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Gröbner Basis Computation of Drazin Inverses with Multivariate Rational Function Entries

Juana Sendra

Dpto. Matemática Aplicada a la I.T. de Telecomunicación.

Research Center on Software Technologies and Multimedia Systems for Sustainability (CITSEM), UPM, Spain

jsendra@euitt.upm.es

J. Rafael Sendra

Dpto. de Fisica y Matemáticas, Universidad de Alcalá

E-28871 Madrid, Spain

rafael.sendra@uah.es

Abstract

In this paper we show how to apply Gröbner bases to compute the Drazin inverse of a matrix with multivariate rational functions as entries. When the coefficients of the rational functions depend on parameters, we give sufficient conditions for the Drazin inverse to specialize properly. In addition, we extend the method to weighted Drazin inverses. We present an empirical analysis that shows a good timing performance of the method.

keywords Drazin inverse, Gröbner bases, symbolic computation.

1 Introduction

Drazin inverses (introduced by M.P. Drazin in [6]) are used in many applications as singular differential equations and difference equations (see [4]), in finite finite Markov chains (see [16]), etc (see [1]). Many authors have addressed the problem of computing Drazin inverses where the matrix entries are indeed polynomials (see e.g. [2], [7], [10], [11]), for special type of matrices (see e.g. [15], [13]), and for weighted Drazin inverses (see [9], [12]).

Some of the above quoted papers provide symbolic algorithms, in the sense that they use exact arithmetic and the output is exact, but techniques from computer algebra are not directly applied. In this paper, using that Drazin inverse computation can be translated into an elimination theory question, we show how to approach the problem by means of Gröbner bases (see e.g. [5] and [14]). Gröbner bases were introduced by B. Buchberger in [3] and, since then, they have been applied to multiple problems both in mathematics and in practical applications. Particularly important is the application of Gröbner bases to solve systems of algebraic equations. In this context, Gröbner bases can be seen as the generalization of gaussian elimination to the non-linear case of algebraic equations. The worst case complexity of Gröbner bases computation is double exponential in the number of variables, and simple exponential if the ideal is zero-dimensional (in the context of solving system of algebraic equations, it means that the number of solutions of the system is finite). Nevertheless, in practice, there are good implementations, and depending on the problem, the computation of a Gröbner basis is feasible.

In this paper, using Gröbner bases, we show how to compute the Drazin inverse of matrices which entries are rational functions in several variables. Moreover, we also deal with the problem when the input matrix has entries being rational functions in several variables with coefficients depending on parameters. In addition, we provide conditions to ensure when the Drazin inverse, of such parametric multivariate rational functions, specialized properly. We also present an empirical analysis that shows that the timings of the presented method has a good performance. We also show how these ideas apply to weighted Drazin inverses.

Throughout this paper we use the following notation. Let $\mathbb{F}$ be the rational function field $\mathbb{C}(\mathbf{z})$ (similarly, if we replace $\mathbb{C}$ by another field), where $\mathbf{z} = (z_1, \dots, z_s)$, and let $\mathcal{M}_n(\mathbb{F})$ be the ring of $n \times n$ matrices over $\mathbb{F}$. For $A \in \mathcal{M}_n(\mathbb{F})$ we denote by A^D the Drazin inverse of A and by index(A) the Drazin index of A.

2 The Multivariate Case

Let $A \in \mathcal{M}_n(\mathbb{F})$ with Drazin index k. In order to approach the computation of the Drazin inverse A^D of A, we consider a generic element $\hat{X}$ in $\mathcal{M}_n(\mathbb{F})$, that is $\hat{X} = (x_{ij})_{1 \le i,j \le n}$ where x_{ij} are new variables. A^D is the unique solution of the algebraic system of equations, in the variables x_{ij},

$$A^{k+1}\hat{X} - A^k = \mathbf{O},\ \hat{X}A\hat{X} - \hat{X} = \mathbf{O},\ A\hat{X} - \hat{X}A = \mathbf{O} \qquad (1)$$

where $\mathbf{O}$ is the null matrix. Therefore, if $\hat{\mathcal{F}}$ is set of polynomials of $\mathbb{F}[\mathbf{x}]$ (where $\mathbf{x} = (x_{11}, \dots, x_{nn})$), provided by (1), the ideal generated by $\hat{\mathcal{F}}$ over $\mathbb{F}[\mathbf{x}]$ is zero-dimensional, and since Gröbner bases algorithm does not extend the ground field, one concludes that $A^D \in \mathcal{M}_n(\mathbb{F})$. More generally, we have that following statement.

