Wave propagation in soils problems using the Generalized Finite Difference Method

J.J. Benito , F. Ureña , E. Saete , A. Muelas , L. Gavete , R. Galindo

A B S T R A C T

The consideration of highly irregular topography or heterogeneities and singularities in the domain are necessary for solving a wide variety of seismic problems, e. g. earthquake site studies. Therefore, the use of a meshless method (MM) with the possibility of employing an irregular grid point distribution can be of interest for modeling this kind of problems. The tradition of using Finite Different Methods for modeling seismic wave propagation problems have allowed researches to solve it and, in fact, a numerical model using the geotechnical calculation software FLAC has been used here to validate the results. It is clear the improvement of using meshless method techniques to incorporate irregularities in a natural way.

Keywords:

Meshless methods
Generalized Finite Difference Method
Moving least squares
Seismic waves propagation
Perfectly matched layer

1. Introduction

During recent years, meshless methods have emerged as a class of effective numerical methods which are capable of avoiding the difficulties encountered in conventional computational mesh-based methods. An important path in the evolution of meshless methods has been the development of the Generalized Finite Difference Method (GFDM), also called meshless finite difference method [1]. The bases of the GFDM were published in the early seventies. Jensen [2] was the first author to introduce a fully arbitrary mesh. He considered Taylor's series expansions interpolated on six-node stars in order to derive the Finite Difference (FD) formulae, approximating derivatives of up to the second order. Perrone and Kao [3] suggested that additional nodes in the six-point scheme should be considered and an averaging out process for the generalization of finite difference with coefficients applied. The idea of using an eight node star and weighting functions, to obtain finite difference formulae for irregular meshes, was first put forward by Liszka and Orkisz [4] using moving least squares (MLS) interpolation, and an advanced version of the GFDM was given [1]. Benito et al. [5] studied the dependency of solution of the Generalized Finite Difference Method on the number of nodes in the cloud, the relative coordinates of the nodes with

respect to the star node and on the weight function employed. An h-adaptive method in GFDM is described in [6–8].

The application of GFDM to the solution of the problem of seismic wave propagation using PML absorbing boundary was introduced by Ureña et al. [9]. These authors have also applied this meshless method to the solution of dynamic problems of beams and plates [10], to solve the advection–diffusion equation [11] and to solve parabolic and hyperbolic equations [12].

Absorbing boundary condition must be used in numerical wave problems to truncate unbounded media without the reflection due to numerical boundaries, and Perfectly Matched Layer (PML) is one of the most effective among them. This method has been widely used for Finite Differences Methods (FDM).

Berenger [13] first created a PML for electromagnetic problems in a Finite Element (FE) numerical model. Their equations are based on a field splitting which results in mathematical expressions that can not be easily manipulated. Chew and Weedou (1994), following Berenger's work, introduced the concept of using complex coordinates in the formulation. Sacks et al. (1995) developed an anisotropic PML also valid to the Finite Element Method (FEM).

Chew and Liu [14] first proposed the PML for elastic waves in solids and proved that reflections are null in a regular elastic medium. PML has become very successful in many fields, and in the context of wave propagation: the PML has been applied to acoustic by Qi and Geers (1998), Hagstrom (1999), Liu and Tao (1997) and Kormann et al. (2008) to underwater acoustic propagation models; it has been applied to wave propagation in poroelastic media by Zeng and Huang [15]; to

elastic problems by Colino and Monk (1998), Colino and Tsogka [16], Basu and Chopra [17]; FE applied to classical soil-structure interaction problems in the frequency domain, Cohen and Fauqueux (2003) and with a FDM scheme Moczo et al. [18], Kirsch [19], Skelton et al. [20], Johnson [21], and for wave equation written as a second-order system in displacements by Komatitsch and Tremp [22] and Benito et al. [23].

This paper shows the application of GFDM to solve these kinds of problems. The scheme used and the analyses of stability and dispersion have been clearly referenced. The use of PML absorbing boundary conditions in the model is also explained and the influence of the loss profile is shown. Finally, the efficiency of the method in solving a variation of Lamb's problem on a domain with a simple topographical feature is illustrated as well as another one including a hole.

2. Explicit generalized differences schemes for the seismic wave propagation problem: second order formulation

The equations of motion for a perfectly elastic, homogeneous, isotropic medium in the domain $\Omega \subset \mathbb{R}^2$ are

$$\rho U_{i,tt} = (\lambda + \mu) U_{j,ji} + \mu U_{i,kk} \quad (1)$$

where U_i are the components of the displacement, ρ is the density, λ and μ are the Lamé elastic coefficients.

In this paper three types of boundary conditions are considered: homogeneous, Dirichlet boundary conditions, free surface and, in some cases, a symmetry.

