Unstable manifold, Conley index and fixed points of flows ☆

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A B S T R A C T

We study dynamical and topological properties of the unstable manifold of isolated invariant compacta of flows. We show that some parts of the unstable manifold admit sections carrying a considerable amount of information. These sections enable the construction of parallelizable structures which facilitate the study of the flow. From this fact, many nice consequences are derived, specially in the case of plane continua. For instance, we give an easy method of calculation of the Conley index provided we have some knowledge of the unstable manifold and, as a consequence, a relation between the Brouwer degree and the unstable manifold is established for smooth vector fields. We study the dynamics of non-saddle sets, properties of existence or non-existence of fixed points of flows and conditions under which attractors are fixed points, Morse decompositions, preservation of topological properties by continuation and classify the bifurcations taking place at a critical point.

1. Introduction

In this paper we are interested in the study of the unstable manifold of an isolated invariant continuum K of a flow $\varphi : M \times \mathbb{R} \to M$ defined on a locally compact metric space M. We shall use the notation $W^u(K)$ for the unstable manifold and we shall often consider the flow $\varphi|W^u(K) : W^u(K) \times \mathbb{R} \to W^u(K)$ restricted to the unstable manifold. The structure of $W^u(K)$ turns out to be very complicated in many cases. By the very definition of unstable manifold, K is a repelling set of the restricted flow $\varphi|W^u(K)$, i.e. $\omega^*(x) \in K$ for every $x \in W^u(K)$, where ω^* is the negative omega-limit. However, in general, K is not stable for negative times, which prevents us from saying that K is a repeller for $\varphi|W^u(K)$. One of the nicest properties of attractors and repellers is that the flow is parallelizable when restricted to the complement of the attractor or the repeller in its basin of attraction or repulsion. However, if we consider the flow $\varphi|W^u(K)$, the structure of $W^u(K) - K$ might be rather wild in many cases and, in particular, the flow might be non-parallelizable in $W^u(K) - K$. Some attempts have been made to give $W^u(K)$ a reasonable structure; however, they pass

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through defining a different topology, the so-called *intrinsic topology*, in $W^u(K)$ (see [32,40,3,37]). This topology does not agree, in general, with the standard topology inherited from the phase space M so the problem remains of studying $W^u(K)$ with its natural topology to detect some regularity in its structure. One of the aims of this paper is to contribute with some knowledge in that direction. In spite of the fact that $\varphi|W^u(K) - K$ is not parallelizable, we see that there exist certain sections S of the flow such that it is parallelizable in an *initial part* of $W^u(K) - K$, i.e. in the part of the flow coming before the section S. From this fact, which is proved in the very general case of flows in locally compact metric spaces, many nice consequences are derived, specially in the case of plane continua, to whose study we devote much of the paper. For instance, we give an easy method of calculation of the Conley index provided we have some knowledge of the unstable manifold. The Conley index is a basic tool in the theory of differential equations, and our method for the calculation of this index also allows the determination of the Brouwer degree of smooth vector fields in $\mathbb{R}^2$. We study the dynamics of non-saddle sets, properties of existence or non-existence of fixed points of flows, in particular conditions under which attractors of flows are fixed points, Morse decompositions, preservation of topological properties by continuation and classify the bifurcations taking place at a critical point.

We shall use through the paper the standard notation and terminology in the theory of dynamical systems. In particular, we shall use the notation $\gamma(x)$ for the *trajectory* of the point x, i.e. $\gamma(x) = \{xt \mid t \in \mathbb{R}\}$. Similarly for the *positive semi-trajectory* $\gamma^+(x) = \{xt \mid t \in \mathbb{R}^+\}$ and the *negative semi-trajectory* $\gamma^-(x) = \{xt \mid t \in \mathbb{R}^-\}$. By the *omega-limit* of a set $X \subset M$ we understand the set $\omega(X) = \bigcap_{t>0} \overline{X \cdot [t, \infty)}$ while the *negative omega-limit* is the set $\omega^*(X) = \bigcap_{t>0} \overline{X \cdot (-\infty, -t]}$. The *unstable manifold* of an invariant compactum K is defined as the set $W^u(K) = \{x \in M \mid \emptyset \neq \omega^*(x) \subset K\}$. Similarly the *stable manifold* $W^s(K) = \{x \in M \mid \emptyset \neq \omega(x) \subset K\}$. An invariant compactum K is *stable* if every neighborhood U of K contains a neighborhood V of K such that $V \cdot [0, \infty) \subset U$. Similarly, K is *negatively stable* if every neighborhood U of K contains a neighborhood V of K such that $V \cdot (-\infty, 0] \subset U$. The compact invariant set K is said to be *attracting* provided that there exists a neighborhood U of K such that $\omega(x) \subset K$ for every $x \in U$ and *repelling* if there exists a neighborhood U of K such that $\omega^*(x) \subset K$ for every $x \in U$. An *attractor* (or *asymptotically stable* compactum) is an attracting stable set and a *repeller* is a repelling negatively stable set. We stress the fact that stability (positive or negative) is required in the definition of attractor or repeller. If K is an attractor, its region (or basin) of attraction $\mathcal{A}$ is the set of all points $x \in M$ such that $\omega(x) \subset K$. It is well known that $\mathcal{A}$ is an invariant open set and that the flow $\varphi|\mathcal{A} - K$ (i.e. the flow restricted to $\mathcal{A} - K$) has compact sections and is parallelizable. By a section of $\mathcal{A} - K$ we understand a set $S \subset \mathcal{A} - K$ such that for every $x \in \mathcal{A} - K$ there exists a unique $t \in \mathbb{R}$ such that $xt \in S$. On the other hand $\varphi|\mathcal{A} - K$ parallelizable means that there exists a set $C \subset \mathcal{A} - K$ such that the map $C \times \mathbb{R} \to \mathcal{A} - K$ defined by $(x, t) \mapsto xt$ is a homeomorphism; in this case C is a section and the map $\sigma : \mathcal{A} - K \to \mathbb{R}$ defined by the property $x\sigma(x) \in C$ for every $x \in \mathcal{A} - K$ is continuous. Of course, the notions of section and parallelizability make sense for any invariant region of the flow.

We shall assume in the paper some knowledge of the Conley index theory of isolated invariant compacta of flows. These are compact invariant sets K which possess a so-called isolating neighborhood, that is, a compact neighborhood N such that K is the maximal invariant set in N, or setting

$$N^+ = \{x \in N : x[0, +\infty) \subset N\}; \qquad N^- = \{x \in N : x(-\infty, 0] \subset N\};$$

such that $K = N^+ \cap N^-$. We shall make use of a special type of isolating neighborhoods, the so-called isolating blocks, which have good topological properties. More precisely, an isolating block N is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and exit sets, satisfying

(1) $\partial N = N^i \cup N^o$,

(2) for every $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M - N$ and for every $x \in N^\bullet$ there exists $\delta > 0$ such that $x(0, \delta] \subset M - N$,
(3) for every $x \in \partial N - N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \mathring{N}$ and for every $x \in \partial N - N^\bullet$ there exists $\delta > 0$ such that $x(0, \delta] \subset \mathring{N}$.