Proposition 1 *Let $\mathbb{L}$ be a field and $A \in \mathcal{M}_n(\mathbb{L})$, then $A^D \in \mathcal{M}_n(\mathbb{L})$.*

We observe that $\mathcal{L} = \{A^{k+1}\hat{X} - A^k = \mathbf{O}, A\hat{X} - \hat{X}A = \mathbf{O}\}$ is a linear system of equations, while $\{\hat{X}A\hat{X} - \hat{X} = \mathbf{O}\}$ is an algebraic system of quadratic polynomials. Furthermore, $\mathcal{L}$ is compatible. Solving $\mathcal{L}$ and substituting the solution in the quadratic system we get a new quadratic (in general) system, equivalent to (1), and having less variables. Let

$$XAX - X = \mathbf{O} \tag{2}$$

be the resulting system. So let $\mathcal{F}$ be the set of polynomials defining (2); note that its ideal is zero-dimensional. The solution of a Gröbner basis of $\mathcal{F}$, w.r.t. a lexicographic order of its variables, jointly with the solution of $\mathcal{L}$, provides A^D. These ideas yield to the following algorithm.

Algorithm-1: Given $A \in \mathcal{M}_n(\mathbb{C}(\mathbf{z}))$ the algorithm computes its Drazin inverse A^D.

1. Compute $k :=$ index(A).
2. Solve the linear system $\mathcal{L} = \{A^{k+1}\hat{X} - A^k = \mathbf{O}, A\hat{X} - \hat{X}A = \mathbf{O}\}$ and substitute its solution $\mathcal{S}$ in $\hat{X}A\hat{X} - \hat{X} = \mathbf{O}$. Let $XAX - X = \mathbf{O}$ be the resulting system and V the set of variables.
3. Compute a Gröbner basis $\mathcal{G}$ of the polynomials defining $XAX - X = \mathbf{O}$ with respect to a lexicographic order of V.
4. Substitute the solution provided by $\mathcal{G}$ and $\mathcal{S}$ in $\hat{X}$ to get A^D.

Let us illustrate this by some examples.

Example 2 *Let*

$$A = \begin{pmatrix} \frac{1}{z_2} & z_1 & 0 \\ 0 & \frac{1}{z_2} & z_1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{C}(z_1, z_2)).$$

index(A) $= 1$. *The solution of $\mathcal{L}$ is*

$$\{x_{1,1} = z_2, x_{1,2} = -z_1{z_2}^2, x_{1,3} = x_{3,3}{z_1}^2{z_2}^2 - 2\,{z_1}^2{z_2}^3, x_{2,1} = 0, x_{2,2} = z_2,$$
$$x_{2,3} = -z_2 z_1 x_{3,3} + z_1{z_2}^2, x_{3,1} = 0, x_{3,2} = 0, x_{3,3} = x_{3,3}\}$$

and the quadractic system (2) turns to be linear, indeed:

$$\{x_{3,3} = 0\}$$

Therefore,

$$A^D = \begin{pmatrix} z_2 & -z_1{z_2}^2 & -2\,{z_1}^2{z_2}^3 \\ 0 & z_2 & z_1{z_2}^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Example 3 *Let*

$$A = \begin{pmatrix} z_2 & 1 & 1 \\ z_2 & z_1+1 & z_1+1 \\ -z_2 & -z_1-1 & -z_1-1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}(z_1,z_2)).$$

index(A) $= 2$. *The solution of* $\mathcal{L}$ *can be expressed as*

$$\Big\{ x_{1,1} = {z_2}^{-1}, x_{1,2} = x_{1,2}, x_{1,3} = x_{1,2}, x_{2,1} = x_{1,2}z_2, x_{2,2} = -\frac{2\,x_{1,2}{z_2}^2 + x_{3,3}z_2 - 2}{z_2},$$

$$x_{2,3} = -\frac{x_{1,2}{z_2}^2 + x_{3,3}z_2 - 1}{z_2}, x_{3,1} = -x_{1,2}z_2, x_{3,2} = \frac{x_{1,2}{z_2}^2 + x_{3,3}z_2 - 1}{z_2}, x_{3,3} = x_{3,3} \Big\}$$