On the free surface the following conditions are imposed

$$\sigma_{ij} n_j = (\lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}) n_j = g_i(t) \Leftrightarrow (\lambda U_{k,k} \delta_{ij} + \mu (U_{ij} + U_{ji})) n_j = g_i(t) \quad (2)$$

where $g_i(t)$ is equal to zero when there are no forces on the surface.

The aim is to obtain explicit linear expressions for the approximation of partial derivatives at the points of the domain. First of all, an irregular grid or cloud of points is generated in the domain $\Omega \cup \Gamma$, where Γ is the boundary of the domain. On defining the central node with a set of nodes surrounding that node, the star then refers to a group of established nodes in relation to a central node. Every node in the domain has an associated star assigned to it. This scheme uses the central-difference form for the time derivative.

Following [1,5,6,8,10], the explicit finite difference formulae for the second spatial derivatives with second and fourth order approximation ($p=2, 4$) for the spatial derivatives are obtained

$$[U_{ijk}^0]_{t=n\Delta t} = -m_{jk}^{0,p} n u_i^0 + \sum_{l=1}^N m_{jk}^{l,p} n u_l^l + \Theta[(h_i)^p] \quad (3)$$

where capital letters are used for exact values and small letters are used for approximated values. The superscript n denotes the time step, the superscripts 0 and l refer to the central node and the rest of star's nodes respectively, N is the number of nodes in the star (in this work, for the second order approximation $N=8$ and the star nodes are selected by using the distance criteria [3], and the fourth order approximation $N=30$ and the star nodes are selected by using the distance criteria) and $h_i^l = x_i^l - x_i^0$.

$m_{jk}^{0,p}$ are the coefficients that multiply the approximate values of the functions U_i at the central node ($^n u_i^0$) for the time $n\Delta t$ in the generalized finite difference explicit expressions for the space derivatives.

$m_{jk}^{l,p}$ are the coefficients that multiply the approximate values of the functions U_i at the other nodes of the star ($^n u_l^l$) for the time $n\Delta t$ in the generalized finite difference explicit expressions for the space derivatives. In all these expression the cross-terms are equal.

The replacement in Eq. (1) of the explicit expressions obtained for the spatial derivatives and the central-difference formula for the time derivatives, leads to the explicit difference scheme

$$\begin{aligned} {}^{n+1} u_i^0 = & 2 {}^n u_i^0 - {}^{n-1} u_i^0 + \frac{(\Delta t)^2}{\rho} \left[(\lambda + \mu) \left(-m_{ji}^{0,p} {}^n u_j^0 + \sum_{l=1}^N m_{ji}^{l,p} {}^n u_l^l \right) \right. \\ & \left. + \mu \left(-m_{kk}^{0,p} {}^n u_i^0 + \sum_{l=1}^N m_{kk}^{l,p} {}^n u_l^l \right) \right] + \Theta[(\Delta t)^2, (h_i)^p] \quad (4) \end{aligned}$$

Imposing boundary conditions is not difficult except for the case of free surface. The values of the function in the nodes of the free surface are unknown but free surface conditions Eq.(2) are known. If U_i^{se} are the displacements of the n_{se} added nodes (Newmann nodes) that belong to a star being U_i^{si} the displacement in its central node and U_i^{sl} the displacements of the remaining n_{si} nodes of the star. Substituting the first order derivatives that appear in the free surface condition Eq. (2) by the explicit expressions Eq. (3) the system of $2n$ equations is obtained

$$\begin{aligned} \left[\lambda \left(-m_k^s {}^n u_k^s + \sum_{l=1}^N m_k^l {}^n u_l^l \right) + \mu \left(-m_j^s {}^n u_j^s + \sum_{l=1}^N m_j^l {}^n u_l^l \right) \right. \\ \left. + \left(-m_i^s {}^n u_j^s + \sum_{l=1}^N m_i^l {}^n u_l^l \right) \right] n_j = g_i(t) \quad (5) \end{aligned}$$

where $N = n_{se} + n_{si}$ and the $2n$ unknowns are the displacements in the n added nodes. These unknowns appear in the summation. By solving this system, the function values ($^n u_i^{se}$) are obtained on the n added Neumann nodes at the time $t = n\Delta t$.

As an explicit method is being used, the stability has been studied and the star stability condition has been obtained in [9]. In this paper, the condition for stability of the star has been established as:

$$\Delta t < \sqrt{\frac{4}{(\alpha^2 + \beta^2)[(|m_{11}^{0,p}| + |m_{22}^{0,p}|) + \sqrt{(m_{11}^{0,p})^2 + (m_{22}^{0,p})^2 + (m_{12}^{0,p})^2}]}} \quad (6)$$

where

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}}$$

are the velocities of P and S waves respectively.