These blocks form a neighborhood basis of K in M. We shall also use the notation $n^+ = N^+ \cap \partial N$ and $n^- = N^- \cap \partial N$. The *Conley index* $h(K)$ of an isolated invariant set K is defined as the homotopy type of the pair $(N/N^\bullet, [N^\bullet])$, where N is any isolating block of K. A crucial fact concerning the definition is, of course, that this homotopy type does not depend on the particular choice of N. If the flow is differentiable, the isolating blocks can be chosen to be differentiable manifolds which contain N^i and $N^\bullet$ as submanifolds of their boundaries and such that $\partial N^i = \partial N^\bullet = N^i \cap N^\bullet$. For flows defined on $\mathbb{R}^2$, the exit set $N^\bullet$ is the disjoint union of a finite number of intervals $J_1, \ldots, J_m$ and circumferences $C_1, \ldots, C_n$ and the same is true for the entrance set N^i. We refer the reader to [9–11,35] for information about the Conley index theory.

We use a minimum of topological notions in the paper. Homotopy and homology theory play an important role in the Conley index theory, however we try to restrict ourselves to the most basic facts. There is a form of homotopy which has proved to be the most convenient for the study of the global topological properties of the invariant spaces involved in dynamics, namely the *shape theory* introduced and studied by Karol Borsuk. We do not use shape theory in this paper. However, it is convenient to know that some topological properties of plane continua have a very nice interpretation in terms of shape. Two compacta are said to be of the same shape if they have the same homotopy type in the homotopy theory of Borsuk (or shape theory). The following result gives a classification of the shapes of all plane continua.

Theorem 1. *(See* K. Borsuk [7]*.) Two continua K and L contained in $\mathbb{R}^2$ have the same shape if and only if they disconnect $\mathbb{R}^2$ in the same number (finite or infinite) of connected components. More generally, the shape of K dominates the shape of L (shortly $Sh(K) \geq Sh(L)$) if and only if the number of connected components of $\mathbb{R}^2 - L$ is less than or equal to the number of components of $\mathbb{R}^2 - K$. In particular, a continuum has trivial shape (the shape of a point) if and only if it does not disconnect $\mathbb{R}^2$. A continuum has the shape of a circle if and only if it disconnects $\mathbb{R}^2$ into two connected components. Every continuum has the shape of a wedge of circles, finite or infinite (Hawaiian earring).*

Although we do not make use of shape theory in our proofs, we may occasionally refer to this theorem and to the terminology derived from it to make it clear that some of the results can be interpreted in that context. For a complete treatment of shape theory we refer the reader to [7,12,13,27,26,38]. The use of shape in dynamics is illustrated by the papers [18,15,19,21,24,32,33,36]. For information about basic aspects of dynamical systems we recommend [5,34,44] and for algebraic topology the books written by Hatcher [22] and Spanier [42] are very useful.

Concerning the Brouwer degree and fixed point theory we suggest Refs. [1] and [31].

2. On the structure of the unstable manifold

In this section we study the general case of a flow $\varphi : M \times \mathbb{R} \to M$ defined on a locally compact metric space M, and we consider an isolated invariant compactum K of the flow. Our aim is to understand the dynamics in $W^u(K)$, the unstable manifold of K. The set $W^u(K) - K$ is called the *truncated unstable manifold of K* (we remark that this terminology has been used with other meaning in [40]). If we consider the restriction $\varphi_0 = \varphi | W^u(K) \times \mathbb{R}$ of the flow to $W^u(K)$ then, in general, K is not negatively stable and, therefore, it is not a repeller of φ_0. Moreover, the flow restricted to the truncated unstable manifold $W^u(K) - K$ is not, in general, parallelizable. However, we shall prove in this section that if we restrict ourselves to an initial part of the truncated unstable manifold (in a sense that will be precised) then we obtain a parallelizable structure.

We start by studying an important particular case in which the flow on the truncated unstable manifold is, indeed, parallelizable. A similar result is contained in our paper [40], however we give here a more direct proof. We recall that an isolating block N is *non-return* if every orbit leaving N (in positive time) never returns to N (see [40]). In Example 1 we shall show that this result does not hold in the absence of non-return isolating blocks.

Theorem 2. *Let K be an isolated invariant compactum and suppose that K has a non-return isolating block N. Then K is a repeller for the flow $\varphi_0 = \varphi|W^u(K) \times \mathbb{R}$ and, as a consequence, for every compact section S of $W^u(K) - K$ the map $h : S \times \mathbb{R} \to W^u(K) - K$ defined by $(x, t) \mapsto xt$ is a homeomorphism (i.e. the truncated unstable manifold is parallelizable).*

Proof. By the definition of unstable manifold, K is a repelling set for $\varphi_0 = \varphi|W^u(K) \times \mathbb{R}$. In order to qualify as a repeller K must also be negatively stable. In order to prove this, we remark that the fact that N is non-return implies that $W^u(K) \cap N = N^-$. Now, if K is not negatively stable, then there exist a neighborhood U of K, a sequence $x_n \in W^u(K)$, $x_n \to x_0 \in K$ and a sequence $t_n \to -\infty$, $t_n < 0$, such that $x_n t_n \notin U$. Since $W^u(K) \cap N = N^-$ we may assume that $x_n \in N^-$ for every n and, since N^- is negatively invariant, $x_n t_n \in N^-$. By the compactness of N^- we may also assume that $x_n t_n \to y \in N^-$. Since $x_n t_n \notin U$ for every n we have that $y \in N^- - K$. Moreover for every $t \in \mathbb{R}$ we have that $t_n + t$ is negative and $x_n(t_n + t) \in N^-$ for almost all n, hence $yt \in N^-$. Thus the trajectory $\gamma(y) \subset N^- - K$, which is in contradiction to the fact that N is isolating. This completes the proof of the theorem. $\square$

If K does not have a non-return isolating block then $W^u(K) - K$ is not, in general, parallelizable. We postpone the proof of this fact to Example 1 since we must establish first some results. Our aim now is to study the general situation and prove that, in spite of this negative feature, certain parts of the truncated unstable manifold admit a parallelizable structure. We start by introducing a definition.

Definition 1. Let K be an isolated invariant compactum and let S be a compact section of the truncated unstable manifold $W^u(K) - K$. Then S is said to be an *initial section* provided that $\omega^*(S) \subset K$.

It is easy to see that if N is an isolating block of K then n^- is an example of initial section. If S is an initial section we define $I^u_S(K) = S(-\infty, 0]$ and we say that $I^u_S(K)$ is an *initial part of the truncated unstable manifold*. Obviously $I^u_S(K) = \{x \in W^u(K) - K : xt \in S \text{ with } t \geq 0\}$. It will be seen that, although $I^u_S(K)$ depends on S, all the initial parts have basically the same structure. In accordance with this terminology we say that $I^u_S(K) \cup K$ is an *initial part of the unstable manifold* of K and we denote it by $W^u_S(K)$.

