Numerical simulation of a climate energy balance model with continents distribution.

Lourdes Tello & Arturo Hidalgo
Universidad Politécnica de Madrid.
Some processes involved in global climate models:
Some processes involved in global climate models:
Physical problem

Solar radiation

Cooling

Equator (0,0)

Cooling

Pole S (-1,0)

Pole N (1,0)

(-1,-H)

(1,-H)

upwelling

downwelling

THE MODEL (Based on Watts-Morantine [1990])

- The model represents the evolution of temperature within an ocean of depth H.
- Spatial variables \((x, z)\): \(x = \sin(\text{latitude})\) and \(-z\) (depth).
- Spatial domain \(\Omega = (-1, 1) \times (0, -H)\)
- Boundary: \(\Gamma = \Gamma_H \cup \Gamma_0 \cup \Gamma_1 \cup \Gamma_{-1}\)

\[
\Gamma_H = \{(x, z) \in \overline{\Omega} : \ z = -H\}
\]
\[
\Gamma_0 = \{(x, z) \in \overline{\Omega} : \ z = 0\}
\]
\[
\Gamma_{-1} = \{(x, z) \in \overline{\Omega} : \ x = -1\}
\]
\[
\Gamma_1 = \{(x, z) \in \overline{\Omega} : \ x = 1\}
\]

- The model considers the average temperature over each parallel as the unknown.

The governing equation for the ocean interior is a heat equation with advective transport (DOM)

\[ U_t - \left( \frac{K_H}{R^2} (1 - x^2) U_x \right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in} \ (0, T) \times \Omega_i, \quad i = 1 \ldots N. \]

**U**: temperature,

**\omega**: vertical velocity,

**K_V**: vertical diffusivity,

**K_H**: horizontal diffusivity,

**R**: radius of the Earth.
Mathematical model

**Boundary condition for ocean bottom**

\[ \omega x U_x + K_v U_z = 0, \quad \text{on} \quad \Gamma_H \times (0,T) \]

**Boundary condition for upper boundary: Energy Balance Model (EBM)**

\[
Du_t - \frac{DK_H}{R^2} \left( (1 - x^2)^2 \left| u_x \right|^{p-2} u_x \right)_x + Bu + C + K_v \frac{\partial U}{\partial z} + \omega x u_x \in \frac{1}{\rho c} QS(x) \beta(u)
\]

\[ \text{on} \quad \Gamma_0 \times [0,T] \]

* u: temperature,
* \( \omega \): velocity,
* \( K_v \): vertical diffusivity,
* \( K_{H0} \): horizontal diffusivity,
* R: radius of the Earth.

**Symbols:**
* D: thickness mixed layer,
* \( \rho \): density,
* c: specific heat coeff.,
* \( \beta(u) \): coalbedo,
* Q: solar constant,
* Bu+C: cooling term,
* S(x): insolation.

1) We consider the case $p=3$ (Stone, 1972)

2) We use the coalbedo, $\beta(u)$, (Budyko model)
Mathematical model

\[
U_t - \left( \frac{K_H}{R^2} (1 - x^2) U_x \right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } \Omega \times (0, T),
\]
\[
\omega x U_x + K_V U_z = 0 \quad \text{in } \Gamma_H \times (0, T),
\]
\[
Du_t - \frac{D K_{H_0}}{R^2} \left( (1 - x^2)^{3/2} \left| u_x \right| u_x \right)_x + K_V \frac{\partial U}{\partial z} + \omega u U_x + Bu + C \in \frac{1}{\rho c} QS(x) \beta(u)
\]
on \Gamma_0 \times (0, T),

\[
U_x = 0, \quad \text{on } \Gamma_{-1} \times [0, T],
\]
\[
U_x = 0, \quad \text{in } \Gamma_1 \times [0, T],
\]
\[
U(x, z, 0) = U_0(x, z), \quad \text{in } \Omega,
\]
\[
u(x, 0) = u_0(x), \quad \text{in } \Gamma_0.
\]

**Final system**

Mathematical model

\[
\frac{K_v \, \partial U}{D \, \partial z}
\]

Represents the coupling atmosphere-ocean in the sense of analyzing the influence of the ocean temperature in the atmosphere. In this work we shall show results with and without this term.

