Infinite invertible arrowhead matrices and applications

J. Abderramán Marrero and Venancio Tomeo

Abstract

Motivated by current applications of the arrowhead matrices of large order, the infinite invertible arrowhead matrices are considered. A method based on a simple but suitable triangular factorization is proposed for obtaining, in the finite-dimensional case, a decomposition of the inverse in $O(n)$ time. This procedure can be applicable to infinite arrowhead matrices factored properly. Some illustrative examples are given.

Key words: Arrowhead matrix, infinite matrix, matrix inverse, matrix factorization.

1 Introduction

There exists a current and increasing interest in the use of the arrowhead matrices of large order in the theory of the hub matrices, with applications in networks, wireless communication, and the world wide web; see [8, 11] and the references therein. The arrowhead matrices are also of use in the symmetric eigenvalue problem [13], and some research was dedicated to their eigenvalues and eigenvectors; see e.g. [7, 10]. Infinite arrowhead matrices arise in physical applications [3, 5]. For details on infinite matrices see e.g. [4, 12].

Motivated by the applications, we proposed a simple but fruitful UDL factorization on the $n \times n$ nonsingular arrowhead matrices that enables us to decompose their inverses as diagonal plus a rank-one matrices in $O(n)$ time. Such factorization is extended to infinite invertible arrowhead matrices UDL factored properly so that associativity of multiplication of the matrices involved succeeds. Unlike the existence of many classical inverses associated to an infinite invertible matrix [4, 12], in particular for the Hessenberg matrices [1], and the tridiagonal matrices [2], we shall see that the classical inverse of an infinite invertible arrowhead matrix with an appropriate UDL factorization has a unique decomposition as a diagonal plus a rank-one matrix. In addition, we can use the limit of finite matrices for studying the infinite arrowhead matrices.

The material is organized as follows. In Section 2 we recall some basic results about inverses of finite arrowhead matrices. We manage a proper UDL triangular factorization for such matrices, focusing in the nonsingular ones. The case of arrowhead matrices with an only zero on their main diagonal is also considered. In Section 3 we study the factorization and inversion of infinite arrowhead matrices. Throughout the text the results are illustrated with appropriate examples. Section 4 is devoted to outline some applications of the arrowhead matrices. Bixon-Jortner's matrix illustrates such applications.

2 Finite arrowhead matrices

An $n \times n$ real or complex matrix A is called *arrowhead* if it is of the form

$$A = \left(\begin{array}{c|cccc} b_0 & c_1 & c_2 & \cdots & c_{n-1} \\ \hline a_1 & b_1 & 0 & \cdots & 0 \\ a_2 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & b_{n-1} \end{array}\right), \tag{1}$$

in short $A = \{a_i, b_i, c_i\}_n$. If A has the zero entries $a_i = 0$, $i = 1, 2 \ldots, n-1$, it is called an *upper triangular arrowhead matrix*. Alternatively, if A has the zero entries $c_i = 0$, $i = 1, 2 \ldots, n-1$, then A is called a *lower triangular arrowhead matrix*. Triangular arrowhead matrices have interesting properties: If A, B are $n \times n$ upper triangular arrowhead matrices over $\mathbb{K}$, a real or complex field, and $\lambda \in \mathbb{K}$, then $A+B$, λA and AB are also upper triangular arrowhead matrices. If A is a nonsingular upper triangular arrowhead matrix, its matrix inverse $A^{-1} = (d_{ij})_{i,j=1}^n$ is also arrowhead and upper triangular. Its entries are given by

$$d_{ij} = \begin{cases} 1/b_i, & \text{if } i = j, \\ -c_i/b_i, & \text{if } i = 1 \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

The same properties are verified for lower triangular arrowhead matrices. A triangular arrowhead matrix is nonsingular if its determinant is nonzero, $\prod_{i=0}^{n-1} b_i \neq 0$. The proofs of these sentences are trivial.

