unsteady force Fourier coefficients, $\hat{f}_n$, depend on the reduced frequency and the inter-blade phase angle, $\hat{f}_n(k, \sigma)$, and are complex numbers.

3.7.1 Aerodynamic work-per-Cycle

The work per oscillation cycle exerted by the fluid on the profile determines whether the flow has a stabilizing or a destabilizing character; in case of the fluid putting work

on the structure the situation is referred to as unstable, and the sign criteria used in this thesis refers the work as positive in such cases. The work-per-cycle performed by the aerodynamic forces can be calculated as:

$$W_{aero} = \int_T \left(\int_{\Sigma} p(x, t) \cdot \mathbf{V}_w \cdot \mathbf{n} \cdot dA \right) dt \quad (3.7.12)$$

where p represents the unsteady pressure, $\mathbf{V}_w$ the airfoil motion, and Σ the airfoil surface (see Fig. 3.7.1). If it is assumed that there is no spatial dependency of the pressure and airfoil motion, i.e.: $p = p(t)$, and the same is true for the airfoil motion, $\mathbf{V}_w(x, t) = \mathbf{V}_w(t)$, then the previous formula is simplified and can be written as:

$$W_{aero} = \int_T \mathbf{f}_{aero}(t) \cdot \mathbf{V}_w(t) dt \quad (3.7.13)$$

If we assume now that the motion is harmonic in time, $\mathbf{x}_w = \hat{\mathbf{x}}e^{i\omega t} \Rightarrow \mathbf{V}_w(x, t) = i\omega\hat{\mathbf{x}}_we^{i\omega t}$ and $\mathbf{f} = \hat{\mathbf{f}}e^{i\omega t}$, then the work per cycle becomes $W_{cycle} = \pi(\hat{f}_{aero,I}\hat{x}_R - \hat{f}_{aero,R}\hat{x}_I)$. If we set the time origin such as $x = \hat{x}_R \cos \omega t$, then:

$$W_{cycle} = \pi\hat{f}_{aero,I}\hat{x}_R \quad (3.7.14)$$

This means that the aerodynamic work per cycle is associated to the imaginary, or out-of-phase, component of the forces. In the general case that the pressure and the mode-shape are a function of the position, the same result may be obtained, if the order of the spatial and temporal integral is exchanged.

If it is recovered the expression obtained in the Chapter 2 for the aerodynamic forces, 2.1.7, an expression for the W_{cycle} as a function of the influence coefficients can be obtained. In general, the ICs are not symmetric, $\Delta\tilde{l}^{*+} \neq 0$ and $\Delta\tilde{l}^{*-} \neq 0$, in which case the expression of the aerodynamic forces is:

$$\tilde{f}_{aero} = \tilde{l}_0 + \Delta\tilde{l}^+ \cos \sigma - \Delta\tilde{l}^{*-} \sin \sigma + i \left(\tilde{l}_0^* + \Delta\tilde{l}^{*+} \cos \sigma + \Delta\tilde{l}^- \sin \sigma \right) \quad (3.7.15)$$

Then, the aerodynamic work per cycle can be written in terms of the influence coefficients as:

$$\tilde{W}_{cycle} = \frac{W_{cycle}}{\frac{1}{2}\pi\rho_c v_c^2 \delta_c^2} = \pi \cdot \left(\tilde{l}_0^* + \Delta\tilde{l}^{*+} \cos \sigma + \Delta\tilde{l}^- \sin \sigma \right) \hat{x}_R \quad (3.7.16)$$

where it can be clearly seen that the destabilizing effect comes from the neighbouring

airfoils, while the contribution to the work per cycle of the airfoil itself is always of stable nature. It can also be seen that the mean value of the work per cycle is associated to the effect of the oscillating airfoil in itself. This expression has a deep physical insight and will be exploited in the next chapters of the thesis.

3.7.1.1 Non dimensional aerodynamic work

The non-dimensionalization of the aerodynamic work per cycle is treated deeply in this thesis, the basics and the classical approach are presented here just to provide a complete vision of the unsteady aerodynamic problem. The aerodynamic work-per-cycle, W_{aero} , may be scaled from a purely aerodynamic point of view using the unsteady pressure, estimated in Eq. 3.7.4, as:

$$\Theta = \frac{W_{cycle}}{2\pi \cdot \rho_c U_c \cdot \omega \cdot \delta_{max}^2 \cdot \max(1, k) \cdot S} \quad (3.7.17)$$

Then, according with the classical approach, the non-dimensional aerodynamic work, Θ , depends on the following parameters:

$$\Theta = \Theta(k, \text{Mode} - \text{shape}, \frac{\delta}{c}, M, Re, \alpha_{in}) \quad (3.7.18)$$

where it can be seen that the non-dimensional aerodynamic work depends on the vibration frequency, the mode-shape and the steady flow-field. The understanding of the dependence of the non-dimensional aerodynamic work with these parameters will be the focus of this thesis. A short overview of the common knowledge up to these days is presented here:

- The reduced frequency, k , is usually used as a target design criterion to fix flutter problems, both in the conceptual and in the detailed design phase. Natural frequencies are raised either increasing the blade chord, the thickness, or both. The most powerful mechanism of controlling the reduced frequency is the blade aspect ratio.
- Aerodynamic work sensitivity with the mode-shape is much larger. The aerodynamic work per cycle, W_{cycle} , changes strongly with the inter-blade phase

angle, $IBPA$, but even when focusing in the most critical $IBPA$, which is the minimum of the $W_{cycle} - IBPA$ curves and represents the least stable situation, the position of the center of torsion of the airfoil may dramatically change the aerodynamic work. The most clear example is the change in aerodynamic work per cycle between flap and edge modes, as it can be seen in Fig. 3.7.2, which represent the non-dimensional work per cycle curves as a function of the $IBPA$ for the flap and edge modes for an LPT for a reduced frequency of $k = 0.1$. It can be seen that, for the same reduced frequency, the flap configuration is much more unstable (about five times) than the edge configuration, and that the sign of the $IBPA$ at which occurs the minimum of the work per cycle is the opposite. The location of the torsion center must be carefully controlled to drive the designs in the right direction.

- Aerodynamic work is, in general, insensitive to the vibration amplitude except in particular non-linear problems, such as shock flutter, where the aerodynamic work may even change of sign with the vibration amplitude.
- Steady aerodynamic flow-field obviously change the aerodynamic work per cycle:
 - For a given application the variability of the Mach number is reduced, since this is fixed by purely aerodynamic criteria, and it is not a free parameter.
 - Sensitivity of the aerodynamic work to the Reynolds number is small, since the unsteady pressures caused by the airfoil motion are mainly an inviscid phenomenon. An exception is the case of compressor stall flutter, where separation is controlled by viscous effects.
 - Incidence change caused by shaft speed variation may heavily change the flow-field and hence the aerodynamic work per cycle, especially in compressors.

The aerodynamic work may also be scaled assuming that it will take part of the airfoil motion equations, which for a single degree of freedom system take the following form:

Figure 3.7.2: NON-DIMENSIONAL WORK-PER-CYCLE AS A FUNCTION OF THE IBPA FOR THE FLAP AND EDGE MODES FOR $k = 0.1$

$$m\ddot{q} + c\dot{q} + kq = 0 \quad (3.7.19)$$

where m is the modal mass, c the viscous damping, and k the airfoil stiffness. Which can be non-dimensionalised by using $\tau = \omega_n t$ and $\dot{q} = q' \omega_n$ to obtain:

$$q'' + 2\xi q' + q = 0 \quad (3.7.20)$$

where $\xi = c/2m\omega_n$ is the critical damping ratio. Viscous damping, c , is related with the aerodynamic forces as $c = \frac{1}{\omega_n} \text{Im} \left(\hat{F}_a / \hat{q}_R \right)$, where $\hat{q}_R$ is the real part of the displacement, that in practice may be translated as the maximum displacement of the airfoil, $\hat{q}_R = \delta_{\max}$. Taking into account that the imaginary part of the aerodynamic force and the aerodynamic work are related by $\hat{F}_a = W_{\text{aero}} / \hat{q}_R$, the critical damping ratio reduces to:

$$\xi = \frac{1}{m\omega_n^2} \frac{W_{\text{aero}}}{2\pi\delta_{\max}^2} \quad (3.7.21)$$

which is another form of non-dimensionalising the aerodynamic work. The aerodynamic work is usually obtained from an unsteady aerodynamic simulation while m , is the modal mass. It can be seen that the critical damping ratio and the non-dimensional work are related by the expression:

$$\xi = \Theta \rho U_c S / m \omega_n = \mu^{-1} \Theta / k \quad (3.7.22)$$

where $\mu = m/\rho cS$ is a measure of the ratio between the mass of the blade and the mass of the air surrounding the airfoil, $\mu \sim m_{blade}/m_{air}$. It must be noticed that a change of the blade material density may change the critical damping ratio, while leaving unaffected the non-dimensional aerodynamic work-per-cycle, which is a purely aerodynamic quantity.

Chapter 4

Analytical and Numerical Models

It is fatal to enter any war without
the will to win it

General Douglas MacArthur

Contents

4.1	Analytical Model	69
4.2	Numerical Model	74

4.1 Analytical Model

4.1.1 Governing Equations

The three-dimensional Reynolds averaged Navier-Stokes equations in conservative form for an arbitrary deformable control volume, in the absence of body forces and heat sources, can be written as:

$$\frac{d}{dt} \int_{\Omega} \mathbf{U} d\Omega + \int_{\Sigma} (\mathbf{F} - \mathbf{U} \frac{\partial f}{\partial t}, \mathbf{G} - \mathbf{U} \frac{\partial g}{\partial t}, \mathbf{H} - \mathbf{U} \frac{\partial h}{\partial t}) d\mathbf{A} = 0 \quad (4.1.1)$$

where $\mathbf{U}$ is the vector of conservative variables, $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ the sum of the inviscid and viscous fluxes, Ω the flow domain, Σ its boundary, $d\mathbf{A}$ the differential

area pointing outward to the boundary and $\partial f/\partial t$, $\partial g/\partial t$, and $\partial h/\partial t$, the three components of the boundary velocity, $\mathbf{v}_c$. The vectors of conservative variables, and convective fluxes are given by:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, F = \begin{bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho uv - \tau_{xy} \\ \rho uw - \tau_{xz} \\ \rho uH - \hat{\tau}_{hx} \end{bmatrix}, G = \begin{bmatrix} \rho v \\ \rho uv - \tau_{xy} \\ \rho v^2 + p - \tau_{yy} \\ \rho vw - \tau_{yz} \\ \rho vH - \tau_{hy} \end{bmatrix}, H = \begin{bmatrix} \rho w \\ \rho uw - \tau_{xz} \\ \rho vw - \hat{\tau}_{yz} \\ \rho w^2 + p - \tau_{zz} \\ \rho wH - \tau_{hz} \end{bmatrix} \quad (4.1.2)$$

where ρ is the density, u , v and w are the velocity components in the x-, y- and z-directions respectively, E is the total energy, H is the total enthalpy, and p is the static pressure. The fluid is assumed to be a perfect gas so that the total energy E is related to the total enthalpy H by:

$$H = E + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2} (u^2 + v^2 + w^2) \quad (4.1.3)$$

The viscous shear stresses contributing to the momentum equation can be written in compact form as:

$$\tau_{\xi\eta} = (\mu + \mu_t) \left(2\gamma_{\xi\eta} - \frac{2}{3}\delta_{\xi\eta}\nabla \cdot \mathbf{v} \right) \text{ with } \gamma_{\xi\eta} = \left(\frac{\partial v_\xi}{\partial \eta} + \frac{\partial v_\eta}{\partial \xi} \right) \quad (4.1.4)$$

where μ and μ_t are the molecular and the turbulent viscosity, respectively, $\mathbf{v}$ the velocity, ξ and η the x, y, and z directions, and $\delta_{\xi\eta}$ the Kronecker delta. The viscous and heat conduction contribution to the energy equation can be written in compact form as:

$$\tau_{h\xi} = u\tau_{\xi x} + v\tau_{\xi y} + w\tau_{\xi z} + (k + k_t) \frac{\partial T}{\partial \xi} \quad (4.1.5)$$

where $k_t = \mu_t c_p / Pr_t$, is the turbulent conductivity, Pr_t the turbulent Prandtl number, and T the temperature. The turbulent viscosity is computed by means of a turbulence model.

4.1.2 Linearisation of the Governing Equations

When the amplitude of vibration of the airfoils is small, it is sufficient to model the small-disturbance behaviour of the unsteady flow, obtaining the linear equations. Because of the linearity of the equations, the unsteady solutions may be superposed. Without loss in generality, it is assumed that the motion of the blades is harmonic in time with temporal angular frequency ω , and that the airfoils vibrate with fixed constant inter-blade phase angle σ . As it was show by [64], the use of a harmonically deforming computational grid which conforms to the motion of the vibrating airfoils greatly improves the accuracy of the computational scheme. Using this approach, the computational coordinates, $\mathbf{x}_0$, are defined. The coordinates of the control volumes correspond with that of the mean position. The computational grid is fixed in the computational coordinate system, but undergoes harmonic deformations in the physical coordinate system, $\mathbf{x}$. The relation between the two coordinate systems, or the deformed and undeformed is given by the following transformation:

$$\mathbf{x} = \mathbf{x}_0 + \hat{\mathbf{x}}' e^{i\omega t} = \mathbf{x}_0 + (\hat{f}(\mathbf{x}_0), \hat{g}(\mathbf{x}_0), \hat{h}(\mathbf{x}_0)) e^{i\omega t} \quad (4.1.6)$$

The small perturbation functions, $\hat{f}$, $\hat{g}$ and $\hat{h}$ are chosen such that the motion of the grid at the airfoil boundaries conforms to the prescribed motion of the airfoils, and are in general complex allowing for phase differences between adjacent airfoils. Analogously, the volume and area of the conservation equations can be written as:

$$\Omega = \Omega_0 + \hat{\Omega}'(\mathbf{x}_0) e^{i\omega t} \text{ and } \mathbf{A} = \mathbf{A}_0 + \hat{\mathbf{A}}'(\mathbf{x}_0) e^{i\omega t} \quad (4.1.7)$$

In a similar way, the flow field may be decompose into two parts: a mean or steady flow, plus a small harmonic perturbation part:

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{U}_0(\mathbf{x}) + \text{Re}(\hat{\mathbf{u}}(\mathbf{x}) e^{i\omega t}) \quad (4.1.8)$$

where $\mathbf{U}_0$ is the vector of conservation variables representing the zeroth-order or mean flow field, and $\hat{\mathbf{u}}$ is the vector of complex amplitudes of the perturbation conservation

variables. The flux vectors are then:

$$\begin{aligned}\mathbf{F}(\xi, \eta, \tau) &= \mathbf{F}_0(\mathbf{U}_0) + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}} e^{j\omega t} \\ \mathbf{G}(\xi, \eta, \tau) &= \mathbf{G}_0(\mathbf{U}_0) + \left(\frac{\partial \mathbf{G}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}} e^{j\omega t} \\ \mathbf{H}(\xi, \eta, \tau) &= \mathbf{H}_0(\mathbf{U}_0) + \left(\frac{\partial \mathbf{H}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}} e^{j\omega t}\end{aligned}\tag{4.1.9}$$

The first term of the right-hand side of the previous equations represents the zeroth-order (mean flow) flux vectors, and the second term is the first-order perturbation due to the unsteadiness in the conservation flow variables, $\hat{\mathbf{u}}$. The mean flow equations are given by:

$$\int_{\Sigma} (\mathbf{F}_0, \mathbf{G}_0, \mathbf{H}_0) \cdot d\mathbf{A}_0 = 0 \tag{4.1.10}$$

and the small-perturbation unsteady flow equations are given by:

$$i\omega \int_{\Omega} \hat{\mathbf{u}} d\Omega + \int_{\Sigma} \left[\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}}, \left(\frac{\partial \mathbf{G}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}}, \left(\frac{\partial \mathbf{H}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}} \right] \cdot d\mathbf{A}_0 = \mathcal{F}(\mathbf{U}_0, \omega, \hat{\mathbf{x}}') \tag{4.1.11}$$

The resulting set of equations for $\hat{\mathbf{u}}$ is essentially identical to that of the standard constant control volume integral Navier-Stokes equations with a forcing term, $\mathcal{F}(\mathbf{U}_0, \omega, \hat{\mathbf{x}}')$, that depends on the mean flow-field, the frequency, and the actual shape of the motion. More specifically:

$$\mathcal{F}(\mathbf{U}_0, \omega, \hat{\mathbf{x}}') = -i\omega \int_{\Omega} \mathbf{U}_0 d\hat{\Omega}' - \int_{\Sigma} (\mathbf{F}_0, \mathbf{G}_0, \mathbf{H}_0) \cdot d\hat{\mathbf{A}}' + i\omega \int_{\Sigma} \mathbf{U}_0 \hat{\mathbf{x}}' \cdot d\mathbf{A}_0 \tag{4.1.12}$$

where the first term represents the variation of the control volume, the second is the variation of the convective fluxes due to the area variation, and the third is the contribution of the velocity of the control volume.

4.1.3 Boundary Conditions

Taking into account that it has been assumed that the blades vibrate with constant frequency, ω , and inter-blade phase angle, σ , the periodicity condition in a single passage has to be generalised to account for the phase-lag between both sides of the boundary. The boundary condition in physical space and cylindrical coordinates

read:

$$\mathbf{U}(x, r, \theta, t) = \mathbf{U}(x, r, \theta + \theta_p, t - \Delta T) \quad (4.1.13)$$

that essentially states that the time signal in one of the periodic boundaries at time t , is the same than the signal in the other periodic boundary, $\theta + \theta_p$, where θ_p is the airfoil pitch, at a previous time instant, $t - \Delta T$,

$$\Delta T = \frac{2\pi n}{N} \frac{1}{\omega} = \frac{\sigma}{\omega} \quad (4.1.14)$$

where n is the nodal diameter, and N the number of airfoils. It is important to highlight that this boundary condition holds as well for non-linear single-passage Navier-Stokes simulations. The unsteady perturbed flow satisfies the complex periodicity condition:

$$\hat{\mathbf{u}}(x, r, \theta + \theta_p) = \hat{\mathbf{u}}(x, r, \theta + \theta_p) e^{j\sigma} \quad (4.1.15)$$

where θ_p is the distance between two adjacent blades. The periodicity conditions allows the computations domain to be reduced to a single passage, hence reducing the size of the computational domain, resulting in a considerable reduction of the computational time.

On the airfoil surface, the flow must satisfy the no-slip condition, which states that the velocity of the fluid must equal the velocity of the airfoil surface. The position of the airfoil at an arbitrary time instant, t , is given by the position vector of the airfoil, $\mathbf{x}_a(s, t)$, where s is the distance along the airfoil surface, and it can also be represented by means of perturbation series, in a similar way that the flow field:

$$\mathbf{x}_a(s, t) = \mathbf{x}_{a0}(s) + \hat{\mathbf{x}}'_a(s) e^{j\omega t} \quad (4.1.16)$$

The first order approximation of the no-slip condition is then:

$$\hat{\mathbf{v}}(s) = j\omega \hat{\mathbf{x}}'_a(s) \quad (4.1.17)$$

where $\hat{\mathbf{v}}$ is the perturbation flow velocity on the airfoil surface.

Non-reflecting boundary conditions based on the theory of characteristics are used at the inflow and the outflow.

4.2 Numerical Model

During the last decades, strong effort has been made to developing computational tools for modeling the unsteady flow in turbomachines, with special focus on the prediction of aeroelastic phenomena, such as flutter or forced response. The earliest efforts ([56], [65], [66], [67]) were based on semi-analytical formulations under some hypotheses (inviscid, incompressible, and irrotational flows). Despite the limitation of these methods to reproduce real situations, they helped to understand some basic mechanisms of the unsteady flow and were widely used in aerodynamic design. Nowadays, modern computational techniques for solving the unsteady flow can be divided in: nonlinear time-marching ([68], [69], [70]), linearized frequency domain ([71], [72], [73], [74], [75]) and non-linear frequency domain methods ([76], [77]).

After all these years of research, it has been shown that linearised frequency domain methods offer a good compromise between computational efficiency and flow accuracy in the absence of highly non-linear phenomena such as strong shock waves or stall conditions. Since the focus of the thesis is not any of the aforementioned problems, a linear frequency domain solver has been used in the numerical validation of the theoretical framework. This numerical code will be briefly described in the following lines.

4.2.1 Numerical Formulation

The suitability of using the linearized Navier-Stokes equations for predicting aerodynamic damping and its range of validity has been demonstrated many times using non-linear simulations [78, 79], and experiments [32, 80]. The range of validity of linearised methods is surprisingly large, even in the presence of shock-waves, provided that the vibration amplitude remains small. [81]

The main advantage of frequency domain linearised methods is two-fold. Firstly, the governing equations do not need to be marched in physical time until a periodic state is reached, and therefore it is much more computationally efficient than a standard

Figure 4.2.1: TYPICAL HYBRID-CELL GRID AND ASSOCIATED DUAL MESH

non-linear method. Secondly, phase-shifted boundary conditions allow the reduction of the computational domain to a single-passage. Both fact together deliver a strong reduction in the computational time with respect classical non-linear time-marching methods. Moreover, the formulation of the problem in the Fourier space allows the implementation of more accurate boundary conditions in the inlet and outlet boundaries.

The linearised discrete set of equations 4.1.11 can be mathematically seen as a system of linear equations with complex coefficients. However, if a pseudo-time, τ , derivative is added, Eq. 4.1.11 can be expressed in compact form as:

$$\left(\frac{d}{d\tau} + i\omega \right) \int_{\Omega} \hat{\mathbf{u}} d\Omega + \int_{\Sigma} \left[\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}}, \left(\frac{\partial \mathbf{G}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}}, \left(\frac{\partial \mathbf{H}}{\partial \mathbf{U}} \right)_0 \hat{\mathbf{u}} \right] . d\mathbf{A}_0 = \mathcal{F}(\mathbf{U}_0, \omega, \hat{\mathbf{x}}') \quad (4.2.1)$$

that can be solved marching in the pseudo-time τ to obtain the steady-state solution, $\hat{\mathbf{u}}$, using the same numerical techniques as the non-linear solver. The frozen turbulent viscosity approach is followed.

4.2.1.1 Spatial Discretisation

The code used in the thesis, $Mu^2s^2T - L$, solves the three-dimensional Navier-Stokes equations in conservative form [32, 82] and has been well validated for vibrating airfoils [32, 80]. The spatial discretisation is obtained linearising the discretised equations of the non-linear version of the code Mu^2s^2T [83], from which the background solution is obtained. The spatial domain is discretised using hybrid

Figure 4.2.2: CLOSE-UP OF THE GRID ABOUT AN LPT AIRFOIL.

unstructured grids, that can contain cells with an arbitrary number of faces, and the solution vector is stored at the vertex of the cells. The code uses an edge-based data structure, a typical grid is discretised by connecting the median dual of the cells surrounding an internal node (Figure 4.2.1). For the node i , the semi-discrete form of Eq.4.2.1 can be written as

$$\frac{d(\Omega_i \hat{\mathbf{u}}_i)}{d\tau} + \sum_{j=1}^{n_{edges}} \frac{1}{2} S_{ij} (\hat{\mathbf{F}}_i + \hat{\mathbf{F}}_j) - \hat{\mathbf{D}}_{ij} = \hat{S}(\hat{\mathbf{u}}_i) \quad (4.2.2)$$

where S_{ij} is the area associated with the edge ij , and n_{edges} the number of edges that surround node j . The resulting numerical scheme is cell centered in the dual mesh, and second-order accurate. It can be shown that for triangular grids the scheme is equivalent to a cell vertex finite volume scheme. A close-up of an example of the type of grids used in this thesis may be seen in Fig. 4.2.2.

A blend of second and fourth order artificial dissipation terms, $\hat{\mathbf{D}}_{ij}$, is added to capture shock waves and prevent the appearance of high frequency modes in smooth flow regions, respectively. The second-order terms are activated in the vicinity of shock waves by means of a pressure-based sensor, and locally the scheme reverts to first order in these regions. The artificial dissipation terms can be written as:

$$\hat{\mathbf{D}}_{ij} = |A_{ij}| S_{ij} [\mu_{ij}^{(2)} (\hat{\mathbf{u}}_j - \hat{\mathbf{u}}_i) - \mu_{ij}^{(4)} (L_j - L_i)] \quad (4.2.3)$$

where $\mu_{ij}^{(2)} = 0.5(\mu_i^{(2)} + \mu_j^{(2)})$ and $\mu_{ij}^{(4)} = 0.5(\mu_i^{(4)} + \mu_j^{(4)})$ are the average of the artificial

viscosity coefficients in the nodes i and j , which are given by:

$$\mu_i^{(2)} = \min(\epsilon_2, k_2 \delta_i), \quad \mu_i^{(4)} = \min(0, \epsilon_4 - k_4 \delta_i) \quad (4.2.4)$$

where δ_i is a pressure based sensor:

$$\delta_i = \frac{|\sum_{j=1}^{n_{edges}} (\hat{p}_j - \hat{p}_i)|}{\sum_{j=1}^{n_{edges}} (\hat{p}_j + \hat{p}_i)} \quad (4.2.5)$$

and ϵ_2 , k_2 , ϵ_4 and k_4 are constants. Typically $\epsilon_2 = 0.5$ and $\epsilon_4 = \frac{1}{128}$. L is a pseudo-Laplacian operator:

$$L(\hat{\mathbf{u}}_i) = \sum_{j=1}^{n_{edges}} (\hat{\mathbf{u}}_j - \hat{\mathbf{u}}_i) \simeq \frac{n_{edges}}{4} (\Delta x^2 \hat{\mathbf{u}}_{xx} + \Delta y^2 \hat{\mathbf{u}}_{yy})_i \quad (4.2.6)$$

where the last approximation is only valid in regular grids. $|A_{ij}|$ is a 5x5 matrix that plays the role of a scaling factor. If $|A_{ij}| = (|u| + c)_{ij} I$, where I is the identity matrix, the standard scalar formulation of the numerical dissipation terms [84] is recovered. When $|A_{ij}|$ is chosen as the Roe matrix [85] the matricial form of the artificial viscosity [86] is obtained. The scalar version of the numerical diffusion terms has been used in this work since the difference between both approaches is negligible for the Mach numbers of interest [87] and is more robust.

The viscous fluxes can be evaluated in a multiple number of ways. The gradients of the flow variables are approximated at the nodes using the divergence theorem in the same way that the convective fluxes are computed. The approximation of the gradients at the midpoint of the edge is obtained by a simple average:

$$\overline{\nabla \mathbf{U}}_{ij} = \frac{1}{2} (\nabla \mathbf{U}_i + \nabla \mathbf{U}_j) \quad (4.2.7)$$

4.2.1.2 Temporal Discretisation

Equation 4.2.1 can be expressed in compact form as

$$\frac{d(\Omega_i \hat{\mathbf{u}}_i)}{d\tau} = \mathbf{R}(\hat{\mathbf{u}}) = \mathbf{C}(\hat{\mathbf{u}}) + \mathbf{H}(\hat{\mathbf{u}}) + \mathbf{S}(\hat{\mathbf{u}}) \quad (4.2.8)$$

where the residual can be split in the convective, $\mathbf{C}$, numerical diffusion and viscous, $\mathbf{H}$, parts and a source term $\mathbf{S}$. The integration in time is performed using an explicit five stage Runge-Kutta scheme, where the artificial viscosity terms are evaluated only

in three stages of the Runge-Kutta.