Theorem 3. *Let K be an isolated invariant compactum and suppose that S is a compact section of the truncated unstable manifold $W^u(K) - K$. If S is initial then the map $h : S \times (-\infty, 0] \to I^u_S(K)$ defined by $(x, t) \mapsto xt$ is a homeomorphism. Conversely, if h is a homeomorphism then S is initial.*

Proof. The map h is, obviously, a continuous bijection, hence we have to prove only that if $x_n t_n \to x_0 t_0$, with $x_n, x_0 \in S$ and $t_n, t_0 \in (-\infty, 0]$ then $x_n \to x_0$ and $t_n \to t_0$. We remark that the sequence t_n is bounded since, otherwise, there exists a subsequence $t_{n_k} \to -\infty$ and, thus, $x_{n_k} t_{n_k} \to x_0 t_0 \in \omega^*(S)$ with $x_0 t_0 \notin K$, in contradiction to the hypothesis that S is an initial section. Now consider a subsequence x_{n_m} of x_n. Suppose that $x_{n_m} \to y \in S$. Since t_{n_m} is also bounded, it has a convergent subsequence as well, say $t_{n_{m_l}} \to s \in (-\infty, 0]$. Hence $x_{n_{m_l}} t_{n_{m_l}} \to ys \in I^u_S(K)$. But $x_{n_{m_l}} t_{n_{m_l}} \to x_0 t_0$ and, as a consequence, $x_0 t_0 = ys$ and, being S a section, $y = x_0$. This proves that every convergent subsequence of x_n converges to x_0 and, since S is compact, $x_n \to x_0$. On the other hand, using that the sequence t_n is bounded, a similar argument shows that t_n converges to t_0.

Suppose now that the map $h : S \times (-\infty, 0] \to I_S^u(K)$ defined by $(x,t) \mapsto xt$ is a homeomorphism. We consider an isolating block N of K such that $N \cap S = \emptyset$. This implies that $N^- \subset I_S^u(K)$. Suppose, to get a contradiction, that there exists $y \in \omega^*(S)$, $y \notin K$. Then, by definition, there exist $x_n \in S$, $t_n \to -\infty$ such that $x_n t_n \to y$. We may assume that $t_n < 0$ for every n. Now, if there is a subsequence $(x_{n_k} t_{n_k}) \subset N^-$ then $x_{n_k} t_{n_k} \to y$ and, hence, $y \in N^-$. But, since $N^- \subset I_S^u(K)$, we have that $y = xt_0$ with $x \in S$ and $t_0 < 0$ and this is in contradiction to the fact that h is a homeomorphism. Then, necessarily, $x_n t_n \notin N^-$ for almost every n and, hence, there is a sequence s_n such that $s_n < t_n$ and $x_n s_n \in n^-$ for almost every n. By the compactness of n^- there is a subsequence $x_{n_k} s_{n_k} \to z \in n^-$ with $s_{n_k} \to -\infty$ and the same argument as before leads to a contradiction. □

In the next result we establish a topological property of $I_S^u(K)$.

Proposition 4. *If S is an initial section of the truncated unstable manifold then the closure of $I_S^u(K)$ in M is contained in $I_S^u(K) \cup K$. As a consequence $W_S^u(K) = I_S^u(K) \cup K$ (initial unstable manifold) is closed in M. In fact, $W_S^u(K)$ is compact.*

Proof. If y is in the closure of $I_S^u(K)$ then $x_n t_n \to y$ with $x_n \in S$, $t_n \leq 0$. We may assume that $x_n \to x \in S$. If t_n is bounded then there exists a convergent subsequence $t_{n_m} \to t$. Hence $x_{n_m} t_{n_m} \to xt = y \in I_S^u(K)$. If t_n is unbounded, then there exists a subsequence $t_{n_k} \to -\infty$ and $x_{n_k} t_{n_k} \to y \in \omega^*(S) \subset K$. This proves the inclusion. Since K is compact, it is obvious that $W_S^u(K)$ is closed in M. Moreover, if N is an isolating block, the fact that S is initial implies the existence of a $t_0 < 0$ such that $S(-\infty, t_0] \subset N^-$. Hence $W_S^u(K) = (W_S^u(K) \cap N) \cup S[0, t_0]$ is compact. □

We see now that all initial sections are homeomorphic and that the homeomorphism can be defined in a very natural way.

Theorem 5. *Let K be an isolated invariant compactum and suppose that S and T are initial sections of the truncated unstable manifold $W^u(K) - K$. Then the map $h : S \to T$ defined by $h(x) = \gamma(x) \cap T$ is a homeomorphism.*

Proof. As we said before, if N is an isolating block of K then n^- is an initial section and there is a $t_0 < 0$ such that $S(-\infty, t_0] \subset N^-$. Now, the exit map of N^- (i.e. the map which assigns to each $x \in N^- - K$ the point $\gamma(x) \cap n^-$) can be used to define a homeomorphism $e : St_0 \to n^-$ and, as a consequence, the map $S \to n^-$ defined by $x \to \gamma(x) \cap n^-$ is also a homeomorphism. The map h in the statement of the theorem is a composition of this homeomorphism and the inverse of the analogous homeomorphism $T \to n^-$. □

All our considerations so far are relative to the unstable manifold of K. It is clear, however, that they can be dualized for the stable manifold $W^s(K)$ so that they are valid for the dual notions of *final section* and *final part of the truncated stable manifold* $W^s(K) - K$, which are defined in the obvious way. We shall use the notations $F_S^s(K)$ and $W_S^s(K)$ for the final part of the truncated stable manifold and final part of the stable manifold respectively, corresponding to the final section S. All the previous results hold for this dual situation and, in particular, Theorem 3 takes the following nice form.

Theorem 6. *Let K be an isolated invariant compactum and suppose that S is a compact section of the truncated stable manifold $W^s(K) - K$. If S is final then the map $h : S \times [0, \infty) \to F_S^s(K)$ defined by $(x,t) \mapsto xt$ is a homeomorphism. Moreover, the restriction $\varphi_0 = \varphi | W_S^s(K) \times \mathbb{R}$ of the flow to the final part of the stable manifold $W_S^s(K)$ defines a semi-dynamical system and K is a global attractor of φ_0.*

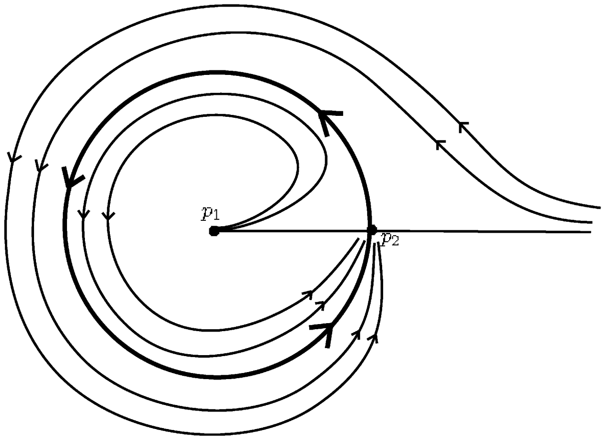

Fig. 1. Mendelson flow.