Expanding successively by its first row or column the determinant of A, $|A|$, a nonsingular arrowhead matrix as given in (1), with $b_j \neq 0$, $j = 1, 2, \ldots, n-1$, we obtain,

$$|A| = b_0 \prod_{i=1}^{n-1} b_i - \sum_{i=i}^{n-1} a_i c_i \prod_{\substack{j=2 \\ j \neq i}}^{n-1} b_j = \prod_{j=1}^{n-1} b_j \left[b_0 - \sum_{i=i}^{n-1} \frac{a_i c_i}{b_i} \right] \neq 0. \tag{2}$$

Theorem 1. *Every nonsingular arrowhead matrix* $A = \{a_i, b_i, c_i\}_n$*, with* $b_i \neq 0$*,* $i = 1, 2, ..., n-1$*, has a* UDL *factorization of the form*

$$A = UDL = \begin{pmatrix} 1 & \frac{c_1}{b_1} & \frac{c_2}{b_2} & \cdots & \frac{c_{n-1}}{b_{n-1}} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} u & & & & \\ & b_1 & & & \\ & & b_2 & & \\ & & & \ddots & \\ & & & & b_{n-1} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ \frac{a_1}{b_1} & 1 & & & \\ \frac{a_2}{b_2} & & 1 & & \\ \vdots & & & \ddots & \\ \frac{a_{n-1}}{b_{n-1}} & & & & 1 \end{pmatrix}, \tag{3}$$

where the entry u *is nonzero,*

$$u = b_0 - \sum_{i=1}^{n-1} \frac{a_i c_i}{b_i} \neq 0. \tag{4}$$

Proof. Considering the value of u, the product of those matrices is trivially A. The proof for uniqueness is simple: if U', D' and L' are matrices as in the establishment and $U'D'L' =$

$$= \begin{pmatrix} 1 & u_1 & u_2 & \cdots & u_{n-1} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} d_0 & & & & \\ & d_1 & & & \\ & & d_2 & & \\ & & & \ddots & \\ & & & & d_{n-1} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ l_1 & 1 & & & \\ l_2 & & 1 & & \\ \vdots & & & \ddots & \\ l_{n-1} & & & & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} d_0 + \sum_{i=1}^{n-1} d_i u_i l_i & b_{12} & b_{13} & \cdots & b_{1n} \\ d_1 l_1 & d_1 & & & \\ d_2 l_1 & & d_2 & & \\ \vdots & & & \ddots & \\ d_{n-1} l_{n-1} & & & & d_{n-1} \end{pmatrix} = A.$$

Therefore for $i = 1, 2, ..., n-1$, we have $d_i = b_i$, $u_i = c_i/b_i$, $l_i = a_i/b_i$. Since A is nonsingular,

$$d_0 = b_0 - \sum_{i=1}^{n-1} d_i u_i l_i = b_0 - \sum_{i=1}^{n-1} \frac{a_i c_i}{b_i} \quad \Leftrightarrow \quad d_0 = u \neq 0.$$

Thus the proposed triangular factorization is unique. □

The matrices U, D, and L, are invertible and the matrix inverse A^{-1} can be factored trivially,

Corollary 1. *The inverse A^{-1} of a nonsingular arrowhead matrix $A = \{a_i, b_i, c_i\}_n$ satisfying Theorem 1 has a triangular factorization of the form, $A^{-1} = L^{-1}D^{-1}U^{-1} =$*

$$= \begin{pmatrix} 1 & & & & \\ \frac{-a_1}{b_1} & 1 & & & \\ \frac{-a_2}{b_2} & & 1 & & \\ \vdots & & & \ddots & \\ \frac{-a_{n-1}}{b_{n-1}} & & & & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{u} & & & & \\ & \frac{1}{b_1} & & & \\ & & \frac{1}{b_2} & & \\ & & & \ddots & \\ & & & & \frac{1}{b_{n-1}} \end{pmatrix} \begin{pmatrix} 1 & \frac{-c_1}{b_1} & \frac{-c_2}{b_2} & \cdots & \frac{-c_{n-1}}{b_{n-1}} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}. \quad (5)$$