The scheme is second-order accurate in time. Implicit residual smoothing and full coarsening multigrid can be used to increase the stability limit of the scheme and speed up its convergence. A more detailed description of the numerical scheme and details of the validation process can be found in [82].

4.2.1.3 Boundary Conditions

The exact 2D unsteady non-reflecting boundary conditions [88] have been used at the inlet and outlet (see Fig. 4.2.3) in a plane by plane basis with the aim of preventing spurious reflections. At the solid walls half volumes associated with the wall nodes are used to march in pseudo time. For viscous cases, the velocity is imposed at the boundary nodes. If we assume that the steady flow is uniform at the boundary, the solution may be decomposed in Fourier series as follows:

$$\mathbf{u}(x, y, t) = \left[\sum_{n=1}^N a_n \mathbf{u}_n^R e^{ik_n x} \right] e^{i(l y - \omega t)} \quad (4.2.9)$$

where l is the corresponding y-Fourier mode, k_n is the n -th root of the dispersion relation and $\mathbf{u}_n^R$ is the corresponding right eigenvector. The exact non-reflecting boundary conditions impose that $a_n = 0$ for each n corresponding to an incoming wave. This condition is equivalent to:

$$\mathbf{v}_n^L \mathbf{u} = 0 \quad (4.2.10)$$

where $\mathbf{v}_n^L$ is the left eigenvector. In the code an invariant decomposition is used to discriminate which waves are entering the computational domain and a Fourier decomposition in y is used to calculate each harmonic reflection-free coefficient. The analytical equations, instead of the numerical approximation of them, are used in the formulation of the problem.

The code uses phase-shifted boundary conditions in the periodic boundaries (see Fig. 4.2.3) to simulate annular cascades of airfoils vibrating sinusoidally with a common angular frequency, ω , and common Inter Blade Phase Angle, IBPA. In the Fourier

Figure 4.2.3: SCHEMATIC SHOWING COMPUTATIONAL DOMAIN AND BOUNDARY CONDITIONS

space they may be written as:

$$\hat{\mathbf{u}}(x, r, \theta) = \hat{\mathbf{u}}(x, r, \theta + \theta_p) e^{j\sigma} \quad (4.2.11)$$

which is basically an exact analytical boundary condition. Where σ is the IBPA and θ_p is the blade pitch.

Chapter 5

Low Reduced Frequency Limit Analysis

There is nothing so practical as a
good theory

James Clerk Maxwell

Contents

5.1	State of the Art	81
5.2	Asymptotic Analysis at Low Reduced Frequency	82
5.3	Calculus of the Unsteady Loading Parameter	92
5.4	Simplified Analytical Model for the Work-per-cycle . . .	100
5.5	Review of Theodorsen's Theory	105
5.6	Conclusions of the Analysis at Low Reduced Frequency	109

5.1 State of the Art

After many years of research, there are three well known parameters correlated to LPT flutter, that are commonly considered when designing LPT blades: the reduced

frequency [6], the steady conditions, and the vibration mode-shape [89, 5]. Panovsky and Kielb [90], created a preliminary design tool for LPT blades, the Tye-Dye (TD) plots, which combined the interdependency of critical reduced frequency (the limit at which the configuration is stable) with vibration mode shape all in a single plot.

There are a number of well know trends of the work-per-cycle curves related with the aforementioned parameters that are commonly accepted and have been proved several times. Among this widely known trends it can be mentioned: the stabilizacion of the work per cycle curves with the reduced frequency, and the differences observed in the shape of the work per cycle curves when considering different vibration mode-shapes. An example of the second trend is that flap modes (and usually torsion modes) are much more unstable than the edge modes [5, 1].

All this facts are based on experience, and there is a lack of sound theoretical support, and/or physical understanding of the basic mechanisms which govern the behavior of the work per cycle curves (which in turn represents the stability of a configuration) with the reduced frequency and the mode-shape.

As a consequence, in recent years, there is a renewed interest in understanding the fundamentals of the aerodynamics of oscillating airfoils, and several authors [91, 92, 93, 94, 95, 96] have made strong efforts to throw some light in the subject; especially in the context of turbine flutter, which is considered the simplest flutter instability of all.

5.2 Asymptotic Analysis at Low Reduced Frequency

5.2.1 Governing Equations

The time-dependent Navier-Stokes equations of an ideal gas in conservative and compact form can be written as:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}_c(\mathbf{U}) = \nabla \cdot \mathbf{F}_v(\mathbf{U}) \quad (5.2.1)$$

where $\mathbf{U} = \{\rho, \rho\mathbf{v}, \rho E\}^T$ is the vector of conservative variables and $\mathbf{F}_c$ and $\mathbf{F}_v$ are the convective and viscous fluxes respectively. Without loss of generality the simplest boundary conditions for a two-dimensional airfoil cascade could be written as: $P_t = P_0$, $T_t = T_0$ and $\alpha = \alpha_{in}$ at the inlet; $p = p_{exit}$ at the exit, null velocity, $\mathbf{v} = 0$, and heat flux, $\partial T / \partial n = 0$, are prescribed at the airfoil's solid walls whereas periodic boundary conditions are enforced at the lateral boundaries of the airfoil passage.

The above equations may be non-dimensionalised using $p_c = P_0$, $T_c = T_0$ and $v_c^2 = 2c_p T_0(1 - (p_{exit}/P_0)^{(\gamma-1)/\gamma})$ as the characteristic pressure, temperature and velocity respectively. The characteristic velocity used here is the isentropic exit velocity that may be readily computed from the boundary conditions. The airfoil chord, c , is used as the characteristic length, L_c . Then, choosing the inverse of ω as the characteristic time, $t_c = \omega^{-1}$, Eq. 5.2.1 can be written in non-dimensional form as:

$$k \frac{\partial \tilde{\mathbf{U}}}{\partial \tau} = -\tilde{\nabla} \cdot \mathbf{F}_c(\tilde{\mathbf{U}}) + \frac{1}{Re} \tilde{\nabla} \cdot \mathbf{F}_v(\tilde{\mathbf{U}}) = -\tilde{\nabla} \cdot \mathbf{F}(\tilde{\mathbf{U}}) \quad (5.2.2)$$

where $\tau = \omega t$ is the non-dimensional time, $\tilde{\mathbf{x}} = \mathbf{x}/c$ the non-dimensional distance and $\tilde{\mathbf{U}} = \{\tilde{\rho}, \tilde{\rho}\tilde{\mathbf{W}}, \tilde{\rho}\tilde{E}\}^T$ the vector of non-dimensional conservative variables, with $\tilde{\rho} = \rho/\rho_0$, $\tilde{p} = p/P_0$, $\tilde{\mathbf{W}} = \mathbf{v}/v_c$ and $\tilde{E} = \frac{1}{2\gamma}\tilde{T}/(1 - \Pi_R^{-(\gamma-1)/\gamma}) + \frac{1}{2}\tilde{\mathbf{W}}^2$. The non-dimensional boundary conditions become $\tilde{P}_t = 1$, $\tilde{T}_t = 1$ and $\alpha = \alpha_{in}$ at the inlet and $\tilde{p} = \Pi_R^{-1}$ at the exit where $\Pi_R = P_0/p_{exit}$ is the pressure ratio of the row or the cascade. The non-dimensional solution of the problem, for a given geometry, depends on:

$$\tilde{\mathbf{U}} = f(\tilde{\mathbf{x}}, \tau, \Pi_R Re, k, \gamma,) \quad (5.2.3)$$

where Re is the Reynolds number based on the characteristic velocity and the chord, and k is the so-called reduced frequency:

$$k = \frac{\omega c}{v_c} \quad (5.2.4)$$

which represents the ratio between the flow-through time and the characteristic time of the motion, or the inverse of the frequency if the motion is harmonic.

Figure 5.2.1: POSITION OF THE AIRFOIL IN AN ARBITRARY TIME INSTANT

For a non-vibrating airfoil the no-slip boundary condition at the solid wall reduces to $\tilde{\mathbf{W}} = 0$ (or $\tilde{\mathbf{W}} \cdot \mathbf{n} = 0$ if the flow is inviscid). However, for moving walls this boundary condition has to be generalized to specify that the velocity of the fluid and the airfoil are the same at the airfoil surface, $\mathbf{x}_c(\xi, t)$. For an inviscid flow, the no-penetration boundary condition in non-dimensional form is:

$$\tilde{\mathbf{W}}(\tilde{\mathbf{x}}_c) \cdot \mathbf{n}(\tilde{\mathbf{x}}_c) = k \cdot \frac{d\tilde{\boldsymbol{\delta}}}{d\tau} \cdot \mathbf{n} \quad (5.2.5)$$

where $\tilde{\boldsymbol{\delta}} = \tilde{\mathbf{x}}_c - \tilde{\mathbf{x}}_{c0}$ is the displacement of the airfoil with respect a reference position, $\mathbf{x}_{c0}(\xi)$, typically the initial or the mean position of the airfoil if the motion is harmonic (see Fig. 5.2.1). The reduced frequency appears both in the governing equations and in the boundary condition. It is important to note that (i) if $k \ll 1$ then at the wall $\tilde{\mathbf{W}} \simeq 0$, or $\tilde{\mathbf{W}}(\tilde{\mathbf{x}}_c) \cdot \mathbf{n}(\tilde{\mathbf{x}}_c) \simeq 0$, exactly as in a non-vibrating airfoil, and that the time derivatives may be neglected in first approximation in the governing equations, (ii) the no-penetration boundary condition (Eq. 5.2.5) is a Lagrangian description of the problem since it is specified at the moving boundary, $\mathbf{x}_c$.

5.2.2 Linearization of the Governing Equations

The non-dimensional governing equations (Eq.5.2.2) and their boundary conditions can be simplified if it assumed that the solution may be decomposed as the sum of a

mean baseline flow, $\tilde{\mathbf{U}}_0(\tilde{\mathbf{x}})$, plus a small unsteady perturbation, $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tau)$:

$$\tilde{\mathbf{U}}(\tilde{\mathbf{x}}, \tau) = \tilde{\mathbf{U}}_0(\tilde{\mathbf{x}}) + \tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tau) \quad (5.2.6)$$

such that $\tilde{\mathbf{u}} \ll \tilde{\mathbf{U}}_0$. It may be shown that this approximation is valid if the vibration amplitude, $\|\boldsymbol{\delta}\|$, is much smaller than the airfoil's chord, i.e.: $\|\boldsymbol{\delta}\| \ll c$, and highly non-linear phenomena, such as shock-waves and separation bubbles, are not present in the mean flow-field. Under these hypotheses the Navier-Stokes equations may be linearized about the mean flow to yield:

$$k \frac{\partial \tilde{\mathbf{u}}}{\partial \tau} + \frac{\partial}{\partial \tilde{\mathbf{x}}} \left(A_0(\tilde{\mathbf{U}}_0) \tilde{\mathbf{u}} \right) = 0 \quad (5.2.7)$$

where $A_0 = (\partial \tilde{\mathbf{F}} / \partial \tilde{\mathbf{U}})_0$ is the non-dimensional Jacobian matrix, and $\tilde{\mathbf{U}}_0$ satisfies $\nabla \cdot \tilde{\mathbf{F}}(\tilde{\mathbf{U}}_0) = 0$. The linearization of the wall boundary conditions deserves special attention and it is addressed in the following sub-section.

5.2.3 Linearization of the no-penetration boundary condition

It has been previously mentioned than the airfoil's no-slip boundary condition (Eq. 5.2.5) is expressed in Lagrangian form because it is imposed in a moving boundary. Linearized Navier-Stokes codes designed to deal with vibrating airfoils usually implement the no-slip boundary condition, and the rest of the algorithm, in a fixed grid, but implicitly compute the forcing terms in a moving grid. Non-linear codes implement the whole algorithm in moving grids, and the wall boundary condition is implemented exactly as specified in Eq. 5.2.5 (see for instance [75, 32]).

However for analytical studies this form hides relevant information. When $k \ll 1$, the condition reduces to $\tilde{\mathbf{W}} \simeq 0$ at $\tilde{\mathbf{x}} = \tilde{\mathbf{x}}_c(\tau)$, concealing the fact that there is a quasi-steady displacement of the airfoil that makes the airfoil loading change with time. This trivial observation is not easy to explicitly state from a mathematical point of view within this formulation.

Alternatively, the no-penetration boundary condition may be linearized and explicitly expressed in Eulerian form in the same way that was done by Verdon [42] in the resolution of the transonic unsteady potential equation. The non-dimensional fluid

velocity field may be expressed as:

$$\tilde{\mathbf{W}}(\tilde{\mathbf{x}}, \tau) = \tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}) + \tilde{\mathbf{w}}(\tilde{\mathbf{x}}, \tau) \quad (5.2.8)$$

where $\tilde{\mathbf{W}}_0(\tilde{\mathbf{x}})$ represents the mean velocity and $\tilde{\mathbf{w}}(\tilde{\mathbf{x}}, \tau)$ the perturbed velocity, with $\tilde{\mathbf{w}} \ll \tilde{\mathbf{W}}_0$. The no-penetration boundary condition can then be written as:

$$[\tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0} + \tilde{\boldsymbol{\delta}}) + \tilde{\mathbf{w}}(\tilde{\mathbf{x}}_{c0} + \tilde{\boldsymbol{\delta}}, \tau)] \cdot [\mathbf{n}(\tilde{\mathbf{x}}_{c0}) + (\tilde{\boldsymbol{\delta}} \cdot \nabla)\mathbf{n}] = k \cdot \frac{d\tilde{\boldsymbol{\delta}}}{d\tau} \cdot \mathbf{n} \quad (5.2.9)$$

where, the term $(\tilde{\boldsymbol{\delta}} \cdot \nabla)\mathbf{n}$, which accounts for the variation of the normal direction of the airfoil due to the rotation of the airfoil in the mean flow field, has been retained. This term does not explicitly appear in the linearization of the no-slip boundary condition, since in this case the normal to the airfoil is not involved in the boundary condition.

The boundary condition on the actual position of the airfoil may be referred to its mean or initial position, $\tilde{\mathbf{x}}_{c0}$. Assuming that $\|\tilde{\boldsymbol{\delta}}\| \ll 1$, expanding $\tilde{\mathbf{W}}$ in Taylor series, neglecting second-order terms, and recalling that $\tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0}) \cdot \mathbf{n} = 0$, the previous equation reduces to:

$$(\tilde{\boldsymbol{\delta}} \cdot \nabla)\tilde{\mathbf{W}}_0 \Big|_{\mathbf{x}=\mathbf{x}_{c0}} \cdot \mathbf{n}(\tilde{\mathbf{x}}_{c0}) + \tilde{\mathbf{w}}(\tilde{\mathbf{x}}_{c0}) \cdot \mathbf{n} + \tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0}) \cdot (\tilde{\boldsymbol{\delta}} \cdot \nabla)\mathbf{n} = k \cdot \frac{d\tilde{\boldsymbol{\delta}}}{d\tau} \cdot \mathbf{n}$$

and the no-penetration linearized boundary condition becomes:

$$\tilde{\mathbf{w}}(\tilde{\mathbf{x}}_{c0}) \cdot \mathbf{n} = k \cdot \frac{d\tilde{\boldsymbol{\delta}}}{d\tau} \cdot \mathbf{n} - (\tilde{\boldsymbol{\delta}} \cdot \nabla)\tilde{\mathbf{W}}_0 \Big|_{\mathbf{x}=\mathbf{x}_{c0}} \cdot \mathbf{n} - \tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0}) \cdot (\tilde{\boldsymbol{\delta}} \cdot \nabla)\mathbf{n} \quad (5.2.10)$$

The important fact is that this boundary condition is imposed in the original undeformed airfoil and physically represents a blowing velocity in the wall boundary. Numerical codes that implement this type of boundary condition do not need to resort to the use of moving grids because they implicitly assume that the airfoil displacements are small. The boundary condition now provides a new term, that was hidden when expressing it in Lagrangian form. This term is what has been called the Unsteady Loading Parameter (ULP), and it represents a new aeroelastic parameter. It can be clearly seen that the term $\tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0}) \cdot (\tilde{\boldsymbol{\delta}} \cdot \nabla)\mathbf{n}$ vanishes for a pure bending mode, since in this case $\nabla\mathbf{n} = 0$ during the airfoil motion. In others words, this term is only important for torsion modes, especially, as it will be shown later on, for

lightly loaded airfoils or plates vibrating in torsion at low reduced frequency, where it becomes the leading term in Eq. 5.2.10, since in this case $\nabla \tilde{\mathbf{W}}_0 \simeq 0$.

For rigid blade motions:

$$\tilde{\boldsymbol{\delta}} = \tilde{\mathbf{h}} + \boldsymbol{\alpha} \times \tilde{\mathbf{r}} \quad (5.2.11)$$

where $\tilde{\mathbf{h}}$ defines the amplitude and direction of blade translations, $\boldsymbol{\alpha}$ is the angle (positive counter clockwise) of blade rotation, and $\tilde{\mathbf{r}}$ is the position vector extending from the axis of rotation to the mean position of the airfoil surface. In this case, the linearized no-penetration boundary condition reduces to:

$$\tilde{\mathbf{w}}(\tilde{\mathbf{x}}_{c0}) \cdot \mathbf{n} = \left[k \cdot \frac{d\tilde{\boldsymbol{\delta}}}{d\tau} - (\tilde{\boldsymbol{\delta}} \cdot \nabla) \tilde{\mathbf{W}}_0 \right]_{\mathbf{x}=\mathbf{x}_{c0}} + \boldsymbol{\alpha} \times \tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0}) \cdot \mathbf{n} \quad (5.2.12)$$

where it is more clearly seen that the third term in the r.h.s. is related with the rotation of the airfoil.

The fact that the rotational term appears only explicitly in the inviscid no-penetration boundary condition does not mean that its effect is not present in viscous flows provided that there is no separation. If the flow remains attached to the airfoil the flow-field can be decomposed in a thin boundary layer close to the airfoil and an external inviscid flow. The boundary conditions for this external flow-field are identical to that of an inviscid case with an additional, steady transpiration velocity at the wall to account for the boundary layer thickness [97, 98].

5.2.4 Asymptotic Analysis of the Linearized Equations

Although perturbation methods are implicitly used in the derivation of the linearized governing equations, with few exceptions they have not been used to seek simplified solutions of the Euler or Navier-Stokes equations in the presence of moving walls. Moore [7] used an asymptotic analysis to derive a quasi-analytical solution for an axially oscillating flat plate with a laminar boundary layer, in the low-reduced frequency limit. Interestingly, it may be anticipated that the mathematical structure of the solution that he obtained, is exactly the same as that it has been found in this thesis, since the oscillating flat plate is just particular case of a more generic class of

problems described here.

Different levels of approximation of the linearized Navier-Stokes equations may be obtained taking advantage of the smallness of k , and assuming that the displacement of the airfoil and the flow are periodic in time, i.e.: $\boldsymbol{\delta} = \hat{\boldsymbol{\delta}}e^{i\omega t}$ and $\mathbf{u} = \hat{\mathbf{u}}e^{i\omega t}$. For the sake of simplicity in the notation from now on, within this section, unless explicitly mentioned otherwise, the term $-\nabla\tilde{\mathbf{W}}_0\Big|_{\mathbf{x}=\mathbf{x}_{c0}} \cdot \tilde{\boldsymbol{\delta}}$ will be used instead of the full expression when referring to the Unsteady Loading Parameter, Υ , which means that the whole term $\Upsilon = -(\tilde{\boldsymbol{\delta}} \cdot \nabla)\tilde{\mathbf{W}}_0\Big|_{\mathbf{x}=\mathbf{x}_{c0}} + \boldsymbol{\alpha} \times \tilde{\mathbf{W}}_0(\tilde{\mathbf{x}}_{c0})$ is substituted by a more compact expression.

5.2.4.1 0th Order Approximation

If it is assumed that $k = 0$ the linearized system of equations reduces to:

$$\frac{\partial}{\partial \mathbf{x}} \left(A_0(\tilde{\mathbf{U}}_0)\tilde{\mathbf{u}}_0 \right) = 0, \text{ with } \tilde{\mathbf{w}}_0(\tilde{\mathbf{x}}_{c0}) \cdot \mathbf{n} = -\nabla\tilde{\mathbf{W}}_0\Big|_{\mathbf{x}=\mathbf{x}_{c0}} \cdot \tilde{\boldsymbol{\delta}} \cdot \mathbf{n} \quad (5.2.13)$$

where the characteristic amplitude of $\tilde{\mathbf{u}}_0$ is of order $\tilde{\mathbf{u}}_c \sim \|\tilde{\boldsymbol{\delta}}\|$, which is small if $\nabla\tilde{\mathbf{W}}_0\Big|_{\mathbf{x}=\mathbf{x}_{c0}} \sim o(1)$. The pressure and the rest of the variables are in-phase with the displacement, since physically they are due to the perturbation of the base flow associated to the static displacement of the airfoil. In this approximation, time plays the role of a parameter. For a harmonic motion whose solution may be expressed in the complex domain, the solution is real and the phase of the solution with the airfoil displacement, $\phi_0 = 0$.

5.2.4.2 1st Order Approximation

The first order correction of the solution with k may be obtained if the velocity is expressed as:

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, \tau) = \tilde{\mathbf{u}}_0(\tilde{\mathbf{x}}, \tau) + \tilde{\mathbf{u}}_1(\tilde{\mathbf{x}}, \tau) \quad (5.2.14)$$

Introducing the above expression in Eqs 5.2.7 and 5.2.10, the 1st order correction for the governing equations and the boundary condition in the frequency domain is

obtained:

$$ik\hat{\mathbf{u}}_0 + \frac{\partial}{\partial \mathbf{x}} \left(A_0(\tilde{\mathbf{U}}_0)\hat{\mathbf{u}}_1 \right) = 0, \text{ with } \hat{\mathbf{w}}_1(\tilde{\mathbf{x}}_{c0}) = ik\hat{\boldsymbol{\delta}} \quad (5.2.15)$$

where for the sake of simplicity the tilde to indicate non-dimensional variables has been omitted, and we have assumed that the motion is harmonic in time, and the displacement has only a real component, $\hat{\boldsymbol{\delta}} = \hat{\boldsymbol{\delta}}_R$, i.e.: it is the origin of the phases and a real mode is being considered. In this case $\hat{\mathbf{u}}_1$ is purely imaginary, $\hat{\mathbf{u}}_1 = \mathbf{u}_{1I} + \text{h.o.t.}$, since $\hat{\mathbf{u}}_0$ is real. The characteristic value of the correction $\mathbf{u}_{1c} \sim k \|\boldsymbol{\delta}\|$ is much smaller than the characteristic value of the zero order approximation $\mathbf{u}_{0c} \sim \nabla \tilde{\mathbf{W}}_{0c} \|\boldsymbol{\delta}\|$, since $k \ll 1$. The 1st order correction of the phase, ϕ_1 , may be estimated as:

$$\tan \phi_1 \sim \frac{\mathbf{u}_{1Ic}}{\mathbf{u}_{0c}} \sim k \frac{(1 + \alpha^* \nabla \tilde{\mathbf{W}}_{0c})}{\nabla \tilde{\mathbf{W}}_{0c}} \quad (5.2.16)$$

which is small as well, since $k \ll 1$ and α^* is a constant of order one, actually $\phi_1 \propto k$ if $\nabla \tilde{\mathbf{W}}_{0c} \sim o(1)$, exactly as it will be seen in the numerical validation chapter.

5.2.4.3 2nd Order Approximation

The next approximation of the solution provides the correction with the frequency of the real part of the solution. As it will become apparent in the following section, this correction provides further insight of the solution for lightly loaded airfoils, of which the flat plate is the limiting case. When the gradient of the velocity of the mean flow vanishes, the quasi-steady motion of the airfoil does not cause any perturbation and the next approximation of the flow is needed.

Following a consistent perturbation analysis, the 2nd order approximation is given by:

$$-k\mathbf{u}_{1I} + \frac{\partial}{\partial \mathbf{x}} \left(A_0(\tilde{\mathbf{U}}_0)\mathbf{u}_{1R} \right) = 0, \text{ with } \hat{\mathbf{w}}_{1R}(\tilde{\mathbf{x}}_{c0}) = 0 \quad (5.2.17)$$

where it has been assumed that $\hat{\mathbf{u}}_1 = \mathbf{u}_{1R} + i\mathbf{u}_{1I}$ and that $\mathbf{u}_{1I}$ is already known from the previous expansion. It can be observed that the time derivative is responsible of coupling the real and imaginary parts of the solution. The characteristic value of $\mathbf{u}_{1R}$ is, according with the previous equation, $u_{1Rc} \sim ku_{1Ic} \sim k^2 \|\boldsymbol{\delta}\|$. The 2nd order

approximation of the phase, ϕ_2 , is then:

$$\tan \phi_2 \sim \frac{u_{1Ic}}{u_{0c} + u_{1Rc}} \sim \frac{k}{a \nabla \tilde{\mathbf{W}}_{0c} + bk^2} \quad (5.2.18)$$

where a and b are real constants of order unity. It may be readily observed that for highly loaded airfoils (i.e.: $\nabla \tilde{\mathbf{W}}_{0c} \sim o(1)$), if $k \ll 1$ then $\phi \sim k$; which is one of the first conclusions of the analysis. However, for lightly loaded airfoils (i.e.: $\nabla \tilde{\mathbf{W}}_{0c} \simeq 0$), $\phi \simeq \pi/2 - dk$ even if $k \ll 1$ (d is as well a real constant of order unity). This is a second major conclusion.

5.2.5 Estimate of the Unsteady Pressure

For highly loaded airfoils the leading term of the previous expansion is $\tilde{\mathbf{u}}_0$, which is real and in phase with the displacement, being its characteristic value $\tilde{\mathbf{u}}_{c0} \sim \|\tilde{\boldsymbol{\delta}}\| \Upsilon$. Since $\tilde{p} = p/P_0$ and $\tilde{p}$ varies from 1 at the inlet to Π_R^{-1} at the exit, the characteristic unsteady pressure is:

$$p'_{c0} \sim \frac{1}{2} \rho_c v_c^2 \left(\frac{\delta_c}{c} \right) \Upsilon \quad (5.2.19)$$

since the perturbation of the flow is purely static and, therefore, the unsteady pressure is proportional to the dynamic head and the displacement.

Its first order correction with k is due to the airfoil vibration velocity and is associated to $\tilde{\mathbf{u}}_1$, whose characteristic value is $\tilde{u}_{1c} \sim k \|\boldsymbol{\delta}\|$, and therefore the correction of the pressure perturbation due to the unsteady terms is of the order of

$$p'_{c0} \sim \frac{1}{2} \rho_c v_c^2 \left(\frac{\delta_c}{c} \right) \left(\frac{\omega c}{v_c} \right) \sim \rho_c v_c \omega \delta_c. \quad (5.2.20)$$

This second scaling is only relevant if either the unsteady aerodynamic loading is low or vanishes. Strictly speaking this is the order of the unsteady pressure when $\Upsilon \ll k$. This means that even for low reduced frequencies, i.e.: $k \ll 1$, if the unsteady loading parameter is small enough, the value of the characteristic pressure is given by 5.2.20. This scaling is also valid for reduced frequencies of order unity (i.e.: $k \sim 1$), since in this case Eqs. 5.2.19 and 5.2.20 are equivalent.