and the corresponding system (2) is

$$\begin{cases} -x_{1,2}{z_2}^2 + 1 = 0 \\ -{x_{1,2}}^2 z_1 {z_2}^4 + 2\,x_{1,2}z_1{z_2}^2 - x_{3,3}{z_2}^2 - z_1 - 1 = 0 \\ -{x_{1,2}}^2 z_1 {z_2}^4 + 2\,x_{1,2}z_1{z_2}^2 - x_{1,2}{z_2}^3 - x_{3,3}{z_2}^2 - z_1 + z_2 - 1 = 0 \\ {x_{1,2}}^2 z_1 {z_2}^4 - 2\,x_{1,2}z_1{z_2}^2 + x_{1,2}{z_2}^3 + x_{3,3}{z_2}^2 + z_1 - z_2 + 1 = 0 \\ {x_{1,2}}^2 z_1 {z_2}^4 - 2\,x_{1,2}z_1{z_2}^2 + 2\,x_{1,2}{z_2}^3 + x_{3,3}{z_2}^2 + z_1 - 2\,z_2 + 1 = 0 \end{cases}$$

that is quadratic in the variable $x_{1,2}, x_{3,3}$. *A Gröbner basis of the above polynomials w.r.t. the lexicographic order, with* $x_{1,2} < x_{3,3}$, *is*

$$\left\{ x_{1,2}{z_2}^2 - 1, x_{3,3}{z_2}^2 + 1 \right\}$$

that yields to

$$A^D = \begin{pmatrix} {z_2}^{-1} & {z_2}^{-2} & {z_2}^{-2} \\ {z_2}^{-1} & {z_2}^{-2} & {z_2}^{-2} \\ -{z_2}^{-1} & -{z_2}^{-2} & -{z_2}^{-2} \end{pmatrix}.$$

3 The Parametric Multivariate Case

We can use Gröbner bases for the computation of Drazin inverse of matrices with multivariate rational functions entries, which coefficients depend on parameters. More precisely, let Λ be a finite set of parameters. We consider the matrix ring $\mathcal{M}_n(\mathbb{C}(\Lambda)(\mathbf{z}))$. Let $A(\Lambda) \in \mathcal{M}_n(\mathbb{C}(\Lambda)(\mathbf{z}))$. We can deal with the Drazin inverse of $A(\Lambda)$; we denote it by $A(\Lambda)^D$. On the other hand, for every value Λ_0 of Λ taken in $\mathbb{C}(\mathbf{z})$, we can deal with the Drazin inverse of $A(\Lambda_0)$; we denote it by $(A(\Lambda_0))^D$. Algorithm-1 is valid for the field $\mathbb{C}(\Lambda)(\mathbf{z})$, and hence it computes $A(\Lambda)^D$. Also, Algorithm-1 is valid for the field

$\mathbb{C}(\Lambda_0)(\mathbf{z})$, and so it also computes $(A(\Lambda_0))^D$. The question is whether we can deduce $(A(\Lambda_0))^D$ from $A(\Lambda)^D$; that is whether $A(\Lambda_0)^D = (A(\Lambda_0))^D$. The next theorem states that for almost all Λ_0 the equality holds. We start with a lemma

Lemma 4 *There exists a non-zero polynomial $H \in \mathbb{C}(\mathbf{z})[\Lambda]$ such that, if $\Lambda_0 \in \mathbb{C}(\mathbf{z})$ satisfies that $H(\Lambda_0)$ is not identically zero, then* $\mathrm{index}(\mathrm{A}(\Lambda_0)) = \mathrm{index}(\mathrm{A}(\Lambda))$.