Taking the matrix product (5), we obtain a decomposition of A^{-1} with complexity $O(n)$,

Corollary 2. *The inverse A^{-1} of a nonsingular arrowhead matrix $A = \{a_i, b_i, c_i\}_n$ satisfying Theorem 1, can be decomposed in $O(n)$ time as a determined diagonal plus rank-one matrix,*

$$A^{-1} = \begin{pmatrix} 0 & & & & \\ & \frac{1}{b_1} & & & \\ & & \frac{1}{b_2} & & \\ & & & \ddots & \\ & & & & \frac{1}{b_{n-1}} \end{pmatrix} + \frac{1}{u} \begin{pmatrix} 1 \\ \frac{-a_1}{b_1} \\ \frac{-a_2}{b_2} \\ \vdots \\ \frac{-a_{n-1}}{b_{n-1}} \end{pmatrix} \begin{pmatrix} 1 & \frac{-c_1}{b_1} & \frac{-c_2}{b_2} & \cdots & \frac{-c_{n-1}}{b_{n-1}} \end{pmatrix}. \quad (6)$$

Example 1. *Given the arrowhead matrix*

$$A = \begin{pmatrix} -9 & -1 & 1 & 3 & 5 \\ -4 & 2 & & & \\ -1 & & 1 & & \\ 6 & & & 3 & \\ -3 & & & & 1 \end{pmatrix},$$

we obtain readily $u = -1$. The UDL factorization of A is trivial. Also for $A^{-1} = L^{-1}D^{-1}U^{-1}$. The diagonal plus rank-one matrix decomposition of A^{-1} is

$$\begin{aligned} A^{-1} &= \begin{pmatrix} 0 & & & & \\ & \frac{1}{2} & & & \\ & & 1 & & \\ & & & \frac{1}{3} & \\ & & & & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & -1 & -1 & -5 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\frac{1}{2} & 1 & 1 & 5 \\ -2 & -\frac{1}{2} & 2 & 2 & 10 \\ -1 & -\frac{1}{2} & 2 & 1 & 5 \\ 2 & 1 & -2 & -\frac{5}{3} & -10 \\ -3 & -\frac{3}{2} & 3 & 3 & 16 \end{pmatrix}. \end{aligned}$$

2.1 Nonsingular arrowhead matrices with a zero diagonal entry

If an arrowhead matrix has two zeros at least on its main diagonal, it cannot be inverted because (2) is not true. However, an arrowhead matrix with solely a zero entry on its main diagonal, e.g. entry $[j,j]$, and $a_j \neq 0$, $c_j \neq 0$, can be inverted, but its inverse cannot be decomposed as in (5) and (6). Nevertheless, the inversion of such class of nonsingular arrowhead matrices is well-known and simple.

Proposition 1. *Let $A = \{a_i, b_i, c_i\}_n$ be an $n \times n$ nonsingular arrowhead matrix with one and only one zero entry, $b_j = 0$, $j \neq 0$, on its main diagonal,*

$$A = \left(\begin{array}{cccc|c|cc} b_0 & c_1 & c_2 & \cdots & c_j & \cdots & c_{n-1} \\ a_1 & b_1 & & & & & \\ a_2 & & b_2 & & & & \\ \vdots & & & \ddots & & & \\ \hline a_j & & & & 0 & & \\ \hline \vdots & & & & & \ddots & \\ a_{n-1} & & & & & & b_{n-1} \end{array}\right).$$

Its inverse A^{-1} is given by

$$A^{-1} = \left(\begin{array}{cccc|c|cc} 0 & 0 & 0 & \cdots & \frac{1}{a_j} & \cdots & 0 \\ 0 & \frac{1}{b_1} & 0 & \cdots & -\frac{a_1}{a_j b_1} & \cdots & 0 \\ 0 & 0 & \frac{1}{b_2} & \cdots & -\frac{a_2}{a_j b_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \hline \frac{1}{c_j} & -\frac{c_1}{c_j b_1} & -\frac{c_2}{c_j b_2} & \cdots & d_j & \cdots & -\frac{c_{n-1}}{c_j b_{n-1}} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{a_{n-1}}{a_j b_{n-1}} & \cdots & \frac{1}{b_{n-1}} \end{array}\right),$$