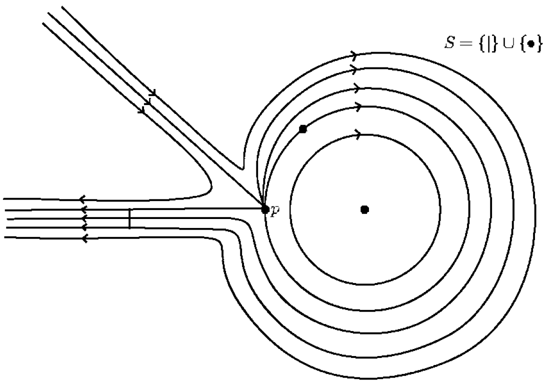

Fig. 2. Non-initial compact section. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

We remark that it is not in general true that K is an attractor for the flow considered in the whole stable manifold $W^s(K)$. This a consequence of the following example.

Example 1. The flow defined by Mendelson in [29] (see Fig. 1) provides an example of an isolated invariant continuum $K = \{p_2\}$ which is an unstable attracting set of $\mathbb{R}^2$ with $W^s(K) = \mathbb{R}^2 - \{p_1\}$ (we remind that the lack of stability means that K does not qualify as an attractor according to our definition). Here the final section S is homeomorphic to a segment (we can take, for instance, a semicircle with center p_2 and radius $r = d(p_1, p_2)/2$ in the lower semiplane) while the truncated stable manifold $W^s(K) - K$ is $\mathbb{R}^2 - \{p_1, p_2\}$. Then $W^s(K) - K$ is not parallelizable since, otherwise, $\mathbb{R}^2 - \{p_1, p_2\}$ would be homeomorphic to $S \times \mathbb{R}$, which is not the case. This proves that K is not an attractor in $W^s(K)$. This example can be dualized to show that, in general, the truncated unstable manifold $W^u(K) - K$ is not parallelizable.

Example 2. The flow described by Fig. 2 provides an example of a compact section of a continuum $K = \{p\}$ which is not initial. The section is marked in red.

Example 3. The following remarkable example (Fig. 3), presented by Campos, Ortega and Tineo in [8], describes a flow in a disk where all points in the boundary are stationary and such that the whole boundary is the ω-limit and the ω^*-limit of every interior point. The boundary K is not isolated and its truncated unstable manifold does not have compact sections. This example shows that the condition of K being isolated is necessary in Theorem 3.

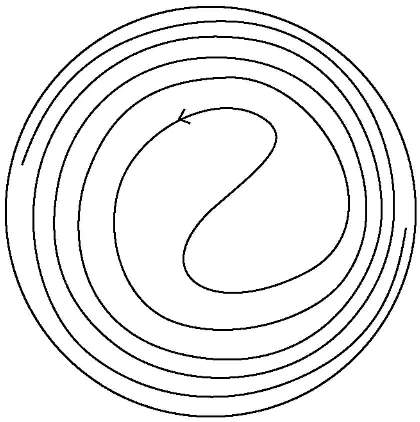

Fig. 3. Flow in a disk.

3. Conley index of plane continua

We start this section by giving a procedure to calculate the Conley index of a plane continuum K by inspection of its unstable manifold together with some topological information on K. We only need to know the number of connected components in which K decomposes $\mathbb{R}^2$ (i.e. the number of components of $\mathbb{R}^2 - K$) and to locate an initial section of $W^u(K) - K$ (we recall that not all compact sections are initial). According to this construction, isolating blocks of K are not necessary to determine the Conley index. First we need the following auxiliary result.

Lemma 7. *Let K be a non-empty isolated invariant continuum of the flow $\varphi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ with $m \geq 1$. Then $\mathbb{R}^m - K$ has a finite number of connected components.*

Proof. We remark first that if U is an open neighborhood of K then all components of $\mathbb{R}^m - K$ except a finite number of them are contained in U (this is valid for every continuum in $\mathbb{R}^m$, even if it is non-isolated). In order to prove it, denote by $A_1, A_2, \ldots, A_n, \ldots$ the connected components of $\mathbb{R}^m - K$, where A_1 is the non-bounded one, and take a closed ball D such that $K \subset \dot{D}$ (and, thus, $A_2 \cup \ldots \cup A_n \cup \ldots \subset \mathring{D}$). If our remark is not true then an easy compactness argument shows that we have points $x_{n_i} \in A_{n_i} - U \subset D - U$, belonging to mutually disjoint components A_{n_i}, with $x_{n_i} \to x \in D - U$; but this is impossible since x must belong to a component, A, which is an open set with empty intersection with the rest of components. Now, if K is isolated, suppose, to get a contradiction, that $\mathbb{R}^m - K$ has an infinite number of components. Then every isolating neighborhood N of K contains a component A_n, which is an invariant set of the flow. Hence N is not isolating for K. $\square$

Theorem 8. *Let K be a non-empty isolated invariant continuum of the flow $\varphi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ and S an initial section of its truncated unstable manifold. Then S has a finite number of connected components. If we denote by n the number of components of $\mathbb{R}^2 - K$, by u the number of components of S (or, equivalently, of an initial part of its truncated unstable manifold $I^u_S(K)$) and by u_c the number of contractible components of S, then $u - u_c \leq n$ and*

(a) *If $u \neq 0$ and $u - u_c < n$ then the Conley index of K is the pointed homotopy type of $(\bigvee_{i=1,\ldots,k} S^1_i, *)$, where $k = n + u_c - 2$ and S^1_i is a pointed 1-sphere based on $*$ for $i = 1, \ldots, k$.*
(b) *If $u - u_c = n$ then K is a repeller and its Conley index is the pointed homotopy type of $(S^2 \vee (\bigvee_{i=1,\ldots,n-1} S^1_i), *)$, where the 2-sphere S^2 and all the S^1_i are pointed and based on $*$.*

(c) *If $u = 0$ then K is an attractor and its Conley index is the pointed homotopy type of $(\bigvee_{i=1,\ldots,n-1} S_i^1 \cup \{\bullet\}, \bullet)$, where the S_i^1 are pointed 1-spheres based on $*$ and $\bullet$ denotes a point not belonging to $\bigvee_{i=1,\ldots,n-1} S_i^1$.*