Some references about global climate
EBM models with or without deep ocean effect:

- Watts & Morantine (1990),
- Xu (1990),
- Hetzer (1990),
- Kim, North & Huang (1992),
- Díaz (1993),
- Schmidt (1994),
- Díaz-Hernández-Tello (1997),
- Arcoya-Díaz-Tello (1998),
- Hetzer (2000),
- Díaz-Tello (2007),
- Bermejo et al (2008),
- ...
We rewrite this problem as advection-reaction-diffusion equations, both for the upper boundary EBM and for the DOM.

**EBM:**

\[
u_t - \left( f(x, u(x,t), u_x(x,t)) \right)_x = \sigma(x, u(x,t)) \frac{\partial U}{\partial z}(x,0,t)\]

with the flux:

\[
f(x, u(x,t), u_x(x,t)) := \frac{K H_0}{R^2} (1 - x^2)^{3/2} |u_x(x,t)| u_x(x,t) - \frac{w}{D} x u(x,t)\]

and the source term:

\[
\sigma(x, u, \frac{\partial U}{\partial n}) := \frac{1}{D} \left(-C + \frac{Q}{\rho c} S(x) \beta(u) + (\omega + x \omega_x - B) u(x,t) - K_v \frac{\partial U}{\partial z}\right)\]
Numerical approximation

**DOM:**

\[
U(x, z, t)_t - (F(x, U_x(x, z, t)))_x - (G(U(x, z, t), U_z(x, z, t)))_z = \Xi(x, U(x, z, t)),
\]

with the fluxes:

\[
F(x, U_x(x, z, t)) := \frac{K_H}{R^2}(1 - x^2)U_x(x, z, t),
\]

\[
G(U(x, z, t), U_z(x, z, t)) := K_VU_z(x, z, t) - wU(x, z, t),
\]

and the source term:

\[
\Xi(x, U(x, z, t)) := \omega_zU(x, z, t).
\]
Numerical approach: finite volume method with Weighted Essentially Non-Oscillatory (WENO) reconstruction in space and third-order Runge-Kutta TVD for time integration.

For each time step, we compute a numerical solution of the EBM model equation for each cell \( u_i^{n+1} \)

\[
Du_t - \frac{DKH_0}{R^2} \left( (1 - x^2)^{\frac{p}{2}} \left| u_x \right|^{p-2} u_x \right)_x + Bu + C + KV \frac{\partial U}{\partial z} + \omega xu_x \in QS(x)\beta(u)
\]

on \( \Gamma_0 \times [0, T] \)

then we use \( u_i^{n+1} \) as a Dirichlet boundary condition for the DOM to obtain \( U_{i,j}^{n+1} \)

\[
U_t - \left( \frac{KH}{R^2} (1 - x^2) \right) U_x (x) - KV U_{zz} + \omega U_z = 0 \quad \text{in } \Omega \times (0, T)
\]
The finite volume framework

Domain: $\Omega$

Upper boundary: $\Gamma_0$

(-1,0)  (1,0)

(-1,-H)  (1,-H)
The finite volume framework

We integrate the equation dividing by the length of the control volume to obtain the following ordinary differential equation (ODE)

\[
\frac{du_i(t)}{dt} = \frac{1}{\Delta x_i} \left( f_{i+1/2} - f_{i-1/2} \right) + \sigma_i(t) \equiv l_i(u(t)),
\]

where

\[
u_i(t) = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t) \, dx \quad \text{integral average of the unknown,}
\]

\[
f_{i+1/2} = f(x, u(x_{i+1/2}, t), u_x(x_{i+1/2}, t)) \quad \text{right intercell numerical flux,}
\]

\[
\sigma_i(t) = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \sigma(x,u, \frac{\partial U}{\partial z}) \, dx \quad \text{integral average of the source term.}
\]

The finite volume framework

We discretize the 2D domain $[-1,1] \times [0,-H]$ in $N_x \times N_z$ control volumes of area $\Delta x_i \times \Delta z_j$