where the entry d_j is

$$d_j = \frac{|A_{j+1,j+1}|}{|A|} = \frac{b_0 \prod\limits_{k=1,k\neq j}^{n-1} b_k - \sum\limits_{i=1}^{n-1} a_i c_i \prod\limits_{k=1,k\neq j,k\neq i}^{n-1} b_k}{-a_j c_j \prod\limits_{k=1,k\neq j}^{n-1} b_k}.$$

Proof. Taking the product, $AA^{-1} = A^{-1}A = I_n$, the identity matrix of order n. □

3 A UDL factorization for infinite arrowhead matrices

Given an infinite matrix of complex numbers $A = (a_{ij})_{i,j=1}^{\infty}$, the matrix $B = (b_{ij})_{i,j=1}^{\infty}$ is a *classical inverse*, or *two-sided inverse*, of A if both A and B satisfy $AB = BA = I$. It is well known that an infinite matrix can have not classical inverse. Also, an infinite invertible matrix can have two classical inverses [12, 1, 2], and then infinite many classical inverses, because if B' and B'' are classical inverses of A, then $\alpha B' + (1-\alpha)B''$ is also a classical inverse of A, for every $\alpha \in \mathbb{C}$.

The infinite arrowhead matrices are of the form

$$A = \left(\begin{array}{c|cccc} b_0 & c_1 & c_2 & c_3 & \cdots \\ \hline a_1 & b_1 & 0 & 0 & \cdots \\ a_2 & 0 & b_2 & 0 & \cdots \\ a_3 & 0 & 0 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \tag{7}$$

and they are denoted as $A = \{a_i, b_i, c_i\}$. We also assume $b_i \neq 0$, $i \in \mathbb{Z}^+$.

Theorem 2. *Let $A = \{a_i, b_i, c_i\}$ be an infinite invertible arrowhead matrix, and $u = b_0 - \sum\limits_{i=1}^{\infty} \frac{a_i c_i}{b_i}$ is a nonzero complex number. Assume the sequences (a_i), (c_i), $\left(\frac{a_i}{b_i}\right)$, and $\left(\frac{c_i}{b_i}\right)$ belong to $\mathbb{C}^\infty$, the space of the complex square summable sequences. The matrix A can be UDL factored in the form*

$$A = \begin{pmatrix} 1 & \frac{c_1}{b_1} & \frac{c_2}{b_2} & \frac{c_3}{b_3} & \cdots \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} u & & & & \\ & b_1 & & & \\ & & b_2 & & \\ & & & b_3 & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & & \\ \frac{a_1}{b_1} & 1 & & & \\ \frac{a_2}{b_2} & & 1 & & \\ \frac{a_3}{b_3} & & & 1 & \\ \vdots & & & & \ddots \end{pmatrix}. \tag{8}$$

Proof. Taking the matrix product UDL, we have

$$(UD)L = U(DL) = \begin{pmatrix} u + \frac{a_1 c_1}{b_1} + \frac{a_2 c_2}{b_2} + \frac{a_3 c_3}{b_3} + \cdots & c_1 & c_2 & c_3 & \cdots \\ a_1 & b_1 & & & \\ a_2 & & b_2 & & \\ a_3 & & & b_3 & \\ \vdots & & & & \ddots \end{pmatrix}, \tag{9}$$

but entry $[1,1]$ is $u + \sum\limits_{i=1}^{\infty} \frac{a_i c_i}{b_i} = b_0$. Therefore, the matrix product is $UDL = A$. $\square$

Corollary 3. *Let $A = \{a_i, b_i, c_i\}$ be an infinite invertible arrowhead matrix under the assumptions of Theorem 2. Its classical inverse A^{-1} is factored in the form $A^{-1} = L^{-1}D^{-1}U^{-1}$,*

$$A^{-1} = \begin{pmatrix} 1 & & & & \\ \frac{-a_1}{b_1} & 1 & & & \\ \frac{-a_2}{b_2} & & 1 & & \\ \frac{-a_3}{b_3} & & & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{u} & & & & \\ & \frac{1}{b_1} & & & \\ & & \frac{1}{b_2} & & \\ & & & \frac{1}{b_3} & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & \frac{-c_1}{b_1} & \frac{-c_2}{b_2} & \frac{-c_3}{b_3} & \cdots \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}. \qquad (10)$$

Such a factorization for A^{-1} is unique.