Proof: Let $k = \mathrm{index}(\mathrm{A}(\Lambda))$ as matrix in $\mathcal{M}_n(\mathbb{C}(\Lambda)(\mathbf{z}))$. For $i = 1, \ldots, k+1$, let T_i denote the gaussian triangularization of A^i. Let us assume that we have expressed all non-zero entries of each T_i with a common denominator. Then, H can be taken as the product of the primitive part w.r.t. Λ of the numerators and denominators of all diagonal elements of $T_1, \ldots, T_{k+1}$. □

Theorem 5 *There exists a non-zero polynomial $H \in \mathbb{C}(\mathbf{z})[\Lambda]$ such that, if $\Lambda_0 \in \mathbb{C}(\mathbf{z})$ satisfies that $H(\Lambda_0)$ is not identically zero, then $A(\Lambda_0)^D = (A(\Lambda_0))^D$.*

Proof: For the theoretical reasoning, we consider that $A(\Lambda)^D$ is computed by taking a Gröbner basis of (1). Now, the proof follows from Lemma 4 and the behavior of Gröbner basis under specializations; see e.g. Exercise 7, p. 283 in [5]. □

The polynomial H in Theorem 5 can be expressed as $H = H_1 H_2 H_3$ where:

- H_1 is the polynomial provided by the index computation (see Lemma 4),
- H_2 is the primitive part, w.r.t. Λ, of the least common multiple of all denominators of the entries of $A(\Lambda)^D$, times the primitive part, w.r.t. Λ, of the least common multiple of all denominators of the entries of $A(\Lambda)$ (note that that the denominators in A^i do not introduce different factors to those considered before), and
- H_3 is the polynomial provided by the Gröbner basis computation.

Although there exist algorithms to compute H_3, and hence to know in advance which evaluations are valid, these algorithm are time consuming. Nevertheless, in our case, since the solution is unique, we have a better result.

Theorem 6 *Let H_1, H_2 as above. If $\Lambda_0 \in \mathbb{C}(\mathbf{z})$ is such that $H_1(\Lambda_0)H_2(\Lambda_0) \neq 0$, then $A(\Lambda_0)^D = (A(\Lambda_0))^D$.*

Proof: Let Λ_0 be as in the statement. We consider the evaluation ring homomorphism $\varphi_{\Lambda_0} : \mathbb{C}[\Lambda, \mathbf{z}] \to \mathbb{C}[\mathbf{z}]$; $\varphi(M(\Lambda, \mathbf{z})) = M(\Lambda_0, \mathbf{z})$. We extend it to rational functions in $\mathbb{C}(\Lambda)(\mathbf{z})$ which denominators do not vanish at Λ_0, and similarly to $\mathcal{M}_n(\mathbb{C}(\Lambda)(\mathbf{z}))$. Because of H_1, $\mathrm{index}(\varphi_{\Lambda_0}(\mathrm{A}(\Lambda))) = \mathrm{index}(\mathrm{A}(\Lambda_0))$ and $\varphi_{\Lambda_0}(A(\Lambda)) = A(\Lambda_0), \varphi_{\Lambda_0}(A(\Lambda)^k) =$

$A(\Lambda_0)^k$, $\varphi_{\Lambda_0}(A(\Lambda)^{k+1}) = A(\Lambda_0)^{k+1}$, and because of H_2, $\varphi_{\Lambda_0}(A(\Lambda))^D = A(\Lambda_0)^D$. Therefore, $A(\Lambda_0)$ has index k and $A(\Lambda_0)^D$ satisfies the equations

$$A(\Lambda_0)^{k+1}X - A(\Lambda_0)^k = \mathbf{O},\; XA(\Lambda_0)X - X = \mathbf{O},\; A(\Lambda_0)X - XA(\Lambda_0) = \mathbf{O}.$$

Now, the result follows from the uniqueness of the solutions of the system. □

Remark 1 *We observe that the method described in Theorem 6 is also applicable for matrices in $\mathcal{M}_n(\mathbb{C}(\mathbf{z}))$ considering $\mathbf{z}$ as parameters, and hence to determine, by specializations, the Drazin inverses of matrices in $\mathcal{M}_n(\mathbb{C})$.*

Example 7 *We consider Example 5.3 in [10]. There, the authors analyze the time for computing the Drazin inverse of the matrix*

$$B(q) = \begin{pmatrix} 1+z & z & 1+z \\ z^q & -1+z & z \\ 1+z & 1 & 1+z \end{pmatrix} \in \mathcal{M}_3(\mathbb{C}(z))$$

when $q \in \mathbb{N}$ takes natural values. We can consider the matrix

$$A(\lambda) = \begin{pmatrix} 1+z & z & 1+z \\ \lambda & -1+z & z \\ 1+z & 1 & 1+z \end{pmatrix} \in \mathcal{M}_3(\mathbb{C}(\lambda)(z)).$$