$\Delta x_i = x_{i+1/2} - x_{i-1/2}$, $\Delta z_j = z_{j+1/2} - z_{j-1/2}$
The finite volume framework

We integrate the equation dividing by the area of the control volume to obtain the following ordinary differential equation (ODE)

\[
\frac{dU_{i,j}}{dt} = \frac{1}{\Delta x_i} \left( F_{i+1/2,j} - F_{i-1/2,j} \right) + \frac{1}{\Delta z_j} \left( G_{i,j+1/2} - G_{i,j-1/2} \right) + \Gamma_{i,j} \equiv L_{i,j}
\]

where

\[
U_{i,j} = \frac{1}{\Delta x_i \Delta z_j} \int_{z_{i-1/2}}^{z_{j+1/2}} \left( \int_{x_{i-1/2}}^{x_{i+1/2}} U(x,z,t) dx \right) dz,
\]

integral average of the unknown,

\[
F_{i+1/2,j} = \frac{1}{\Delta z_j} \int_{z_{j-1/2}}^{z_{j+1/2}} F \left( x_{i+1/2} , U_x \left( x_{i+1/2} , z , t \right) \right) dz,
\]

Spatial integral average of intercell fluxes,

\[
G_{i,j+1/2} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} G \left( U \left( x , z_{j+1/2} , t \right) , U_z \left( x , z_{j+1/2} , t \right) \right) dx,
\]

\[
\Gamma_{i,j} (t) = \frac{1}{\Delta x_i \Delta z_j} \int_{z_{j-1/2}}^{z_{j+1/2}} \left( \int_{x_{i-1/2}}^{x_{i+1/2}} \Xi(x,U(x,z,t)) dx \right) dz,
\]

integral average of the source term.
EBM:

\[ u^{k,1} = u^n + \Delta t l(u^n), \quad u^{k,2} = \frac{3}{4} u^n + \frac{1}{4} u^{k,1} + \frac{1}{4} \Delta t l(u^{k,1}), \]

\[ u^{n+1} = \frac{1}{3} u^n + \frac{2}{3} u^{k,2} + \frac{2}{3} \Delta t l(u^{k,2}). \]

DOM:

\[ U^{k,1} = U^n + \Delta t L(U^n), \]

\[ U^{k,2} = \frac{3}{4} U^n + \frac{1}{4} U^{k,1} + \frac{1}{4} \Delta t L(U^{k,1}), \]

\[ U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{k,2} + \frac{2}{3} \Delta t L(U^{k,2}). \]
WENO reconstruction

**EBM**

1) For intercell fluxes

For an order of accuracy $r$ we have $r$ candidate stencils each one of them with $r$ cells

$$\{S_{i-r+1}, S_{i-r+2}, \ldots, S_i\}, \{S_{i-r+2}, S_{i-r+3}, \ldots, S_{i+1}\}, \ldots, \{S_i, S_{i+1}, \ldots, S_{i+r-1}\}$$

For each stencil we consider a $(r-1)$th degree interpolating polynomial

$$p_l(x), \quad l = 0, \ldots, r - 1$$

Each one of the polynomials considered must be conservative:

$$\frac{1}{\Delta x_k} \int_{S_k} p_l(x)dx = u_k(t), \quad 0 \leq l \leq r - 1, \quad 0 \leq k \leq r - 1$$

Remark: In this work we have used $r=4$. Therefore, the candidate stencils are:

$$\{S_{i-3}, S_{i-2}, S_{i-1}, S_i\}, \{S_{i-2}, S_{i-1}, S_i, S_{i+1}\},$$

$$\{S_{i-1}, S_i, S_{i+1}, S_{i+2}\}, \{S_i, S_{i+1}, S_{i+2}, S_{i+3}\}.$$
Mapping using reference element:

\[
x = x_{i-1/2} + \Delta x_i \xi
\]
\[
z = z_{j-1/2} + \Delta z_j \eta
\]
Integrals are approximated using Gaussian numerical quadrature