Proof. By the classical theorem of Gutiérrez [20], according to which every continuous flow on $\mathbb{R}^2$ is topologically equivalent to a differentiable flow, we may assume that φ is differentiable. Denote the unbounded component of $\mathbb{R}^2 - K$ by A_1 and the bounded components by $A_2, \ldots, A_n$. As a consequence of the results proved by Conley and Easton in [10] (see also [14]) on the structure of isolating blocks we may assume that there exists an isolating block, N, of K in $\mathbb{R}^2$ which is a connected surface with boundary and, hence, N can be represented, up to homeomorphism, as $N = D_1 - (\mathring{D}_2 \cup \ldots \cup \mathring{D}_n)$ where the D_i are closed disks with $D_2 \cup \ldots \cup D_n \subset \mathring{D}_1$ (i.e. N is homeomorphic to a perforated disk, where $\mathring{D}_2, \ldots, \mathring{D}_n$ are the holes). On the other hand, for every $i = 2, \ldots, n$, the disk D_i is contained in the bounded connected component A_i of $\mathbb{R}^2 - K$ and $\mathring{D}_i = A_i - (N \cap A_i)$. Concerning the unbounded component, we remark that $\mathbb{R}^2 - D_1 = A_1 - (N \cap A_1)$. Moreover, the boundary ∂N agrees with $N^i \cup N^o$ where N^o is a disjoint union of a finite number of intervals and circumferences and similarly is N^i. The intersection $N^i \cap N^o$ consists of the extremes of the intervals. Since all initial sections are homeomorphic, we can consider the particular case $S = n^-$, where $n^- = \partial N \cap N^-$. This kind of isolating neighborhoods will be used several times in the sequel.

In order to prove the theorem, we shall show first that every component L of N^o contains exactly one component of n^-. Consider the case when L is contained in the circle $C_1 = \partial D_1$. We may assume that $L \neq C_1$ (otherwise $C_1 \subset n^-$ and, thus, $L \subset n^-$), hence L is a topological interval. Suppose $x, y \in L \cap n^-$. We claim that every point $z \in L$ lying between x and y also belongs to n^-. Otherwise the trajectory of z abandons N at a negative time $t < 0$ with $z[t, 0] \subset N$ and $zt \in N^i$. Then $z[t, 0]$ disconnects D_1 into two components and we can express $D_1 = D_1^a \cup D_1^b$ where D_1^a and D_1^b are homeomorphic to closed discs and $D_1^a \cap D_1^b = z[t, 0]$. Suppose D_1^b is the disc containing K. Then one of the points x, y, say x, is in D_1^a. Since the trajectory of x cannot meet $z[t, 0]$, this trajectory is forced to leave D_1, and hence N, in the past, which is in contradiction to the fact that $x \in n^-$. This proves that L contains at most one component of n^-. The discussion for the discs D_i lying in the bounded components of $\mathbb{R}^2 - K$ is only slightly different and we leave it to the reader.

We shall show now that $L \cap n^-$ is non-empty. We consider again the case when L is contained in the circle $C_1 = \partial D_1$ and $L \neq C_1$. Suppose first that $N^o \cap C_1$ consists of at least two components, where L is one of them. Then L is adjacent to two components of $N^i \cap C_1$ and we denote by J one of them. The set of points of $\mathring{L}$ (i.e. the interval excluding the extremes) which leave N in the past through J is open and non-empty. So is the set of points which leave N through the union of the other components of $N^i \cap C_1$ different from J. Hence (by connectedness) not all points of $\mathring{L}$ leave N in the past and at least one of them stays for all negative times and, thus, it belongs to n^-. If $N^o \cap C_1$ consists of exactly one component L (different from the whole circle C_1) then $N^i \cap C_1$ has exactly one component L'. Suppose, to get a contradiction, that $n^- \cap L$ is empty. If we represent by B the union of all the bounded components of $\mathbb{R}^2 - K$ (i.e. $B = A_2 \cup \ldots \cup A_n$) then the entrance map $e : D_1 - (K \cup B) \to L'$ defines a strong deformation retraction, which is impossible since L' is contractible and $D_1 - (K \cup B)$ is not. This proves that $L \cap n^-$ is non-empty. The discussion for the discs D_i lying in the bounded components of $\mathbb{R}^2 - K$ is again slightly different and we leave it to the reader.

Our discussion, so far, shows that the components of n^- are in bijection with the components of N^o, and that this bijection is induced by the inclusion. Hence u is finite and, since $N^o \subset \partial N$, there are, at most, n non-contractible components of n^-, which proves that $u - u_c \leq n$.

We proceed now to the calculation of the Conley index. We discuss first the case when $u \neq 0$ and $u - u_c < n$. Since $\mathbb{R}^2 - K$ has n components then N is a perforated disk with $n - 1$ holes and, thus, has the homotopy type of $\bigvee_{i=1,\ldots,n-1} S_i^1$. Consider the components C_i of ∂N which are entirely contained in N^o. If

$i \neq 1$, the effect of collapsing the component C_i to a point, say c_i, amounts to fill a hole of N and, hence, to subtract a copy of S^1 in the former wedge. If $i = 1$, then D_1 becomes a sphere after identifying its boundary to a point c_1, and the effect on the wedge is the same. However, since all those components collapse to the *same point,* we must identify all the points c_i to a single point $*$, which produces new copies of S^1. The result, after the identifications are carried out, amounts to subtracting a unit to $n-1$, getting $n-2$ copies of S^1 in the former wedge of circles. Now, the rest of the components of $N^\bullet$ are in bijection with the contractible components of n^- and, thus, there are u_c of them. Each one contributes, after identification with $*$, a copy of S^1. Hence we obtain $n + u_c - 2$ copies of S^1. If there are no components of ∂N entirely contained in $N^\bullet$ then we have to identify u_c contractible components to the point $*$ and the result is the same again. The discussion of the cases (b) and (c) is similar. We have only to remark that in the case (b) we have that $n^- = \partial N$ and, thus, K is a repeller and, when all the components of ∂N are collapsed to a point, we get a sphere S^2 with $n-1$ loops attached. The case (c) is the easiest one since $N^\bullet = n^-$ is empty. $\square$

A nice consequence of Theorem 8 is the following result, which establishes a relation between the Brouwer degree and the number of contractible components of the initial sections on the unstable manifold.

Corollary 9. *Let X be a smooth vector field on $\mathbb{R}^2$ and suppose that the flow φ is generated by $\dot{x} = -X(x)$. Let K be a non-empty isolated invariant continuum of φ and N an isolating block for K. Then* $\deg(X, \dot{N}) = 2 - n - u_c$.

Proof. It is known (see [43,28,23]) that $\deg(X, \dot{N}) = \chi(h(K))$, where χ stands for the Euler characteristic and $h(K)$ is the Conley index of K. Now, the Euler characteristic of the Conley index of K is, according to Theorem 8, $2 - n - u_c$. $\square$

4. Dynamics of plane continua

In this section we present several results about the dynamics of plane continua (or near plane continua). In many of them we make use of the structure of the unstable manifold studied in Section 2. We start by discussing to what extent the numbers u and u_c determine the dynamics. In coherence with our previous notation, we denote by u' the number of components of a final section of the truncated stable manifold $W^s(K) - K$ and by u'_c the number of contractible components.