The dispersion for the phase and group velocities has also been studied in depth in [9].

3. Benchmark test

To analyse the obtained approximation, the method has been applied to the solution of Eq. (1) in a square of unitary length. We have used models with 441 and 1681 nodes, Young's modulus $\lambda=0.5$, a shear modulus $\mu=0.25$, a density $\rho=1$ and Dirichlet boundary conditions at every side. The exact solution is

$$\begin{cases} U_x(x, y, t) = \cos\left(\sqrt{\frac{2\mu}{\rho}} t\right) \sin x \sin y \\ U_y(x, y, t) = \cos\left(\sqrt{\frac{2\mu}{\rho}} t\right) \cos x \cos y \end{cases} \quad (7)$$

Fig. 1 shows the exact solution and the values obtained using the GFD scheme at the node with coordinates (0.5, 0.5) for 500 time steps that are equivalent to 0.5 s. The solution has been obtained with the schemes with second and fourth order approximation for the space second order derivatives.

To validate the results obtained by the application of the GFDM a numerical model using the geotechnical calculation software FLAC [24] is done. An elastic medium of horizontal dimension, in

Fig. 1. Displacements using a second order formulation with explicit finite difference formulae for the spatial derivatives with second order and fourth order accuracy. Two models with 441 nodes (21×21) and 1681 nodes (41×41) have been used.

distance units, equal to 5 and a vertical dimension of 1.5, formed by a grid of equally spaced points in both directions (distance equal to 0.02) is considered. Therefore, there are a total of 19,076 points. The elastic properties of the considered medium are, in stress units, $\lambda=18$ and $\mu=9$. A density equal to one is considered. A vertical dynamic step-like load (in stress units) is applied on a point located at the vertical symmetry axis above the upper boundary of the defined geometry

$$p(t) = \begin{cases} 0.5, & 0 \leq t \leq 0.012 \\ 0, & t > 0.012 \end{cases} \quad (8)$$

The model described is equally reproduced in FLAC, choosing the same geometry, distribution and separation points, elastic properties and acting load. Also the same boundary conditions and the same calculation time step are used.

The points $P_i(x, y)$, where x corresponds to the horizontal distance and y to the vertical, being the origin at the upper left corner of the model, $P_1=(2.5, 0.5)$, $P_2=(3.0, 0.0)$ and $P_3=(3.207, 0.707)$ are used to compare the components of both methodologies.

As the P-wave velocity is equal to 6, the P-waves reach the point P_1 at time $t=0.083$ s. A maximum value of the vertical displacement of 5×10^{-5} produced by the P-wave in a range immediately subsequent to time t_1 is shown in Fig. 2. From the comparison of the vertical displacements history at point P_1 , it is observed that the GFDM gives very close values compared to the results obtained by FLAC. At the same time, practically null horizontal displacements are obtained at that point P_1 .

P-waves reach the point P_2 in 0.083 s, S-waves in 0.166 s and Rayleigh waves in 0.175 s. Coinciding with the time instants immediately following these, maximum local values are shown in Fig. 3. The comparison of the values obtained by both methods is satisfactory, although there are small deviations for higher times when S and Rayleigh waves reach that point. From the simultaneous observation of the displacements evolution, it can be seen that the maximum vertical

Fig. 2. Vertical displacement versus time at the point (2.5, 0.5) using step-like load.

displacement corresponds to the instant of time where the horizontal displacement is canceled (0.21 s), and also the maximum value of the horizontal displacement corresponds to a value of almost zero vertical displacement at the instant of time equal to 0.19 s. This correspondence shows the propagation path that is characteristic of Rayleigh waves.

Finally, P_3 corresponds to a point of the grid at a distance of 1 from the point of application of the load (located in the increasing direction of the x coordinate) and placed at equal distance from the upper boundary and the vertical symmetry axis of the model. At this point P_3 the P-wave arrives at a time of 0.166 s (see Fig. 4) obtaining very similar values in vertical and horizontal displacements because the horizontal and vertical components have the same projection angle with respect to the propagation direction. These results are similar for both numerical methods as seen in Fig. 4.

4. Schemes in GFDM for PML in x -direction and y -direction

It is assumed that the space far from the region of interest is homogeneous, linear and time invariant. Then, the radiating

Fig. 3. Vertical and horizontal displacement versus time at the point (3.0,0.0) using step-like load.

Fig. 4. Vertical and horizontal displacement versus time at the point (3.207,0.707) using step-like load.

solution in infinite space can be considered as a superposition of plane waves. As the field variables are analytic functions of x and y , it can be analytically continued, evaluating the solution at complex values of x and y . To avoid this inconvenient, a change of variables is performed in the PML and it can be conceptually assumed as a single transformation of the original equation. Then, for PML in direction x where the spatial derivatives appear in the wave equations, they are replaced in the form

$$\begin{cases} \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \tilde{x}} \left(1 + i \frac{\delta_x(x)}{\omega}\right)^{-1} \\ \frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial \tilde{y}} \left(1 + i \frac{\delta_y(x)}{\omega}\right)^{-1} \end{cases} \quad (9)$$

where the frequency (ω) has been included to have a loss profile in the PML independent of frequency.