\[
F_{i+1/2,j} = \frac{1}{\Delta z_j} \int_{z_{j-1/2}}^{z_{j+1/2}} F(x_{i+1/2}, z, t^n) \, dz = \int_0^1 \hat{F}(1, \eta, t^n) \, d\eta \approx \sum_{k=1}^{NG} \gamma \hat{F}(1, \beta_k, t^n)
\]

\[
F_{i-1/2,j} = \frac{1}{\Delta z_j} \int_{y_{j-1/2}}^{y_{j+1/2}} F(x_{i+1/2}, z, t^n) \, dz = \int_0^1 \hat{F}(0, \eta, t^n) \, d\eta \approx \sum_{k=1}^{NG} \gamma \hat{F}(0, \beta_k, t^n)
\]

\[
G_{i,j+1/2} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} G(x, z_{j+1/2}, t^n) \, dx = \int_0^1 \hat{G}(\xi, 1, t^n) \, d\xi \approx \sum_{k=1}^{NG} \gamma \hat{G}(\alpha_k, 1, t^n)
\]

\[
G_{i,j-1/2} = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} G(x, z_{j-1/2}, t^n) \, dx = \int_0^1 \hat{G}(\xi, 0, t^n) \, d\xi \approx \sum_{k=1}^{NG} \gamma \hat{G}(\alpha_k, 0, t^n)
\]
We must compute a unique flux at each control volume interface

\[
F_{i+1/2, j} = \frac{1}{2} (F_{i+1/2, j}^{i,j, \text{RIGHT}} + F_{i+1/2, j}^{i+1,j, \text{LEFT}})
\]

\[
G_{i+1/2, j} = \frac{1}{2} (G_{i, j+1/2}^{i,j, \text{UP}} + G_{i, j+1/2}^{i,j+1, \text{DOWN}})
\]

**REMARKS:**
1) When considering advective terms, other types of flux averaging or Riemann problem solutions must be introduced: Force, Rusanov, Osher, Roe…
2) In the diffusive averaging of the flux a term that accounts for the jump can be added.
If we use 2 integration points (NG=2) the values are:

\[
\gamma_{0,0}^{x,z} = \gamma_{1,1}^{x,z} = 1
\]

\[
\alpha_0 = \beta_0 = -\frac{\sqrt{3}}{3}
\]

\[
\alpha_1 = \beta_1 = \frac{\sqrt{3}}{3}
\]
In order to obtain the solution and gradients at Gaussian points we need to perform a reconstruction procedure. This give rise to a piecewise polynomial function whose restriction to cell $T_{ij}$ is the polynomial:

$$w_{ij}(\xi, \eta, t^n) = \sum_{k=1}^{M+1} \sum_{l=1}^{M+1} \hat{w}_{ij}^{k,l}(t^n) \phi_k(\xi) \phi_l(\eta)$$

which must be conservative:

$$\int_0^1 \int_0^1 w_{ij}(\xi, \eta, t^n) d\xi d\eta = Q^n_i \Rightarrow \sum_{k=1}^{M+1} \sum_{l=1}^{M+1} \hat{w}_{ij}^{k,l}(t^n) \int_0^1 \int_0^1 \phi_k(\xi) \phi_l(\eta) d\xi d\eta = Q^n_i$$

And its gradient

$$\begin{pmatrix}
\frac{\partial w_{ij}}{\partial \xi} \\
\frac{\partial w_{ij}}{\partial \eta}
\end{pmatrix} = \sum_{k=1}^{M+1} \sum_{l=1}^{M+1} \begin{pmatrix}
\hat{w}_{ij}^{k,l}(t^n) \phi_k'(\xi) \phi_l(\eta) \\
\hat{w}_{ij}^{k,l}(t^n) \phi_k(\xi) \phi_l'(\eta)
\end{pmatrix}$$
Substencils for obtaining nonlinear reconstruction operator

\[ S_{ij}^{s,x} = \bigcup_{p=i+s-e}^{i+s+e} T_{pj} \quad \text{and} \quad S_{ij}^{s,y} = \bigcup_{q=j+s-e}^{j+s+e} T_{qj} \]

If we consider 3 cells in the stencils: \( e=1 \) and \( s=-e \) for the left-sided stencil, \( s=0 \) for the central stencil and \( s=e \) for the right-sided stencil.