5.2.6 Summary

According with the previous perturbation analysis, some conclusions may be drawn for vibrating airfoils at low reduced frequency. In this context low means much smaller than one. From a mathematical point of view it is necessary to have $k \sim 0.1$ to formally support the analysis although, as it will be shown in the numerical validation chapter, the conclusions hold as well for higher values of k , being the actual threshold, for which the current analysis is valid, case dependent. Flutter boundaries for cantilever LPTs are in the range of $k \sim 0.1 - 0.2$ [1] and, as a consequence, the following conclusions may be applied to realistic aeroelastic configurations:

1. The influence coefficient phase of highly loaded airfoils, i.e. $\Upsilon \sim o(1)$, changes linearly with k , i.e.: $\phi \propto k \ll 1$, and therefore is small. The unsteady pressure induced by the vibration is given by Eq. 5.2.19 and therefore the influence coefficients are in first approximation independent of k and proportional to the dynamic head and the unsteady loading parameter. The influence coefficient of the j -th airfoil, l_j , can be written as:

$$l_j = \frac{1}{2} \rho_c v_c^2 \left(\frac{\delta_c}{c} \right) \Upsilon c \left(\tilde{l}_{0j} + \tilde{l}_{1j} \frac{k}{\Upsilon} \right) \quad (5.2.21)$$

if the dependence with k is retained. This means that in second order approximation the influence coefficient varies linearly with k .

2. The phase of lightly loaded airfoils, i.e. when $\Upsilon \simeq 0$ or more precisely $\Upsilon \ll k$, departs from $\pi/2$ and varies linearly with k , i.e.: $\phi - \pi/2 \propto k$. This fact has a deep impact on the work-per-cycle curves as it will be shown later on. The unsteady pressure induced by the vibration is given in this case by Eq. 5.2.20 and hence the influence coefficients scale with k for lightly loaded airfoils and can be written as:

$$l_j = \rho_c v_c \omega \delta_c c \tilde{l}_{0j} \quad (5.2.22)$$

that can be seen as the limit of Eq. 5.2.21 when $\Upsilon \rightarrow 0$.

5.3 Calculus of the Unsteady Loading Parameter

5.3.1 Integral Unsteady Loading Parameter

The order of magnitude of the second and third terms in the r.h.s. of Eq.5.2.12 can be estimated if the mean flow-field is known. The second term is directly related with the steady loading of the airfoil, and is typically of order unity since usually the velocity changes of the order of itself in distances of the order of the chord. The third term is also of order unity since the velocity has been non-dimensionalized with the exit velocity.

However, both $\nabla \tilde{\mathbf{W}}_0$ and $\tilde{\mathbf{W}}_0$, are point-wise variables since they change along the airfoil chord. In order to quantitatively predict the effect of the unsteady loading in the work-per-cycle curves, the Unsteady Loading Parameter (ULP) has been defined as:

$$\Upsilon = \frac{1}{2\tilde{\delta}_{max}} \left\{ \oint \tilde{\rho} \tilde{\mathbf{W}}_0^2 (\tilde{\delta} \cdot \nabla) \tilde{\mathbf{W}}_0 \cdot \mathbf{n} d\tilde{s} + \oint \alpha \tilde{\rho} \tilde{\mathbf{W}}_0^3 d\tilde{s} \right\} \quad (5.3.1)$$

where the integrals extend along the whole airfoil surface, $\tilde{\delta}_{max}$ is the maximum of the non-dimensional displacements, and $d\tilde{s}$ is the non-dimensional arc-length. The first term of the r.h.s. represents the loading contribution, ULP_L , while the second term is the rotational contribution, ULP_R , which is zero for bending modes and a constant contribution for any torsion mode, independently of its torsion center.

It may be noticed that both contributions are weighted with the local non-dimensional dynamic pressure, $\frac{1}{2}\tilde{\rho}\tilde{\mathbf{W}}_0^2(\tilde{s})$, since the unsteady pressure associated to the airfoil vibration and the work-per-cycle are proportional to the dynamic head. Therefore, the ULP accounts for the fact that the unsteady pressure generated in the vicinity of the stagnation point is much smaller than that created in the suction side of the airfoil, where the peak Mach number is usually attained at design conditions.

Moreover, the ULP is mode dependent since $\tilde{\delta}(\tilde{s})$ depends on the actual mode-shape. The beauty of the ULP is that it is able, as it will be shown later on, to account for the effect of the mode-shape and the aerodynamic loading distribution, in the low reduced frequency limit, without performing any unsteady simulation.

The loading contribution to the ULP may be obtained in practice as:

$$\Upsilon_L = \oint \frac{1}{2\tilde{\delta}_{max}} \tilde{\rho} \tilde{\mathbf{W}}^2_0 \frac{\partial \tilde{\mathbf{W}}_0}{\partial n} \cdot \tilde{\boldsymbol{\delta}}_n d\tilde{s} \quad (5.3.2)$$

where $\tilde{\boldsymbol{\delta}}_n$ is the airfoil displacement in the normal direction.

In order to calculate the integral Unsteady Loading Parameter for viscous cases, it is enough to substitute in Eq. 5.3.1 the velocity at the airfoil surface, $\tilde{\mathbf{W}}_0$, that is actually zero for a viscous case, by the isentropic velocity at the airfoil surface. In this thesis, for the sake of simplicity, inviscid cases are mostly considered, but the influence coefficient modulus and phase of the unsteady pressure, as well as the work-per-cycle curves, of vibrating airfoils do not considerably differ between viscous and inviscid cases, provided that the flow remains attached to the airfoil. Figure 5.3.1 displays the non-dimensional work per cycle curves as a function of the IBPA for an LPT airfoil vibrating in flap mode at design conditions for a reduced frequency $k = 0.1$. It can be seen that the impact of viscous effects is very small for classical flutter. Obviously there are aeromechanic instabilities that are driven by viscous effects, such as stall flutter [18], and in this case the conclusions would have to be revisited, but it will be shown that the ULP derived in the present thesis is strongly correlated with the unsteady aerodynamics of oscillating airfoils at low reduced frequency provided that the flow remains attached to the airfoil.

5.3.2 Interpretation of the Unsteady Loading Parameter

Let us consider the momentum equation of a steady inviscid flow:

$$\rho \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p \quad (5.3.3)$$

If the previous equation is expressed in its components along the streamline direction, t , and its normal, $\mathbf{n}$, it is obtained that :

$$\begin{aligned} \mathbf{t} : \quad & \rho V \frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s} \\ \mathbf{n} : \quad & \rho \frac{V^2}{R} = -\frac{\partial p}{\partial n} \end{aligned} \quad (5.3.4)$$

Figure 5.3.1: NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA FOR THE LPT CASE VIBRATING IN FLAP ($M = 0.74$, $k = 0.1$) FOR A VISCOUS AND AN INVISCID CASE.

In the incompressible limit, $\rho = \rho_0$, the momentum equation along the streamlines reduces to the Bernoulli equation:

$$P_0 = p + \frac{1}{2}\rho_0 V^2 \quad (5.3.5)$$

where P_0 is the stagnation pressure of that streamline. If the flow is homeotropic and all the streamlines have the same stagnation pressure, Eq. 5.3.5 may be injected in the normal component of the momentum equation (5.3.4) to obtain:

$$\frac{1}{V} \frac{\partial V}{\partial n} = -\frac{1}{R} \quad (5.3.6)$$

where R is the local radius of curvature of the streamline. For an attached flow, the airfoil is a streamline and, therefore, the tangent and normal of the streamline are indistinguishable from those of the airfoil. This means that: $\nabla \tilde{\mathbf{W}}_0 \cdot \mathbf{n} \simeq \tilde{\mathbf{W}}_0 / R$.

A similar result may be obtained for compressible flows using the isentropic condition.

In this case, it may be shown that:

$$\frac{1}{M} \frac{dM}{dn} = -\frac{1}{R} \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{-1} \quad (5.3.7)$$

which is the compressible counter part of Eq.5.3.6. A similar expression to 5.3.6 can

be obtained for the compressible case realizing that:

$$\frac{1}{V} \frac{\partial V}{\partial n} = \frac{1}{Ma} \frac{d(Ma)}{dn} = \frac{1}{M} \frac{dM}{dn} + \frac{1}{a} \frac{da}{dn} \quad (5.3.8)$$

where a is the sound speed and a_0 the stagnation sound speed. Taking into account that $\frac{a_0}{a} = (1 + \frac{\gamma-1}{2} M^2)^{1/2} = \Gamma^{1/2}$, the second term of the right hand side can be written as:

$$\frac{1}{a} \frac{\partial a}{\partial n} = -\frac{1}{2}(\gamma - 1)\Gamma^{-1} M \frac{dM}{dn} \quad (5.3.9)$$

and then:

$$\frac{1}{V} \frac{\partial V}{\partial n} = \frac{1}{M} \frac{dM}{dn} \left[1 - \frac{\gamma-1}{2\Gamma} M^2 \right] = -\frac{1}{R} \left[\Gamma^{-1} - \frac{\gamma-1}{2} \Gamma^{-2} M^2 \right] \quad (5.3.10)$$

If is considered $M = 0.7$, then $\frac{\gamma-1}{2} M^2 \simeq 0.1 \ll 1$, and then:

$$\frac{1}{V} \frac{\partial V}{\partial n} \simeq -\frac{1}{R} \left[1 - \frac{\gamma-1}{2} M^2 \right] = -\frac{1}{R} (1 - \epsilon), \quad (5.3.11)$$

where the error, ϵ , with respect the expression 5.3.6, is $\epsilon < 10\%$, if $M < 0.7$.

If the displacements induced by bending and torsion are comparable, i.e.: $\tilde{\delta} \sim \alpha$, the ratio between the loading and rotational contributions is the radius of curvature. Actually, $\nabla_n \tilde{\mathbf{W}}$ could be estimated as: $\nabla_n \tilde{\mathbf{W}} \simeq \tilde{\mathbf{W}}/R$.

5.3.3 Examples

With the aim of illustrating the meaning of the different components of the ULP, an LPT airfoil (see Fig. 5.3.2), which will be described in detail in the numerical validation chapter, has been selected. The design operating point is defined by an inlet and exit angle of $\alpha_1 = 44^\circ$ and $\alpha_2 = 60^\circ$, respectively, exit Mach number of $M_{exit} = 0.75$ and a pitch to chord ratio of $s/c = 0.96$.

Figure 5.3.3 displays the distribution along the airfoil of the normal gradient of the velocity divided by the velocity, computed using three different approaches: (i) computing the flow-field velocity gradient in the normal projection into the airfoil, which is considered as the exact method (■ symbols) (ii) the incompressible approximation, i.e.: the inverse of the non-dimensional radius of curvature, $1/\tilde{R} =$

Figure 5.3.2: SNAPSHOT OF THE UNSTEADY PRESSURE CAUSED BY THE VIBRATION OF THE CENTRAL BLADE IN AN LPT AIRFOIL FOR $k = 0.3$.

Figure 5.3.3: DISTRIBUTION OF THE NORMAL GRADIENT OF THE VELOCITY ALONG AN LPT AIRFOIL COMPUTED USING THREE DIFFERENT APPROACHES

Figure 5.3.4: UNSTEADY LOADING PARAMETER DISTRIBUTION (LOADING CONTRIBUTION, ULP_L , AND ROTATIONAL CONTRIBUTION, ULP_R) ALONG AN LPT AIRFOIL FOR DIFFERENT MODE-SHAPES.

c_{ax}/R , ($\triangle$ symbols) and (iii) the compressible correction to $1/\tilde{R}$, i.e.: 5.3.8, ($\blacktriangle$ symbols). First of all, it may be noticed that the distribution of the normal gradient of the velocity along the airfoil resembles the distribution of $1/\tilde{R}$, confirming the physical interpretation outlined in the preceding sub-section. Moreover, large peaks can be seen both, in the leading and trailing edges caused by the small radius of curvature of the l.e. and t.e., that give rise to very large local velocity gradients. These peaks are not only undesirable, but have low physical sensitivity, since they occur in regions of low velocity, that generate very low levels of unsteady pressure. For this reason, the forcing terms generated in the wall no-penetration boundary condition (Eq. 5.2.12), and that form the core of the ULP integrand, are weighted with the local dynamic head to properly scale its impact on the final result.

Figure 5.3.4 depicts the ULP distribution for the same airfoil for two modes: flap and torsion about the l.e. The peak of the ULP for the flap mode is close to the location of the maximum unsteady pressure modulus, as it can be seen in Fig. 5.3.2, which shows the unsteady pressure modulus for the LPT configuration under study, vibrating in flap mode at a reduced frequency $k = 0.3$; obviously the rotational contribution to the ULP is zero for this case. It can be seen that the ULP distribution is able to predict

Figure 5.3.5: ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD FOR THE LPT AND THE NACA65 CASES

where will be located the maximum of the unsteady pressure, which constitutes a very surprising result since, the ULP has been obtained only from the steady flow field and the given vibration mode shape, without using any unsteady data. The maximum of the loading contribution to the ULP for a torsion mode about the l.e. occurs downstream of the peak of the flap mode, since the local vibration amplitude increases with the axial length in this case, causing a downstream shift with respect the peak of the previous case. The rotational component of the ULP parameter is fairly constant along the chord, but lower than the loading contribution. The ratio between both, the loading and rotational contributions is consistent with Eq. 5.3.8 that predicts a factor of $1/\tilde{R}$, that in this case is roughly 3 (see Fig. 5.3.3).

The contribution of the suction side is much larger than that of the pressure side, by a factor of about 100. This is consistent with the fact that the isentropic velocity in the suction side is about five times greater than in the pressure side (see the zero incidence isentropic Mach number distribution in Fig. 5.3.5), and that the integrand of the ULP scales with the cube of the non-dimensional velocity. The accumulated contribution to the ULP can be also seen in Fig, 5.3.6 ($\diamond$). It can be noticed that the contribution increases smoothly along the chord, and that the influence of the l.e. is negligible, except by a small contribution to the p.s. that gives rise to a small off-set

Figure 5.3.6: UNSTEADY LOADING PARAMETER DISTRIBUTION

Figure 5.3.7: UNSTEADY LOADING PARAMETER DISTRIBUTION FOR THE NACA65 AIRFOIL. (A): $M_1 = 0.7$. (B): $M_1 = 0.8$.

respect to zero.

Figure 5.3.7 displays the ULP distribution for the NACA65 airfoil operating at subsonic (left) and transonic (right) conditions, for the flap mode. The subsonic operating point (see [9, 99]) is defined by an inlet angle $\alpha_1 = 55^\circ$, exit angle $\alpha_2 = 40^\circ$, inlet and exit isentropic Mach number, $M_{in} = 0.7$ and $M_{exit} = 0.45$, respectively, and airfoil spacing, $s/c = 1$; and for the transonic case the inlet isentropic Mach number is raised to $M_{in} = 0.8$. The isentropic Mach number distributions can be seen in Figure 6.2.1. When looking at the ULP distributions, it can be seen that, on the contrary than in the LPT case, the l.e. contribution is very high, actually most of the ULP is created in this region in both, the subsonic and the transonic cases. The transonic condition gives rise to a large discontinuity in the region where the shock wave is sitting on the airfoil ($x/c \simeq 0.2$), that nearly balance that created on the l.e. The contribution to the ULP of the region downstream of the shock-wave is almost negligible. When looking to Fig. 5.3.8, which represents the imaginary part of the unsteady pressure for the NACA65 operating at subsonic and transonic conditions, vibrating in flap mode, at reduced frequency $k = 0.1$, it can be seen that, as it happens with the LPT case, the ULP correctly predicts the region where the maximum of the imaginary part of unsteady pressure, which means the region that will mostly contribute to the work per-cycle, will be located.

The different behavior of the NACA65 and the LPT airfoils is due to the distinct distributions and values of the radius of curvature on both airfoils, that is much higher and sharp in the l.e. region of the compressor airfoil.

5.4 Simplified Analytical Model for the Work-per-cycle

The previous assertions have direct implications on the shape of the work-per-cycle curves. It has been previously reported [5, 100], that for LPT airfoils, and in general for highly loaded subsonic airfoils, vibrating in flap and edge-wise modes, only the

Figure 5.3.8: IMAGINARY PART OF THE UNSTEADY PRESSURE OF THE NACA65 COMPRESSOR FOR $k = 0.1$, FOR $M_{in} = 0.7$ (A) AND FOR $M_{in} = 0.8$ (B)

vibrating airfoil, $j = 0$, and the adjacent airfoils, $j = \pm 1$, contribute significantly to the work-per-cycle, W_{cycle} . The same assumption has been adopted here. It may be shown that the work-per-cycle can be expressed as a function of the inter-blade phase angle, σ , and the influence coefficients in the following form:

$$W_{cycle} = \pi p'_c \delta_c c \left[\tilde{l}_0 \sin \phi_0 + \tilde{l}_{+1} \sin(\phi_1 - \sigma) + \tilde{l}_{-1} \sin(\phi_{-1} + \sigma) \right] \quad (5.4.1)$$

where $\tilde{l}_j$ and ϕ_j are respectively the modulus and phase of the non-dimensional ICs, and δ_c the vibration amplitude.

5.4.1 Loaded Airfoils

Taking into account that the phase of the ICs of highly loaded airfoils ($\Upsilon \sim o(1)$) is small and reduces with the unsteady loading:

$$\tan \phi \simeq \phi \sim \frac{k(1 + \alpha^* \Upsilon)}{\Upsilon} \quad (5.4.2)$$

where α^* is a constant,

for unsteady loaded airfoils vibrating at $k \ll 1$ the scaling of W_{cycle} is as follows:

$$W_{cycle} = \mathbb{C} \left[\tilde{l}_0 \phi_0 + \tilde{l}_{\pm 1} \sin(\phi_{\pm 1} \mp \sigma) \right] = \mathbb{C} \left[\tilde{l}_0 \alpha_0 \frac{k(1 + \alpha^* \Upsilon)}{\Upsilon} + \tilde{l}_{\pm 1} \sin(\alpha_{\pm 1} \phi \mp \sigma) \right] \quad (5.4.3)$$

where $\mathbb{C} = \frac{1}{2} \pi \rho_c v_c^2 \delta_c^2 \Upsilon$. Since $\phi_0 \simeq \alpha'_0 k$ and $\phi_{\pm 1} \simeq \alpha'_{\pm 1} k$, where α'_0 and $\alpha'_{\pm 1}$ are constants of order unity, and the non-dimensional IC moduli, $\tilde{l}_0$ and $\tilde{l}_{\pm 1}$, are constant with a small correction proportional to k , then the mean value of the work-per-cycle, $\bar{W}_{cycle-mean}$, scales with k whereas the variation with σ is constant in first approximation, i.e.:

$$W_{cycle} = \bar{W}_{cycle-mean} + W'_{cycle}(\sigma, k) = k W_{cycle}^0 + W'_{cycle}(\sigma, k) \quad (5.4.4)$$

where $\max_{k, \sigma} W'_{cycle}$ is bounded with k in first order approximation, as it can be seen in the numerical results shown in Fig. 5.4.1 for an LPT airfoil vibrating in flap mode. Numerical experiments have been carried out in the numerical verification chapter, , to confirm that this is a generic result independent of the geometry. If more ICs were needed to be retained to accurately represent the solution, as it is sometimes the

Figure 5.4.1: MEAN VALUE AND 1st HARMONIC OF THE NON-DIMENSIONAL WORK AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE LPT CASE

case for some torsion modes, the conclusions would be the same but more harmonics would have to be retained in Eq. 5.4.3.

The validity of this model for the three fundamental modes (i.e., flap, edge and torsion) is implicitly assured since the theory does not make any hypothesis about the mode-shape of the airfoil. Nevertheless results for the three different modes are presented in the thesis.

It is interesting to see that while the mean value of the work-per-cycle, $\bar{W}_{cycle}$, increases slightly, but not proportionally, with the unsteady loading, the variation of W_{cycle} with σ , W'_{cycle} , scales directly with it

5.4.2 Lightly Loaded or Unloaded Airfoils

For unloaded airfoils vibrating at $k \ll 1$ the scaling of the W_{cycle} is as follows:

$$\begin{aligned} W_{cycle} &= \frac{1}{2} \pi \rho_c v_c^2 \delta_c^2 k \left[\tilde{l}_0 + \tilde{l}_{+1} \sin\left(\frac{\pi}{2} - \sigma\right) + \tilde{l}_{-1} \sin\left(\frac{\pi}{2} + \sigma\right) \right] \\ &= \frac{1}{2} \pi \rho_c v_c^2 \delta_c^2 k \left[\tilde{l}_0 + (\tilde{l}_{+1} + \tilde{l}_{-1}) \cos \sigma \right] \end{aligned} \quad (5.4.5)$$

since all the phases are approximately $\pi/2$ (see Eq. 5.4.2 when $\Upsilon \rightarrow 0$). The first observation is that the work-per-cycle scales linearly with the reduced frequency and, what is more striking, that the damping curves are symmetric with respect $\sigma = 0^\circ$,

i.e.:

$$W_{cycle} = k W_{cycle}^0(\sigma) = kW_{cycle}^0(-\sigma) \quad (5.4.6)$$

It is interesting to observe that (i) this symmetry is possible even if $\tilde{l}_{+1} \neq \tilde{l}_{-1}$ and (ii) increasing or decreasing the reduced frequency does not produce any noticeable effect on the stability, i.e.: the sign of W_{cyc} does not change with k . This behaviour will be confirmed using numerical experiments. It is important to highlight that for lightly loaded airfoils it may well happen that more ICs are needed to represent the mode. It is enough to retain the complete expansion of the W_{cycle} as a function of the ICs in Eq. 5.4.5 to conclude that in any case the W_{cycle} scales linearly with k , whereas the symmetry with σ may be spoiled if the phase of these ICs is different from $\pi/2$.

5.4.3 Symmetric Cascades

The general form of the W_{cycle} , Eq. 5.4.1, can be particularized for symmetric cascades, where $l_{-1} = -l_{+1}$ and $\phi_{-1} = \pi + \phi_{+1}$, in which case Eq. 5.4.1 reduces to:

$$W_{cycle} = \pi p'_c \delta_c c \left[\tilde{l}_0 \sin \phi_0 + 2\tilde{l}_{+1} \sin \phi_{+1} \cos \sigma \right] \quad (5.4.7)$$

It has been shown that for loaded airfoils vibrating at $k \ll 1$, $\phi \propto k$, while the influence coefficients are fairly independent of k , (see Eq. 5.2.21) and then:

$$W_{cycle} = \frac{1}{2} \pi \rho_c v_c^2 \delta_c^2 \nabla \tilde{\mathbf{W}}_{0c} k \left[\alpha_0 \tilde{l}_{00} + 2\alpha_{+1} \tilde{l}_{+1} \cos \sigma \right] \quad (5.4.8)$$

Two important conclusions can be drawn at this point. The first is that for a symmetric case the W_{cycle} curves are symmetric, i.e. $W_{cycle}(\sigma) = W_{cycle}(-\sigma)$, as it could be expected because of the symmetry of the problem. The second conclusion is that the work-per-cycle scales linearly with the reduced frequency for all the inter-blade phase angles, i.e.:

$$W_{cycle}(\sigma) \propto k \quad (5.4.9)$$

Since the W_{cycle} curve of all the unloaded airfoils are symmetric and scale with k , as well, it can be concluded that all the symmetric cases, for different reasons, fulfill these two properties.

In the next section, the main aspects of the theory derived by Theodorsen are firstly presented and finally some conclusions applied to the study of unsteady aerodynamics in turbomachinery airfoils are directly extracted from the formulation he derived.

5.5 Review of Theodorsen's Theory

Unsteady aerodynamic theories have been developed for many years for a variety of uses. The aerodynamics theory developed by Theodorsen, [6], is based on potential incompressible flow for the lift and pitching moment on an isolated flat plate oscillating with a frequency ω in a translation in the y direction and in a rotation about an arbitrary torsion center. The theory assumes small perturbations and harmonic motion. An analytical solution was obtained for incompressible and inviscid flow, which gives the loads on the airfoil in terms of the kinematics and the Theodorsen's function, that is a function of the reduced frequency, defined as $k = \omega c / U_\infty$, where c is the airfoil chord and U_∞ the velocity of the free stream. Theodorsen was able to show that the reduced frequency was the correct measure of the unsteadiness of the flow because Theodorsen's function is solely a function of reduced frequency.

Theodorsen obtained the aerodynamic force as sum of two contributions: the first is the non circulatory force, determined by a distribution of sources and sinks along the airfoil surface; and the second is the circulatory force, obtained by superimposing a vorticity distribution on the body surface (bound vorticity) and in the wake (free vorticity).

Figure 5.5.1: SKETCH OF THE DISPLACEMENTS OF AN ISOLATED OSCILLATING PLATE

5.5.1 Non-circulatory contribution

Theodorsen achieved the solution by distributing sources (upper sides) and sinks (lower side) of equal strength. According with Theodorsen's results, the non-circulatory part of the lift is:

$$L_{NC} = \pi c^2 (\ddot{h} + U_{\infty} \dot{\alpha} - \beta c \ddot{\alpha}) \quad (5.5.1)$$

where c represents the length of the plate, h is the vertical displacement in z direction, β is the distance between the torsion axis and the origin, and α is the torsion angle. Figure 5.5.1 represents the plate's displacement. According with this, a vertical displacement at constant velocity, $\dot{h}$, does not have unsteady effects on the non-circulatory part of the lift. It is equivalent to the effect of a change in the effective incidence, α , such that $\dot{h} = U\alpha$.

5.5.2 Circulatory contribution

The non-circulatory part is not able to fulfill the Kutta condition at the trailing edge. Theodorsen obtained the solution for the circulatory contribution by superimposing a vorticity distribution on the body surface (bound vorticity) and in the wake (free

vorticity). He obtained that the circulatory part of the lift is:

$$L_C = 2\pi c U_\infty C(k) \left[\dot{h} + U_\infty \alpha + c \left(\frac{1}{2} - \beta \right) \dot{\alpha} \right] \quad (5.5.2)$$

where $C(k)$ is the Theodorsen function, that is a function of the reduced frequency.