The PML loss profile is:

$$\delta_x(x) = \delta_y(y) = \delta(x) = \delta = \frac{1}{\Delta t} \sigma \left(\frac{x_i^F - x_i}{x_i^F - x_i^I} \right)^Q \quad (10)$$

where x_i^I and x_i^F are the initial and final x -coordinates of PML and exponent Q will take the value 2.

The equations of motion in the velocity-stress formulation are

$$\begin{cases} \rho v_{i,t} = \sigma_{ij,j} \\ \sigma_{ij,t} = \lambda \delta_{ij} v_{k,k} + \mu (v_{ij} + v_{ji}) \end{cases} \quad (11)$$

After applying there transformations Eq. (9), the stability analysis for the continuum is carried out following Meza-Fajardo and Papageorgiu [25,26]. The transformed set of equations can be written as

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{A} \mathbf{U} \quad (12)$$

where $\mathbf{U} = \{ \sigma_{xx} \ \sigma_{xy} \ \sigma_{yy} \ v_x \ v_y \}^T$ and the coefficients matrix (with $\delta_x(x) = \delta_y(y) = \delta(x) = \delta$) is

$$\mathbf{A} = \begin{pmatrix} -\delta & 0 & 0 & ik_x(\lambda + 2\mu) & ik_y\lambda \\ 0 & -\delta & 0 & ik_y\mu & ik_x\mu \\ 0 & 0 & -\delta & ik_x\lambda & ik_y(\lambda + 2\mu) \\ \frac{ik_x}{\rho} & \frac{ik_y}{\rho} & 0 & -\delta & 0 \\ 0 & \frac{ik_x}{\rho} & \frac{ik_y}{\rho} & 0 & -\delta \end{pmatrix} \quad (13)$$

We consider our system as stable if all the eigenvalues χ_i of $\mathbf{A}$ have negative real parts. The eigenvalues are

$$\chi = \left\{ -\delta, -\delta \pm i \sqrt{(k_x^2 + k_y^2) \frac{\mu}{\rho}}, -\delta \pm i \sqrt{(k_x^2 + k_y^2) \frac{\lambda + 2\mu}{\rho}} \right\} \quad (14)$$

Therefore the PML proposed is stable.

At the star corner the properties of the overlapping layers are superimposed.

Substituting the explicit finite difference formulae Eq. (3) and Eq. (10) into Eq. (11) and operating, the scheme in GFDM for the PML part is obtained

$$\begin{aligned} {}^{n+1}u_i^0 &= \frac{1}{1 + \delta \Delta t} \left[(2 - \delta^2 (\Delta t)^2) {}^n u_i^0 + \frac{(\Delta t)^2}{\rho} \left[(\lambda + \mu) (-m_{ji}^{0,p} {}^n u_j^0 \right. \right. \\ &\quad \left. \left. + \sum_{l=1}^N m_{ji}^{l,p} {}^n u_l^l \right) \right] \\ &\quad \left. + \mu \left(-m_{kk}^{0,p} {}^n u_i^0 + \sum_{l=1}^N m_{kk}^{l,p} {}^n u_l^l \right) \right] \\ &\quad - (1 - \delta \Delta t) {}^{n-1} u_i^0 + \Theta[\Delta t^2, (h_i)^p] \end{aligned} \quad (15)$$

4.1. Stability of GFD scheme for PML

For the stability analysis of GFD scheme for PML Eq. (15) the first idea is to make a harmonic decomposition of the approximated solution at grid points and at a given time level (n)

$$\begin{aligned} {}^n u_x^0 &= A \xi^n e^{i \mathbf{k}^T \mathbf{x}_0}; \quad {}^n u_x^j = A \xi^n e^{i \mathbf{k}^T \mathbf{x}_j}; \quad {}^n u_y^0 = B \xi^n e^{i \mathbf{k}^T \mathbf{x}_0}; \quad {}^n u_y^j \\ &= B \xi^n e^{i \mathbf{k}^T \mathbf{x}_j} \quad \forall j = 1, \dots, N \end{aligned} \quad (16)$$

where N is the number of nodes in the star, and

$$\mathbf{x}_j = \mathbf{x}_0 + \mathbf{h}_j; \quad \xi = e^{-i \omega \Delta t}; \quad \mathbf{k} = \begin{Bmatrix} \kappa_x \\ \kappa_y \end{Bmatrix} = \kappa \begin{Bmatrix} \cos \phi \\ \sin \phi \end{Bmatrix} \quad (17)$$

Fig. 5. The domain with elastic and PML part.