The matrix (λ) is regular unless $\lambda = z$ (similarly, $B(q)$ is regular unless $q = 1$), indeed,

$$\det(A(\lambda)) = -(1+z)(-1+z)(\lambda - z).$$

So, $H_1 = \lambda - z$. The Drazin index is 0 and the Drazin inverse of $A(\lambda)$, as matrix in $\mathcal{M}_3(\mathbb{C}(\lambda, z))$, is

$$A(\lambda)^D = \begin{pmatrix} -\frac{z^2-z-1}{(1+z)(-1+z)(\lambda-z)} & (\lambda - z)^{-1} & -\frac{1}{(1+z)(-1+z)(\lambda-z)} \\ (-1+z)^{-1} & 0 & -(-1+z)^{-1} \\ -\frac{-z^2+\lambda+1}{(1+z)(-1+z)(\lambda-z)} & -(\lambda-z)^{-1} & \frac{z\lambda-z^2+1}{(1+z)(-1+z)(\lambda-z)} \end{pmatrix}.$$

So, $H_2 = (\lambda - z)$. Therefore, for $\lambda_0 \in \mathbb{C}(z) \setminus \{z\}$ one has that $A(\Lambda_0)^D = (A(\Lambda_0))^D$. For $\lambda_0 = z$ the Drazin inverse is

$$\begin{pmatrix} -\frac{z^3-1}{(z^2-z-2)(1+z)(z-2)} & \frac{2\,z^2-1}{(1+z)(z-2)^2} & -\frac{z^3-1}{z^4-2\,z^3-3\,z^2+4\,z+4} \\ -\frac{z}{(1+z)(z-2)} & 2\,(z-2)^{-1} & -\frac{z}{(1+z)(z-2)} \\ \frac{2\,z^3-2\,z^2-z+1}{(z-2)(z^3-3\,z-2)} & -\frac{3\,z^2-z-3}{(1+z)(z-2)^2} & \frac{2\,z^3-2\,z^2-z+1}{(z^2-z-2)(1+z)(z-2)} \end{pmatrix}$$

In particular, this implies that $A(z^q)^D = B(q)^D$ for any integer $q > 1$.

Example 8 *Let*

$$A(\lambda) = \begin{pmatrix} \lambda - z + 1 & 1 & 0 & 0 \\ z - 1 - z^{-1} & -1 + \lambda - z^{-1} & 0 & 0 \\ -\lambda + z^{-1} + z & -\lambda + z^{-1} + z & z & z \\ -z & -z & -z & -z \end{pmatrix} \in \mathcal{M}_4(\mathbb{C}(\lambda)(z)).$$

$\mathrm{index(A(\lambda))} = 2$ *and*

$$A(\lambda)^D = \begin{pmatrix} \frac{\lambda z - z - 1}{(\lambda z - 1)(\lambda - z)} & -\frac{z}{(\lambda z - 1)(\lambda - z)} & 0 & 0 \\ -\frac{z^2 - z - 1}{(\lambda z - 1)(\lambda - z)} & \frac{z(\lambda - z + 1)}{(\lambda z - 1)(\lambda - z)} & 0 & 0 \\ -\frac{z}{\lambda z - 1} & -\frac{z}{\lambda z - 1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $H_2 = (\lambda z - 1)(\lambda - z)$*. The triangularization* T_i *of* A^i*,* $i = 1, 2, 3$ *are*

$$T_1 = \begin{pmatrix} -z & -z & -z & -z \\ 0 & \lambda - z & -\frac{z^2 - z - 1}{z} & -\frac{z^2 - z - 1}{z} \\ 0 & 0 & \frac{\lambda z - 1}{z} & \frac{\lambda z - 1}{z} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; T_2 = \begin{pmatrix} -\frac{(\lambda z - 1)^2}{z^2} & -\frac{(\lambda z - 1)^2}{z^2} & 0 & 0 \\ 0 & (\lambda - z)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} -\frac{(\lambda z - 1)^3}{z^3} & -\frac{(\lambda z - 1)^3}{z^3} & 0 & 0 \\ 0 & (\lambda - z)^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $H_1 = (\lambda - z)(\lambda z - 1)$*. Therefore, for* $\lambda_0 \in \mathbb{C}(z) \setminus \{z, z^{-1}\}$*, one has that* $A(\lambda_0)^D = (A(\lambda_0))^D$*. For* $\lambda_0 = z$ *or* $\lambda_0 = z^{-1}$ *one can apply Algorithm-1.*