The vanishing of the coefficient u_c turns out to be related with a property introduced and studied by N.P. Bhatia in [4], namely the property of an invariant set being *non-saddle.*

Definition 2. A compact invariant set K of a flow $\varphi : M \times \mathbb{R} \to M$ is said to be saddle provided that there exists a neighborhood U of K in M such that for every neighborhood $V \subset U$ of K in M there is a point $x \in V$ such that $\gamma^+(x) \cap (M - U) \neq \emptyset$ and $\gamma^-(x) \cap (M - U) \neq \emptyset$ (i.e. the orbit of x leaves U in the past and in the future). K is said to be non-saddle if it is not saddle.

Non-saddle sets have also been studied in [16], and they turn out to have very nice dynamical and topological properties; attractors and repellers are particular types of non-saddle sets. The first part of the next result characterizes non-saddleness. The second part can be interpreted as a form of time duality in terms of the stable and unstable manifolds.

Theorem 10. *Let K be an isolated invariant continuum of a plane flow φ. Then*

(1) $u_c = 0$ *if and only if K is non-saddle.*

(2) *The coefficients* u_c *and* u'_c *agree. Hence the initial sections of the truncated unstable manifold and the final sections of the truncated stable manifold have the same homotopy type if and only if they have the same number of connected components (i.e. if and only if* $u = u'$*).*

Proof. Suppose, to get a contradiction, that K is non-saddle but $u_c \neq 0$. Consider an isolating block N of K as in the proof of Theorem 8. Then, N^o has at least one connected component E which is a (topological) interval. Denote $E_0 = E \cap n^-$, which is also an interval or a point. Since $E_0 \neq E$, there is a sequence $x_n \in E - E_0$ such that $x_n \to x_0 \in E_0$. Obviously, $\gamma^+(x_n)$ is not contained in N and, since $x_n \notin n^-$, the negative semiorbit $\gamma^-(x_n)$ is not contained in N either. On the other hand, since $\omega^*(x_0) \subset K$ and $x_n \to x_0$ then for every $\varepsilon > 0$ there is an x_n and a $t > 0$ such that $x_n[-t,0] \subset N$ and $d(x_n(-t), K) < \varepsilon$. The orbit of the point $x_n(-t)$ must leave N in the past and in the future and this contradicts the fact that K is non-saddle. This proves that $u_c = 0$ if K is non-saddle. Conversely, if $u_c = 0$, consider an isolating block N of K as in before. The neighborhood N can be chosen arbitrarily small. Since $u_c = 0$, all the connected components of N^o, and also of N^i, are circles, which implies that $N^o = n^-$ and $N^i = n^+$. Hence, every orbit through ∂N stays in N either for all positive times or for all negative times. This implies that K is non-saddle.

Concerning the second statement, the numbers u_c and u'_c can be calculated using an isolating block as indicated before. This block has a form of symmetry in the following respect: if we consider a component of ∂N not entirely contained either in N^o or in N^i then the number of intervals of N^o lying in this component is exactly the same as the number of intervals of N^i lying in the same component. Since u_c and u'_c are the sums of the respective numbers for all components of ∂N, we get that $u_c = u'_c$. Hence $u = u'$ if and only if the number of non-contractible components of the initial section agrees with the number of components of the final one and from this readily follows the statement. □

As a consequence of our previous discussion we see that if K is non-saddle then, given a component A of $\mathbb{R}^2 - K$, it happens that K has either an attracting behavior or a repelling behavior towards the points of A which are close to K. In fact, K is either an attractor or a repeller of the restricted flow $\varphi|A \cup K$. The first kind of components, which are the components of $\mathbb{R}^2 - K$ having empty intersection with $W^u(K)$, will be called *a-components* and the second kind, i.e. those with empty intersection with $W^s(K)$ will be called *r-components*. A consequence of the previous remark is that every bounded a-component A contains a dual repeller R of the flow $\varphi|A \cup K$ whose basin of repulsion is A. This dual repeller is the largest compact invariant set contained in A, and an easy consequence of this is that it does not disconnect $\mathbb{R}^2$ (i.e. R has trivial shape). Similarly, every bounded r-component contains an attractor of trivial shape whose basin of attraction is the whole component. If we fill all the holes of K we get a continuum $\hat{K}$, which is the union of K with all the bounded components of $\mathbb{R}^2 - K$. Obviously $\hat{K}$ does not disconnect $\mathbb{R}^2$ (and, hence, is of trivial shape) and it is either an attractor or a repeller of the flow, depending on the nature of φ in the unbounded component. We call $\hat{K}$ the *saturation of* K. The family of attractors and repellers just described, together with K, define a Morse decomposition $\mathcal{M}$ of $\hat{K}$ whose Morse equations contain a great deal of information both about the global topology of K and the dynamics near K. To be more precise, we denote by $M_1, \ldots, M_k$ the attractors contained in the r-components of $\mathbb{R}^2 - K$, we take $M_{k+1} = K$ and denote by $M_{k+2}, \ldots, M_n$ the repellers contained in the a-components. Then $\mathcal{M} = \{M_1, \ldots, M_k, M_{k+1}, \ldots, M_{k+2}, \ldots, M_n\}$ is a Morse decomposition of $\hat{K}$, which we call the *natural Morse decomposition of* $\hat{K}$. For general information on Morse decompositions and their corresponding Morse equations we refer the reader to [9,35,25].

Theorem 11. *Suppose* K *is an isolated non-saddle continuum of a flow* φ *in* $\mathbb{R}^2$ *which is neither an attractor nor a repeller. Suppose that the number of bounded r-components of* $\mathbb{R}^2 - K$ *is* k *and that the unbounded component is also an r-component. Then the Morse equations of* φ *for the natural Morse decomposition* $\mathcal{M}$ *of* $\hat{K}$ *(the saturation of* K*) are:*

$$k + (n-2)t + (n-k-1)t^2 = t^2 + (1+t)Q(t)$$

where n is the number of components of $\mathbb{R}^2 - K$ and the coefficients of $Q(t)$ are non-negative integers.

In the same situation, but assuming now that the unbounded component is an a-component, the equations are:

$$k + (n-2)t + (n-k-1)t^2 = 1 + (1+t)Q^*(t)$$

where $Q^*(t)$ has also non-negative coefficients.

Hence the Morse equations completely determine the shape of K and the dynamical structure near K.