The amplitude of $\mathbf{k}$ is the wave number, ξ is the amplification factor whose value will determine the stability condition and ω is the angular frequency.

Substituting Eq. (16) into Eq. (15), and after cancellation of $\xi^n e^{i \mathbf{k}^T \mathbf{x}_0}$, we obtain

$$\begin{cases} A\xi(t)1c4\delta\Delta t = (2 - \delta^2(\Delta t)^2)A - \frac{A}{\xi}(1 - \delta\Delta t + (\Delta t)^2) \left[\alpha^2 \left(-Am_0 + A \sum_{j=1}^N m_j e^{i \mathbf{k}^T \mathbf{h}_j} \right) + \beta^2 \left(-A\eta_0 + A \sum_{j=1}^N \eta_j e^{i \mathbf{k}^T \mathbf{h}_j} \right) + (\alpha^2 - \beta^2) \left(-B\zeta_0 + B \sum_{j=1}^N \zeta_j e^{i \mathbf{k}^T \mathbf{h}_j} \right) \right] \\ B\xi(1 + \delta\Delta t) = (2 - \delta^2(\Delta t)^2)B - \frac{B}{\xi}(1 - \delta\Delta t + (\Delta t)^2) \left[\beta^2 \left(-Bm_0 + B \sum_{j=1}^N m_j e^{i \mathbf{k}^T \mathbf{h}_j} \right) + \alpha^2 \left(-B\eta_0 + B \sum_{j=1}^N \eta_j e^{i \mathbf{k}^T \mathbf{h}_j} \right) + (\alpha^2 - \beta^2) \left(-A\zeta_0 + A \sum_{j=1}^N \zeta_j e^{i \mathbf{k}^T \mathbf{h}_j} \right) \right] \end{cases} \quad (18)$$

where

$$m_0 = \sum_{j=1}^N m_j; \quad \eta_0 = \sum_{j=1}^N \eta_j; \quad \zeta_0 = \sum_{j=1}^N \zeta_j \quad (19)$$

Substituting Eq. (19) in Eq. (18), and if B is obtained from the second equation and is included in the first equation and operating, the following condition is obtained

$$\begin{aligned} & 2 \left[\cos w\Delta t + 2i\delta\Delta t \sin w\Delta t - (2 - \delta^2(\Delta t)^2) + (\Delta t)^2 \left(\alpha^2 \sum_{j=1}^N m_j (1 - e^{i \mathbf{k}^T \mathbf{h}_j}) + \beta^2 \sum_{j=1}^N \eta_j (1 - e^{i \mathbf{k}^T \mathbf{h}_j}) \right) \right] \\ & \times = (\Delta t)^4 (\alpha^2 - \beta^2)^2 \left[\sum_{j=1}^N \zeta_j (1 - e^{i \mathbf{k}^T \mathbf{h}_j}) \right]^2 \end{aligned} \quad (20)$$

Operating with the Eq. (20) and canceling terms with conservative criteria, the conditions for stability of the star are obtained as

$$\begin{cases} 0 < \delta\Delta t < 1 \\ \Delta t < \sqrt{\frac{4}{(\alpha^2 + \beta^2)(|m_0| + |\eta_0|) + \sqrt{(m_0 + \eta_0)^2 + \zeta_0^2 + \delta^2 \Xi^2}}} \end{cases} \quad (21)$$

where

$$\Xi = \frac{\alpha^2 \beta^2 (m_0 + \eta_0)^2 + m_0 \eta_0 (\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2 (m_0 + \eta_0)}$$

This condition of stability is 5% smaller than in the elastic case.

4.2. Influence of the parameters

In this section the influence of the σ parameter corresponding to the PML loss profile Eq. (12) in the solution for the PML domain will be analyzed.

Fig. 5 shows the domain used to analyze this influence. It is the model of a semi-infinite elastic medium with a horizontal free surface.

$$\Omega = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 5; 0 \leq y \leq 1.5\} \quad (22)$$

The corresponding PML domain to simulate the infinite condition is given by:

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 0.5; 0 \leq y \leq 1.0\} \quad (23)$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 / 4.5 \leq x \leq 5; 0 \leq y \leq 1.0\} \quad (24)$$

$$\Omega_3 = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 5; 1.0 \leq y \leq 1.5\} \quad (25)$$

The boundary conditions are:

$$\begin{cases} U_x(x, 1.5, t) = U_y(x, 1.5, t) = 0 \\ U_x(0, y, t) = U_y(0, y, t) = 0 \\ U_x(5, y, t) = U_y(5, y, t) = 0 \\ \left. \frac{\partial U_x(x, y, t)}{\partial y} \right|_{(x, 0)} = \left. \frac{\partial U_y(x, y, t)}{\partial y} \right|_{(x, 0)} = 0 \quad 0 \leq x \leq 5 \end{cases} \quad (26)$$

A regular mesh with 19,076 nodes (node spacing 0.02) has been used.