4 Weighted Drazin Inverses

The ideas developed in Sections 2 and 3 can be extended to the case of weighted Drazin inverses (see [9]). More precisely, let $A \in \mathcal{M}_{n\times m}(\mathbb{F})$ and $W \in \mathcal{M}_{m\times n}(\mathbb{F})$ and let $k = \mathrm{index(AW)}$. In order to approach the computation of the W-Weighted Drazin inverse $A^{D,W}$ of A, we consider a generic element $\hat{X}$ in $\mathcal{M}_{n\times m}(\mathbb{F})$, that, is

$\hat{X} = (x_{ij})_{1\leq i\leq n, j\leq m}$ where x_{ij} are new variables. $A^{D,W} \in \mathcal{M}_{n\times m}(\mathbb{F})$ is the unique solution of the algebraic system of equations, in the variables x_{ij},

$$(AW)^{k+1}\hat{X}W - (AW)^k = \mathbf{O},\ \hat{X}WAW\hat{X} - \hat{X} = \mathbf{O},\ AW\hat{X} - \hat{X}WA = \mathbf{O} \qquad (3)$$

where $\mathbf{O}$ is the null matrix. Reasoning as in Proposition 1, one deduces that

Proposition 9 *Let $\mathbb{L}$ be a field and $A \in \mathcal{M}_{n\times m}(\mathbb{L})$, $W \in \mathcal{M}_{m\times n}(\mathbb{L})$, then $A^{D,W} \in \mathcal{M}_{n\times m}(\mathbb{L})$.*

Reasoning as in Section 2, we get the following algorithm.

<u>Algorithm-1W</u>: $A \in \mathcal{M}_{n\times m}(\mathbb{C}(\mathbf{z}))$ and $W \in \mathcal{M}_{m\times n}(\mathbb{C}(\mathbf{z}))$ the algorithm computes its Drazin weighted inverse $A^{D,W}$.

1. Compute $k :=$ index(AW).
2. Solve the linear system $\mathcal{L} = \{(AW)^{k+1}\hat{X}W - (AW)^k = \mathbf{O}, AW\hat{X} - \hat{X}AW = \mathbf{O}\}$ and substitute its solution $\mathcal{S}$ in $\hat{X}WAW\hat{X} - \hat{X} = \mathbf{O}$. Let $XWAWX - X = \mathbf{O}$ be the resulting system and V the set of variables.
3. Compute a Gröbner basis $\mathcal{G}$ of the polynomials defining $XWAWX - X = \mathbf{O}$ with respect to a lexicographic order of V.
4. Substitute the solution provided by $\mathcal{G}$ and $\mathcal{S}$ in $\hat{X}$ to get $A^{D,W}$.

Example 10 *We consider the matrices*

$$A = \begin{pmatrix} z_1 & z_1 & z_1 & z_1 \\ z_2 & z_2 & z_2 & z_2 \\ 1 & 1 & 1 & 1 \end{pmatrix},\ W = \begin{pmatrix} {z_2}^{-1} & z_2 & z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The index of AW is 1. Applying Algorithm1W we get

$$A^{D,W} = \begin{pmatrix} \frac{z_1{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{z_1{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{z_1{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{z_1{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} \\ \frac{{z_2}^3}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{{z_2}^3}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{{z_2}^3}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{{z_2}^3}{({z_2}^3+{z_2}^2+z_1)^2} \\ \frac{{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} & \frac{{z_2}^2}{({z_2}^3+{z_2}^2+z_1)^2} \end{pmatrix}.$$

5 Empirical Analysis

The complexity for computing Gröbner bases is, in general, double exponential. In our case, since the ideal is zero-dimensional, the worst case complexity is $2^{\mathcal{O}(n^2-r)}$, where r is the rank of the matrix associated with the linear system $\mathcal{L}$ (see Theorem 3 in [8]). Nevertheless, in this section we show by means of some empirical experiments that the actual computing time of the algorithm has a good time performance. We consider three experiments. In the first one, the number of variable z_i is fixed, in fact 2, however, the order of the matrix, as well as the degree of its rational function entries, increase. In the second experiment, the degree of the rational function entries is fixed, in fact 2, however, the order of the matrix, as well as the number of variables, increase. The third experiment is similar to the second by neither the number of variables nor the degree is fixed. The computations has been done on Intel(R) Core(TM) i7-2630QM at 2.00GHz 16GB. Time is measured in second and the computations have been executed using the mathematical software Maple 17.