Proof. The product of these matrices is associative, $AA^{-1} = (UDL)(L^{-1}D^{-1}U^{-1}) = I$, and $A^{-1}A = (L^{-1}D^{-1}U^{-1})(UDL) = I$, then A^{-1} has a unique $L^{-1}D^{-1}U^{-1}$ factorization. □

Uniqueness of the classical inverse A^{-1} is an interesting question because other infinite matrices has infinite many classical inverses, as Hessenberg, tridiagonal and pentadiagonal matrices. The class of infinite invertible arrowhead matrices UDL factored properly is analogous to the class of infinite invertible triangular matrices or diagonal matrices having a unique classical inverse.

Taking the product $L^{-1}D^{-1}U^{-1}$, we obtain a decomposition for the inverses of arrowhead matrices analogous to the finite-dimensional case.

Corollary 4. *Let $A = \{a_i, b_i, c_i\}$ be an infinite invertible arrowhead matrix under the assumptions of Theorem 2. Its classical inverse A^{-1} can be decomposed as a diagonal plus a rank-one matrix of the form*

$$A^{-1} = \begin{pmatrix} 0 & & & & \\ & \frac{1}{b_1} & & & \\ & & \frac{1}{b_2} & & \\ & & & \frac{1}{b_3} & \\ & & & & \ddots \end{pmatrix} + \frac{1}{u} \begin{pmatrix} 1 \\ \frac{-a_1}{b_1} \\ \frac{-a_2}{b_2} \\ \frac{-a_3}{b_3} \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & \frac{-c_1}{b_1} & \frac{-c_2}{b_2} & \frac{-c_3}{b_3} & \cdots \end{pmatrix}. \qquad (11)$$

Example 2. *Given the infinite arrowhead matrix*

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ \frac{1}{4} & \frac{1}{8} & & & \\ \frac{1}{8} & & \frac{1}{16} & & \\ \frac{1}{16} & & & \frac{1}{32} & \\ \vdots & & & & \ddots \end{pmatrix}, \text{ with the value for } u = \frac{1}{4} - \sum_{i=1}^{\infty} \frac{a_i c_i}{b_i} = -\frac{1}{2}.$$

The UDL factorization for A and the $L^{-1}D^{-1}U^{-1}$ factorization for A^{-1} are straightforward. The decomposition for A^{-1} of Corollary 4 is

$$A^{-1} = \begin{pmatrix} 0 & & & & \\ & 8 & & & \\ & & 16 & & \\ & & & 32 & \\ & & & & \ddots \end{pmatrix} - 2 \begin{pmatrix} 1 \\ -2 \\ -2 \\ -2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 & -1 & \cdots \end{pmatrix}.$$

3.1 Infinite invertible arrowhead matrices with a zero diagonal entry

For an infinite invertible arrowhead matrix with one and only one zero entry on its main diagonal, the lack of the triangular factorization (8) and the representation of its classical inverse is completely analogous to the finite case given in Proposition 1. For such class of infinite arrowhead matrices, nothing is said about uniqueness of their classical inverses.

