Solutions of general linear difference equations, nested sums and applications

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Abstract

The equivalence between the celebrated combinatorial and determinantal representations for the solutions of linear difference equations with variable coefficients, of both n-th and unbounded order, is considered. A more explicit representation is proposed using determined nested sums of their variable coefficients, which enables us to manage adequately all the sum of products involved in the previous representations. Some particular applications are also outlined.

Key words: Hessenbergian, linear difference equation, nested sum

1 Extended abstract

Closed representations for the solutions of linear difference equations with variable coefficients [3], LDE for short, of both finite and unbounded order are of interest in many branches of science and engineering. Some approaches for representing the solutions of LDE have been considered in the literature. Among these, most noteworthy have been the determinantal representations [4, 6], using hessenbergians [1] of submatrices of a single solution (Hessenberg) matrix, and the combinatorial one [5] based in determined combinations of sums of products of their variable coefficients.

The nested sums have resulted to be useful for obtaining explicit representations of complex combinatorial formulas, e.g. with binomial, Gaussian binomial, or Stirling-like coefficients. These nested structures have been applied on the expansion of transcendental functions and multiscale multiloop integrals, on orthogonal polynomials, linear non-autonomous area-preserving maps, representations for the inverses of tridiagonal matrices, and also on continued fractions; see [2] and the references therein. Relative to the LDE, the nested sums are suitable for the representations of the solutions of parameterized LDE [7], and of second order homogeneous LDE [2].

Our aim is to discuss the equivalence of the hessenbergian representation for the solutions of LDE with respect to the combinatorial one. It is also of use to establish more simple representations of such solutions. Therefore, the suitability of the nested sums to represent the solutions of LDE of both n-th and unbounded order is considered. As examples of their potential applications, compact representations for hessenbergians, inverses of triangular matrices, the multinomial distribution, and Roger-Szegö's polynomials, are also illustrated.

1.1 Compact representations for the solutions of LDE and their equivalence

A LDE of unbounded order can be formulated without loss of generality in the form [4],

$$\sum_{i=1}^{k} p(k,i) y_i = f(k), \tag{1}$$

where the coefficients $p(k,i)$, and the nonhomogeneous terms $f(k)$, are known functions. Here the coefficients $p(k,k)$ satisfy $p(k,k) \neq 0$, for every $k \in \mathbb{Z}^+$. Since y_n depends only on $y_1, y_2, \cdots, y_{n-1}$, the first n equations allow us to attain y_n. Defining the ratios $x_i = \frac{f(i)}{p(i,i)}$, $b_{ki} = -\frac{p(k,i)}{p(k,k)}$, a determinantal representation for the solution, completely analogous to the given in [4], is

$$y_n = \sum_{i=1}^{n} C_{i,1}^{(n)} x_i, \quad \text{with} \tag{2}$$

$$C_{i,1}^{(n)} = \begin{vmatrix} b_{i+1,i} & -1 & 0 & \dots & 0 \\ b_{i+2,i} & b_{i+2,i+1} & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ b_{n-1,i} & b_{n-1,i+1} & b_{n-1,i+2} & \dots & -1 \\ b_{n,i} & b_{n,i+1} & b_{n,i+2} & \dots & b_{n,n-1} \end{vmatrix}. \tag{3}$$

Notice $C_{n,1}^{(n)} = 1$. If we compare (2) and (3) with the combinatorial form given in [5], equations (2)-(3), the equivalence between both representations for the solutions of the difference equation (1) comes out.

Proposition 1. *The hessenbergians* (3) *have the compact representations,*

$$C_{i,1}^{(n)} = \begin{cases} b_{n,i} + \sum\limits_{j=2}^{n-i} \sum\limits_{\substack{(l_1,l_2,\cdots,l_j) \\ l_1,l_2,\cdots,l_j \geq 1 \\ l_1+l_2+\cdots+l_j=n-i}} b_{n,n-l_1} \left[\prod\limits_{m=2}^{j} b_{n-\sum\limits_{k=1}^{m-1} l_k \, , \, n-\sum\limits_{k=1}^{m} l_k} \right], & \text{if } 1 \leq i < n-1; \\ b_{n,n-1}, & \text{if } i = n-1; \\ 1, & \text{if } i = n. \end{cases} \quad (4)$$

The equivalence of the representation given in [4] and [5] for the solutions of the n-th order LDE will be also provided.