The total stencils are given by the union of the substencils

\[ S_{ij}^{x} = \bigcup_{s} S_{ij}^{s,x} \quad \text{and} \quad S_{ij}^{z} = \bigcup_{s} S_{ij}^{s,z} \]
WENO RECONSTRUCTION

$$S_{ij}^{s,x} = \bigcup_{e=i-L}^{i+R} I_{ej}, \quad S_{ij}^{s,z} = \bigcup_{e=j-L}^{j+R} I_{ei}$$

M=even
(M=2)

M=odd
(M=3)

Three stencils

Four stencils

M+1 cells in each stencil
Dimension-by-Dimension WENO reconstruction

Let us denote the one-dimensional polynomial WENO reconstruction in x direction as:

\[ \hat{\omega}_l(t^n) = R_x(T^n_{pj}), \quad T^n_{pj} \in S_{ij}^x \]

And the one-dimensional polynomial WENO reconstruction in z direction as:

\[ \hat{\omega}_l(t^n) = R_z(T^n_{iq}), \quad T^n_{iq} \in S_{ij}^z \]

Application of the reconstruction operator in x-direction to the cell averages yields the coefficients of the reconstruction polynomial in x-direction.

\[ \hat{\omega}_{ij}^{l,0}(t^n) = R_x(T^n_{pj}), \quad T^n_{pj} \in S_{ij}^x \]

Since the reconstruction operator in z-direction \( R_y(T^n_{iq}) \) acts on averages in z-direction, it can be applied to each single coefficient of the reconstruction polynomial in x-direction.

\[ \hat{\omega}_{ij}^{l,m}(t^n) = R_z(\hat{\omega}_{iq}^{l,0}(t^n)), \quad \forall 0 \leq l \leq NG, \quad T^n_{iq} \in S_{ij}^z \]
Dimension-by-Dimension WENO reconstruction

In this way we obtain all the necessary coefficients of the 2D tensor-product reconstruction polynomial

\[ w_{ij}(\xi, \eta, t^n) = \sum_{k=1}^{M+1} \sum_{l=1}^{M+1} \hat{w}_{ij}^{k,l}(t^n)\phi_k(\xi)\phi_l(\eta) \]

Remarks:
This way to proceed is different to the classical point-wise WENO approach, since entire polynomials are obtained instead of piecewise values (as in the original WENO Of Jiang and Shu).

Some references of polynomial WENO reconstruction:

“High order space-time adaptive ADER-WENO finite volume schemes for non-conservative hyperbolic systems“, M. Dumbser, A. Hidalgo, O. Zanotti

Dimension-by-Dimension WENO reconstruction

\[ \hat{u}(1, \beta_k, t^n) = \sum_{k=1}^{M+1} \omega_k w_{ij}(1, \beta_k, t^n); \quad u(0, \beta_k, t^n) = \sum_{k=1}^{M+1} \omega_k w_{ij}(0, \beta_k, t^n) \]

\[ \hat{u}(\alpha_k, 1, t^n) = \sum_{k=1}^{M+1} \omega_k w_{ij}(\alpha_k, 1, t^n); \quad u(\alpha_k, 0, t^n) = \sum_{k=1}^{M+1} \omega_k w_{ij}(\alpha_k, 0, t^n) \]

\[ \nabla \hat{u}(1, \beta_k, t^n) = \sum_{k=0}^{r-1} \omega_k \nabla w_{ij}(1, \beta_k, t^n); \quad \nabla \hat{u}(0, \beta_k, t^n) = \sum_{k=0}^{r-1} \omega_k \nabla w_{ij}(0, \beta_k, t^n) \]

\[ \nabla \hat{u}(\alpha_k, 1, t^n) = \sum_{k=0}^{r-1} \omega_k \nabla w_{ij}(\alpha_k, 1, t^n); \quad \nabla \hat{u}(\alpha_k, 0, t^n) = \sum_{k=0}^{r-1} \omega_k \nabla w_{ij}(\alpha_k, 0, t^n) \]
Numerical example without latent heat: \( \gamma(U)=U \)