Example 3. *The infinite invertible arrowhead matrix*

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ \frac{1}{2} & 1 & & & & \\ \frac{1}{4} & & 1 & & & \\ \frac{1}{8} & & & 0 & & \\ \frac{1}{16} & & & & 1 & \\ \vdots & & & & & \ddots \end{pmatrix},$$

has a classical inverse

$$A^{-1} = \begin{pmatrix} 0 & & & 8 & & & \\ & 1 & & -4 & & & \\ & & 1 & -2 & & & \\ 8 & -4 & -2 & d_3 & -\frac{1}{2} & -\frac{1}{4} & \cdots \\ & & & -\frac{1}{2} & 1 & & \\ & & & -\frac{1}{4} & & 1 & \\ & & & \vdots & & & \ddots \end{pmatrix}, \text{ with the entry } d_3 = \frac{|A_{44}|}{|A|} = \frac{-131}{64}.$$

4 Some applications of the arrowhead matrices

Some interesting applications of the finite arrowhead matrices are outlined. The arrowhead matrices appear in the modeling of the properties of wireless communication and some network systems. The capacity of such systems are related and determined by the eigenstructure of an arrowhead matrix $A = B^T B$; where B is a hub matrix. The hub matrix B

can also represent hyperlinks of a web page ranking, or a channel matrix, e.g. the trivial matrix pair

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 5 & 2 & 1 & 2 \\ 2 & 2 & & \\ 1 & & 1 & \\ 2 & & & 2 \end{pmatrix},$$

where B is a hub matrix, and $A = B^T B$ is its associated arrowhead matrix. A hub matrix theory has been proposed in [8, 11], with applications to multiple-input and multiple-ouput (MIMO) communications systems. The results given here enable us to consider the arrowhead matrices in potential applications on infinite-dimensional networks and systems.

The arrowhead matrices have been also applied on some problems of modern physics. We shall illustrate with an infinite eigenvalue problem in intramolecular radiationless transitions [3]. The arrowhead matrices also appear in the study of localized vibrational modes in Fermi liquids [5], and in engineering realizations of spin chains for efficient state transfers in quantum information devices [9].

4.1 The Bixon-Jortner matrix

The arrowhead matrices appear in mathematical physics by the use of a self-adjoint operator (observable) T defined from an arrowhead matrix, the Bixon-Jortner matrix A; see [3], Equation (12). It is applied on the solution of the eigenvalue problem, $H\Psi_n = T\Psi_n = E_n\Psi_n$ of the energy levels of the wavefunctions Ψ_n of vibronic states of two electronic states from a particular molecule. Such wavefunctions were expressed as $\Psi_n = \xi_n\psi_s + \sum_k \xi_k^n \psi_k$, linear combinations of vibronic states $\{\psi\}$ in the so-called adiabatic approximation, with energy levels uniformly spaced. The value $k = 0$ is given for the nearest state from below to ψ_s; see [3] for approximation details and notation. Using its matrix representation,

$$T\Psi_n = \left(\begin{array}{c|c} E_s & v' \\ \hline v & D_E \end{array}\right) = E_n\Psi_n, \quad \text{with} \quad D_E = \operatorname{diag}(E_s - \alpha, E_s - \alpha \pm k\epsilon),$$

with $k \in \mathbb{Z}^+$, $\alpha = E_s - E_0$, the sequence $v = (v_i)$ ($v_i =< T\psi_i, \psi_s > \neq 0$ of vibronic coupling matrix elements, and ϵ the constant energy gap between two consecutive levels of the vibronic states. The arrowhead operator T is unbounded because $|E_s - \alpha \pm k\epsilon|$ is not bounded when k increases. Condition $u = E_s - v_0^2/E_0 - \sum_{k=1}^{\infty} v_k^2/(E_0 \pm k\epsilon) \neq 0$ is assumed.

For regular values E_n^*, the resolvent $(T - E_n^* I)^{-1}$ is

$$(T - E_n^* I)^{-1} = \operatorname{diag}(0,\ D_{E-E_n^*}^{-1}) + \frac{1}{u} < \cdot, \overline{x}_0 > x_0, \text{ with } x_0 = \left(1 - \left(D_{E-E_n^*}^{-1} v\right)'\right)' \in \mathcal{H}.$$

Since $(T - E_n^* I)^{-1} = \operatorname{diag}(0, D_{E-E_n^*}^{-1}) + F_1$ is a sum of two self-adjoint compact operators, one diagonal and F_1 a finite-rank operator, such resolvent (as well as T^{-1}) are self-adjoint compact operators. Since T^{-1} is compact, we can establish some basic properties, with $\xi \in \mathbb{C}$ so that $||x_{E_n}|| = 1$. It is a particular case of Theorem 6.1 from [6].