5.5.3 Conclusions of Theodorsen theory at $k \ll 1$

The theory derived by Theodorsen can be analyzed in more detail and some practical conclusions can be drawn from the expressions he obtained for the lift force. If it is assumed that the motion of the plate is harmonic in time, then the displacement, h , and the torsion angle, α , can be expressed as:

$$h = h_0 + \Delta h \sin(\omega t) \text{ and } \alpha = \alpha_0 + \Delta \alpha \sin(\omega t) \quad (5.5.3)$$

where h_0 and α_0 represent the mean value and Δh and $\Delta \alpha$ the variations with the time. The non-circulatory lift can then be written as:

$$L_{NC} = \rho \pi c^2 (-\Delta h \omega^2 \sin(\omega t) + U_\infty \Delta \alpha \omega \cos(\omega t) + \beta c \omega^2 \Delta \alpha \sin(\omega t)) \quad (5.5.4)$$

and in non-dimensional form:

$$\hat{L}_{NC} = \frac{L_{NC}}{\frac{1}{2} \rho U_\infty^2 c} = 2\pi \left[k^2 (\beta \Delta \alpha - \tilde{\Delta h}) \sin \tau + k \Delta \alpha \cos \tau \right] \quad (5.5.5)$$

where $\tau = \omega t$ is the non-dimensional time, and $\tilde{\Delta h}$ is the non-dimensional displacement. In a similar way, the circulatory part of the lift can be written as:

$$L_C = \frac{1}{2} \rho U_\infty^2 4\pi c \frac{\omega}{U_\infty} C(k) \left[\Delta h \cos(\omega t) + \frac{U_\infty}{\omega} \Delta \alpha \sin(\omega t) + \left(\frac{1}{2} - \beta \right) c \Delta \alpha \cos(\omega t) \right] \quad (5.5.6)$$

which in non-dimensional form is:

$$\hat{L}_C = \frac{L_C}{\frac{1}{2} \rho U_\infty^2 c} = 4\pi C(k) \left[\Delta \alpha \sin \tau + \left(\tilde{\Delta h} + \left(\frac{1}{2} - \beta \right) \Delta \alpha \right) k \cos \tau \right] \quad (5.5.7)$$

Taking into account the expressions obtained for the non-dimensional lift, and assuming that $k \ll 1$ (which directly implies that $C(k) \simeq 1$), different order corrections of the lift with the reduced frequency can be obtained. Table 5.1 summarise the different order corrections to the circulant and non-circulant

	Circulant		Non-circulant	
	$\sin \tau$	$\cos \tau$	$\sin \tau$	$\cos \tau$
k^0	$2\pi\Delta\alpha$	0	0	0
k^1	0	$2\pi \left[\Delta\tilde{h} + \Delta\alpha \left(\frac{1}{2} - \beta \right) \right]$	0	$\pi\Delta\alpha$
k^2	0	0	$\pi \left[\Delta\alpha\beta - \Delta\tilde{h} \right]$	0

Table 5.1: CIRCULANT AND NON-CIRCULANT CONTRIBUTIONS TO THE NON-DIMENSIONAL LIFT LIFT

contributions to the total lift, separating them in its sinusoidal contribution (in phase with the plate displacement) and its cosinusoidal contribution (out of phase with the plate displacement).

5.5.3.1 Calculus of the Lift Phase

As it has been mentioned, the total lift has an in-phase, L_s , and an out-of-phase, L_c , contribution with the plate's motion. The phase of the lift with respect the plate's displacement, ϕ , can be calculated as: [6]

$$\tan \phi = \frac{L_c}{L_s} = \frac{k \left(\Delta\alpha + 2 \left[\Delta\tilde{h} + \Delta\alpha \left(\frac{1}{2} - \beta \right) \right] \right)}{2\Delta\alpha + k^2 \left(\beta\Delta\alpha - \Delta\tilde{h} \right)} \quad (5.5.8)$$

- For torsion modes, i.e.: if $\Delta\alpha \neq 0$ and $\Delta\tilde{h} = 0$, the phase of the lift for low k is:

$$\tan \phi \simeq k(1 - \beta) \quad (5.5.9)$$

- For purely bending modes, i.e.: if $\Delta\alpha = 0$ and $\Delta\tilde{h} \neq 0$, the phase of the lift is:

$$\tan \phi = -\frac{2}{k} \text{ and then, if } k \ll 1 \Rightarrow \phi \rightarrow \pi/2 \quad (5.5.10)$$

Flat plate torsion modes behave as loaded airfoils because at low reduced frequencies the plate is able to generate a quasi-steady component of the lift due to an effective change of the angle of attack, whereas this mechanism is not possible for bending modes in the normal direction of the plate, which are not able to perturb the flow in the quasi-stationary regime, where the plate lift is zero in first order approximation.

Both, the torsion and bending modes, give rise to an in-phase component of the lift, that scales with k^2 , see Table 5.1, and that is not associated to the generation of circulation. This term, as it was outlined in the previous chapter, 3, is the “*apparent*” mass of the flat plate and does not generate any damping.

The results obtained from Theodorsen’s theory for the unsteady pressure phase are the same than that obtained in the asymptotic analysis performed in this thesis. It is important to note that the strong points of the theory presented in this thesis are: (i) the way in which the results are derived is very simple and does not require a strong mathematical effort to achieve the solution and (ii) the theory is applicable to loaded (real) airfoils.

5.6 Conclusions of the Analysis at Low Reduced Frequency

The unsteady aerodynamics of vibrating airfoils and its impact in the work-per-cycle curves in the low reduced frequency regime has been analytically studied by means of a perturbation analysis that takes advantage of the smallness of k . The non-dimensional unsteady loading parameter (ULP) and the symmetry or asymmetry of the cascade have been identified as two key parameters to understand the different trends of the influence coefficient modulus and phase, as well of the work-per-cycle curves with the reduced frequency (Fig. 5.6.1 summarizes the different behavior of the mean and the minimum of the work per cycle as a function of the new aeroelastic parameters). It has been reported that the ULP can predict the different trends with the reduced frequency of the influence coefficient moduli, and phases of the unsteady pressure, and the work-per-cycle curves. The ULP has been defined in a quantitative manner and its influence will be numerically quantified in the following chapter, using a well validated linearized Navier-Stokes solver on a multiplicity of two-dimensional airfoils, operating at high subsonic and transonic regimes, design and off-design conditions, vibrating with different mode-shapes.

Figure 5.6.1: SKETCH OF THE MAIN CONCLUSIONS

The main conclusions for airfoils vibrating at low reduced frequency are the following:

1. For highly steady loaded airfoils, independently of the mode shape, and for steady lightly loaded airfoils vibrating in torsion mode, which effectively means $\Upsilon \sim O(1)$, the phase of the unsteady pressure with respect to the airfoil displacement is proportional to the reduced frequency while the modulus remains constant with this parameter. The higher the ULP the lower the slope of the phase with the reduced frequency.
2. For lightly steady loaded airfoils vibrating in bending mode, which means unsteady unloaded airfoils ($\Upsilon \simeq 0$), the phase of the unsteady pressure at low reduced frequency is $\pi/2$, and the modulus is proportional to the reduced frequency.
3. A new aeroelastic parameter, the Unsteady Loading Parameter (ULP) has been identified. This new parameter, encompasses the effect of both, the steady loading and the vibration mode-shape onto the flutter characteristics of oscillating airfoils, at low reduced frequency. The classical non-dimensional

analysis states that the non-dimensional work-per-cycle depends on:

$$\Theta = \Theta \left(\overbrace{(k, \text{Modeshape})}^{\text{High Influence}}, \underbrace{C_p}_{\text{Steady}}, \overbrace{(Re, M)}^{\text{Low Influence}} \right).$$

$\underbrace{k}_{\text{Unsteady}}$
 $\underbrace{C_p}_{\text{Steady}}$
 $\underbrace{Re, M}_{\text{Steady}}$

The new understanding derived in this thesis suggests that the non-dimensional aerodamping depends on:

$$\Theta = \Theta \left(\overbrace{(k, \Upsilon)}^{\text{High Influence}}, \overbrace{(Re, M)}^{\text{Low Influence}} \right)$$

$\underbrace{k, \Upsilon}_{\text{Unsteady}}$
 $\underbrace{Re, M}_{\text{Steady}}$

where the number on parameters which the non-dimensional work depends on has been reduced.

4. The ULP has two contributions: the first accounts for the airfoil motion through a spatially varying mean velocity field, and which is proportional to the steady loading of the airfoil. This first contribution has been named the loading contribution of the ULP. The second is due to the airfoil rotation in the mean velocity field (the rotational contribution).
5. The rotational contribution is independent of the centre of torsion and it only depends on the local variation of the normal to the airfoil surface with the blade motion and the velocity distribution along the airfoil surface. This contribution is zero for a pure bending mode, whereas for a lightly loaded blade vibrating in torsion, is the only contribution to the total unsteady loading. The ULP can be computed making use solely of the isentropic velocity distribution, its normal gradient along the airfoil, and the mode-shape. It does not require to perform any unsteady simulation and therefore the parameter is well suited for engineering conceptual studies and to predict the result of design changes in the aerodynamic damping. The academic implications are not negligible either.
6. The work-per-cycle of lightly unsteady loaded airfoils ($\Upsilon \simeq 0$) scales linearly as well with the reduced frequency (see Fig. 5.6.1) and exhibits a symmetry with the IBPA. The same behavior applies to symmetric cascades, independently of whether they are made up of loaded or unloaded airfoils (see Fig. 5.6.1). As

a direct consequence, the stability of this type of airfoils does not change with the reduced frequency, i.e., stable configurations remain stable whatever the reduced frequency is.

7. For highly loaded airfoils, the mean value of the work-per-cycle scales linearly with k (see Fig. 5.6.1), whereas the minimum value increases linearly with k but does not scale directly proportional with it.
8. The mean value of the work-per-cycle is always proportional to k , independently of the airfoil loading or symmetry of the cascade.
9. It is important to recall that the critical damping ratio, ξ , which is frequently used to quantify the aerodynamic damping, is related with the purely non-dimensional damping as:

$$\xi = \frac{\Theta}{\mu k}$$

therefore, the conclusion that the mean non-dimensional aerodynamic damping is proportional to k , $\Theta \propto k$, implies that the mean critical damping ratio is constant at low reduced frequency, $\xi_{mean} \simeq \xi_0$.

10. These conclusions are in first approximation independent of the type of mode.

The usefulness of the ULP, as defined in this thesis, has been outlined. A more exhaustive analysis, including the discussion of numerical results, is postponed to the next chapter.

Chapter 6

Numerical Validation of the Theory

Facts are stupid until brought into
connection with some general law

Louis Agassiz

Contents

6.1	Relevance of the Rotational Contribution	118
6.2	Steady Loading Effects	123
6.3	Mode shape effects	132
6.4	Geometry Effects	137
6.5	Mach Number Effect	139
6.6	Symmetric cascades	147
6.7	Physical Interpretation	151
6.8	Conclusions of the Numerical Validation	164

As it was highlighted in the introduction, the unsteady pressure field caused by airfoil vibration plays a major role in flutter and forced response of turbomachinery blade rows. However, despite the fact that both problems have been around for decades in the turbomachinery community, little understanding and theoretical support have been gained, even of the most well established trends. The previous chapter was an

attempt to throw some light in the most fundamental aspects of the aerodynamics of vibrating airfoils, by using a formal perturbation analysis of the linearised Navier-Stokes equations, in the limit of low-reduced frequency, $k \ll 1$, which has both, a practical and a theoretical interest.

It was concluded that a new parameter, the non-dimensional unsteady loading and the symmetry or asymmetry of the cascade are two key parameters to understand the different trends of the influence coefficient modulus and phase, as well as of the work-per-cycle curves, with the reduced frequency. The new parameter derived, encompasses the effect of both, the steady loading and the vibration mode-shape onto the flutter characteristics of oscillating airfoils, at low reduced frequency. Other authors [92, 91] performed thoroughly systematic studies to investigate the influence of the exit Mach number on the flutter characteristics of a turbine section, which is a particular case of aerodynamic loading variation, but it was difficult to derive clear conclusions from such studies.

It was derived that, for unsteady unloaded airfoils, the influence coefficient phase departs from $\pi/2$ and varies linearly with k , (i.e.: $\phi - \pi/2 \propto k$), whereas the influence coefficient modulus scales with k and, as a consequence, the work-per-cycle curves have a cosinusoidal shape, are symmetric with respect $IBPA = 0$, and scale proportionally with the reduced frequency. On the other hand, for highly unsteady loaded airfoils, the influence coefficient phase is proportional to the reduced frequency, while its modulus is constant, in first order approximation in k , and, therefore, the mean value of the work-per-cycle curves scales proportionally with the reduced frequency, whereas its variation with the inter-blade phase angle (IBPA) is fairly insensitive to the reduced frequency. The perturbation analysis also showed that for symmetric cascades, the work-per-cycle curves are proportional to the reduced frequency and symmetric with respect the IBPA.

The analysis, that was formally derived assuming that $k \ll 1$, can be used beyond its formal range of validity to guess the trends for larger values of k . In particular, it may be inferred that the phase of the ICs for high values of the reduced frequency tends to

$\pi/2$, and hence the work-per-cycle curves have a cosine-like shape, which matches the experience obtained running simulations for compressor blades, that actually operate at much higher reduced frequencies than LPT cantilever blades. It is important to highlight that this shape of W_{cycle} curves is due solely to a reduced frequency effect, and it is not associated with the actual shape of the airfoil.

Although analytical theories based on simplifying hypothesis are of great value for constructing engineering models, a lot of attention has to be paid to not oversimplify the model and derive misleading conclusions. Traditionally, analytical models have been verified against experimental data, with a mutual positive feed-back between them. Today, it is widely admitted that well validated numerical tools may overtake some of the role traditionally reserved for experiments. There are many advantages in using numerical experiments to corroborate analytical models; among them, the following can be highlighted: (i) Parametric studies may be easily conducted (ii) Results can be inspected in full detail (iii) Non-physical models may be constructed (iv) High level of consistency between the analytical and the numerical models. Moreover, in many occasions, experiments are difficult, or even impossible, to perform at affordable costs. In this thesis, an iterative process has been followed between the analytical work and the numerical experiments. Every time a theoretical conclusion was derived, was immediately verified by means of numerical simulations. Whenever both results were not considered consistent, both approaches were revisited until the conclusions obtained by both were congruent.

This chapter numerically verifies the analytical conclusions derived in the previous chapter using a flat plate linear cascade, and rows of, a low-pressure turbine section, the NACA012 and NACA65 profiles, operating at different flow conditions and vibrating in three fundamental mode shapes. A sketch of all the configurations studied can be seen in Fig. 6.0.1 (b).

The operating points and the isentropic Mach number distributions along the airfoil chord of these sections will be described in the corresponding subsections, nevertheless, Fig. 6.0.1 (a) shows the isentropic Mach number distributions along

Figure 6.0.1: ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD (A) AND SKETCH OF THE CONFIGURATIONS (B)

the non-dimensional airfoil chord, for the design operating points, of all the sections studied. It can be seen that they are quite different among them, both in terms of geometry and static pressure distribution, since they are intended to cover a wide spectrum of flow conditions including off-design and transonic cases.

The Mu^2s^2T suite of codes has been used in the present thesis. This set of numerical tools has been well validated for vibrating airfoils in the past [32] and more recently in [80]. An idea of the level of accuracy that can be expected using these numerical tools is given in the chapter 7.1 of this thesis. The flow model, which is based on the linearized Navier-Stokes equations, is described in detail in the chapter 4. The suitability of using the linearized Navier-Stokes equations for [41, 79] predicting aerodynamic damping and its range of validity has been demonstrated many times using non-linear simulations and experiments [32, 80].

The Unsteady Loading Parameter (ULP), derived in the theoretical analysis resorting to the analysis of the no-penetration boundary condition at the wall, is an attempt to grasp the unsteadiness created by the airfoil motion in a simple parameter. The ultimate goal now is to correlate the actual value of the ULP with the flutter

characteristic of an airfoil, such as the phase of the influence coefficients, the minimum and mean damping, etc.

The objective of this chapter is to quantitatively evaluate the Unsteady Loading Parameter (ULP), to elucidate if the ULP is able to quantitatively predict the influence of different aerodynamic loadings, reduced frequency and mode-shapes on the aerodynamic damping. It is important to highlight that computing the ULP does not require to perform unsteady simulations and, therefore, the use of this parameter in the conceptual design phase can provide a quick assessment of the flutter characteristics of bladed-disks. More interestingly, the prediction capability of this parameter confirms the rationale of a large body of knowledge related with the aerodynamics of oscillating airfoils and its impact on the aerodynamic damping of turbomachinery airfoils.

The correlation of the ULP with the work-per-cycle curves is not only a clear indication of the understanding of the problem, but a conceptual tool to predict the aerodynamic damping variations associated to design changes. In this section, the problem of correlating the ULP with changes in:

1. Incidence and Mach number
2. Mode-shape
3. Airfoil geometry

is addressed, All the trends derived from the asymptotic analysis have been proved, and the ULP is quantitatively correlated with the flutter characteristics of oscillating airfoils. Both, the Influence Coefficient (IC) and the Traveling Wave (TW) approaches have been used in combination. The cascade constructed to run the simulations using the IC formulation is made up of five airfoils in all the cases. The influence of the number of airfoils of a linear cascade on the unsteady aerodynamics of oscillating airfoils was investigated in [82] Nevertheless parallel studies were conducted to investigate that five airfoil were enough to avoid the presence of artificial perturbations transmitted through the periodic boundaries.

It has been concluded that all the aforementioned theoretical conclusions are consistent with the numerical results presented here.

6.1 Relevance of the Rotational Contribution

For lightly steady loaded airfoils ($\nabla \tilde{\mathbf{W}}_0 \simeq 0$), the rotational contribution plays a major role. For bending modes, the ULP is small ($\Upsilon \simeq 0$), strictly zero for a flat plate where $\nabla \tilde{\mathbf{W}}_0 = 0$. In this case the theory states, the IC moduli vary proportionally with the reduced frequency, while the IC phases are constant, $\phi = \pi/2$, which directly implies that the work-per-cycle curves have a cosinusoidal shape. Nevertheless, for steady lightly or unloaded airfoils vibrating in a torsion mode, (i.e.: $\alpha \neq 0$), the rotation contribution to the ULP is not zero, and then $\Upsilon \neq 0$, which directly implies that this type of cases should behave sanalogously to the loaded cases (i.e.: constant l_j with k , and small ϕ , which varies $\phi \propto k$, and sinusoidal shape of the work-per-cycle curves). In order to verify this assertion, an asymmetric flat plate linear cascade, vibrating in both flap and torsion has been studied, and the results for the work per-cycle curves are presented together. The flow model, which is based on the linearized Euler equations is described in the chapter 4.

6.1.1 Asymmetric Flat Plate Linear Cascade

The limiting case of a steady unloaded airfoil is a flat plate. For this reason, a linear cascade of inviscid flat plates vibrating in torsion about the l.e. has been chosen as the vehicle to verify that the ULP presented in this thesis can predict the behaviour of vibrating airfoils at low reduced frequency, even when $\nabla \tilde{\mathbf{W}}_{0c} = 0$. The flat plate stagger angle is $\theta = 45^\circ$, the $M_{exit} = 0.45$, and the pitch to chord ratio $s/c = 0.5$. The cascade constructed to run the simulations using the IC formulation is made up of five plates. This cascade has been used to investigate the bending mode as well.

When the steady loading is zero, and the surface normal does not change during the blade vibration (pure bending modes), the ULP=0, and then, the phase of the

Figure 6.1.1: (A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE FLAT PLATE CASE VIBRATING IN FLAP (FILLED SYMBOLS) AND TORSION AROUND THE L.E. (OPEN SYMBOLS)

influence coefficients is always close to $\pi/2$, the influence coefficients modulus is proportional to k , and then, the work per cycle curves are cosinusoidal. A direct consequence of the dependence of the ICs and phases with the reduced frequency is that the work-per-cycle scales linearly with the reduced frequency, $W_{cycle}(\sigma) \propto k$, and that the curves are symmetric, $W_{cycle}(\sigma) = W_{cycle}(-\sigma)$, despite of the fact that the cascade is asymmetric ($\theta = 45^\circ$). The reader can find a sketch of the configuration in Fig. 6.1.3.

On the other hand, when considering a zero steady loading case vibrating in a torsion mode, $\nabla \mathbf{n} \neq 0$ and then, the ULP is not zero, actually the $ULP=1$, since the rotational contribution, ULP_R , is not zero, and then the flutter characteristics are very similar to that of a loaded airfoil, i.e: the phase of the influence coefficients is small and varies linearly with k , the influence coefficient moduli are constant with k , and the work per cycle curves are sinusoidal. The qualitatively different behaviour of flat plates vibrating in bending and torsion is predicted by the ULP in conjunction with the general theory described in the previous chapter.

Figure 6.1.1, shows the non-dimensional IC modulus (a) and phase (b) for a flat plate vibrating in flap mode (filled symbols) and in torsion about the l.e. (open symbols). It can be seen that the behaviour of the torsion and flap modes is completely different since, according with the theory, a steady unloaded case vibrating in torsion behaves

Figure 6.1.2: NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA FOR AN ASYMMETRIC CASCADE OF FLAT PLATES VIBRATING IN FLAP MODE (FILLED SYMBOLS) AND TORSION ABOUT THE PLATE MIDPOINT (OPEN SYMBOLS) ($M = 0.45, \theta = 45^\circ, s/c = 0.5, k = 0.03$)

exactly as a loaded case, due to the rotational term of the ULP. In this case, the IC modulus remains nearly constant with k , whereas the phases of the ICs are small and vary linearly. As a consequence, the work-per-cycle curves are sinusoidal. When considering the torsion mode, it can be seen that the behaviour predicted for an unsteady unloaded cases is fulfilled: i) all the ICs vary linearly with the reduced frequency and are null for $k = 0$, and ii) the phases of the influence coefficient remain fairly constant in the frequency range of interest for all the airfoils in the cascade, showing only a small correction with the reduced frequency. The phase of the largest ICs, those corresponding to the 0^{th} and -1^{st} airfoils, is either $\pi/2$ or $-\pi/2$, exactly as predicted by the theory as well.

It can noticed that in both, the bending and the torsion case, the largest contributions to the W_{cycle} come from the 0^{th} and -1^{st} airfoils (see Figs. 6.1.3 and 6.1.4 for an overall view of the flow-field) . Moreover, it can be seen that the range of validity of the theory extends well beyond the $k \ll 1$ limit.

The different shape of the work per cycle curves for torsion and bending modes can be confirmed by looking at Fig. 6.1.2, which plots the non-dimensional work-per-cycle,

Θ_1 , defined as

$$\Theta_1 = \frac{W_{cycle}}{\frac{1}{2}\pi\rho_c v_c^2 \delta_c^2 k}. \quad (6.1.1)$$

as a function of the IBPA, computed using the travelling-wave approach, for the same plate, vibrating in bending (filled symbols), and in torsion about the midpoint of the plate (open symbols), for different values of reduced frequency in the flap case and for a reduced frequency of $k = 0.03$ for the torsion case. It can be seen that while for the torsion mode the curves are of sinusoidal shape, for the bending mode, the work-per-cycle curves are cosinusoidal in a wide range of reduced frequencies.

It may also be observed that for the bending case ($ULP = 0$), the damping curves corresponding to a wide range of reduced frequencies nicely collapse into a single universal curve, as predicted by the theoretical analysis, showing surprisingly that it is not possible to destabilise the cascade decreasing the reduced frequency. It may be seen that the theoretical predictions are valid well beyond the $k \ll 1$, since even the $k = 1$ case exhibits a good fit with the rest curves. The contribution in this case of cascade airfoils different from the $+0$ and -1 is not small (see Fig. 6.1.1 (a)) , however in this case it may be observed that the phases of the ± 2 airfoils are roughly $\phi_{\pm 2} = \pi/2 \pm 3\pi/4$, and hence do not contribute to spoil the symmetry of the damping curve.

It can be concluded that the different behaviour of a flat plate oscillating in bending and torsion is strongly correlated with ULP and the theory derived for low-reduced frequency, and is due to the contribution of the rotational part to the ULP.

The iso-contours of the real and imaginary, or in-phase and out-of-phase, parts of the unsteady pressure of a flat plate vibrating in a torsion about the l.e. at $k = 0.1$ are displayed in Fig. 6.1.3. It can be noticed that, the real component is much higher than the imaginary one, which is consistent with the fact that the unsteady pressure phase is small. On the contrary, when considering the bending mode (see Fig. 6.1.4 for more details), it can be seen that now the real component is much smaller than the imaginary one, which is consistent with the fact that the phase is $\pm\pi/2$. This different behaviour of the real and imaginary part between a bending

Figure 6.1.3: REAL (A) AND IMAGINARY (B) COMPONENTS OF THE UNSTEADY PRESSURE FOR AN ASYMMETRIC CASCADE OF FLAT PLATES VIBRATING IN TORSION ABOUT THE L.E. ($M_{in} = 0.45, \theta = 45^\circ, s/c = 0.5, k = 0.1$)

Figure 6.1.4: (A) REAL AND (B) IMAGINARY COMPONENTS OF THE UNSTEADY PRESSURE FOR AN ASYMMETRIC CASCADE OF FLAT PLATES VIBRATING IN BENDING ($M_{in} = 0.45, \theta = 45^\circ, s/c = 0.5, k = 0.1$)

and a torsion case, even when considering the same cascade, with the same steady loading,, is exactly as predicted by the theory.

6.2 Steady Loading Effects

In order to investigate the effect of the aerodynamic loading in the aerodynamic damping, i.e.: changes in $\nabla \tilde{\mathbf{W}}_0$, different operating points have been studied for an LPT airfoil and for the NACA 65 configuration. The NACA65 configuration will also be used to explain the trends at reduced frequencies $k \sim O(1)$. It will be seen that, although the way to obtain conclusions at high values of reduced frequency is not rigorous, since the theoretical analysis has been done under the assumption that $k \ll 1$, it will be seen that the theory is able to explain many behaviours of the work-per-cycle curves at values of reduced frequency $k \sim O(1)$, which is a surprising fact. The vibration mode has been defined for all the cases as a rigid body flap mode in a direction orthogonal to the engine axis. The trends derived from the ULP are compared against that obtained solving the linearised Euler equations numerically.

6.2.1 LPT case off-design effect

An LPT airfoil has been considered in first place to check the ability of the ULP to predict the aerodynamic damping of vibrating blades at low reduced frequency. The design operating point is defined by an inlet and exit angle of $\alpha_1 = 44^\circ$ and $\alpha_2 = 60^\circ$, respectively, $M_{exit} = 0.75$, and an airfoil spacing of $s/c = 0.96$. The aerodynamic loading is changed by modifying the incidence with respect the aerodynamic design point (ADP). Different operating points (OPs) have been studied, the ADP, a positive off-design of 10 deg (name ADP+10), and a negative off-design of -24 deg (named ADP-24). Linearised inviscid simulations were carried out using a grid containing about 32,000 nodes per passage.

The results obtained for the ULP for the three OPs are shown in Table 6.1, while Fig. 6.2.1 shows the isentropic Mach number distribution along the airfoil chord for

Incidence	$i = 0^\circ$	$i = 10^\circ$	$i = -24^\circ$
ULP	1.3	1.75	1.07

Table 6.1: UNSTEADY LOADING PARAMETER VALUE FOR THE LPT CONFIGURATION FOR DIFFERENT INCIDENCE ANGLES

Figure 6.2.1: ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD FOR THE LPT CASE FOR THE DESIGN AND OFF DESIGN CONDITIONS

the three cases. It can be observed that the velocity distributions only differ in the front part of the airfoil, whereas the loading in the aft part of the blade remains unchanged, as it was expected. It is important to remark that, in this case, the spurious contribution to the ULP due to the acceleration around the rounded t.e. of an inviscid flow (that is estimated in 0.7 for all the operating points) has been manually removed. First of all, it can be noticed that the results obtained for the ULP are consistent with the isentropic Mach number distributions: the highest value of the ULP is the corresponding to the positive off design, whereas the lowest one is the corresponding to the negative off design. The difference between the design case and the negative off design case is smaller than the difference between the design case and the positive off design case. According to the theory, the influence coefficient modulus is proportional to the unsteady loading parameter, whereas the phase is proportional to its inverse, which directly implies that the higher the ULP is, the more unstable the configuration is, from a work-per-cycle point of view.