First experiment

We consider matrices constructed as follows. Let $J_{\rho,\ell}$ be the $\ell \times \ell$ Jordan block of $\rho \in \mathbb{C}(\mathbf{z})$; with 1 over the diagonal. Then, we take $A(n_1, n_2) = Q \cdot P \in \mathcal{M}_{n_1+n_2}(\mathbb{C}(z_1, z_2))$ where

$$P = \begin{pmatrix} J_{z_2,n_1} & M \\ \mathbf{O} & J_{0,n_2} \end{pmatrix},$$

$M = (m_{i,j})$ is such that $m_{i,j} = 1, \forall i, j$, and $Q = (q_{i,j}) \in \mathcal{M}_{n_1+n_2}(\mathbb{C}(\mathbf{z}))$ is such that $q_{i,i} = 0, q_{i,j} = i/z_2$ if $i < j$ and $q_{i,j} = jz_1z_2$ if $i > j$. For instance, for $n_1 = n_2 = 2$ we have

$$P = \begin{pmatrix} z_1 & 1 & 1 & 1 \\ 0 & z_1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 2\,z_2z_1 & 3\,z_2z_1 & 4\,z_2z_1 \\ 2\,{z_2}^{-1} & 0 & 3\,z_2z_1 & 4\,z_2z_1 \\ 3\,{z_2}^{-1} & 3\,{z_2}^{-1} & 0 & 4\,z_2z_1 \\ 4\,{z_2}^{-1} & 4\,{z_2}^{-1} & 4\,{z_2}^{-1} & 0 \end{pmatrix}$$

and

$$A(2,2) = \begin{pmatrix} 0 & 2\,{z_1}^2z_2 & 2\,z_2z_1 & 5\,z_2z_1 \\ 2\,\frac{z_1}{z_2} & 2\,{z_2}^{-1} & 2\,{z_2}^{-1} & 2\,{z_2}^{-1}+3\,z_2z_1 \\ 3\,\frac{z_1}{z_2} & 3\,{z_2}^{-1}+3\,\frac{z_1}{z_2} & 6\,{z_2}^{-1} & 6\,{z_2}^{-1} \\ 4\,\frac{z_1}{z_2} & 4\,{z_2}^{-1}+4\,\frac{z_1}{z_2} & 8\,{z_2}^{-1} & 12\,{z_2}^{-1} \end{pmatrix}.$$

Table 5 shows the computing times for determining $A(i,j)^D$ when $i, j \in \{1, 2, 3, 4\}$, as well as the degree and number of terms in $A(i,j)^D$. More precisely, if $A(i,j)^D = (\delta_{k_1,k_2})_{1\leq k_1,k_2\leq i+j} \in \mathcal{M}_{i+j}(\mathbb{C}(\mathbf{z}))$ then we denote by $\deg(A^D_{i,j}) = \max\{\deg_{\mathbf{z}}(\delta_{k_1,k_2}) \,|\, 1 \leq k_1, k_2 \leq i+j\}$, where the degree of a rational function is the maximum of its numerator degree and its denominator degree. Also, we denote by terms$(A(i,j)^D)$ the maximum number of non-zero terms in the rational functions entries of $A(i,j)^D$.

	time	N_1	N_2	N_3		time	N_1	N_2	N_3
(1,1)	0.078	2×2	2	3	(3,1)	0.468	4×4	26	39
(1,2)	0.063	3×3	8	6	(3,2)	1.513	5×5	26	54
(1,3)	0.140	4×4	12	10	(3,3)	5.257	6×6	28	63
(1,4)	0.281	5×5	18	14	(3,4)	11.248	7×7	34	72
(2,1)	0.078	3×3	18	14	(4,1)	3.323	5×5	34	76
(2,2)	0.281	4×4	18	25	(4,2)	4.821	6×6	34	95
(2,3)	0.951	5×5	20	33	(4,3)	16.957	7×7	36	107
(2,4)	2.215	6×6	26	39	(4,4)	45.677	8×8	42	119