Corollary 5. *Let T be an unbounded self-adjoint arrowhead operator densely defined (from Bixon-Jortner's matrix) in $\mathcal{D}(T) \subset \mathcal{H}$, an adequate infinite Hilbert space. The following statements hold:*

a) *T is simple and has infinitely many countable simple eigenpairs $\{E_n, x_{E_n}\}$. The set $\{x_{E_n}\}$ is a total orthonormal set (basis) of $\mathcal{H}$ and $\lim_{n\to\infty} |E_n| = \infty$. The space Span$(x_{E_n})$ is the one-dimensional eigenspace associated with E_n, $n = 1, 2, \ldots$.*

b) $\mathcal{D}(T) = \{x \in \mathcal{H} : \sum_n |E_n|^2 | < x, x_{E_n} > |^2 < \infty\}$.

c) $Tx = \sum_n E_n < x, x_{E_n} > x_{E_n}$, $x \in \mathcal{D}(T)$.

Between two nearer entries of the diagonal of D_E there is one and only one eigenvalue of T, and Cauchy's interlacing property for the eigenvalues also holds.

Acknowledgements

This work has been partially supported by a research grant of the UPM-CAM in Madrid, Spain.

References

[1] J. Abderramán Marrero, V. Tomeo, E. Torrano, *On inverses of infinite Hessenberg matrices*, J. Comp. Appl. Math. **275** (2015) 356-365.

[2] J. Abderramán Marrero, V. Tomeo, E. Torrano, *Inversion of infinite tridiagonal matrices*, Proceedings of the 14th International Conference on Computational and Mathematical Methods in Science and Engineering , CMMSE 2014, Vol. I, 5-16.

[3] M. Bixon, J. Jortner, *Intramolecular radiationless transitions*, J. Chem. Phys. **48** (1968) 715-726.

[4] R. G. Cooke, *Infinite matrices and sequence spaces*, Dover Publications, New York, U.S.A. 1965.

[5] J. M. Gadzuk, *Localized vibrational modes in Fermi liquids, general theory*, Phys. Rev. B. **24** (1981) 1651-1663.

[6] I. Gohberg, S. Goldberg, M. A. Kaashoek, *Basic classes of linear operators*, Birkauser Verlag, Basel, Switzerland. 2003.

[7] N. Jakovcevic Stor, I. Slapnicar, J. L. Barlow, *Accurate eigenvalues decomposition of real symmeric arrowhead and applications*, Linear Algebra Appl. **464** (2015) 62-89.

[8] H. T. Kung, B. W. Sutter, *A hub matrix theory and applictions to wireless communications*, EURASIP J. Adv. Sgnal Process. (2007) Article ID 13659, 8 pages.

[9] D. Mogilevtsev, A. Maloshtan, S. Kilin, L. E. Oliveira, S. B. Cavalcanti, *Spontaneous emission and qubit transfer in spin-1/2 chains*, J. Phys. B **43** (2010) Article ID 095506, 10 pages.

[10] D. P. O'Leary, G. W. Stewart, *Computing the eigenvalues and eigenvectors of symmetric arrowhead matrices*, J. Comput. Phys. **90** (1990) 497-505.

[11] L. Shen, B. W. Sutter, *Bounds for eigenvalues of arrowhead matrices and their applications to hub matrices and wireless communications*, EURASIP J. Adv. Sgnal Process. (2009) Article ID 379402, 12 pages.

[12] P. N. Sivakumar, K. C. Shivakumar, *A review of infinite matrices and their applications*, Linear Algebra Appl. **430** (2009) 976-998.

[13] J. H. Wilkinson, *The algebraic eigenvalue problem*, Oxford University Press, New York, USA, 1965.