Proof. None of the attractors and repellers involved in the Morse decomposition disconnects $\mathbb{R}^2$, and the same is true for $\hat{K}$. On the other hand, K is a non-saddle set disconnecting $\mathbb{R}^2$ into n components. With these data, we can calculate the Conley index of all the elements of the Morse decomposition by using Theorem 8. In particular, the Conley index of K is the pointed homotopy type of a wedge of $n-2$ circles. Since the coefficients of the Morse equations are obtained from the Betti numbers of the homological Conley indices we readily get the equations in the statement of the theorem. In particular, K is responsible for the term $(n-2)t$, the k attractors in the r-components give the term k and the $(n-k-1)$ repellers in the a-components contribute with the term $(n-k-1)t^2$. The difference between the two equations lies in the repelling or attracting character of the saturation of K. In the first case we have the term t^2 and in the second case, the term 1 in the second member of the equation. □

The non-saddleness property turns out to be related to the non-existence of fixed points. In fact, we have the following result, which gives necessary conditions for the non-existence of fixed points contained in isolated continua.

Theorem 12. *Let X be a smooth vector field on $\mathbb{R}^2$ and suppose that the flow φ is generated by $\dot{x} = -X(x)$. Let K be an isolated invariant continuum of φ. Suppose that K does not contain fixed points. Then K is a non-saddle set which disconnects the plane into two components. Therefore it must be either a limit cycle or homeomorphic to a closed annulus bounded by two limit cycles.*

Proof. If K does not contain fixed points then it follows from Corollary 9 that $2 - n - u_c = 0$. Therefore we have only the possibilities $n = 1$, $u_c = 1$ and $n = 2$, $u_c = 0$. The first possibility must be excluded since it leads to the following situation: the ω-limit of every point of K is a periodic orbit whose interior is in K (otherwise K would disconnect the plane and n would be greater than 1) but this implies the existence of a fixed point in K. If $n = 2$ and $u_c = 0$ then K is a non-saddle set disconnecting the plane into two components A and B. Suppose A is the unbounded one and suppose it is an a-component (the argument is the same for r-components). Then if we take $x \in A$ sufficiently close to K, $\omega(x)$ is a periodic orbit contained in K, that we denote by γ. Moreover B is contained in the interior of γ (otherwise we would have a fixed point in K). By the same argument, there is a point $y \in B$ whose ω- or ω^*-limit is a periodic orbit γ' contained in K. If $\gamma \neq \gamma'$ the orbits γ and γ' bound a plane region C homeomorphic to an annulus. C is contained in K since, otherwise K would disconnect the plane in more than two components. On the other hand, we prove now that there are no points $z \in K - C$. Suppose, to get a contradiction, that $z \in K$ is in the unbounded component of $\mathbb{R}^2 - C$ (the other case is only slightly different). Then $\omega(z)$ is a periodic orbit, γ'', containing γ in its interior since, otherwise, the interior of γ'' would be entirely contained in K and, thus, it would contain a fixed point of K. Since γ is in the interior of γ'', γ cannot be a limit orbit of points of A. This contradiction establishes that $C = K$. If $\gamma = \gamma'$, an easier argument proves that $K = \gamma = \gamma'$. □

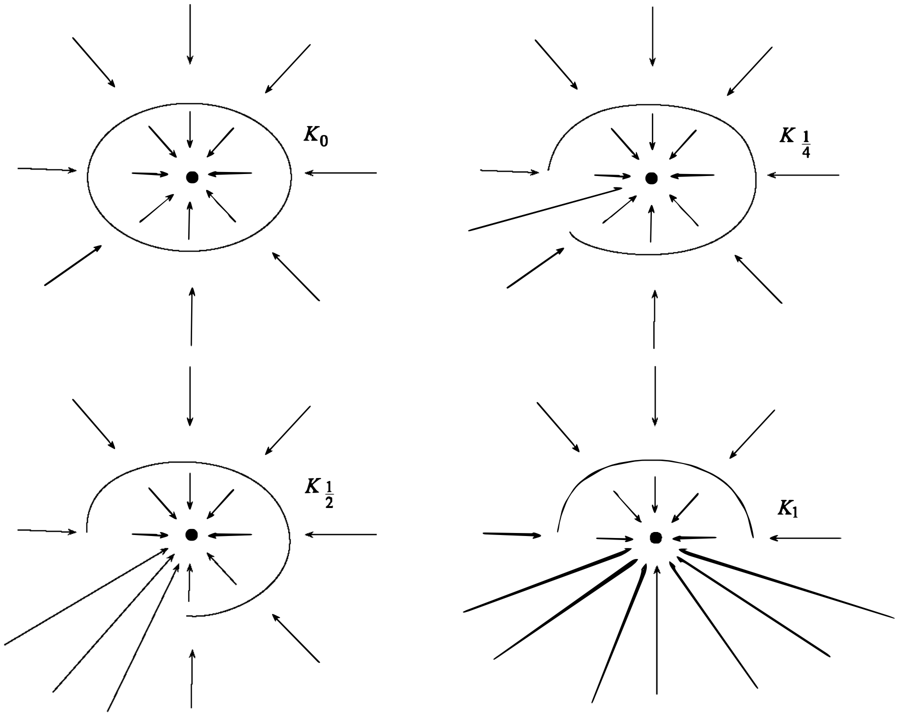

Fig. 4. Continuation.

Remark 1. According to Theorem 12 every isolated periodic orbit γ is a non-saddle set. If γ is neither an attractor nor a repeller, it follows from our previous discussion that $W^u(\gamma)$ is homeomorphic to a punctured disk, while every initial part of its unstable manifold $W^u_S(\gamma)$ is homeomorphic to an annulus with γ as one of the boundary components. On the other hand, if p is an isolated equilibrium which is neither an attractor nor a repeller then $u = u_c$, and it follows from Theorem 10 that the initial parts of the truncated unstable manifold, $I^u_S(p)$, and the final parts of the truncated stable manifold, $F^s_S(p)$, have the same homotopy type. As a matter of fact, it can be readily seen that the unstable manifold $W^u(p)$ is the bijective continuous image (*although not necessarily the homeomorphic image*) of a set of $\mathbb{R}^2$ composed of a finite union of rays from 0 plus a finite union of closed plane sectors with vertex at 0.

We shall discuss in the sequel some matters using the point of view of continuation, a central notion in the Conley index theory. We refer the reader to the papers [9,35,17] for information on basic facts about this notion. In Fig. 4 we reproduce an example from [16] which shows that there exist a parametrized family φ_λ of flows in the plane and a continuation $(K_\lambda)_{\lambda \in I}$ of an isolated invariant continuum K_0 such that $Sh(K_\lambda) \neq Sh(K_0)$ for every $\lambda \geq 0$. Therefore shape is not necessarily preserved by continuation.

In the following result we show that if the shape is not preserved then the global complexity of isolated invariant continua can only decrease through small perturbations, i.e. the shape of the continuation K_λ is dominated by the shape of the initial continuum K_0 for small values of λ. On the other hand, the preservation of shape implies a strong rigidity of the truncated unstable manifold towards deformations of the flow.