A step-like pulse is applied at (2.5, 0.0) the intersection between the symmetry axis and the upper boundary (see Fig. 5). The variation of the wave amplitude inside the PML domain will be analyzed as a function of the σ parameter.

The parameters of the elastic domain are:

$$\lambda = 2 \text{ N/m}^2; \quad \mu = 1 \text{ N/m}^2; \quad \rho = 1 \text{ kg/m}^3 \quad (27)$$

The range of σ values considered is from 0.01 up to 0.15. A higher value of the σ parameter attenuates the wave amplitude in a sharp way. But, a wave reflection takes place at the boundary (see Fig. 6).

A low σ value diminishes the wave reflection but also reduces the attenuation of the wave inside the PML domain (see Fig. 6a). Therefore, there is a need to balance both phenomena (wave reflection at the elastic domain-PML interface, and wave attenuation inside the PML). In order to analyze the optimum value for the σ parameter, the evolution of the wave amplitude has been tracked along the y -coordinate.

Fig. 7 shows the ratio between the source wave amplitude A_0 and the initial amplitude A_1 for different values of σ and y -coordinate. The elastic domain-PML interface corresponds to $y = 1.0$. A detail for this ratio inside the PML can be seen in Fig. 7.

As it may be observed, a step attenuation occurs for $\sigma \geq 0.06$. On the contrary, the wave attenuation is not complete for $\sigma = 0.01$, since the wavelet reaches the PML boundary, causing some reflection.

Fig. 8 shows the ratio between the amplitude of the incident wave A_1 and the reflected at the PML boundary one A_2 at four points along the symmetry axis ($y = 0.2, y = 0.7, y = 0.8, y = 0.96$) and six values of σ . In this graph it can be seen that the ratio A_2/A_1 does not reduce significantly for σ values higher than 0.06. On the contrary, the higher decrease rate is observed for σ values lower than 0.03.

Taking into account both analyses (amplitude attenuation within PML domain, and reflected wavelet amplitude ratio), it can be deduced that the optimum σ value lays between 0.02 and 0.03 for the length of the PML considered in the example.

In case the length of the adopted PML is higher, a lower value of the σ parameter may be chosen, as long as no reflection phenomena take place at the PML external boundary.

5. Applications

This section shows the effects of topography and domain irregularities solving a variation of Lamb's problem [27–30] on a domain with a simple topographical feature and also with a hole

Fig. 6. (a, b) Reflection and attenuation on the PML interface for $\sigma=0.15$ and $\sigma=0.01$.

Fig. 7. Ratio (A_1/A_0) versus y-coordinate for different values of σ .

located inside the domain. In all cases, a base model as described in Section 4.1 is used, with: $\lambda=18$, $\mu=9$ and $\rho=1$. The PMLs have $\sigma=0.02$ and $Q=2$ in both directions. The excitation is a vertical step-like pulse Eq. (8) applied at the center of the free surface (point (2.5, 0.0)). A regular initial cloud with only 251×76 nodes (space between nodes 0.02) has been used. Boundary conditions are free surface conditions on the top side and Dirichlet homogeneous conditions on the faces located behind the PML.

5.1. Half-space with a hole inside. Cases of a circle and a rectangle.

As a first step, nodes that later will serve to define the boundary of the circular hole have been added in the model, as shown in Fig. 9. The added points will add an irregular behavior to the scheme by generating different star patterns to those defined in the rest of the cloud of nodes.

Analyzing the real influence of having an irregularity in the mesh is the main objective of this initial model. Fig. 10a) shows the magnitude of the displacements at time $t=0.137$ s (a time step increment of 0.0003 s has been considered). At this time the P wave has already passed and S and Rayleigh waves pass through the irregularity. It is possible to observe there is no noticeable

Fig. 8. Ratio (A_2/A_1) versus four points along the axis of symmetry for six values of σ .

distortion, although nodes are clearly visible due to the results have been drawn on them. For a better assessment Fig. 10b), shows the same results but instead of plotting the cloud of points directly, a surface has been generated by triangularization. The vertices of each triangle are a point in the cloud, and the triangles have been

Fig. 9. Geometry of the half-space with added nodes in a circular shape.

Fig. 10. (a) Magnitude of the displacements on an irregularity, plotting the nodes. (b) Magnitude of the displacements on an irregularity, plotting interpolated surfaces.

colored by interpolating the values at the corners (points of the cloud of nodes).