**Physical parameters:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Scaled Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_H )</td>
<td>0.049</td>
</tr>
<tr>
<td>( K_{H0} )</td>
<td>0.555 \times 10^{-3}</td>
</tr>
<tr>
<td>( K_V )</td>
<td>0.0125</td>
</tr>
<tr>
<td>( C, B )</td>
<td>190, 2</td>
</tr>
<tr>
<td>( c, \rho )</td>
<td>1, 1</td>
</tr>
<tr>
<td>( Q )</td>
<td>340</td>
</tr>
<tr>
<td>( D )</td>
<td>60</td>
</tr>
</tbody>
</table>

\[ S = 1 - \frac{1}{2} P_2(x) \]

**Space and time discretization:**

\[ \Delta x = 2 / 60; \quad \Delta z = 1 / 60 \]

\[ \Delta t = \min \left( \alpha \Delta x^2 \left( (1 - x^2) K_H \right)^{-1}, \alpha \Delta z^2 \left( K_V \right)^{-1}, \alpha \Delta x^2 \left( (1 - x^2) K_{H0} \frac{du}{dx} \right)^{-1} \right), (\alpha = 0.3) \]

**Initial condition:**

\[ U(x, z, 0) = 18e^{-x^2-z^2} + 80e^{-x^2} - 60 \]
DOM solution. Output time = 1.0

WITH influence of deep ocean on atmosphere.

\[
\frac{K_v}{D} \frac{\partial U}{\partial n} \neq 0
\]

WITHOUT influence of deep ocean on atmosphere.

\[
\frac{K_v}{D} \frac{\partial U}{\partial n} = 0
\]

EBM solution. Output time = 1.0

Mathematical model with land-sea distribution

\[ U_t \left( -\frac{K_H}{R^2} (1 - x^2)U_x \right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in} \ (0,T) \times \Omega_i, \quad i = 1...N. \]

\[ Du_t - \frac{DK_{H_0}}{R^2} \left( (1 - x^2)^{p/2} |u_x|^{p-2} u_x \right)_x + \sum_{i=1}^{M} \left( \chi_{u_i} K_V \frac{\partial U}{\partial z} \right) + wxu_x + Bu + C = \]

\[ = \frac{1}{\rho c} QS(x)\beta(x,u) \quad \text{in} \ (0,T) \times \Gamma_0, \]

\[ wxU_x + K_V U_z = 0 \quad \text{in} \ (0,T) \times \Gamma_H^i, \quad i = 1...N. \]

\[ (1 - x^2)^{p/2} |u_x|^{p-2} u_x = 0 \quad \text{in} \ x = \pm 1, \]

\[ U_x = 0 \quad \text{in} \ ((0,T) \times \Gamma_{-1}^i) \cup ((0,T) \times \Gamma_1^i), \]

\[ U(0,x,z) = U_0(x,z) \quad \text{in} \ \Omega_i, \]

\[ U(0,x,0) = u_0(x) \quad \text{in} \ \Gamma_0. \]

**Final system**
Continental zones
Mathematical model with land-sea distribution

<table>
<thead>
<tr>
<th>Physical parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_H (m^2 c^{-1}))</td>
<td>0.049</td>
</tr>
<tr>
<td>(K_{H0} (m^2 c^{-1}))</td>
<td>(0.555 \times 10^{-3})</td>
</tr>
<tr>
<td>(K_V (m^2 c^{-1}))</td>
<td>0.0125</td>
</tr>
<tr>
<td>(C, B)</td>
<td>190.2</td>
</tr>
<tr>
<td>(Q (Wm^{-2}))</td>
<td>340</td>
</tr>
<tr>
<td>(c_w (J(kg^\circ C)^{-1}))</td>
<td>3900</td>
</tr>
<tr>
<td>(c_a (J(kg^\circ C)^{-1}))</td>
<td>1004</td>
</tr>
<tr>
<td>(\rho_w (kgm^{-3}))</td>
<td>1030</td>
</tr>
<tr>
<td>(\rho_a (kgm^{-3}))</td>
<td>1.225</td>
</tr>
</tbody>
</table>
Temperature in the deep ocean for $t=5$.