Figure 6.2.2: (A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE OF THE CENTRAL AIRFOIL AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE LPT CASE VIBRATING IN FLAP MODE FOR THE DIFFERENT OPERATING CONDITIONS

Figure 6.2.2 (a) represents the influence coefficients moduli for the vibrating airfoil of the cascade, for the three operating points, whereas the phases of the influence coefficient are shown in Fig.6.2.2 (b). It can be seen that the slope of the $\phi - k$ curves and the IC moduli correlate very well with the unsteady loading parameter, since the highest influence coefficient modulus and the lowest phase correspond to the highest unsteady loading parameter (ADP+10), and the lowest influence coefficient modulus and the highest phase are associated to the lowest unsteady loading parameter (ADP-24). More precisely, the ratio of influence coefficient phases (or the ratio of the influence coefficient modulus) at design and off-design conditions, e.g.: $\frac{\phi_{0,i=+0s}}{\phi_{0,i=+10s}} \simeq 1.3$, is approximately the same as the ratio of the ULPs, i.e. $\frac{ULP_{0,i=+0s}}{ULP_{0,i=+10s}} \simeq 1.3$. Note that in 6.2.2 (b) the slope of the $\phi - k$ curves at the origin are represented by means of dashed lines, as well as its numerical values, to ease the understanding of the trends in the low reduced frequency limit. The results for the ± 1 airfoils of the cascade have not been shown in this case in order to not overcrowd the plot, but the ULP is also able to predict their behaviour in a qualitative manner, and it will be checked later on for other cases.

Figure 6.2.3 shows the non dimensional work per cycle, Θ_2 , defined as:

$$\Theta_2 = \frac{W_{cycle}}{\frac{1}{2}\pi\rho_c v_c^2 \delta_c^2} \quad (6.2.1)$$

Figure 6.2.3: NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA AND THE STEADY FLOW CONDITIONS FOR THE LOW PRESSURE TURBINE CASE VIBRATING IN FLAP ($M = 0.74$, $k = 0.1$)

as a function of the IBPA, for the three operating points. In the same line as for the IC approach, the damping curves follow the ULP trends, which can predict the stability trend of the W_{cycle} curves. The ADP+10 operating point has the largest ULP, and therefore is the most unstable configuration, whereas the ADP-24 operating point, which has the smallest UDP, is the most stable.

The sensitivity of the W_{cycle} to the $ULP = \Upsilon$ is much smaller than that of the IC modulus or phase. It is important to recall at this point (see [101]) that:

$$\phi \sim \frac{k(1 + \alpha^* \Upsilon)}{\Upsilon} \quad (6.2.2)$$

whereas the non-dimensional IC moduli scales as $\tilde{l}_j \sim \Upsilon$, hence the mean value of the non-dimensional W_{cycle} is: $\Theta_{mean} \sim k(1 + \alpha \Upsilon)$, and therefore its sensitivity to the ULP is smaller than that of ϕ or $\tilde{l}_j$, as it can be noticed in Fig. 6.2.3. The damping variation with the IBPA, however, scales as $\Delta\Theta \sim \Upsilon$, and Fig. 6.2.3 is consistent as well with this scaling. In general, the low sensitivity of the aerodamping of LPTs with the incidence is in accordance with the existent experience.

6.2.2 NACA 65 Compressor

6.2.2.1 Off-design effects

The NACA65 compressor airfoil, [9, 99], has been chosen as the vehicle to prove that the ULP can predict the behaviour of different types of geometries and also of airfoils operating at transonic conditions, as long as they are in the low reduced frequency range. Although compressors typically operate at higher reduced frequencies, it is still interesting to understand the behaviour in the limit case of $k \ll 1$. Linearised inviscid simulations were carried out using a grid containing about 16,000 nodes per passage. The subsonic operating point (see [9, 99]) is defined by an inlet angle $\alpha_1 = 55^\circ$, exit angle $\alpha_2 = 40^\circ$, inlet and exit isentropic Mach number, $M_{in} = 0.7$ and $M_{exit} = 0.45$, respectively, and airfoil spacing, $s/c = 1$. In this operating point the airfoil was operating at a positive off-design and then, the authors have decided to study the design case by reducing the inlet angle by 3 deg ($i = -3^\circ$) with respect to the standard case.

Figure 6.2.4 summarizes the isentropic Mach number distributions along the airfoil chord for all the NACA65 cases, including the two subsonic operating conditions. It can be seen that a variation of 3 deg in the incidence gives rise to a noticeable loading modification in the front part of the airfoil. Table 6.2 shows the unsteady loading parameter for the different off-design conditions. Figures 6.2.5 (a) and 6.2.5 (b) display the non-dimensional influence coefficient moduli and phase for the NACA65 airfoil. It can be seen that, for the central airfoil, the influence coefficient modulus of the design case is about twice that of the standard off-design case ($l_{0,i=-3deg} \simeq 2 * l_{0,i=0deg}$) and that the slope of the $\phi - k$ curves is $\phi_{0,i=0deg} \simeq 2 * \phi_{0,i=-3deg}$, exactly as predicted by the ULP, since $\Upsilon_{i=-3deg} \simeq 2 * \Upsilon_{i=0deg}$. It can be concluded that the ULP is strongly correlated with the slope of the phase of the IC, and its modulus, when the isentropic Mach number distribution is changed due to a change of the incidence. The same behavior was obtained for the LPT airfoil, which has a completely different shape and loading distribution. It is concluded that this observation is quite general and that the ULP can quantitatively predicts off-design effects in the low reduced

Incidence	$i = 0^\circ$	$i = -3^\circ$
ULP	0.3	0.7

Table 6.2: UNSTEADY LOADING PARAMETER FOR THE NACA65 CONFIGURATION FOR DIFFERENT INCIDENCE ANGLES AT $M_1=0.7$

Figure 6.2.4: ISENTROPIC MACH NUMBER DISTRIBUTIONS ALONG THE AIRFOIL CHORD FOR THE NACA65 COMPRESSOR FOR SUBSONIC DESIGN AND OFF DESIGN CONDITIONS, AND FOR DESIGN TRANSONIC CONDITIONS

frequency limit.

6.2.2.2 Behaviour at $k \sim O(1)$

Figure 6.2.6 (a) represents the modulus of the ICs as a function of the reduced frequency. It may be observed that reduced frequency range of the plot extends well beyond the low reduced frequency regime. When $k \leq 0.1 - 0.2$ the ICs are fairly constant, (see the inset of Fig. 6.2.6 (a)) however when $k \sim O(1)$ the ICs grow proportionally with k , exactly as it predicts the theory derived for $k \ll 1$. It is important to recall that the unsteady pressure associated solely to the vibration velocity of the airfoil, not to the loading, is proportional to k . It can be seen as well that only the 0^{th} and -1^{st} airfoils contribute significantly to the W_{cycle} at low reduced frequency.

Figure 6.2.5: (A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 CASE VIBRATING IN FLAP MODE FOR THE $i = 0^\circ$ (FILLED SYMBOLS) AND $i = -3^\circ$ (OPEN SYMBOLS) CASES

Figure 6.2.6 (b) shows the non-dimensional work per cycle, Θ_2 defined as

$$\Theta_2 = \frac{W_{cycle}}{\frac{1}{2} \pi \rho_c v_c^2 \delta_c^2} \quad (6.2.3)$$

as a function of the IBPA and the reduced frequency. It can be clearly seen that for $k \ll 1$ all curves collapse into a single one, and have a sinusoidal shape. It can also be seen that for $k \sim o(1)$ the curves have a co-sinusoidal shape, what is a direct consequence of the fact that when k is increased the phase of the ICs tends to $\pi/2$, as can be seen in Fig. 6.2.7, that displays the phase of the ICs as a function of k .

For loaded airfoils the mean value of the damping increases with k . At low reduced frequency, i.e $k \ll 1$, it is due to the fact that $\phi \propto k$, whereas at higher values of k (for $k \sim o(1)$) the phase is $\phi \simeq \pi/2$ but $p'_c \propto k$. Simultaneously the sinusoidal shape of the W_{cycle} curve at $k \ll 1$ becomes a cosine-like curve for $k \sim o(1)$ because of $\phi \rightarrow \pi/2$. This behaviour is the same when considering a low-pressure turbine geometry and it is consistent with the theory presented in the previous chapter. This actually means that this is a generic behaviour independent of the airfoil shape. Actually it becomes apparent that the cosine-like shape typical of the damping curves of compressor airfoils is not due to the airfoil's shape or flow conditions but it is a direct consequence of the fact that their reduced frequency is close to one, since at $k \ll 1$ compressors have as well damping curves with a sine-like shape.

Figure 6.2.6: (A) NON-DIMENSIONAL IC MODULUS OF THE 0^{th} AND $\pm 1^{st}$ AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY AND (B) NON-DIMENSIONAL DAMPING AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE NACA65 COMPRESSOR ($M_{in} = 0.7$)

Figure 6.2.7: PHASES OF THE 0^{th} AND ± 1 AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 COMPRESSOR ($M_{in} = 0.7$)

M_1	$M_1 = 0.7$	$M_1 = 0.8$
ULP	0.3	-0.17

Table 6.3: UNSTEADY LOADING PARAMETER FOR THE NACA65 CONFIGURATION FOR DIFFERENT INLET MACH NUMBER

6.2.3 NACA 65 Mach number effects

In order to check that the ULP can predict changes in the loading distribution associated to variations in the Mach number, the inlet Mach number was raised from $M_{in} = 0.7$ to $M_{in} = 0.8$, driving the airfoil into the transonic regime.

Figure 6.2.4 displays the isentropic Mach number distributions along the airfoil chord for both inlet Mach numbers, whereas Table 6.3 shows the corresponding ULP for these two cases.

The transonic case exhibits a strong shock wave in the suction side that is clearly visible in the isentropic Mach number distribution. It is very interesting to notice that the ULP for the transonic case is significantly lower than that of the subsonic case. This fact is reflected in the influence coefficient moduli and phases (Fig. 6.2.8 (a) and (b)). It can be seen that, in the region where the theory is applicable ($k \ll 1$), the influence coefficient modulus of the vibrating airfoil is the double for the subsonic case than for the transonic case, and that the influence coefficient moduli of the other airfoils of the cascade is smaller for the transonic than for the subsonic conditions. The influence coefficient phases follow the expected trends as well: the slope of the $\phi - k$ curves is higher for the subsonic than for the transonic case.

For the transonic case, the low reduced frequency theory is only valid for very low values of the reduced frequency because the ULP is very low. Remember that the theory states that an airfoil behaves as a loaded airfoil if $\tilde{\nabla} \tilde{W}_c \sim \text{ULP} \gg k$. If the unsteady loading is small, the range of k at which the airfoil behaves as a loaded airfoil (constant IC with k , and small pressure phase) is very narrow. In our case, the IC modulus does not remain constant with the reduced frequency and the phase of the airfoil 0 is $-\pi/2$. The negative sign of the ULP predicts that, on the contrary than for the subsonic case, the highest contribution to the work-per-cycle comes from the +1

Figure 6.2.8: (A) NON-DIMENSIONAL INFLUENCE COEFFICIENT MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 CASE VIBRATING IN FLAP MODE. $M_1 = 0.8$: FILLED SYMBOLS AND $M_1 = 0.7$: OPEN SYMBOLS.

airfoil (the blade numbering sign criterion is defined in Fig. 6.2.9). Figure 6.2.9 (a) displays the real and imaginary part of the unsteady pressure for the transonic case. It can be seen that the presence of the shock wave generates a high level of unsteady pressure in its vicinity, and then, the unsteady pressure distribution completely differs from that observed for the subsonic case (see Fig. 6.2.9 (b))

6.3 Mode shape effects

The capability of the ULP to predict the effect of the bending direction on the modulus and phase of the ICs has been studied for the LPT case operating at design conditions and for the NACA 65 compressor operating at subsonic conditions. The motion of a blade as a rigid body about an arbitrary torsion centre can be seen as a linear combination of a flap, an edge-wise, and a torsion mode. The predictions of the analytical trends for the flap and edge modes, are compared against numerical results for the LPT, and the NACA65 configurations. For the LPT, the torsion mode is also studied. The motion of an airfoil as a rigid body about an arbitrary torsion centre can be seen as a linear combination of a flap, an edge-wise, and a torsion mode [90]. The predictions based on the ULP are compared against the actual numerical results for the LPT and the NACA65 configurations oscillating in

Figure 6.2.9: REAL (LEFT) AND IMAGINARY (RIGHT) PART OF THE UNSTEADY PRESSURE OF THE NACA65 COMPRESSOR FOR $k = 0.1$, FOR $M_{in} = 0.8$ (A) AND FOR $M_{in} = 0.7$ (B)

ULP Effect	F	E	LE	c/4	c/2	3c/4
Loading	1.93	0.92	1.39	0.3	0.11	-0.5
Rotation	0	0	1.15	1.15	1.15	-1.15

Table 6.4: UNSTEADY LOADING PARAMETER FOR THE LPT CONFIGURATION FOR THE TWO FUNDAMENTAL BENDING DIRECTIONS AND DIFFERENT TORSION CENTRES

the bending and edge-wise modes.

6.3.1 LPT Design Case

The ULP has been evaluated for the edge-wise, flap, and several torsion modes. According to the results, summarised in Table 6.4 (note that the values for the ULP summarised in this table include the t.e. contribution, that was removed in the subsection “Steady Loading Effects”). In accordance with the ULP values computed for the flap and edge-wise modes (First two columns in Table 6.4) we should obtain that $l_{0,FLAP} \simeq 2 \cdot l_{0,EDGE}$ and $\phi_{0,EDGE} \simeq 2 \cdot \phi_{0,FLAP}$; and, as a consequence, the airfoil is more unstable vibrating in the bending than in edge-wise direction.

This statement can be confirmed by looking at the non dimensional work-per-cycle curves, represented in Fig. 6.3.1, for the reduced frequency $k = 0.1$. It can be seen that the mean values are almost the same for the two modes whereas the amplitude of the sinusoidal variation of the aerodynamic damping with the IBPA for the flap mode is about twice as big as that corresponding to the edge mode, as predicted by the theory. The different trend of the IC modulus and phase (see Fig. 6.3.2 (a) and (b) respectively) of the flap and edgewise modes with the reduced rfrequency correlates very well quantified with the ULP for the central airfoil, and it is also well predicted for its neighbouring airfoil.

Torsion modes have also been investigated. Different torsion centres, all of them with a fixed vibration amplitude of 1 deg, have been considered, namely: a torsion about the l.e. and rotations about centres placed at a distance of $c/2$, $c/4$ and $3c/4$ from the l.e., where here c represents the true chord. The ULP values are presented together with the corresponding to the bending modes, in Table 6.4. The total unsteady loading parameter for the torsion mode is the sum of the two contributions (the

Figure 6.3.1: NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA FOR THE LPT CASE VIBRATING IN FLAP, EDGE AND DIFFERENT TORSION MODES ($M = 0.74$, $k = 0.1$)

Figure 6.3.2: (A) NON-DIMENSIONAL IC MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE LPT CASE VIBRATING IN FLAP (FILLED SYMBOLS) AND EDGE MODE (OPEN SYMBOLS).

steady loading contribution and the rotational contribution), and then the total ULP parameters are $\Upsilon_{TORSION_{LE}} \simeq 2.46$, $\Upsilon_{TORSION_{c/4}} \simeq 1.47$, $\Upsilon_{TORSION_{c/2}} \simeq 1.3$ and $\Upsilon_{TORSION_{3c/4}} \simeq -1.65$.

When looking at the non dimensional work per cycle curves (Fig. 6.3.1) it can be seen that the instability level is perfectly aligned with the simplified results predicted using the ULP. For the torsion about the centre placed at a distance of $3c/4$ from the l.e., the ULP is negative; as it has been previously mentioned, this is an indication that the neighbouring airfoil with the largest contribution to the work per cycle is located at the opposite side of the central airfoil, in this case -1 profile. This fact directly implies that the minimum damping occurs at IBPA=-90° instead of at IBPA=90°. This behaviour can be confirmed by looking at the damping curves, Fig. 6.3.1. The effect of the sign of the ULP onto the neighbour airfoils and, as a consequence, onto the W_{cycle} curves was reported for the NACA65 configuration when comparing the subsonic and the transonic conditions, in the subsection “Steady Loading Effects”.

6.3.2 NACA65 Subsonic Case

Analogously to the LPT case, the ULP is now evaluated for the NACA 65 operating at subsonic conditions for different mode-shapes. Analytical results for the ULP are presented in Table 6.5. The analytical solution shows that $\Upsilon_{EDGE} \simeq 5.5 \cdot \Upsilon_{FLAP}$. According to this, it should be obtained from the linearised Euler solver that $l_{0,EDGE} \simeq 5.5 \cdot l_{0,FLAP}$ and $\phi_{0,FLAP} \simeq 5.5 \cdot \phi_{0,EDGE}$; and then, contrary to the general behaviour of the LPT airfoils, the edge mode should be more unstable than the flap mode. It is very important to note that the negative sign of the ULP denotes that the highest contribution to the work per cycle comes from the opposite neighbouring airfoil i.e: for the flap mode the -1 airfoil is the one contributing the most, while for the edge mode the +1 airfoil has the highest contribution. This trend can be confirmed by analyzing Fig. 6.3.3, which represent the non-dimensional IC moduli and phase, obtained from the numerical simulations. It can be seen that, from the IC point of view, there is a very good correlation between the influence coefficient

Mode shape	Flap	Edge
<i>ULP</i>	0.294	-1.63

Table 6.5: UNSTEADY LOADING PARAMETER FOR THE NACA65 CONFIGURATION FOR THE TWO FUNDAMENTAL BENDING DIRECTIONS

Figure 6.3.3: (A) NON-DIMENSIONAL IC MODULUS AND (B) PHASE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA65 CASE VIBRATING IN FLAP (FILLED SYMBOLS) AND EDGE MODE (OPEN SYMBOLS)

modulus and phase, even for the neighbouring airfoils, with the ULP, that actually predicts very well their trends.

It can be seen that the theory is valid for a larger range of k for the edge mode than for the flap: i.e the range of k at which ϕ_0 varies linearly and l_0 remains constant; this is a direct consequence that the ULP of the flap mode is much smaller than that of the edge mode.

6.4 Geometry Effects

In the previous subsections, the effect of the aerodynamic loading and the vibration mode onto the unsteady aerodynamics of oscillating airfoils at low reduced frequency has been investigated. It has been proved that the ULP can predict with an acceptable degree of accuracy, the variations onto the ICs modulus and phase, and the work-per-cycle curves when a modification, either in the design operating conditions or in the vibration mode shape, is introduced. It is necessary the reader

Mode	ULP		m	
	LPT	NACA65	LPT	NACA65
FLAP	1.3	0.3	2	17
EDGE	0.92	-1.63	3	3

Table 6.6: UNSTEADY LOADING PARAMETER AND SLOPE OF THE $\phi - k$ CURVES FOR THE LPT AND THE NACA65 CONFIGURATIONS FOR THE TWO FUNDAMENTAL BENDING DIRECTIONS

Figure 6.4.1: MEAN (FILLED SYMBOLS) AND MINIMUM (OPEN SYMBOLS) DAMPING AS A FUNCTION OF THE REDUCED FREQUENCY OF THE LPT AND NACA65 (SUBSONIC AND TRANSONIC) CONFIGURATIONS.

paid attention to the fact that the ULP can also be used to establish comparisons among different airfoil geometries. Table 6.6 summarises the values obtained for the ULP and the slopes, m , of the $\phi_0 - k$ curves for the LPT and the NACA65 sections. It can be seen that there is a qualitative good agreement between the ULP and m ratios, even when considering different airfoil geometries.

In the same line, a summary of the numerical results obtained for the LPT profile and the NACA65 airfoil, operating at transonic and subsonic conditions, and vibrating in flap mode, is graphically sketched in Fig. 6.4.1. The mean value of the LPT is higher than the corresponding to the NACA65, whereas for the NACA65 it does not change when operating at transonic or subsonic conditions, as expected, since the mean value only increases slightly with the unsteady loading. According to the theory, the minimum damping increases proportionally to the unsteady loading, therefore, Fig.

6.4.1 shows that the most unstable configuration is the LPT airfoil, because it is the one with the largest ULP, and that the NACA 65 airfoil is more unstable operating at $M_{in} = 0.7$ than at $M_{in} = 0.8$, because of the ULP parameter is higher for the subsonic conditions than for the transonic conditions. It can be seen that in the same scale, the variation of the minimum damping for the NACA65 when moving from the subsonic to the transonic conditions is small, when it is compared with the corresponding value for the LPT. This fact is consistent with the values calculated for the ULP, since $\Upsilon_{LPT} \simeq 6 \cdot \Upsilon_{NACA65,transonic}$ and $\Upsilon_{NACA65,subsonic} \simeq 1.6 \cdot \Upsilon_{NACA65,transonic}$.

It can be concluded that the ULP can qualitatively compare different airfoils, but is much better when comparing the same airfoil at different operating points and mode-shapes.

6.5 Mach Number Effect

6.5.1 Phasing Variation

The effect of the Mach number in the conclusions derived in this thesis is investigated by means of the LPT configuration previously presented. The vibration mode shape chosen is a flap mode, although all the conclusions presented here apply to other mode-shapes. The results presented within this section correspond to viscous cases. The phase of the unsteady pressure of the central and +1 airfoils with respect the airfoil displacement, ϕ , plays a central role in the overall work of the system and therefore it has been inspected in great detail. Figure 6.5.1 shows the dependence of the influence coefficient phase of the central, ϕ_0 , and +1, ϕ_{+1} , airfoils as a function of the reduced frequency and Mach number. It may be seen that both, in a large range of reduced frequencies, scale linearly, as predict by the theory. It can also be seen that the sensitivity to the Mach number is fairly slow, so it can be concluded that compressibility is a second order effect, which only takes place through a slight modification of the airfoil loading shape, as it will be explained later on in the “Physical Interpretation” section. The effect of the change in the dynamic head is

Figure 6.5.1: PHASE OF THE INFLUENCE COEFFICIENTS OF THE 0TH AND +1ST AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY AND MACH NUMBERS FOR A LPT AIRFOIL.

well captured and the secondary role of Mach number effects, neglected in the theory, confirmed.

As it was already explained in the previous chapter, if it is assumed that the influence of all the airfoils, except that of the central and adjacent airfoils, can be neglected in the work per cycle (see [5] and Fig.6.5.2) then:

$$W_{cycle} = -\pi p'_c \delta_c c \left[\tilde{l}_0 \sin \phi_0 + \tilde{l}_{+1} \sin(\phi_1 + \sigma) + \tilde{l}_{-1} \sin(\phi_{-1} - \sigma) \right] \quad (6.5.1)$$

where $\tilde{l}_j$ and ϕ_j are respectively the non-dimensional modulus and phase of the j^{th} influence coefficient, and δ_c the modulus of the vibration amplitude. Moreover if the reduced frequency is small enough, and therefore $\phi_j \ll 1$, and if $\tilde{l}_{-1} \ll \tilde{l}_{+1}$ (which is frequently seen in aerodynamically loaded oscillating airfoils), then the expression 6.5.1, neglecting second order terms, reduces to:

$$W_{cycle} = -\pi \delta_c^2 \tilde{l}_{+1} \left[\left(\frac{\tilde{l}_0}{\tilde{l}_{+1}} \right) \phi_0 + \sin(\phi_1 + \sigma) \right], \quad (6.5.2)$$

Where, it can be seen that the phase of the 0th influence coefficient is responsible of the work-per-cycle increase with k (the stabilisation of the work per cycle curves with the reduced frequency), which in first approximation is linear since the phase is linear with the reduced frequency; actually it is $\phi_0 \simeq 2k$ for this case. The phase of the +1 airfoil is responsible of the shift of the minimum work-per-cycle location.

Figure 6.5.2: MODULUS OF THE FORCE INFLUENCE COEFFICIENTS AS A FUNCTION OF THE REDUCED FREQUENCY FOR $M_2 = 0.75$

The small values of the phase associated to the low reduced frequency regime are the ultimate reason of the sinusoidal shape of the work-per-cycle as a function of the IBPA curve. Moreover if the contribution of the -1 airfoil is much smaller than that of the +1 airfoil the minimum of the curve takes place at $\sigma \simeq 90^\circ$.

6.5.2 Work-per-cycle Model

Considering all the previous results it is easy to derive a simple model for the work-per-cycle based on the expression 6.5.2, that relates the work-per-cycle with the influence coefficients. Since it has already been stated that $\phi_0 \simeq 2k$ and $\phi_1 \simeq k$, to close the model an expression for the ratio between the influence coefficients, $\tilde{l}_0/\tilde{l}_{+1}$, and an expression for $\tilde{l}_{+1}$ are needed. The ratio has a direct impact in the determination of the stability region, the critical reduced frequency, while the actual value of the IC affects only the severity of the instability.

Figure 6.5.3 represents the $r = \tilde{l}_0/\tilde{l}_{+1}$ ratio as a function of the reduced frequency and Mach number. It may be observed that in the range of interest this ratio is fairly constant, increasing slightly with the reduced frequency, about 10% when k is increased from 0 to 0.4. The variation with the Mach number is smooth as well,

Figure 6.5.3: RATIO BETWEEN THE MODULUS OF INFLUENCE COEFFICIENT 0 AND +1 AS A FUNCTION OF THE REDUCED FREQUENCY AND MACH NUMBER

the ratio drops about 10% when M_{exit} is changed from 0.2 to 0.75. Hence, the ratio between the ICs can be modelled as $r = r_0 + \alpha k$, with $r_0 \simeq 1.1$ and $\alpha \simeq 0.2$ for this particular case.

For a flapping mode the exit angle of all the airfoils of the cascade is the same and then, the total lift generated by them must be constant, in a quasi-steady regime. The lift increase generated in the central airfoil when it moves downwards and the pitch of the suction side passage is increased has to be balanced by the lift decrease of the +1 airfoil, that sees a decrease of the pitch of its suction side passage. Therefore, $\tilde{l}_0 \simeq \tilde{l}_{+1}$ in first approximation, irrespectively of the reduced frequency and the Mach number. The missing lift (around 10% in this case) is compensated by the contribution of the rest of the airfoils.

The critical reduced frequency, k_c , may be derived from:

$$(r_0 + \alpha k_c)2k_c - 1 = 0. \quad (6.5.3)$$

The most important conclusion is that the stability limit is weakly dependent on the Mach number.

Figure 6.5.4 describes the variation of the non-dimensional influence coefficient

Figure 6.5.4: VARIATION OF THE INFLUENCE COEFFICIENT OF THE CENTRAL BLADE WITH THE SQUARE OF THE MACH NUMBER AND THE REDUCED FREQUENCY

modulus of the central airfoil with k and M . It may be seen that the variation with the Mach number squared, or the dynamic head, is properly absorbed by the non-dimensionalisation since the variation of the IC with M_{exit}^2 in the range $M_{exit} \in [0.4 - 0.7]$ is of second order compared with the variation of k , that is small as well, of the order of 10% in the range $k \in [0 - 0.2]$.

The previous effect is a clear indication that the unsteady pressure scales proportionally to the dynamic head, exactly as it was predicted by the theoretical analysis. The effect of the Mach number is then in first approximation to increase the severity of the instability due to the fact that $l_0 \propto M^2$, or more precisely, $l_0 \propto \frac{1}{2} \rho_c U_c^2 f(M)$, but it does not significantly change the stability limit, since ratio of the influence coefficient modulus, $r = \tilde{l}_0 / \tilde{l}_1$ is fairly constant with the Mach number.

The dependence of the IC with the reduced frequency may be derived from the same figure, and is of the form: $l_0 = (l_0)_{k=0} + (dl_0/dk)_{k=0} k$, where the linear correction with k is small for loaded airfoil and low reduced frequencies, as it has been demonstrated theoretically.