Fig. 11 corresponds to the magnitude displacements at $t=0.177$ s (a time step increment of 0.0003 s has been considered), when a circular hole has been included in the model. The inner nodes to the circumference have been removed and Dirichlet homogeneous boundary conditions have been imposed at the circumference. Several wavefronts reflected within the domain and on the surface in the direction of the central zone can be seen.

Finally, a group of nodes within a square area have been removed from the model [31] and free surface boundary conditions at the sides have been imposed. Fig. 12, shows the magnitude displacement at time $t=0.165$ s (a time step increment of 0.0003 s has been considered) when the P and S waves have passed. At this time, it is possible to appreciate how a small perturbation starts to appear at the area surrounding the hole.

5.2. Half-space with a small mountain

In this subsection, a different irregularity is shown, a small mountain on the free surface has been introduced [32]. In Fig. 13a, the magnitude displacement at time $t=0.14$ s (a time step increment of 0.00015 s has been considered), just when the Rayleigh

waves pass, and it is shown how a reflection on the surface at both sides of the irregularity is initiated. Fig. 13b shows the situation a little later, at $t=0.182$ s, and it is possible to see how this small mountain acts as a scatterer. The figure clearly shows the new family of backscattered waves.

6. Conclusions

This paper shows a scheme in generalized finite differences to the solution of the problem of seismic wave propagation in 2D with Perfectly Matched Layers absorbing boundaries in the model. Some variations of Lamb's problem are analyzed, showing the influence of irregularities on P, S, Rayleigh and Von Schmidt waves.

The recursive equations for two directions have been obtained and the condition for stability, obtained in previous works, is shown. The influence of nodes and the order of approximation have been illustrated by an example. The results obtained show how the solution is not affected during the first steps and how it improves by increasing the number of nodes and the order of approximation. Nevertheless, passing from the second to the fourth order of approximation involves increasing, by approximately seven, the time needed to solve the problems on a computer. The comparison between the displacements

Fig. 11. Magnitude of the displacements around a circular hole with Dirichlet boundary conditions ($t=0.177$ s).

Fig. 12. Magnitude of the displacement around a square hole with free surface boundary conditions ($t=0.165$ s).

Fig. 13. Magnitude of the displacement in a half-space with a small mountain at (a) $t=0.14$ s and (b) $t=0.184$ s.

historical obtained in several nodes in the domain using the calculation software FLAC and the GFDm to validate the latter has been satisfactory.

Introduction of the PML absorbing boundaries in the model is explained and the influence of the fundamental parameters of the used loss profile equation is analyzed. The reflection at the interface with the PML decreases when the value of σ parameter decreases, but the attenuation of wave amplitude in the boundary layer also decreases, being necessary to increase its thickness to avoid reflexions at the Dirichlet boundary. Therefore, it is necessary to find a balance for the different parameters involved. It is shown how the introduction of small irregularities in the distribution of nodes in the model has a negligible influence on the solution.

The influence of including in the domain cases of a circular hole with Dirichlet homogeneous boundary conditions and a square hole with free boundary conditions have been studied. The reflected waves at the walls of the mentioned holes can be clearly seen although a very coarse mesh has been used. The case of a small mountain on the half-space, which acts as a scatterer, clearly illustrates the importance of taking into account the topography.

Acknowledgement

The authors acknowledge the support of the Escuela Técnica Superior de Ingenieros Industriales (UNED) of Spain, Project 2014-IFC02.