A) First ocean;  B) Second Ocean.
Solution for $t = 5$. **Full green line**: land-sea model; **dotted red line**: only Continental model; **dash-dotted blue line**: Only Ocean model

L. Tello y A. Hidalgo. IMAC2015. UJI.
March 30th. 2015.
Validation of the numerical scheme with land-sea distribution

Manufactured solution:

\[ U(t, x, z) = 10 \frac{(x^2 - 1)^2 x^2 (1 + z)^2}{1 + t} \]
Validation of the numerical scheme with land-sea distribution

\[ U_t - \left( \frac{K_H}{R^2} (1 - x^2) U_x \right)_x - K_V U_{zz} + \omega U_z = \Phi(t, x, z), \quad \text{in } (0,T) \times \Omega_i, \quad i = 1, 2 \]

\[ wxU_x + K_V U_z = 0 \quad \text{in } (0,T) \times \Gamma^i_H, \quad i = 1, 2 \]

\[ Du_t - \frac{DK_{H0}}{R^2} \left( (1 - x^2)^{3/2} |u_x| u_x \right)_x + \sum_{i=1}^{2} \left( \chi_{\sigma_i} K_V \frac{\partial U}{\partial z} \right) + wxU_x + C + Bu = \]

\[ = \frac{1}{\rho c} QS(x) \beta(x, u) + \psi(t, x), \]

\[ (1 - x^2)^{3/2} |u_x| u_x = 0 \quad \text{in } x = \pm 1, \]

\[ U_x = 0 \quad \text{in } ((0,T) \times \Gamma^i_{-1}) \cup ((0,T) \times \Gamma^i_1), \]

\[ U(0, x, z) = 200 \left( x^2 - 1 \right)^2 \left( x + 0.8 \right)^2 \left( x + 0.4 \right)^2 \left( x - 0.4 \right)^2 \left( x - 1 \right)^2 \left( 1 + z \right)^2, \]

\[ u(0, x) = 200 \left( x^2 - 1 \right)^2 \left( x + 0.8 \right)^2 \left( x + 0.4 \right)^2 \left( x - 0.4 \right)^2 \left( x - 1 \right)^2 \quad \text{in } \Gamma_0. \]
Validation of the numerical scheme with land-sea distribution

<table>
<thead>
<tr>
<th>Number of cells in x direction</th>
<th>$L_2$-norm</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>$4.93 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>$1.27 \times 10^{-2}$</td>
<td>1.95</td>
</tr>
<tr>
<td>87</td>
<td>$3.08 \times 10^{-3}$</td>
<td>1.98</td>
</tr>
<tr>
<td>167</td>
<td>$8.84 \times 10^{-4}$</td>
<td>1.90</td>
</tr>
</tbody>
</table>

\[
\| \epsilon \|_{L_2} = \sqrt{\Delta x \sum_{i=1}^{N_x} \left(u_i(t^n) - \tilde{u}_i(t^n)\right)^2}
\]


Other results (latent heat)

Multiple solutions and numerical analysis to the dynamic and stationary models coupling a delayed energy balance model involving latent heat and discontinuous albedo with a deep ocean
J. I. Díaz, A. Hidalgo, L. Tello
WITH LATENT HEAT (t=5)

WITHOUT LATENT HEAT (t=5)

L. Tello y A. Hidalgo. IMAC2015. UJI.
March 30th. 2015.
Conclusions and further research

• We have obtained the numerical solution of a 1D energy balance model with nonlinear diffusion, coupled with a 2D deep ocean model in a rectangular domain.
• The method used is a finite volume method with 3rd order Runge-Kutta TVD.
• It has been obtained the evolution of the temperature in the deep ocean and also in the surface, due to the combination of melting ice, heating-cooling of the surface of the ocean.
• The results show the thermostatic effect of the ocean.
• The effect of the land-sea distribution has been considered in the problem.
• A verification of the accuracy of the scheme has been carried out solving an auxiliary problem with known analytical solution.

• More “realistic” values of parameters.
• 3D extension.