Table 1: Results of Experiment 1: $N_1 = \text{order}(A(i,j))$, $N_2 = \deg(A^D_{i,j})$, $N_3 = \text{terms}(A(i,j)^D)$

Second experiment

Given natural numbers n, m, with $n > 2$, we consider matrices constructed as follows. Let $\Delta_{n-2} = (\delta_{i,j})$ be the $(n-2)\times(n-2)$ matrix where

$$\delta_{i,j} = i\sum_{\ell=1}^{m} z_\ell + \frac{j}{z_2},$$

and let $Q = (q_{i,j})$ be the $n\times n$ matrix where

$$q_{i,j} = -1 \text{ if } i > j \text{ and } q_{i,j} = 1 \text{ if } 1 \leq j.$$

Then, we take

$$A(m,n) = Q\cdot\begin{pmatrix} \Delta_{n-2} & \mathbf{O} \\ \mathbf{O} & J_{0,2}\end{pmatrix}\cdot Q^{-1} \in \mathcal{M}_n(\mathbb{C}(z_1,\ldots,z_m))$$

with $J_{0,2}$ the 2×2 Jordan block associated with 0. Table 5 shows the computing times of determining $A(i,j)^D$ when $i\in\{1,2,3,4,5,6,7,8,9,10\}$ and $j\in\{3,4,5\}$, as well as the number of variables and terms$(A(i,j)^D)$; note that the degree here is always 2.

Third experiment

We consider the same construction as in the previous experiment, but taking

$$\delta_{i,j} = i\sum_{\ell=1}^{m} z_\ell^i + \frac{j}{z_2}.$$

Table 5 shows the results of the experiment.

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	time	N_1	N_2	N_3		time	N_1	N_2	N_3
(1,3)	0.078	3×3	2	4	(6,3)	0.125	3×3	6	13
(1,4)	0.156	4×4	2	4	(6,4)	0.421	4×4	6	13
(1,5)	0.422	5×5	2	4	(6,5)	2.902	5×5	6	13
(2,3)	0.063	3×3	2	5	(7,3)	0.156	3×3	7	13
(2,4)	0.172	4×4	2	5	(7,4)	0.515	4×4	7	15
(2,5)	0.702	5×5	2	5	(7,5)	7.629	5×5	7	15
(3,3)	0.094	3×3	3	6	(8,3)	0.156	3×3	8	15
(3,4)	0.281	4×4	3	7	(8,4)	0.577	4×4	8	17
(3,5)	1.076	5×5	3	7	(8,5)	6.880	5×5	8	17
(4,3)	0.094	3×3	4	7	(9,3)	0.171	3×3	9	17
(4,4)	0.281	4×4	4	7	(9,4)	0.765	4×4	9	19
(4,5)	5.039	5×5	4	9	(9,5)	13.869	5×5	9	19
(5,3)	0.125	3×3	5	11	(10,3)	0.203	3×3	10	19
(5,4)	0.343	4×4	5	11	(10,4)	0.858	4×4	10	21
(5,5)	1.716	5×5	5	11	(10,5)	23.307	5×5	10	21

Table 2: Results of Experiment 2: $N_1 = \text{order}(A(i,j))$, N_2=Number of variables, $N_3 = \text{terms}(A(i,j)^D)$

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	time	N_1	N_2	N_3	N_4		time	N_1	N_2	N_3	N_4
(1,3)	0.062	3×3	2	4	2	(3,3)	0.094	3×3	3	34	7
(1,4)	0.218	4×4	2	5	3	(3,4)	0.328	4×4	3	34	7
(1,5)	1.014	5×5	2	11	6	(3,5)	62.603	5×5	3	71	7
(2,3)	0.063	3×3	2	11	6	(4,3)	0.093	3×3	4	71	7
(2,4)	0.249	4×4	2	11	6	(4,4)	0.437	4×4	4	71	7
(2,5)	3.276	5×5	2	34	7	(4,5)	185.751	5×5	4	121	7

Table 3: Results of Experiment 3: $N_1 = \text{order}(A(i,j))$, N_2=Number of variables, $N_3 = \text{terms}(A(i,j)^D)$, $N_4 = \deg(A(i,j)^D)$.

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