These two facts may be verified deriving directly the damping as a function of the *IBPA* for different Mach numbers. Figure 6.5.5 represents the variation with the

Mach Number of the non-dimensional work, obtained using the TW formulation. It may be observed that while the effect of changing only the reduced frequency but not the Mach Number is a shift in the work-per-cycle curve and an increase in the mean work, the effect of the Mach number is in first approximation very small, much less obviously than that of the dimensional influence coefficient displayed in Fig. 6.5.4.

The weak effect of changing the M onto the $\tilde{W}_{cycle}$ curves is due to the fact that the effect of the Mach on the dynamic pressure has already been retained in the non-dimensionalisation of the influence coefficient moduli, while the effect of the acoustics (the effect of the M on the influence coefficient phase) has been shown to be small and therefore all the curves tend to coalesce in a single one, especially in the minimum damping region.

Since the variation of $\tilde{W}_{cycle}$ with M is solely due to the variation of $r = \tilde{l}_0/\tilde{l}_1$ with M , which is small as well, the curves for different values of M still exhibit a small scaling effect, since changing the Mach number for a given geometry gives rise to a significant change of the airfoil loading, that is superimposed to the Mach number change (see Fig. 6.5.7). However, the stability region is fairly unaffected, as it was claimed before. Figure 6.5.6 shows the amplitude of the mean and 1st harmonic (the variation with the IBPA) of the non-dimensional work as a function of the square of the Mach number. It can be still seen that both scale weakly with Mach Number square.

Note that since $k \ll 1$ and $\alpha \simeq 0.2 \Rightarrow \alpha k \ll r_0 \Rightarrow r \simeq r_0$; therefore, a model for the work per cycle for this mode can be written according to the following formula:

$$W_{cycle} = -\pi\beta q_0^2 [2r_0k + \sin(k + \sigma)] \quad (6.5.4)$$

where $\beta = (P_0 - p_{exit})c = P_0 c \mathcal{F}(M_{exit}^2)$ and, if $M_{exit}^2(\gamma - 1)/2 \ll 1$, then

$$\beta \simeq \frac{\gamma}{2} p_0 M_{exit}^2 \simeq \frac{1}{2} \rho_{exit} V_{exit}^2 \quad (6.5.5)$$

(note that, for $M = 0.7$, $M_{exit}^2(\gamma - 1)/2M_{exit}^2 \simeq 0.1 \ll 1$). Note that the W_{cycle} is proportional to the dynamic head at the exit, that scales as the square of the Mach number, exactly as shown in Fig. 6.5.4, where the explicit dependence with M_{exit}^2

Figure 6.5.5: NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE MACH NUMBER

Figure 6.5.6: MEAN VALUE AND VARIATION WITH THE IBPA OF THE NON-DIMENSIONAL WORK AS A FUNCTION OF THE MACH NUMBER

Figure 6.5.7: ISENTROPIC MACH NUMBER DISTRIBUTION NORMALIZED WITH THE EXIT MACH NUMBER FOR DIFFERENT EXIT MACH NUMBERS.

has been absorbed by the non-dimensionalisation. This is a comprehensive model for W_{cycle} that retains the influence of both the Mach and the reduced frequency.

Figure 6.5.8 represents a sketch which summarises the effect of both, the reduced frequency and the Mach number onto the stability and the trends of the work-per-cycle curves. This figure can be explained as follows: departing from a cas in the $k \ll 1$ range, whose work-per-cycle curve is be similar to the blue line, if the reduced frequency is increased and the Mach number is maintained constant, the effects on the curve can be seen looking at marron curve, which are an increase in the mean value and a shift in the $IBPA$ where occurs the minimum, towards the $IBPA = 0^\circ$. Departing from this situation, if the reduced frequency is frozen and the Mach number is increased, it can be seen that the effect is an increase in the severity of the instability (see pink curve).

Figure 6.5.8: EFFECT OF THE REDUCED FREQUENCY AND THE M ONTO THE WORK-PER-CYCLE CURVES

6.6 Symmetric cascades

6.6.1 NACA 0012 Symmetric Case

A NACA 0012 section has been chosen to build a linear symmetric cascade and verify the theoretical conclusions derived for symmetric loaded configurations. Both the inlet and outlet flow angles are $\alpha_{1,2} = 0^\circ$, the exit isentropic Mach number is $M_{exit} = 0.45$, and the pitch to chord ratio $s/c = 1$. For the asymmetric case, constructed with the same airfoil, the boundary conditions are the same, except for the fact that an inlet angle of 10° is considered. Linearised inviscid simulations have been carried out using grids containing about 45,000 nodes per passage.

Symmetric cascades with symmetric airfoils are a particular case, which implies that with a general character, $\tilde{l}_{+1} = -\tilde{l}_{-1}$ and $\phi_{-1} = \pi + \phi_{+1}$. As it has been already outlined in the previous chapter, $W_{cycle}(\sigma) \propto k$ and the W_{cycle} curves are symmetric, i.e. $W_{cycle}(\sigma) = W_{cycle}(-\sigma)$. Figure 6.6.2 (b) shows the dependence of the influence coefficient phase of the central, ϕ_0 , and ± 1 , airfoils, $\phi_{\pm 1}$, as a function of k . It may be seen that all of them, in a large range of reduced frequencies, scale linearly with

Figure 6.6.1: SNAPSHOT OF THE UNSTEADY PRESSURE FOR THE NACA0012 SYMMETRIC CASE FOR $k = 0.2$

the reduced frequency and that $\phi_{-1} = \pi + \phi_{+1}$, as expected for a symmetric case. In the same line, Fig. 6.6.1, which represents a snapshot of the unsteady pressure for a $k = 0.2$ case, shows that the unsteady behavior of this case is symmetric.

The moduli of the ICs, represented in Fig. 6.6.2 (a), follow two different trends because the NACA 0012 is a lightly loaded section compared with the LPT or NACA 65 sections. Figure 6.0.1 shows how the Mach number or the velocity gradient along the airfoils chord, which is actually only a part of the contribution to the gradient, is very small for the NACA 0012, being this case an intermediate case between a highly loaded case and a flat plate with strictly zero loading. This means that the range of reduced frequencies at which the ICs remain constant is much narrower than in the NACA 65 or LPT cases. As a consequence, the ICs exhibit a small constant region for $k < 0.1$, in the same way than highly loaded profiles, and for $k > 0.1$ a linear increase with k , exactly as in the flat plate (unloaded) case.

For a loaded symmetric profile the work-per cycle scales with the reduced frequency

Figure 6.6.2: (A) NON-DIMENSIONAL IC MODULUS AND (B) PHASE OF THE 0TH AND ± 1 ST AIRFOILS AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE NACA 0012 SYMMETRIC (FILLED SYMBOLS) AND ASYMMETRIC (OPEN SYMBOLS) CASES.

as follows:

$$\Theta_1 = \frac{W_{cycle}}{\frac{1}{2} \pi \rho_c v_c^2 \delta_c^2 k} \quad (6.6.1)$$

The non-dimensional W_{cycle} curves for different values of k are displayed in Fig. 6.6.3 (a), where it can be clearly seen that all curves nicely collapse into a single cosine-like shape curve, in a wide range of reduced frequencies, which tightly follows the theoretical predictions well beyond the low reduced frequency limit.

The region $-45^\circ < \sigma < 45^\circ$ exhibit a series of peaks whose amplitude is determined by its separation to the resonant conditions. Since resonances change with k , while the matrix of computed IBPAs is constant, the same IBPA is affected in a different extent depending on the actual value of k . Moreover the resonance peaks are shifted with k and therefore curve collapsing is not possible in the resonances.

6.6.2 NACA 0012 off-Design Case

Figure 6.6.3(b) shows the non-dimensional work-per-cycle curves for a symmetric linear cascade of NACA 0012 airfoils for different values of k , operating at an incidence of 10° , which directly implies that the steady flow field is not symmetric anymore. Most of the conclusions that were obtained for the LPT and NACA 65 cases are re-confirmed once again, as it could be expected, since we are dealing with a loaded

Figure 6.6.3: NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE NACA 0012 SYMMETRIC (A) AND ASYMMETRIC (B) CASES.

profile. The damping curves have a sinusoidal shape, which is a direct indication that only the airfoils adjacent to the central one significantly contribute to the work-per-cycle, and that the phases of the ICs are small.

For this configuration the -1^{st} airfoil is the one that has the largest contribution, whereas the influence of the $+1^{st}$ airfoil is much smaller (see Fig. 6.6.2 (a), and 6.6.4, which displays a snapshot of the unsteady pressure of a 5-airfoil cascade made up with the NACA0012 airfoils operating at off-design conditions, where only the 0th airfoil moves in bending motion at $k = 0.2$), in a similar manner that it is observed in the NACA 65 airfoil. Therefore the curve is a $+\sin$, instead of a $-\sin$, as it typically occurs for LPTs when vibrating in flap mode, where the $+1^{st}$ airfoil is the dominant adjacent airfoil. It may be appreciated how the effect of the off-design is small in the central airfoil, but significantly increases and decreases the unsteady pressure fluctuations in the upper (-1^{st}) and lower ($+1^{st}$) airfoils respectively, being the mean IC of both airfoils similar to that of the purely symmetric case.

It may be noticed that the phases of all the airfoils are weakly affected by the off-design conditions. The increase in loading of the suction side is nearly balanced by a similar decrease in the pressure side (see Fig. 6.0.1), only some effect may be seen in the -1 airfoil that detects the increase in loading of the suction side of the $+0^{th}$ airfoil,

Figure 6.6.4: SNAPSHOT OF THE UNSTEADY PRESSURE FOR THE NACA0012 ASYMMETRIC CASE FOR $k = 0.2$

giving rise to a smaller slope of the phase with k . It should be remember that the larger the loading is, the smaller is the slope of the ICs phase angle, ϕ , vs k curves.

6.7 Physical Interpretation

The aim of this section is to interpret previous results in physical terms. Figure 6.7.1 displays a snapshot of the unsteady pressure of a 5-airfoil cascade made up with the LPT airfoils described in this chapter, where only the 0^{th} airfoil moves in bending motion at $k = 0.3$. It may be seen first that only the 0^{th} and the $+1^{st}$ airfoils are perturbed by the motion of the central blade and that the highest unsteady pressure is located in the region of the highest Mach numbers, as it can be seen in Fig. 6.7.2, which represent the Isentropic Mach number and the unsteady pressure distributions along blade surface of the 0^{th} (Fig. 6.7.2(a)) and $+1^{st}$ (Fig. 6.7.2 (b)) airfoils. At this point is important to highlight that:

1. For this mode, only the suction sides of the 0 and +1 airfoils are affected by the perturbation. This is a purely quasi-steady effect since the peak Mach number

Figure 6.7.1: SNAPSHOT OF THE UNSTEADY PRESSURE CAUSE BY THE VIBRATION OF THE CENTRAL BLADE IN A LPT AIRFOIL FOR $k = 0.3$

is the profile's location with the highest sensitivity to variations in the airfoil location (see Fig. 6.7.2)

2. The unsteady pressure in the suction side of the airfoil +1 is not the result of an acoustic perturbation originated in the pressure side of the 0^{th} airfoil since actually the unsteady pressure in the pressure side of this airfoil is negligible. There are no tracks in Fig. 6.7.1 of perturbations generated in the PS of the central blade reaching the SS of the neighbouring airfoil. Variations in the static pressure are caused in this case by the pitch variation due to the bending of the central airfoil. The wave-length of an acoustic perturbation originated by the vibrating airfoil, λ , would be $\lambda = T.a$, where a is the speed of sound and T the vibration period; then $\lambda/c = 2\pi(a/\omega c) = 2\pi/(M.k)$. If $M = 0.6$ and $k = 0.3$; then $\lambda/c \simeq 35$. This wave-length does not fit neither in the computational domain nor in any linear vibrating cascade, and may not be seen in the contour plots, actually the wave-length of the perturbations in the pitch-wise direction is much shorter. This is a clear indication that acoustic waves play a secondary

Figure 6.7.2: ISENTROPIC MACH NUMBER AND UNSTEADY PRESSURE MODULUS DISTRIBUTIONS ALONG BLADE SURFACE FOR THE: (A) 0TH AIRFOIL AND (B) +1ST AIRFOIL

role in this type of flows.

3. The SS of the central and the +1st airfoil are in first approximation in anti-phase. When the central blade move downwards (as it happens in the time instant displayed in Fig. 6.7.1) the loading of the SS increases and therefore the static pressure decreases. The opposite is true for the SS of the -1 airfoil.
4. The basic idea is that the flapping motion of the central airfoil in the y-direction does not change in first approximation the exit angle of the adjacent passages, and therefore the total lift of the cascade remains constant. This effectively means that in a quasi-steady motion if the lift of one of the airfoils increases the lift of the rest of the airfoils must decrease accordingly, in order that the global lift remains constant. In this particular mode the balance is obtained exchanging the lift of the central and +1 airfoil.

All these observations are aligned with the behaviour previously described for the work-per-cycle as a function of the reduced frequency and the different scaling of the mean and first harmonic with the frequency. However, to obtain the whole picture it is necessary to inspect the phasing of the airfoils.

Phase behaviour reveals a number of interesting physical phenomena. At very low reduced frequencies ($k \ll 1$) the phase tends either to zero or 180° . This is consistent

with the quasi-steady approach in which the perturbations are either in phase or anti-phase, i.e.: pressure increases or decreases with the motion instantaneously. It is important to recall that the reduced frequency:

$$k = \frac{c/v_c}{\omega^{-1}} = \frac{t_{\text{residence}}}{t_{\text{characteristic}}} \quad (6.7.1)$$

is the ratio between the time of residence, $t_r = c/v_c$, and the characteristic time, ω^{-1} . When $k \ll 1$ fluid particles traverse the domain as many as k^{-1} times during a vibration cycle, and therefore the flow is quasi-steady and $\phi = 0$ along the whole airfoil (see Fig. 6.7.3).

When the frequency is increased, it may be appreciated in Fig 6.7.3 how the phases of both, the airfoil 0 and +1, vary nearly linearly along the chord. This effectively means that there is a travelling wave moving from the leading to the trailing edge of the airfoil. If it is assumed that $\Delta p = p'(x)e^{i(\omega t + \phi)}$ and that $\phi = \phi_{L.E.}(1 - x/c)$ then the velocity at which the pressure perturbations propagate is, $v_p = \omega c / \phi_{L.E.}$. Non-dimensionalising v_p using the characteristic velocity, v_c , then $v_p/v_c = k / \phi_{L.E.}$. The fact that the airfoil's phase scales linearly with the reduced frequency is a direct consequence that the velocity of the perturbation scales with the convection velocity, v_c . Comparing the phases of the central and +1 airfoils it may be concluded that the events in the central airfoil take place first in time since this is the blade that is actually moving. The phase difference between the leading and the trailing edge of either the central or the +1 airfoil, for the case $k = 0.3$, is about 60° (see Fig. 6.7.3). This means that the residence time of the perturbations running downstream of the airfoil is $t_{\text{residence}}/T \simeq 1/6$, taking into account that $t_{\text{residence}}/T = (1/2\pi)(c_{\text{true}}/c)k \simeq 1/12$, where it has been assumed that the ratio between the true and the axial chord, c_{true}/c , is roughly 1.5. Both estimations are consistent and show that the velocity of propagation is of the order of the convection velocity. The variation in the circulation due to the airfoil motion is adjusted by vortex shedding in the trailing-edge, that ultimately is nothing else that vorticity which is convected downstream, approximately with the exit velocity, according to the vorticity equation (Chapter 3, Eq. 3.2.10). It is concluded that all the evidences

Figure 6.7.3: PHASE DISTRIBUTION ALONG THE 0-TH (LEFT) AND +1-TH (RIGHT) AIRFOILS FOR DIFFERENT REDUCED FREQUENCIES. SOLID LINE: SUCTION SIDE. DASHED LINE: PRESSURE SIDE

point out that unsteadiness is caused by an adjustment of the circulation of the central airfoil, whose response time is of the order of the chord divided by the mean convection velocity.

Pressure side phase exhibits a more complex behaviour (see Fig. 6.7.3, dashed lines), in general the phase variation along the airfoil is much steeper at least in the 0th airfoil. Following the previous argument, this is a direct consequence of the fact that the propagation velocity, which is associated to the mean convection velocity, is significantly lower in the separated region of the PS than in the SS, as it may be seen in Fig.6.7.4, this causes an abrupt change in the phase in the reattachment point (Fig. 6.7.3, left) around $x/c \simeq 0.5$. The phase of the PS of +1 airfoil is irrelevant in practise since the pressure in that face is very small, but it follows a similar behaviour than that of the 0th airfoil with steep variations of the phase caused by the low momentum separated flow.

These conclusions are considered of a general nature, although no special attempt has been made to present the same study for other airfoils. However, it should be kept in mind that running the same airfoil for different Mach numbers is equivalent to a change in the airfoil loading distribution and equivalent to running different geometries for the same Mach number. The change in the pressure distribution is outlined in Fig. 6.5.7. It may be appreciated how the pressure distribution is

Figure 6.7.4: MACH NUMBER ISO-CONTOURS AND STREAMLINES

significantly aft loaded when the Mach number is increased. The same effect could be obtained by keeping constant the Mach number and changing the airfoil stagger angle. This is a clear indication that the conclusions are not linked to a particular pressure distribution.

6.7.1 Mode-shape variation

It is well known that the sensitivity of the unsteady aerodynamics of vibrating airfoils to the mode shape is very high [90, 1]. The objective of this section is to understand how general the physical interpretation presented in the previous section for the flap mode is. Two more modes have been investigated: the edge-wise mode, represented as a vibration in the x direction; and a torsion mode about a point located in the airfoil's mean line at the maximum thickness location. The same parametric study that was conducted for the flap mode has been performed for both modes. For the sake of brevity, just a reduced set of data will be presented here.

6.7.1.1 Edgewise Mode

As we have seen in the previous sections, the conclusions that were drawn for the flap mode also apply to the edgewise mode. The non dimensional work as a function of the IBPA and the reduced frequency may be seen in Fig. 6.7.5. It may be appreciated that, at low reduced frequencies, the damping curves have a sine like shape, which

Figure 6.7.5: NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE EDGE MODE

is a direct indication that only the airfoils adjacent to the central contribute to the work per cycle; however, conversely that in the flap mode, both adjacent airfoils are equally important (see Fig. 6.7.6, which represent the ratio between the modulus of the influence coefficients, i.e. $\frac{\lambda_{0,-1}}{\lambda_1}$).

The trend of the mean value and the first harmonic of the work per cycle with the reduced frequency is the same as for the flap mode; therefore, they also require the same differential scaling (see Fig. 6.7.7). It is important to highlight that, to derive this figure, the contribution of the spikes associated to the resonances have been removed.

The influence coefficients scale quadratically with the Mach number, whereas the ratios l_1/l_0 and $l_1/l_{-1} \simeq 1$ are fairly constant with the Mach number and the reduced frequency, even in a wider range than for the flap mode. Figure 6.7.8 displays the phases of the zeroth and ± 1 airfoils. It may be seen that the phases of all the airfoils vary linearly with the reduced frequency and the dependence with the Mach number is very weak but higher than for the flap mode.

The edgewise mode exhibits some differences with respect to the flap mode. Although the only two significant airfoils from the point of view of the modulus of the pressure

Figure 6.7.6: RATIO BETWEEN THE MODULUS OF INFLUENCE COEFFICIENT -1,0 AND +1 AS A FUNCTION OF THE REDUCED FREQUENCY AND MACH NUMBER. FYLLED SYMBOLS: AIRFOIL -1, EMPTY SYMBOLS: AIRFOIL 0

Figure 6.7.7: MEAN VALUE AND 1st HARMONIC (VARIATION WITH THE IBPA) OF THE NON-DIMENSIONAL WORK-PER-CYCLE AS A FUNCTION OF THE REDUCED FREQUENCY FOR THE EDGE MODE

Figure 6.7.8: PHASES OF THE ZEROth AND 1 AIRFOIL (FILLED SYMBOLS), AND -1 AIRFOIL (SOLID SYMBOLS) AS A FUNCTION OF k AND M .

perturbation in the flap mode, the zeroth and the positive-first airfoils, are dominated by vorticity waves, and are therefore fairly insensitive to Mach number variations, the zeroth and negative-first airfoils of the edgewise mode show a weak sensitivity to the Mach number. This presence of acoustic waves helps to explain why the suction side of the zeroth and the pressure side of the negative-first airfoil are in anti-phase for any Mach number and reduced frequency (note that, in Fig. 6.7.8, $\phi_0 - k$ departs from 0 deg, whereas the $\phi_{-1} - k$ departs from π). This is only possible if the effective velocity of propagation (the sound speed) is much larger, and therefore both surfaces are synchronised at any instant and flow condition. Vorticity waves, superimposed with the aforementioned acoustic waves, may also be seen in the zeroth airfoil; however, the precise contribution of each of them cannot be done without a proper mode decomposition.

The influence of acoustic waves in the edgewise mode may also be appreciated in the work-per-cycle curves (Fig. 6.7.5), where the spikes associated to the inlet and outlet resonance conditions may be seen. This is only possible if the pressure perturbations associated to the acoustic waves are at least of the same order as those associated to the vorticity waves. Note that, under the same conditions, these spikes are not

Figure 6.7.9: NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE MACH NUMBER FOR THE EDGEWISE MODE.

visible in the flap mode; that is solely controlled by vortex shedding mechanisms.

Although the resonance conditions are mode independent, their excitation is not, which is the ultimate reason why these are clearly seen in some modes while not in others. Acoustic resonances require excitement by an acoustic mode that is not present in the flex mode, which is the contrary to the edgewise mode. In any case, the sensitivity to the Mach number is low, especially at the lowest reduced frequencies. Figure 6.7.9 displays the dependence of the non dimensional aerodynamic work per cycle with the Mach number. It may be appreciated that all the curves collapse in a single but, in this case, conversely that in the flap mode, a slight dependence with M_2 exists, since the phase of the ICs exhibits a weak dependence with M_2 (see Fig. 6.7.8). Summarising, the functional form for W_{cycle} [Eq. 6.5.4] derived for the flex mode is still valid, although its ability to retain the variation with the Mach number is smaller. The constants of the model for this mode are obviously different, and it has to be recalled that, in this case, the ratio l_{-1}/l_1 needs to be retained because it is not negligible.

Figure 6.7.10: NON-DIMENSIONAL WORK AS A FUNCTION OF THE IBPA AND THE REDUCED FREQUENCY FOR THE TORSION MODE.

6.7.1.2 Torsion Mode

This torsion mode (which represents a torsion about the midpoint of the airfoil) is more difficult to interpret than the flap and edgewise modes. The effort here will be concentrated on describing just the main differences that may be summarised in the following two observations. The first is that the contribution to the work per cycle of the 2 airfoils far away to the vibrating one is relevant. This may be concluded by just realising that the work per cycle as a function of the IBPA is a discontinuous sawtooth function (see Fig. 6.7.10) with a rich harmonic content.

Figure 6.7.11 displays the moment influence coefficients, and it may be appreciated that the contribution of airfoils far away from the central one is still significant, and that the moment influence coefficient of the blade i on the blade zeroth, l_i , is $l_i \propto 1/i$, as corresponds to the Fourier coefficients of a sawtooth function.

It may be seen (Fig. 6.7.12) that the modulus of the unsteady pressure in the zeroth and the negative-first airfoil is much larger than that of the rest of the airfoils. However, due to the torsion nature of the motion, the unsteady pressure contribution to the moment of the front and rear parts of the airfoil cancels out, giving rise to a

Figure 6.7.11: NON-DIMENSIONAL MOMENT INFLUENCE COEFFICIENTS OF THE TORSION MODE AT TWO DIFFERENT REDUCED FREQUENCIES AND $M_2 = 0.75$.

relatively low contribution of these airfoils. On the contrary, -2 , -3 , and -4 have a relatively small unsteady pressure in the trailing-edge region that is amplified when the moment with respect to the central part of the airfoil is computed. The same is true for 2 , 3 , and 4 airfoils but with the leading- edge region.

The torsion mode motion, contrary to the flap and edgewise modes, changes the exit angle, and therefore the lift of the cascade. The system reacts, creating upstream running pressure waves departing from the domain exit, that can be clearly seen in the simulations. When the exit angle of the central airfoil is changed, the whole cascade reacts, adjusting the mass flow. This information is transmitted upstream by means of pressure waves. It may be seen in Fig. 6.7.13 how the phase of the central airfoil is independent of the Mach number; this means that the unsteady pressure is controlled by vortex shedding. However, the phases of the rest of the airfoils, in anti-phase with the central one, vary with the Mach number, which is a clear indication that they are dominated by acoustic waves.

The second difference is that the Mach number sensitivity of the 2 and 1 airfoil phases is noticeable; therefore, the sensitivity of the non dimensional work per cycle to the Mach number is higher than for the flap mode, where ϕ_i remains unaltered with M

Figure 6.7.12: MODULUS OF THE UNSTEADY PRESSURE DISTRIBUTION ALONG THE AIRFOILS OF A LINEAR CASCADE COMPOSED OF NINE AIRFOILS ($M_2 = 0.75$).

Figure 6.7.13: PHASES OF THE INFLUENCE COEFFICIENTS OF THE TORSION MODE FOR DIFFERENT MACH NUMBERS.

but lower than for the edge mode, where ϕ_0 and ϕ_1 vary with the Mach number. The Mach number sensitivity of the airfoils located far away from the central, and which are dominated by the acoustic waves, is a general trend for all the modes, but is only relevant for the torsion mode. Moreover, the reduced-frequency range for which the phase trend is linear with k is narrower. As a consequence, the work-per-cycle model described in Eq. 6.5.2 needs to be extended to include more ICs and to take into account Mach number effects.

6.8 Conclusions of the Numerical Validation

All the theoretically derived conclusions have been numerically verified using a well-validated linearised Navier–Stokes solver on a multiplicity of two-dimensional airfoils operating at design/off-design and high subsonic/transonic conditions, vibrating with different mode-shapes. Both the travelling-wave and the influence coefficient techniques have been used, showing consistent results. The numerical study has been focused on the behaviour of the bending mode but a some results for the three fundamental modes have also been shown to prove that the conclusions are independent of the mode- shape.

The unsteady aerodynamics of vibrating airfoils and its impact on the work-per-cycle curves in the low reduced frequency regime has been analytically studied by means of a new non-dimensional airfoil unsteady loading parameter (ULP). It has been proved by means of numerical experiments that by using the ULP to characterise the loading it is possible to correlate the different trends with the reduced frequency of the influence coefficient moduli, the IC phases, and the work-per-cycle curves in a quantitative manner.

The main conclusions for airfoils oscillating at low reduced frequency are the following:

1. The unsteady aerodynamic loading parameter (ULP) is able to predict and to quantify the behaviour of the influence coefficient modulus and phase (and then

- of the stability trends of the work per cycle curves) to changes in the operating conditions, mode-shape and (in a qualitative way) airfoil geometry.
2. The sign of the ULP predicts, from the point of view of the influence coefficient technique, which of the adjacent airfoils has the highest contribution to the work per cycle, i.e. which airfoil has the highest unsteady pressure. This fact has a direct implication on the sign of IBPA where the minimum damping occurs.
 3. High ULP values implies that the theory is valid for a wide range of reduced frequencies, while small values lead to a more restricted range.
 4. The ULP can be used to quantitatively predict the influence of different aerodynamic loadings, reduced frequency and mode-shapes in the aerodynamic damping, not only for a given airfoil geometry, but also to establish qualitative comparisons among different airfoils.
 5. The unsteady pressure in the flap mode is caused mainly by convective vorticity modes in the central and adjacent airfoils. Acoustic perturbations do not play any role in the low reduced- frequency regime. This is supported by the fact that the phases of the influence coefficients are fairly independent of the Mach number, although their moduli scale with the Mach number. The linear increase of the phase with the reduced frequency is a direct consequence of the convective nature of the unsteady perturbations.
 6. The effect of the Mach number on the flap mode is a pure scaling of the damping curves; therefore, it does not change the stability region but instead changes the severity of the instability, that increases with the Mach number. The scaling of the Mach number is directly absorbed by the non-dimensionalisation.
 7. The physical behaviour of the flap mode is different for the edgewise and torsion modes. The edgewise mode shows a weak dependence with the Mach number, whereas for the torsion mode, the contribution of more ICs is needed to reconstruct the work per cycle.