1.2 Representation of the solutions of LDE with nested sums

Theorem 1. *The hessenbergians* (3) *have the compact representations with nested sums,*

$$C_{i,1}^{(n)} = \begin{cases} b_{n,i} + \sum\limits_{j=2}^{n-i} \sum\limits_{k_1=i+j-1}^{k_0-1} \sum\limits_{k_2=i+j-2}^{k_1-1} \cdots \sum\limits_{k_{j-1}=i+1}^{k_{j-2}-1} \prod\limits_{m=1}^{j-1} b_{k_{m-1},k_m} b_{k_{j-1},i}, & \text{if } i \leq n-1; \\ \\ 1, & \text{if } i = n. \end{cases} \quad (5)$$

Although it presents a more delicate problem, a representation for the solutions of the n-th order LDE with nested sums will be also provided.

1.3 Some applications of the solutions of LDE

1.3.1 Stimulus-response relations among triangular matrices and their inverses

Stimulus-response relationships among the entries of triangular matrices and those of their inverses, in particular lower Toeplitz matrices $\mathbf{L}$, with $l_{ij} = l_{i+k,j+k} = t_{i-j}$, $i \geq j$, and 0 otherwise, can be fully controlled using nested sums and the symbolic *Maple*® package. Indeed, $(\mathbf{L}^{-1})_{ij} = \frac{1}{t_0} C_{1,1}^{(i-j+1)}$, if $i \geq j$, and 0 otherwise. Then we can apply (5) and *Maple*®.

1.3.2 The multinomial distribution

The multinomial expansion can be also handled with nested sums and *Maple*®,

$$\begin{aligned} \left(\sum_{r=1}^{n} p_r \right)^m &= \sum_{\substack{0 \leq l_1,l_2,\cdots,l_n \leq m \\ l_1+l_2+\cdots+l_n=m}} \frac{m!}{l_1!l_2!\cdots l_n!} p_1^{l_1} p_2^{l_2} \cdots p_n^{l_n} \\ &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_m=1}^{n} \prod_{j=1}^{m} p_{k_j}. \end{aligned} \quad (6)$$

1.3.3 Roger-Szegö's polynomials

Representations of orthogonal polynomials with nested sums are available [2]. We illustrate with Roger-Szegö's polynomials, also known as q-deformed Hermite polynomials, with $x = \frac{z}{-\sqrt{q}}$,

$$H_m(x;q) = \sum_{r=0}^{m} \left[\begin{array}{c} m \\ r \end{array} \right]_q x^r, \text{ where } \left[\begin{array}{c} m \\ r \end{array} \right]_q = \frac{(1-q)(1-q^2)\cdots(1-q^m)}{(1-q)\cdots(1-q^r)(1-q)\cdots(1-q^{m-r})},$$

are the q-binomial (Gaussian binomial) coefficients. These polynomials are of use in applied mathematics and they are q-orthogonal on the unit circle, $z = e^{i\varphi}$, with respect to the (weight) Jacobi function $\vartheta_3(q;\varphi)$. The three-term recurrence relation

$$H_{m+1}(x;q) = (x+1)H_m(x;q) + (q^m - 1)xH_{m-1}(x;q), \quad (m \geq 0)$$

with customary initial conditions $H_{-1}(x;q) = 0$, $H_0(x;q) = 1$, allows us to obtain its determinantal form. Hence, we can manage those using nested sums and symbolic computation,

$$H_m(x;q) = \sum_{r=0}^{m-1} \left[\sum_{k_1=0}^{r} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-r}=0}^{k_{m-r-1}} q^{k_1+k_2+\cdots+k_{m-r}} \right] x^r + x^m.$$

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