References

- [1] Orkisz J. Finite difference method (Part, III). In: Kleiber M, editor. *Handbook of computational solid mechanics*. Berlin: Spriger-Verlag; 1998.
- [2] Jensen PS. Finite difference technique for variable grids. *Comput Struct* 1972;2:17–29.
- [3] Perrone N, Kao R. A general finite difference method for arbitrary meshes. *Comput Struct* 1975;5:45–58.
- [4] Liszka T, Orkisz J. The finite difference method at arbitrary irregular grids and its application in applied mechanics. *Comput Struct* 1980;11:83–95.
- [5] Benito JJ, Ureña F, Gavete L. Influence several factors in the generalized finite difference method. *Appl Math Model* 2001;25:1039–53.
- [6] Benito JJ, Ureña F, Gavete L, Alvarez R. An h-adaptive method in the generalized finite difference. *Comput Methods Appl Mech Eng* 2003;192:735–59.
- [7] Benito JJ, Ureña F, Gavete L. *Leading-Edge applied mathematical modelling research* (chapter 7). New York: Nova Science Publishers; 2008.
- [8] Benito JJ, Ureña F, Gavete L, Alonso B. Application of the generalized finite difference method to improve the approximated solution of pdes. *Comput Model Eng Sci* 2009;38:39–58.
- [9] Ureña F, Benito JJ, Salete E, Gavete L. A note on the application of the generalized finite difference method to seismic wave propagation in 2-D. *J Comput Appl Math* 2011;236(12):3016–25.
- [10] Ureña F, Salete E, Benito JJ, Gavete L. Solving third and fourth order partial differential equations using GFDM. Application to solve problems of plates. *Int J Comput Math* 2012;89(3):366–76.
- [11] Ureña F, Benito JJ, Gavete L. Application of the generalized finite difference method to solve the advection–diffusion equation. *J Comput Appl Math* 2011;235:1849–55.
- [12] Benito JJ, Ureña F, Gavete L, Alonso B. Solving parabolic and hyperbolic equations by generalized finite difference method. *J Comput Appl Math* 2007;209:208–33.
- [13] Berenger JP. A perfectly matched layer for the absorption of electromagnetic waves. *J Comput Phys* 1994;114:185–200.
- [14] Chew WC, Liu QH. Perfectly matched layer for elastodynamics; a new absorbing boundary condition. *J Comput Acoust* 1996;4:341–59.
- [15] Zheng Y, Huang X. *Anisotropic perfectly matched layers for elastic waves in cartesian and curvilinear coordinates*. Cambridge, MA: Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute Technology; 2002. p. 1–18.
- [16] Collino F, Tsogka C. Application of the perfectly matched layer absorbing layer model to the linear elastodynamic problem in anisotropic heterogeneous media. *Geophysics* 2001;66(1):294–307.
- [17] Basu U, Chopra AK. Perfectly matched layers for elastic waves and application to earthquake analysis of dams. In: *Proceedings of the 16th ASCE engineering mechanics conference*. University of Washington Seattle; 2003. pp. 16–18.
- [18] Moczo P, Kristek J, Galis M, Pazak P, Balazoviech M. The finite-difference and Finite-element modeling of seismic wave propagation and earthquake motion. *Acta Phys Slovaca* 2007;57(2):177–406.
- [19] Kirsh C. *Non-reflecting boundary conditions for the numerical solution of wave propagation problems in unbounded domains* (Inaugural dissertation). zur Erlangung der Wurde eines Doktors der Philosophie vorgelegt der Philosophisch. Zurich: Naturwissenschaftlichen Fakultät; 2005.
- [20] Skelton EA, Adams SDM, Craster RV. Guided elastic waves and perfectly matched layers. *Wave Motion* 2007;44:573–92.
- [21] Johnson SG. Notes on perfectly matched layers (PMLs). Courses 18.369 and 18.336 at MIT; created August (2007); updated March 2010.
- [22] Komatitsch D, Tromp J. A perfectly matched layer (PML) absorbing condition for the second-order elastic wave equation. *Geophys J Int* 2003;154:146–53.
- [23] Benito JJ, Ureña F, Gavete L, Salete E, Muelas A. AGFDM with PML for seismic wave equations in heterogeneous media. *J Comput Appl Math* 2013;252:40–51.
- [24] Itasca Consulting Group. Inc FLAC, Fast Lagrangian Analysis of Continua, Version 5.0. Fluid-mechanical interaction. Minneapolis; 2005.
- [25] Meza Fajardo KC, Papageorgiou AS. A non-convolutional split-field perfectly matched layer (PML) for wave propagation in isotropic and anisotropic elastic media-stability analysis. *Bull Seismol Soc Am* 2008;98(4):1811–36.
- [26] Meza Fajardo KC, Papageorgiou AS. On the stability of a non-convolutional perfectly matched layer for isotropic elastic media. *Soil Dyn Earthq Eng* 2010;30:68–81.
- [27] Lamb H. *On the propagation of tremors over the surface of an elastic solid*. London: Philosophical Transactions of the Royal Society; 1904. p. 1–42.
- [28] Guangyu L, Kaixin L. Exact solution for a two-dimensional Lamb's problem due to a strip impulse loading. *Acta Mech Solida Sin* 2007;20(3):258–65.
- [29] Guenfoud S, Amranev MN, Bokasov SV, Ouelaa N. Semi-analytical evaluation of integral forms associated with Lamb's problem. *Soil Dyn Earthq Eng* 2009;29:438–43.
- [30] Johnson LR. Green's function for Lamb's problem. *Geophys J R Astron Soc* 1974;34:99–131.
- [31] Schoeder CT, Scott Jr. WR. *Finite-difference time-domain model for elastic waves in the ground*. Atlanta, USA: School of Electrical and Computer Engineering, Georgia Institute of Technology; 1999. p. 1–14 GA 30332-0250, USA.
- [32] Appelö D, Anders Petersson N. A stable finite difference method for the elastic wave equation on complex geometries with free surfaces. *Commun Comput Phys* 2003;5:84–107.