	Airfoils			
Mode	ϕ_0	ϕ_{+1}	ϕ_{-1}	ϕ_{others}
Flap	V	V	0	0
Edge	A	V	A	0
Torsion	V	A	A	A

Table 6.7: MACH NUMBER EFFECT IN THE PHASES OF THE INFLUENCE COEFFICIENTS OF THE AIRFOILS. V: VORTEX SHEDDING DOMINATED. A: ACOUSTICALLY INFLUENCED.

8. The unsteady pressure of airfoils located far away from the central vibrating airfoil is controlled by acoustic perturbations. A summary of the Mach number impact on the airfoil influence coefficient phases is given in Table 6.7, where V and A stand for vorticity and acoustic dominated mode, respectively.

Simulations at higher values of the reduced frequencies, of order unity, show that the phase of the ICs tends to $\pi/2$ when k is increased, while the modulus of the ICs varies linearly with k . Therefore, the work-per-cycle curves exhibit a cosine-like shape, which matches the experience obtained running simulations for compressor blades, that in practise operate at much higher reduced frequencies than LPT cantilever blades. It is important to highlight that this shape of W_{cycle} curves is due solely to a reduced frequency effect, and it is not associated to the actual shape of the airfoil. The second conclusion (i.e., the mean value is proportional to k) can be extended to values of reduced frequency of order unity as well, because even though the phase remains constant and close to $\pi/2$, the ICs keep on growing with k .

Chapter 7

Stabilisation Mechanism of Blade Pairs: Numerical and Experimental Evidences

If everyone is thinking alike, then
somebody isn't thinking

G. S. Patton

Contents

7.1	Context and Problem Description	167
7.2	Experimental Setup	170
7.3	Numerical Validation	174
7.4	Discussion of the Stabilisation Mechanism	181
7.5	Concluding Remarks	183

7.1 Context and Problem Description

As it has been already mentioned in the introduction, modern Low Pressure Turbine (LPT) bladed-disk are made up of very slender rotor blades, and flutter prone. The

Figure 7.1.1: LAYOUT OF DIFFERENT BLADED-DISC ASSEMBLY CONFIGURATIONS

capability of control their vibration is a major engineering issue. In order to control the vibration of shrouded LPT rotor blades, three different configurations have been taken into consideration along the history: cantilever, interlock and welded in pair configurations. Figure 7.1.1 (top) represents a sketch of the shrouds of the three configurations, whereas a global view of the whole assembly can be seen in bottom of the same figure.

The main problem of cantilever blades is that, as long as flat sided shrouds can vibrate freely, even for very small clearances, and especially for low inter-blade phase angles (IBPA), they provide little control over the vibration characteristics of bladed-disks. As an alternative, interlock configurations (consisting on z-shaped shrouds in contact with the neighbouring airfoils) were designed with the aim of remain tight during the whole flight envelope. This design significantly modifies the vibration characteristics of cantilever blades, introducing three new problems, namely, the coupling between the bending and torsion, giving rise to complex modes, the high sensitivity of mode-

shapes and natural frequencies to the details of the contact region, and finally the difficulty in evaluating the frequency characteristics along the engine life due to the deterioration of the contact surfaces. As a consequence of the limitations of both, the cantilever and the interlock configurations, pairs of blades welded in the tip-shroud (see Fig. 7.1.2) were introduced as a practical alternative to control the vibration characteristics of LPT bladed-discs, and can be observed in many large civil turbofan engines.

This latter configuration substantially modifies the mode-shapes and eigenfrequencies of the cantilever and interlock configurations. The influence of blade mode-shape and reduced frequency on the flutter boundaries of LPTs has been numerically studied [90] and supported by experiments [5] in the past. The beneficial effect of grouping the airfoils in pairs was firstly demonstrated by Whitehead and Evans [102]. Kahl [103] studied the stability of a LPT sector of three airfoils, and later on Chernysheva [104] extended it to a packet of six vanes using the Panovsky and Kielb (PK) method. More recently, Corral et al. [1] studied the influence of pairing the rotor blades on the aerodynamic damping of a typical LPT bladed-disc, and compared the results with the equivalent interlock configuration. Moreover it has been shown that the potential beneficial effect of grouping the blades in sector vanes is mostly spoiled due to structural coupling [2, 105]. Finally, the unsteady aerodynamics of blade clusters has been recently investigated experimentally [106].

In this context, the main motivation of this chapter is twofold. Firstly, to validate and explain the working principles of the welded-in-pair configuration, combining experimental data obtained at the EPFL non-rotating annular test facility, and new numerical simulations; and secondly, to serve as an indication of the level of accuracy that can be obtained with the linearised code used in this thesis. Both, the experiments and the simulations were conducted in the framework of the european project FUTURE, that funded my work during three years. This is the ultimate reason of include this chapter in the thesis.

The stabilization mechanism by means of welding the rotor blades in pairs had been

Figure 7.1.2: 1st FLAP MODE-SHAPE OF A WELDED-IN-PAIR ROTOR BLADE CONFIGURATION

previously described recurring solely to analytical methods [1]. Here these studies are extended using recently available experimental data. The numerical simulations have been performed by means of a well validated linearized Navier-Stokes solver, known as $Mu^2s^2T - L$, described in the chapter “Analytical and Numerical Models”.

7.2 Experimental Setup

The experimental data used for the validation of the numerical simulations was obtained in a dedicated experimental campaign performed at the Ecole Polytechnique Fédérale de Lausanne (EPFL). The test facility consists on a non-rotating annular cascade, presented by Bölcs [107]. This facility allows steady and unsteady testing of both, compressor and turbine rotor cascades, in several subsonic, transonic and supersonic flow conditions, with a large variation of inlet flow angles. The incoming air is swirled while the rotor cascades are at rest, obtaining in this way the same relative flow conditions as in a real turbomachinery. This original concept is achieved with the use of a radial-axial nozzle in combination with variable pitch pre-swirl vanes, as schematically represented in Fig. 7.2.1, on the left. The main advantage of this non-rotating setup is the ease of measurement and data acquisition, the direct optical access and the possibility of using extensive measurement equipment and controls without the use of costly telemetry systems.

Figure 7.2.1: SCHEMATIC VIEW OF THE NON ROTATING ANNULAR TEST FACILITY (A) AND TURBINE ROTOR CASCADE INSTRUMENTATION, VIEW FROM DOWNSTREAM SIDE (B)

7.2.1 Test Facility

A 2.25 MW four-stage centrifugal compressor supplies the test facility with compressed air in an open loop. The air exiting the compressor is cooled by two heat exchangers which allow to adjust the air temperature entering the test facility.

Figure 7.2.1 (a) represents the test facility and it can be seen that the incoming air in the radial direction (1,2) is pre-swirled by two sets of pre-swirl vanes (5,6), before accelerating and turning in an axial direction in the radial-axial nozzle. At this point, the flow approaches the test section (7). Afterwards, the flow turns in a radial direction and exits (12) the test facility. With this setup, a quasi-two-dimensional flow at mid-channel height in the test section can be achieved. The flow quantities (total pressure, outlet static pressure and inlet flow angle) can be adjusted via the inlet (1,2) and outlet (12) valves and the pre-swirl vanes (5,6). The total temperature of the flow at inlet is measured in the upstream settling chamber (3,4).

The measuring cascade, represented in Fig. 7.2.1 (b), is composed by twenty prismatic turbine blades, each one has its own independent suspension system for controlled vibration, as shown in Fig. 7.2.2, on the left. The modular blade suspension system setup facilitates the potential change or reposition of components.

The blade vibration frequency is controlled by the stiffness of a spring integrally

machined with the airfoil and the rest of the suspension system. The airfoil mode-shape is controlled by the shape of the spring, which is nothing else than a relatively thin and flexible part of the bar, either bending or torsion modes can be accommodated. Since the stiffness of the spring is much lower than that of the rest of the system, the mode-shape displacements in the airfoil region correspond in first approximation to a rigid body motion, and vary linearly from hub to tip.

Only vibration in one translation direction is enabled. Each individual blade system is oscillated by means of electromagnets with an independent vibration amplitude and phase, whereas the vibration frequency is the same for all blades. The vibration frequency has to be close to the natural frequency of the modular blade suspension systems. The displacement in each blade is controlled by means of an inductive proximity transducer.

The steady state flow-field is determined by means of aerodynamic 5-hole L-shaped probes located at upstream and downstream positions of the measuring rotor cascade. Figure 7.2.2, on the right, represents the traversing locations of each probe with respect to the leading and trailing edges. Two probe holders together with two precise industrial linear displacement systems allow controlling the radial position of each probe. Each probe is staggered along the mean flow direction at the corresponding flow measuring location. The combination of the radial positioning of the probes and the rotation indexing of the measuring cascade allows precise flow mapping of the in- and outflow conditions. One measurement per degree is taken in circumferential direction for 18° in total to cover one pitch of a cascade built by 20 blades, and 14 measurement positions are taken along the channel height.

Six blades are equipped with steady-state pressure taps in order to measure the static surface pressure. These taps are located at three different blade heights (25%, 50% and 90%), as pairs of PS and SS measuring blades (see Fig. 7.2.1, on the right), allowing in this way the validation of the quasi-two-dimensional flow assumption as well as the computation of the steady loading. There are twelve measuring locations on the SS measuring blades and nine measuring locations on the neighboring PS

Figure 7.2.2: BLADE SUSPENSION SYSTEM (LEFT) AND PROBE TRAVERSE LOCATIONS (RIGHT)

measuring blades with an identical LE measurement location on both blades, that can be seen on Fig. 7.2.2, (right). This is useful to verify the symmetry of the incoming flow (angle of attack), as the PS and SS transducers are not located on the same blade, resulting in a channel measurement in a blade passage.

The unsteady flow field is measured by means of cylindrical high response piezo-resistive absolute pressure transducers, mounted in the blades at mid-channel height at the same locations as the steady pressure taps, but in different blades, allowing the measurement of the pressure fluctuations on the blade surface.

The acquisition technique used at EPFL enables to record the blade displacement and the unsteady pressures simultaneously, which allows to determine the aerodynamic blade damping. Taking into account the possibilities of the electromagnetic vibration system and the aforementioned measurement options, all possible inter-blade phase angles (IBPA) can be investigated.

Furthermore, the characteristics of the facility allow the investigation of packet of blades by setting the vibration amplitude and phase of the neighboring blades accordingly.

7.3 Numerical Validation

7.3.1 Problem Description

The mode-shapes of the first modes of a LPT welded-in-pair rotor blade can be described in first approximation as pairs of airfoils vibrating in phase, whose two-dimensional sections are undeformed and can be considered as two-dimensional rigid bodies (see Fig. 7.1.2). These mode-shapes, typical of high aspect ratio rotor blades are well suited to be represented in the EPFL annular cascade, whose airfoils are vibrated as rigid bodies with an individual control of the blade phase.

The EPFL annular vibrating cascade provides a large degree of flexibility to implement different types of modes, since the amplitude and phase of each airfoil can be selected independently while having a constant vibration frequency for all airfoils.

To investigate the damping associated to the welded-in-pair configurations, first of all the bending and torsion mode of the corresponding cantilever configuration have been reproduced in the test facility, by vibrating all airfoils with an identical angular frequency, ω , amplitude, δ , and *IBPA*. To represent the mode-shapes of the welded-in-pair configuration a basic sector of two blades instead of one is chosen. Inside the pair, the two airfoils move in phase with the same amplitude, while all pairs move with the same angular frequency, ω , and Inter-sector Sector Phase Angle (ISPA).

The *IBPA* represents the phasing between neighboring blades, and is commonly defined as $\sigma = 2\pi m/N_s$, where $m = 0, \dots, \pm N_s/2$ and N_s is the number of fundamental sectors. According with the previous definition, $N_s = N_b$ for the single blade case, where N_b represents the number of blades, and for the welded in pair configuration $N_s = N_b/2$. Therefore the number of Inter Sector Phase Angles (*ISPA*) is half of the number of possible *IBPAs*. In the convention used in this thesis a positive *IBPA* physically means a forward traveling-wave. i.e.; a wave moving in the same direction than the rotor.

The most unstable modes of a high aspect ratio cantilever low-pressure turbine are

Figure 7.3.1: BENDING DIRECTION AND TORSION AXIS

the first bending and torsion modes. The frequency and mode-shape of the bending mode are fairly unaffected by welding the blades in pairs [1], however the torsion mode lowers slightly its frequency and the torsion center of the airfoil is moved from the mid-chord to about the center of the passage.

Traveling wave mode experiments for a packet of two blades and for a single blade, both performing an oscillation in a bending direction orthogonal to the blade chord, representing the first flap mode, and a torsion about the z-axis have been considered. Figure 7.3.1 represents the position of the bending axis with respect to the blade chord and the position of the torsion axis. All possible Inter Sector Phase Angles (ISPA) for the packet configuration, and all the possible Inter Blade Phase Angles for the single blade configuration have been considered.

As it has been mentioned before, due to the design of the airfoil mounting system, the displacements on the airfoil correspond to a rigid body motion, which is a good description of the mode-shape in a two-dimensional section of a high aspect ratio airfoil. The selection of the mode-shape of the pair is fully justified for the bending mode, where the motion of each individual airfoil of the packet is very similar in practice to that of the single airfoil.

The mode-shape of the torsion mode is less justified since both airfoils should have a common torsion center located around the middle of the passage. To properly represent

	Unit	Inlet	Outlet
M	[-]	0.34	0.68
β	[°]	46.0	-58.0
p_t	[kPa]	135.0	130.6
p_s	[kPa]	124.3	95.6
T_t	[K]	301	-

Table 7.1: UPSTREAM AND DOWNSTREAM TEST CONDITIONS AT 50% SPAN

this mode a new set of airfoils with a modified design of the torsion springs would have to be manufactured; instead, the same set of airfoils was used and vibrated in phase. This motion retains the main characteristic of the mode-shapes of welded-in pair rotor blades, namely the negligible relative motion between the airfoils of the same pair.

The blade was constructed extruding in the radial direction a two-dimensional section representative of a high-pressure turbine. However, it is believed that all the conclusions derived for this case can be extended to the flow about an LPT, although the pressure distribution is not fully representative. Eventhough the geometry is purely 2D, it will be shown that the flow is highly 3D, because the aspect ratio of the cascade is low ($H/c = 0.52$), and the influence of the unavoidable tip clearance flow, typical of a vibrating cascade.

The steady-state flow conditions were chosen to be identical for all test cases. Table 7.1 lists inlet and outlet flow conditions corresponding to measurements at mid-channel height, arithmetically averaged over one pitch. The inlet and exit Mach numbers are representative of those that can be seen in modern LPTs.

The reduced frequency of the cascade was chosen to be $k = 0.7$ for both modes, which is much higher than what it is typically observed in realistic LPTs. The reason for this is that this cascade was designed to investigate the damping of HPTs, that usually have a lower aspect ratio than LPTs, and therefore higher reduced frequencies. This fact will deserve special attention. Since the annular cascade consists of 20 airfoils, all the possible 20 IBPAs and 10 ISPAs of the single blade and pair configurations, respectively, were measured and computed.

7.3.2 Results

7.3.2.1 Work-per-Cycle Computation

The work-per-cycle, W_{cycle} , of the airfoil is usually presented in non-dimensional form according to the following formula:

$$\Theta_1 = \frac{W_{cycle}}{\frac{1}{2}\rho_c v_c^2 \delta^2 H} \quad (7.3.1)$$

where W_{cycle} is the work per cycle over the blade, ρ_c and v_c are, respectively, the density and velocity at the cascade inlet, δ is the maximum airfoil displacement, and H is the blade height. It is assumed that the characteristic unsteady pressure is a fraction of the inlet dynamic head, i.e.: $p_{c-unsteady} = \frac{1}{2}\rho_c v_c^2 \frac{\delta}{c}$ and the characteristic surface $S = H c$, where c is the airfoil chord.

In the present study the work-per-cycle has been computed in a different way in the experiments than in the simulations. Since the cascade airfoils are equipped with unsteady pressure transducers only in the mid-section of the airfoil, therefore only the unsteady pressure corresponding to the mid-section has been used to compute the W_{cycle} , whereas for the simulations the entire airfoil surface has been employed. The work-per-cycle for a packet of blades is computed considering the aerodynamic surfaces of all the airfoils, and then normalizing the result by the number of airfoils in the packet, two in this case, to allow a comparison of the results with the single blade case.

7.3.2.2 Steady State Results

The annular cascade consists of 20 airfoils with an aspect ratio $H/c = 0.52$ and a relative tip gap $g/c = 0.7\%$. The cascade exhibits an inter-platform gap in the platforms to allow the blades to vibrate freely, resulting in the presence of a small leakage in the hub region.

A semi-unstructured grid with 25 radial planes and 370,000 points was created. No provision was made neither for the tip-clearance nor for the inter-platform gaps, but

Figure 7.3.2: ISENTROPIC MACH NUMBER DISTRIBUTION FOR DIFFERENT SPAN HEIGHTS

it was previously checked in a separate set of simulations that its impact was low. The mesh is sufficiently fine to retain all the unsteady features, but slightly marginal from a steady point of view. The reason for this is that the same setting was used for a much larger simulation campaign.

The flow was assumed to be fully turbulent since the Reynolds number based on the true chord and exit velocity is $Re \simeq 0.96 \times 10^6$. Figure 7.3.2 displays the isentropic Mach number distribution for different span heights. It can be seen that the matching between the experimental and numerical results is good, except in the tip region where the tip gap has not been retained in the simulations. The hub to tip variation of the Mach number distribution is small. Figure 7.3.3 displays the surface streamlines of the PS and SS. It can be seen that the flow pattern is highly three-dimensional close to the inner and outer walls, due to flow migration caused by the secondary flows. The cascade aspect ratio is low and the mid-section is affected by the blockage of the secondary vortex system.

7.3.2.3 Single Blade Case

The purpose of testing the individual blade case is to serve as reference case, and validate the fluid solver. The annular cascade consists of 20 airfoils, and the 20 possible IBPAs have been measured and computed for the bending and torsion modes. Figure 7.3.4 (down triangles) displays the non-dimensional work-per-cycle of the bending (a) and torsion (b) modes measured at the EPFL cascade and computed by the $Mu^2s^2T - L$ linear code, at $k = 0.7$, for the single blade. The vibration amplitude at the tip is $\delta/c = 0.3\%$ and the hub to tip linear variation of the airfoil

Figure 7.3.3: SURFACE STREAMLINES AND ISOCONTOURS OF THE UNSTEADY PRESSURE MODULUS (FLOW MOVES FROM LEFT TO RIGHT) OF THE SUCTION (LEFT) AND PRESSURE (RIGHT) SIDES FOR THE BENDING MODE AND IBPA=90°

Figure 7.3.4: COMPARISON OF EXPERIMENTAL AND ANALYTICAL RESULTS OF THE NON-DIMENSIONAL WORK PER CYCLE AS A FUNCTION OF THE IBPA FOR: (A) BENDING MODE AND (B) TORSION MODE , FOR SINGLE BLADE (∇), AND BLADE-PAIR (Δ) CONFIGURATIONS, AT $k = 0.7$

displacement was retained in the simulations.

A good agreement between the simulations and the experimental data can be observed for both modes, especially for the torsion mode. It can be concluded that the numerical tool is able to accurately reproduce this problem. To support this statement, Fig. 7.3.5 displays the experimental and numerical unsteady pressure distribution along the mid-section for the bending mode and IBPA=-108°. It can be seen that the matching between both distributions is fairly good, showing that the experimental data is not only reproduced at an integral but also at a point-wise level by the simulations. The highest response of the airfoil is at the suction side ($s > 0$),

Figure 7.3.5: EXPERIMENTAL (FILLED SYMBOLS) AND NUMERICAL (OPAQUE SYMBOLS) UNSTEADY PRESSURE MODULUS DISTRIBUTION ALONG THE BLADE CHORD FOR SINGLE BLADE AND BLADE PAIR PACKAGE CONFIGURATIONS FOR $IBPA = -108^\circ$ AND $ISPA = -108^\circ$, RESPECTIVELY AT $k = 0.7$ FOR A BENDING MOTION

in the peak Mach number region, as it could be expected. The unsteady pressure level at the pressure side ($s < 0$) is much lower, and therefore it can be concluded that the major contribution to the W_{cycle} is due to the airfoil SS.

Figure 7.3.3 shows that, although the flow is highly three-dimensional, the unsteady pressure variation in the spanwise direction is small. Hence a section by section investigation of the unsteady pressure is applicable, analogously as for a high aspect ratio LPT rotor blade.

7.3.2.4 Welded-Pair Case

In order to experimentally verify the effect of welding the rotor blades in pairs, the same blades for the same steady boundary conditions were vibrated in bending and torsion, forcing that the pair of airfoils vibrate in phase. Since the cascade can contain only 10 pairs, 10 ISPA were measured and computed.

Figure 7.3.4 (a) and (b) (up triangles) display the work-per-cycle measured at the EPFL cascade and computed by the $Mu^2s^2T - L$ linear code, at $k = 0.7$, for the bending (a) and the torsion (b) modes. The first observation is that the overall damping of the system is lower than that of the single blade configuration. This is

Figure 7.3.6: UNSTEADY PRESSURE MODULUS OBTAINED FROM THE SIMULATIONS FOR THE TORSION MODE AT $k = 0.7$. IBPA/ISPA=180°. (A) SINGLE BLADE CONFIGURATION AND (B) PAIR CONFIGURATION

due to the reduced level of activity on the SS of the airfoil 2 of the pair (the one that has its SS in the pair passage). The effect is bigger for the largest ISPAs, that have as well a higher relative movement of the blades, while the vicinity of IBPA=0 is unaffected by the pair configuration.

It can be concluded that the numerical method is able to reproduce the experimental data, including the decrease in the unsteady pressure field of the 2nd blade caused by the shielding of the 1st blade with the neighbouring sectors (see Figs. 7.3.5 and 7.3.6).

7.4 Discussion of the Stabilisation Mechanism

The physics of the stabilization effect of welding blades in pairs was reported by Corral et al. [1]. They identified an increase in damping as a result of welding the blades in pairs. This behaviour is due to the fact that only one out of every two airfoils contributes significantly to the total work, and hence halving the variation of the aerodynamic damping with the IBPA, which is associated to the activity of the neighbouring airfoils. This phenomenon can be understood by looking at Figure 7.3.6,

which displays a simulation of the torsion mode for the $IBPA=180^\circ$, for the single blade and pair configuration. It can be seen that the unsteady pressure modulus is high along the suction side of all the cascade airfoils, in the single configuration, while for the welded in-pair configuration the unsteady pressure inside the channel is negligible, due to the shielding effect of the neighbouring rotor. This behaviour is confirmed in Figure 7.3.5, where the experimental unsteady C_p distribution is compared with the calculations at the mid-section.

It had already been reported that the stabilization effect of the welded in pair configuration is higher for the torsion mode than for the bending mode [1]. A similar effect can be observed in Fig. 7.3.4 (b) that clearly shows the beneficial effect from a stability point of view of grouping the blades in pairs for the torsion mode. The minimum damping, which is relevant from an aeroelastic point of view, which takes place at $IBPA \simeq 90^\circ$, is clearly increased at the expense of decreasing the maximum damping. However, the bending mode (see Fig. 7.3.4 (a)) decreases the maximum damping while the minimum damping, which takes place at $IBPA \simeq 0^\circ$, remains essentially unaltered since the low IBPAs are weakly affected by the configuration in pairs.

The reduced frequency of a realistic LPT rotor blade oscillates between 0.1-0.2, which is significantly lower than that of our experiment. It has been repeatedly shown in this thesis that the effect of increasing the frequency in the W_{cycle} is an increase of the mean damping and a shift in the location of the minimum damping towards the region of $IBPA = 0$. For this reason, the beneficial effect of grouping the blades in pairs in bending mode can not be fully appreciated in the EPFL's experimental set up, that has a much higher reduced frequency ($k = 0.7$).

With the aim of re-assessing numerically the impact of grouping the blades in pairs at the frequency range of interest, simulations has been repeated at reduced frequency $k = 0.1$. Figure 7.4.1 (a) and (b) show the non-dimensional W_{cycle} for the single and the pair configurations for both the bending and torsion modes, respectively, at $k = 0.1$. It has to be highlighted: first that the curves are sinusoidal, and that both

Figure 7.4.1: NON-DIMENSIONAL WORK PER CYCLE AS A FUNCTION OF THE IBPA FOR SINGLE BLADE AND BLADE PAIR PACKAGE CONFIGURATIONS FOR: (A) BENDING AND (B) TORSION MODE, AT $k = 0.1$.

modes are now unstable. This is an expected result when decreasing the reduced frequency. It can be clearly seen that the effect of grouping the blades in pairs is to decrease the level of severity of the instability since the minimum damping is increased, for both modes.

The effect of grouping the blades in pairs in the unsteady pressure is the same for both reduced frequencies, but the impact in the W_{cycle} is different, due to corrections in phase induced by the change in k , that need to be retained in the computation of the W_{cycle} .

7.5 Concluding Remarks

The effect of grouping turbine blades in pairs has been investigated, and experimental data have been compared with numerical simulations for the first time. The following conclusions can be drawn:

1. The degree of matching between the experimental and numerical results obtained independently by two different appraisals (work-per-cycle and unsteady pressure distributions) is high.
2. The stabilizing effect on the torsion mode is seen both in the experiments and

in the simulations.

3. The origin of the stabilization is mainly due to the suppression of the unsteadiness in the suction side of the airfoil whose suction side is facing the passage of the pair, due to the shielding effect of the pressure side of the other airfoil of the pair, since the relative motion between both is null.
4. The stabilization effect on the bending mode is less evident, both in the experiments and the simulations, because the experiment was conducted at a reduced frequency much higher than that of a realistic application.
5. The pairing effect in the bending mode at high reduced frequencies is to reduce the maximum damping, and essentially leave unaltered the minimum damping.
6. To extrapolate these results at the range of frequencies of interest, the previously validated numerical code was applied to estimate the damping of a single blade and a welded-in-pair configuration at a much lower reduced frequency. It is concluded that the bending mode is also stabilized with respect to the single blade case due to the pairing of the blades.

In overall, it is concluded that the widely extended practice of welding the rotor blades in pairs is well justified and gives rise to a stabilisation effect from an aeroelastic point of view.

Bibliography

- [1] Corral, R., Gallardo, J. M., and Vasco, C., 2007. “Aeroelastic stability of welded-in-pair low pressure turbine rotor blades: A comparative study using linear methods”. *J. Turbomach*, **129**, January, pp. 72–83.
- [2] Corral, R., Gallardo, J. M., and Martel, C., 2009. “A conceptual flutter analysis of a packet of vanes using a mass-spring model”. *J. Turbomach*, **131**, April, pp. 021016–1–7.
- [3] Buffum, D., and Fleeter, S., 2016. “The aerodynamics of an oscillating cascade in a compressible flow field”. *J. Turbomach*, **138**(2), February, p. 021004.
- [4] He, L., 1998. “Unsteady flow in oscillating turbine cascades : Part 2-computational study”. *J. Turbomach*, **120**(2), pp. 269–275.
- [5] Nowinski, M., and J.Panovsky, 2000. “Flutter mechanisms in low pressure turbine blades”. *ASME J. of Engineering for Gas Turbines and Power*, **122**, pp. 89–98.
- [6] Theodorsen, T., 1935. General theory of aerodynamic instability and the mechanism of flutter. Tech. Rep. 496, NACA.
- [7] Moore, F., 1951. Unsteady laminar boundary layer flow. Tech. Rep. TN-2471, NACA.
- [8] Fleeter, S., 1973. “Fluctuating lift and moment coefficients for cascaded airfoils in a nonuniform compressible flow”. *Journal of Aircraft*, **10**, pp. 93–96.

- [9] Bölcs, A., and Fransson, T. H., 1986. Aeroelasticity in turbomachines: Comparison of theoretical and experimental cascade results. Tech. rep., Laboratoire de Thermique Appliquee et de Turbomachines, EPFL.
- [10] Buffum, D. H., and Fleeter, S., 1990. Aerodynamics of a linear oscillating cascade. TM 103250, NASA.
- [11] Vogt, D., and Fransson, T., 2006. “Experimental investigation of mode shape sensitivity of an oscillating low-pressure turbine cascade at design and off-design conditions”. *Journal of Engineering for Gas Turbines and Power*, **129**(2), pp. 530–541.
- [12] Groth, P., Martensson, H., and Edin, N., 2009. “Experimental and computational fluid dynamics based determination of flutter limits in supersonic space turbines”. *J. Turbomach*, **132**(1).
- [13] Collar, A., 1946. “The expanding domain of aeroelasticity”. *Journal of the Royal Aeronautical Society*, **122**, pp. 467–476.
- [14] Bréard, C., Green, J. S., and Imregun, M., 2003. “Low-engine-order excitation mechanism in axial-flow turbomachinery”. *Journal of Propulsion and Power*, **19**, pp. 704–712.
- [15] North Atlantic Treaty Organization, a. G. f. A. R., and Development, 1980. *Multilingual aeronautical dictionary*. AGARD.
- [16] Clark, S., Kielb, R., and Hall, K. C., 2012. “Developing a reduced-order model to understand nonsynchronous vibration (nsv) in turbomachinery”. In Proceedings of the ASME Turbo Expo 2012.
- [17] Corral, R., Gallardo, J., and Martel, C., 2007. “A conceptual flutter analysis of a packet of vanes using a mass-spring model”. In ASME paper GT2007-27090, Proceedings of the 52nd ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.

- [18] Vahdati, M., Simpson, G., and Imregun, M., 2011. “Mechanisms for wide-chord fan blade flutter”. *J. Turbomach*, **133**.
- [19] Sisto, F., 1987. *AGARD Manual on Aeroelasticity in Axial-Flow Turbomachines*, Vol. 1. AGARD, ch. VII: Stall Flutter, pp. 7–1 7–11.
- [20] Adamczyk, J. J., and Goldstein, M., 1978. “Unsteady flow in a supersonic cascade with subsonic leading edge locus”. *AIAA Journal*, **16**, pp. 1248–1254.
- [21] Ekaterinaris, J. A., and Platzer, M. F., 1995. “Progress in the analysis of blade stall flutter”. In *Unsteady Aerodynamics and Aeroelasticity of Turbomachines*, Y. Tanida and M. Namba, eds.
- [22] Vahdati, M., Sayma., A. I., Marshall, J. G., and Imregun, M., 2001. “Mechanisms and prediction methods for fan blade stall flutter”. *Journal of Propulsion and Power*, **17**(5).
- [23] Adamczyk, J. J., 1978. Analysis of supersonic stall bending flutter in axial-flow compressor by actuator disk theory. Tech. rep., NASA.
- [24] Snyder, L. E., and Commerford, G. L., 1974. “Supersonic unstalled flutter in fan rotors; analytical and experimental results”. *J. Eng. Gas Turbines Power*, **96**(4), October, pp. 379–386.
- [25] Sanders, A., Hassan, K., and Rabe, D., 2004. “Experimental and numerical study of stall flutter in a transonic low-aspect ratio fan blisk”. *J. Turbomach*, **126**(1), pp. 169–178. Es de la gente de Honeyweel.
- [26] Zhai, Y., Bladh, R., and Dyverfeldt, G., 2012. “Aeroelastic stability assessment of an industrial compressor blade including mistuning effects”. *J. Turbomach*, **134**(6), p. 060903.
- [27] Kielb, R., Hall, K., Spiker, A. M., and Thomas, J., 2006. Non-synchronous vibration of turbomachinery airfoils. Tech. rep., Duke University.

- [28] Clark, S. T., Besem, F. M., Kielb, R., and Thomas, J. P., 2015. “Developing a reduced-order model of nonsynchronous vibration in turbomachinery using proper orthogonal decomposition methods”. *Journal of Engineering for Gas Turbines and Power*, **137**(5), pp. 052501–11.
- [29] Crawley, E., 1984. “Aeroelastic formulations for turbomachines and propellers”. In Proceedings of the third International Symposium on Unsteady Aerodynamics of Turbomachines and Propellers.
- [30] Martel, C., Corral, R., and Llorens, J. M., 2008. “Stability increase of aerodynamically unstable rotors using intentional mistuning”. *J. Turbomach*, **133**, January, pp. 1–11.
- [31] Kielb, J. J., and Abhari, R. S., 2003. “Experimental study of aerodynamic and structural damping in a full-scale rotating turbine”. *J. Eng. Gas Turbines Power*, **125**(1), December, pp. 102–112.
- [32] Corral, R., Escribano, A., Gisbert, F., Serrano, A., and Vasco, C., 2003. “Validation of a linear multigrid accelerated unstructured navier-stokes solver for the computation of turbine blades on hybrid grids”. In AIAA Paper 2003-3326, 9th AIAA/CEAS Aeroacoustics Conference.
- [33] Vogt, D., 2005. “Experimental investigation of three-dimensional mechanisms in low-pressure turbine flutter”. PhD thesis, KTH Royal Institute of Technology.
- [34] Szechenyi, E., Cafarelli, I., Notin, C., and Girault, J. P., 1984. “A straight cascade wind-tunnel study of fan blade flutter in started supersonic flow”. In Proceedings of the Third International Symposium on Unsteady Aerodynamics of Turbomachines and Propellers, Cambridge, pp. 447–458.
- [35] Crawley, E., and Hall, K. C., 1985. “Optimization and mechanisms of mistuning in cascades”. *J. Eng. Gas Turbines Power*, **2**, pp. 418–426.

- [36] He, L., 1995. “Rotating-stall/stall-flutter prediction using a fluid/structure coupled method”. In *Unsteady Aerodynamics and Aeroelasticity of Turbomachines*, pp. 597–607.
- [37] Vahdati, M., and Imregun, M., 1995. “Nonlinear aeroelasticity analysis using unstructured dynamics meshes”. In *Unsteady Aerodynamics and Aeroelasticity of Turbomachines*.
- [38] Sayma, A. I., Vahdati, M., Green, J. S., and Imregun, M., 1998. “Whole-assembly flutter analysis of a low pressure turbine blade”. In *8th International Symposium in Unsteady Aerodynamics and Aeroelasticity of Turbomachines*, T. Fransson, ed., pp. 347–359.
- [39] Gottfried, D. A., and Fleeter, S., 1999. “Aerodynamic damping predictions for turbomachine blade rows using a three-dimensional time marching simulation”. In *AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit*.
- [40] Sadeghi, M., and Liu, F., 2005. “Computation of cascade flutter by uncoupled and coupled methods”. *International Journal of Computational Fluid Dynamics*, pp. 559–569.
- [41] Corral, R., and Gallardo, J., 2014. “Nonlinear dynamics of bladed disks with multiple unstable modes”. *AIAA Journal*, **56**(6), June, pp. 1124–1132.
- [42] Verdon, J., and Caspar, J., 1984. “A linearized unsteady aerodynamic analysis for transonic cascades”. *Journal of Fluid Mechanics*, **149**, pp. 403–429.
- [43] Giles, M. B., 1991. “A framework for multi-stage unsteady flow calculations”. In *Proceeding of the Sixth International Symposium on Unsteady Aerodynamics, Aeroacoustics and Aeroelasticity of Turbomachines and Propellers*, H. M. Atassi, ed.
- [44] Chen, T., and He, L., 2001. “Analysis of unsteady blade row interaction using nonlinear harmonic approach”. *Journal of Propulsion and Power*, **17**(3), May-June, pp. 651–658.

- [45] Lane, F., 1956. "System mode shapes in the flutter of compressor blade rows". *Journal of the Aeronautical Sciences*, **23**(1), January, pp. 54–66.
- [46] Wagner, H., 1925. *Über die Entstehung des dynamischen Auftriebes von Tragflügeln*. VDI-Verl.
- [47] Garrick, I., 1937. Propulsion of a flapping and oscillating airfoil. Tech. Rep. 496, NACA.
- [48] Mikolajczak, A. A., 1975. The practical importance of unsteady flow. Tech. rep., AGARD CP-177.
- [49] Platzer, M. F., 1977. Unsteady flow in turbomachines - a review of current developments. Tech. rep., AGARD CP-227.
- [50] Sisto, F., 1978. *A Modern Course in Aeroelasticity*. E.H. Dowell.
- [51] Whitehead, D. S., 1980. Unsteady aerodynamics in turbomachinery. in special course on unsteady aerodynamics. Tech. rep., AGARD 679.
- [52] Tijdeman, H., and Seebass, R., 1980. "Transonic flow past oscillating airfoils". *Annual Review of Fluid Mechanics*, **12**, pp. 181–222.
- [53] Verdon, J., 1987. Linearized unsteady aerodynamic theory. Manual on aeroelasticity in axial-flow turbomachines, AGARD.
- [54] Howe, M. S., 1998. *Acoustics of Fluid-Structure Interactions*. Cambridge University Press.
- [55] Saffman, P., 1992. *Vortex Dynamics*. Cambridge Monographs on Mechanics and Applied Mathematics.
- [56] Whitehead, D., 1970. Vibration and sound generation in a cascade of flat plates in subsonic flow. Tech. rep., Aeronautical Research Council.
- [57] Whitehead, D., 1987. *Classical Two-Dimensional Methods, AGARD Manual on Aeroelasticity in Axial-Flow Turbomachines*. AGARD.

- [58] Tyler, J. M., and Sofrin, T. G., 1961. “Axial flow compressor noise studies”. *SAE*, **70**.
- [59] Lighthill, M., 1954. “The response of laminar skin friction and heat transfer to fluctuations in the stream velocity”. *Proceedings of the Royal Society of London*, **224**, pp. 1–23.
- [60] Phillips, J., and Avkerberg, R. C., 1972. “A numerical method for integrating the unsteady boundary-layer equations when there are regions of backflow”. *Journal of Fluid Mechanics*, **58**, pp. 561–579.
- [61] McCroskey, W., and Philippe, J., 1975. “Unsteady viscous flow on oscillating airfoils”. *AIAA Journal*, **13**, pp. 71–79.
- [62] Cebeci, T., 1977. “Calculation of unsteady two-dimensional laminar and turbulent boundary layer with fluctuations in external velocity”. *Proceedings of the Royal Society of London*, **355**, pp. 225–238.
- [63] Crawley, E., 1987. *Manual on Aeroelasticity in Axial Flow Turbomachines-Vol. 2, Structural Dynamics and Aeroelasticity*. AGARD, ch. Aeroelastic Formulation of Tuned and Mistuned Rotor, pp. 19–1 19–24.
- [64] Hall, K., and Clark, W., 1993. “Linearized euler predictions of unsteady aerodynamic loads in cascades”. *AIAA Journal*, **31**, pp. 540–550.
- [65] Lee, C., 1986. “An iterative procedure for non-linear flutter analysis”. *AIAA Journal*, **24**, pp. 833–840.
- [66] Namba, M., 1972. Lifting surface theory for a rotating subsonic or transonic blade row. Tech. rep., Aeronautical Research Council.
- [67] F. Sisto, W. Wu, S. T., and Jonnavithula, S., 1989. “Computational aerodynamics of oscillating cascades with the evolution of stall”. *AIAA Journal*, **27**, pp. 462–471.

- [68] Erdos, J., Alzner, E., and McNally, W., 1977. "Numerical solution of periodic transonic flow through a fan stage". *AIAA Journal*, **15**(11), pp. 1559–1568.
- [69] Giles, M., 1990. "Stator/rotor interaction in a transonic turbine". *Propulsion and Power*, **6**, pp. 621–627.
- [70] He, L., and Denton, J., 1993. "Inviscid-viscous coupled solution for unsteady flows through vibrating blades. part i: Description of the method". *Turbomachinery*, **115**, pp. 94–100.
- [71] Lindquist, D., and Giles, M. B., 1991. "On the validity of linearized unsteady euler equations with shock capturing". In Proceedings of the AIAA.
- [72] Montgomery, M., and Verdon, J., 1997. "A 3d linearized euler analysis for bladerows. part i: Aerodynamic and numerical formulations". In Proceedings of the 8th International Symposium on Unsteady Aerodynamics and Aeroelasticity of Turbomachines, pp. 427–444.
- [73] Ning, W., and He, L., 1998. "Computation of unsteady flows around oscillating blades using linear and non-linear harmonic euler methods". *ASME J. Turbomach*, **120**, pp. 714–722.
- [74] Clark, W. S., and Hall, K., 2000. "Atime linearized navier-stokes analysis of stall flutter". *ASME J. Turbomach*, **122**, pp. 467–476.
- [75] Sbardella, L., and Imregun, M., 2001. "Linearized unsteady viscous turbomachinery flows using hybrid grids". *J. Turbomach*, **123**, pp. 568–582.
- [76] K.C. Hall, J. T., and Clark, W., 2002. "Computation of unsteady non-linear flows in cascades using a harmonic balance technique". *AIAA Journal*, **40**, pp. 879–886.
- [77] J. Crespo, R. C., and J.Pueblas, 2015. "An implicit harmonic balance method in graphics processing units for oscillating blades". *ASME J. Turbomach*, **138**, p. 031001.

- [78] Corral, R., and Gallardo, J., 2011. “Non-linear dynamics of multi-mode unstable bladed-disks: Part i - description of a canonical model and phenomenological results”. In ASME Paper 2011-GT-22943.
- [79] Crespo, J., Corral, R., and Pueblas, J., 2016. “An implicit harmonic balance method in graphics processing units for oscillating blades”. *J. Turbomach*, **138**(3), p. 031001.
- [80] Vega, A., Corral, R., Zanker, A., and Ott, P., 2014. “Experimental and numerical assessment of the aeroelastic stability of blade pair packages”. In ASME Paper GT2014-25607, Proceedings of the 59st ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.
- [81] Lindquist, D. R., and Giles, M. B., 1994. “Validity of linearized unsteady euler equations with shock capturing”. *AIAA Journal*, **32**(1), pp. 46–53.
- [82] Corral, R., and Gisbert, F., 2003. “A numerical investigation on the influence of lateral boundaries in linear vibrating cascades”. *J. Turbomach*, **125**, July, pp. 433–441.
- [83] Burgos, M., Corral, R., and Contreras, J., 2011. “Efficient edge based rotor/stator interaction method”. *AIAA Journal*, **41**(1), January, pp. 19–31.
- [84] Jameson, A., Schmidt, W., and Turkel, E., 1981. “Numerical solution of the euler equations by finite volume techniques using runge-kutta time stepping schemes”. In AIAA Paper 81-1259.
- [85] Roe, P., 1981. “Approximate riemann solvers, parameters, vectors and difference schemes”. *Journal of Computational Physics*, **43**, pp. 357–372.
- [86] Swanson, R. C., and Turkel, E., 1992. “On central-difference and upwinding schemes”. *Journal of Computational Physics*, **101**, pp. 292–306.
- [87] Corral, R., Burgos, M. A., and Garcia, A., 2000. “Influence of the artificial dissipation model on the propagation of acoustic and entropy waves”. In ASME

- Paper 2000-GT-563, Proceedings of the 45th ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.
- [88] Giles, M. B., 1990. “Non-reflecting boundary conditions for euler equation calculations”. *AIAA Journal*, **28**(12), pp. 2050–2057.
- [89] Kirshner, A., Pelet, C., and Gyamathy, G., 1976. “Investigation of bblade flutter in a subsonic turbine”. *Revue Francaise de Mécanique*, **numéro spécial**, pp. 97–104.
- [90] Panovsky, J., and Kielb, R., 2000. “A design method to prevent low pressure turbine blade flutter”. *Journal of Engineering for Gas Turbines and Power*, **122**, pp. 89–98.
- [91] Waite, J., and Kielb, R., 2015. “Physical understanding and sensitivities of low pressure turbine flutter”. *Journal of Engineering for Gas Turbines and Power*, **137**(1), p. 012502.
- [92] Waite, J. J., and Kielb, R. E., 2015. “The impact of blade loading and unsteady pressure bifurcations on low-pressure turbine flutter boundaries”. *J. Turbomach*, **138**(4), December, pp. 041002–041002–9.
- [93] Corral, R., and Vega, A., 2016. “The low reduced frequency limit of vibrating airfoils - part i: Theoretical analysis”. *J. Turbomach*, **138**(2), February, p. 021004.
- [94] Vega, A., and Corral, R., 2016. “The low reduced frequency limit of vibrating airfoils - part ii: Numerical experiments”. *J. Turbomach*, **128**(2), February, p. 021005.
- [95] Corral, R., and Vega, A., 2016. “Physics of vibrating turbine airfoils at low reduced frequency”. *AIAA Journal of Propulsion and Power*, **32**(2), March, pp. 325–336.

- [96] Barbarossa, F., Parry, A. B., Green, J. S., and di Mare, L., 2016. “An aerodynamic parameter for low-pressure turbine flutter”. *J. Turbomach*, **138**(5), May, p. 051001.
- [97] Lemmerman, L., and Sonnad, V., 1979. “Three-dimensional viscous-inviscid coupling using surface transpiration”. *Journal of Aircraft*, **16**(6), pp. 353–358.
- [98] Wolff, J., and Fleeter, S., 1994. “Unsteady inviscid-viscous analysis of oscillating aerodynamics”. In 30th Joint Propulsion Conference and Exhibit, Joint Propulsion Conferences, AIAA, p. AIAA Paper 2797.
- [99] Fransson, T. H., and Verdon, J. M., 1992. Updated report on standard configurations for unsteady flow. Tech. rep., KTH.
- [100] Vega, A., and Corral, R., 2013. “Physics of vibrating airfoils at low reduced frequency”. In ASME paper GT2013-94906, Proceedings of the 58th ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.
- [101] Corral, R., and Vega, A., 2016. “The low reduced frequency limit of vibrating airfoils. part iiii: Theoretical quantification and influence of unsteady loading”. In ASME Paper GT2016-57213, Proceedings of the 61th ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.
- [102] Whitehead, D., and Evans, D., 1992. “Flutter of grouped turbine blades”. In ASME Paper 92-GT-227.
- [103] Kahl, G., 1995. “Application of the time linearized euler method to flutter and forced response calculations”. In ASME Paper 95-GT-123, Proceedings of the 40th ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.
- [104] Chernysheva, O., Fransson, T., Kielb, R., and Barter, J., 2003. “Effect of sector mode shape variation on the aerodynamic stability of a low-pressure turbine sectored vane”. In ASME Paper 2003-GT-38632, Proceedings of the 48nd ASME Gas Turbine and Aero engine Congress, Exposition and Users Symposium.

- [105] Antona, R., Corral, R., and Gallardo, J., 2012. “Effect of the structural coupling in the flutter onset of a sector of low-pressure turbine vanes”. *J. Turbomach*, **134**, September, pp. 051007–1 051007–8.
- [106] Zanker, A., Ott, P., and Calza, P., 2013. “Experimental aeroelastic investigation of vibrating turbine blade cluster”. In 10th European Conference on Turbomachinery, J. Bakman, G. Bois, and O. Leonard, eds., pp. 210–221.
- [107] Bolcs, A., 1983. “A test facility for the investigation of steady and unsteady transonic flows in annular cascades”. In ASME 83-GT-34.

Appendix A

Orthogonality of Aeroelastic Modes

Orthogonality of Travelling Wave Modes

The motion of a bladed-disc made up of N rotor blades vibrating as a traveling wave of n nodal diameters ($\sigma_n = 2\pi n/N$) and angular frequency ω_n may be written as:

$$\mathbf{x}_j^n = \mathbf{x}_0^n \sin(\omega_n t + j\sigma_n) \quad (\text{A.0.1})$$

where $\mathbf{x}_j^n$, are the displacements of the surface of the j -th airfoil. This motion induces a force that, in first approximation, has as well the same travelling-wave form: $f_j^n = f_0^n \sin(\omega_n t + j\sigma_n + \phi)$, where ϕ is the phase shift between the airfoil motion and the force generated. In general, this force can contain higher spatial and temporal harmonics but this fact does not change the argumentation. The work per cycle exerted on the j -th rotor blade by the force $\mathbf{f}_j^n$ in another traveling-wave, with another nodal diameter, m , and another frequency, ω_m :

$$\mathbf{x}_j^m = \mathbf{x}_0^m \sin(\omega_m t + j\sigma_m) \quad (\text{A.0.2})$$

can be expressed as:

$$w_j = \int_0^T \mathbf{f}_j^n \cdot \dot{\mathbf{x}}_j^m dt = \mathbf{f}_0^n \mathbf{x}_0^m \omega_m \int_0^T \sin(\omega_n t + j\sigma_n + \phi) \cos(\omega_m t + j\sigma_m) dt \quad (\text{A.0.3})$$

which after some operations leads to:

$$\begin{aligned}
 w_j = & \frac{1}{2} \mathbf{f}^n_0 \mathbf{x}_0^m \omega_m \int_0^T \sin \left\{ (\omega_n + \omega_m)t + j \frac{2\pi(n+m)}{N} + \phi \right\} dt \\
 & + \frac{1}{2} \mathbf{f}^n_0 \mathbf{x}_0^m \omega_m \int_0^T \sin \left\{ (\omega_n - \omega_m)t + j \frac{2\pi(n-m)}{N} + \phi \right\} dt
 \end{aligned} \tag{A.0.4}$$

If $\omega_n \neq \omega_m$ the integral in time of the harmonic signal for each blade is null, since Eq. A.0.5 represents the integration in time of a periodic signal. Only If $\omega_n = \omega_m$, the second term of the integral gives rise to a non-null contribution:

$$w_j = \pi \mathbf{f}^n_0 \mathbf{x}_0^m \sin \{j(\sigma_n - \sigma_m) + \phi\}. \tag{A.0.5}$$

However, the work-per-cycle of the mode-shape is made up from the contributions of all the rotor blades. Since Eq. A.0.5 represents a spatially harmonic signal, it can be shown that

$$W_{cycle} = \sum_{i=1}^N w_j = 0 \tag{A.0.6}$$

It can be concluded that the traveling waves are orthogonal among them, and then, the eigenfunctions of the aeroelastic problem are travelling waves. In particular the work per cycle of a forward pressure TW with IBPA= σ in a backward TW with IBPA= $-\sigma$ is zero. Now, the question that arises is whether a standing wave can also be solution of the aeroelastic problem (similarly to what happened in the purely structural problem).

Non-orthogonality of Standing Wave Modes

The motion of a bladed-disc made up of N rotor blades vibrating as a standing-wave of n nodal diameters ($\sigma_n = 2\pi n/N$) and angular frequency ω_n may be written as:

$$\mathbf{x}_j^n = \mathbf{x}_0^n \sin(\omega_n t) \sin(j\sigma_n) \tag{A.0.7}$$

where $\mathbf{x}_j^n$, are the displacements of the surface of the j -th airfoil. This motion induces a force that, in first approximation, has as well the same standing-wave form with a phasing, both in space and time:

$$\mathbf{f}_j^n = \mathbf{f}_0^n \sin(\omega_n t + \phi) \sin(j\sigma_n + \varphi) \tag{A.0.8}$$

The work per cycle on the j -th rotor blade of $\mathbf{f}_j^n$ in another standing-wave, with another nodal diameter, m , and frequency, ω_m :

$$\mathbf{x}_j^m = \mathbf{x}_0^m \sin(\omega_m t) \sin(j\sigma_m + \psi) \quad (\text{A.0.9})$$

can be expressed as:

$$w_j = \int_0^T \mathbf{f}_j^n \cdot \dot{\mathbf{x}}_j^m dt = \mathbf{f}_0^n \mathbf{x}_0^m \omega_m \int_0^T \sin(\omega_n t + \phi) \sin(j\sigma_n + \varphi) \cos(\omega_m t) \sin(j\sigma_m + \psi) dt \quad (\text{A.0.10})$$

The phasing ψ , allows to take into account simultaneously the sine and cosine components. Equation A.0.10 can in turn be written as:

$$\begin{aligned} w_j &= \beta \int_0^T [\cos((\omega_n - \omega_m)t) - \cos((\omega_n + \omega_m)t)] \cos \phi dt \\ &+ \beta \int_0^T [\sin((\omega_m - \omega_n)t) + \sin((\omega_m + \omega_n)t)] \sin \phi dt \end{aligned} \quad (\text{A.0.11})$$

where $\beta = \frac{1}{2} \mathbf{f}_0^n \mathbf{x}_0^m \omega_m \sin(j\sigma_n + \varphi) \sin(j\sigma_m + \psi)$. If $\omega_n \neq \omega_m$ then w_j is obviously null, however if $\omega_n = \omega_m$, $w_j = 2\pi\beta \cos \phi / \omega_m$. Exactly as in the TW case, the mode-shape involves all the blades, and hence the total work, $W_{cycle} = \sum_j w_j$, has to be evaluated.

$$W_{cycle} = \beta' \int_0^{2\pi} \sin(n\theta + \varphi - \psi) \sin(m\theta) d\theta \quad (\text{A.0.12})$$

with $\beta' = \pi \cos \phi \mathbf{f}_0^n \mathbf{x}_0^m$. The previous expression can be expanded as

$$\begin{aligned} W_{cycle} &= \frac{\beta}{2} \int_0^{2\pi} [\cos(n-m)\theta - \cos(n+m)\theta] \cos(\varphi - \psi) d\theta \\ &+ \frac{\beta}{2} \int_0^{2\pi} [\sin(n-m)\theta + \sin(n+m)\theta] \sin(\varphi - \psi) d\theta \end{aligned} \quad (\text{A.0.13})$$

Obviously if $n \neq m$, $W_{cycle} = 0$. But, if $n = m$, it is $W_{cycle} \neq 0$ for both, the sine ($\psi = 0$) and cosine ($\psi = \pi/2$). This effectively means that the vibration of a standing-wave of n nodal diameters, $\mathbf{x}_j^n = \mathbf{x}_0^n \sin(\omega_n t) \sin(j\sigma_n)$, produces work not only in the sine component of the vibration but also in the cosine component $\mathbf{x}_j^n = \mathbf{x}_0^n \sin(\omega_n t) \cos(j\sigma_n)$. This means that the standing waves are not eigenfunctions of the aeroelastic problem. This is the ultimate reason of why a linear code for aeroelastic computations is formulated in travelling wave terms.