Theorem 13. *Let $(\varphi_\lambda)_{\lambda \in I}$ be a parametrized family of flows in $\mathbb{R}^2$ and let K_0 be an isolated invariant continuum for φ_0. Suppose that the family of continua $(K_\lambda)_{\lambda \in I}$ continues K_0. Then there exists $\lambda_0 \leq 1$ such that $Sh(K_0) \geq Sh(K_\lambda)$ for every $\lambda \leq \lambda_0$. Moreover, if $Sh(K_0) = Sh(K_\lambda)$ for every $\lambda \in I$, then the initial parts of the truncated unstable manifolds of K_0 and K_λ have the same homotopy type.*

Proof. Suppose K_0 decomposes the plane into n components and consider an isolating block N of K_0 as in the proof of Theorem 8; in particular, N decomposes the plane also into n connected components. Since

$(K_\lambda)_{\lambda\in I}$ is a continuation of K_0, then there exists a $\lambda_0 \leq 1$ such that N is an isolating neighborhood for every K_λ with $\lambda \leq \lambda_0$. Since $K_\lambda \subset N$ then $\mathbb{R}^2 - N \subset \mathbb{R}^2 - K_\lambda$. If the relation $Sh(K_0) \geq Sh(K_\lambda)$ does not hold for some $\lambda \leq \lambda_0$ then $\mathbb{R}^2 - K_\lambda$ has a greater number of connected components than $\mathbb{R}^2 - K_0$ and, thus, there are components of $\mathbb{R}^2 - K_\lambda$ with empty intersection with $\mathbb{R}^2 - N$. As a consequence they are contained in N. Since these components are invariant by the flow φ_λ, the union of K_λ with all of them is an invariant compactum of φ_λ contained in N and N is not an isolating neighborhood of K_λ. This contradiction establishes the first part of the theorem.

If $Sh(K_0) = Sh(K_\lambda)$ then $\mathbb{R}^2 - K_0$ and $\mathbb{R}^2 - K_\lambda$ have the same number of components, say n. We discuss the case $n = 1$ and leave to the reader the slightly more complicated general case. By the preservation of the Conley index by continuation, the numbers u and u_c remain the same for all $\lambda \in I$. This means that the initial sections of K_0 and K_λ, and also the initial parts of their truncated unstable manifolds, have the same homotopy type. $\square$

In the next result we show that very strong dynamical consequences are derived from the topological property of connectedness of the initial sections.

Theorem 14. *Let K be an isolated invariant continuum of a flow in $\mathbb{R}^2$ and let S be an initial section of the truncated unstable manifold $W^u(K) - K$. Suppose S is connected and denote by A the component of $\mathbb{R}^2 - K$ which contains S. Then in every bounded component $B \neq A$ of $\mathbb{R}^2 - K$ there is a repeller $R \subset B$ whose basin of repulsion is B. Moreover, the repeller R contains a critical point of the flow.*

Proof. Suppose B is a bounded component of $\mathbb{R}^2 - K$ different from the component A which contains S. If N is an isolating block of K as described in the proof of Theorem 8 then $N^o \subset A$ since, otherwise, S would meet other components of $\mathbb{R}^2 - K$ and would not be connected. Hence, the component C of ∂N lying in B is totally contained in N^i. The circle C is also the boundary of a disk D contained in B and, since every orbit through C enters N (in the future) and remains there, the disk D is negatively invariant by the flow. As a consequence, in the interior of D there is a repeller R which repels the whole disk. Moreover, since N is isolating, every point of $N \cap B$ goes to D in the past (and remains there), which implies that the basin of repulsion of R is all B. On the other hand, since D is negatively invariant then, for every fixed $t \leq 0$, the correspondence $x \to \varphi(x,t)$ defines by restriction a map $\varphi_t|D : D \to D$ and, by Brouwer's fixed point theorem, there exists a sequence of points $x_n \in D$ and a sequence of numbers $t_n < 0$, $t_n \to 0$ such that $\varphi(x_n, t_n) = x_n$. By the compactness of D there is a convergent subsequence x_{n_i} whose limit $x \in D$ is a fixed point of the flow. $\square$

The following nice result by Alarcón, Guíñez and Gutiérrez gives a relation between global asymptotic stability of a critical point and non-existence of additional critical points in the case of discrete dynamical systems.

Theorem 15. *(See* Alarcón, Guíñez, Gutiérrez [2].*) Assume that $h \in \mathcal{H}_+$ (homeomorphisms of $\mathbb{R}^2$ conserving the orientation) is dissipative and p is an asymptotically stable fixed point of h. The following conditions are equivalent:*

(a) *p is globally asymptotically stable.*
(b) *$Fix(h) = \{p\}$ and there exists an arc $\gamma \subset S^2$ with end points at p and ∞ such that $h(\gamma) = \gamma$.*

The proof in [2] is based on Brouwer's theory of fixed point free homeomorphisms of the plane. Ortega and Ruiz del Portal give in [30] an alternative proof based on the theory of prime ends.

Inspired by Theorem 15, we present a result on continua K which are attractors of dissipative flows in the plane.

Theorem 16. *Let K be a connected attractor of a dissipative flow φ in $\mathbb{R}^2$. The following conditions are equivalent:*

(a) *K is a global attractor.*
(b) *There are no fixed points in $\mathbb{R}^2 - K$ and there exists an orbit γ connecting ∞ and K (i.e. such that $\|\gamma(t)\| \to \infty$ when $t \to -\infty$ and $\omega(\gamma) \subset K$).*

Proof. Since φ is dissipative, then there exists a global attractor K' of the flow and, thus, $K \subset K'$. We must prove that $K = K'$. Otherwise, there exists a point $x \in K' - K$, and we consider $\omega^*(x)$. By the invariance and the compactness of K', we have that $\omega^*(x) \subset K'$ and, since K is an attractor, $\omega^*(x) \cap K = \emptyset$. Hence $\omega^*(x)$ does not contain fixed points and, by the Poincaré–Bendixson theorem, $\omega^*(x)$ is a periodic orbit. Moreover, K is not contained in the interior of this orbit since, in that case, γ would meet $\omega^*(x)$. Hence in the interior of the periodic orbit $\omega^*(x)$ must exist a fixed point not belonging to K, which is a contradiction. This establishes the implication (b) $\Rightarrow$ (a); the converse implication is trivial. $\square$

The following result, which is a consequence of Theorem 16 and a theorem by Bhatia, Lazer and Szego in [6], gives a nice characterization of globally attracting fixed points.

Corollary 17. *Let K be a minimal attractor of a dissipative flow in $\mathbb{R}^2$. The following conditions are equivalent:*

(a) *K is a globally attracting fixed point.*
(b) *There are no fixed points in $\mathbb{R}^2 - K$ and there exists an orbit connecting ∞ and K.*

Proof. It is a consequence of Theorem 16 and Bhatia, Lazer and Szego's Theorem 4.1 in [6] according which minimal global attractors in $\mathbb{R}^2$ are fixed points. $\square$

We shall be concerned now with bifurcations at critical points of the flow. Suppose that we have a continuous family of flows $\varphi_\lambda : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, with $\lambda \in I$, such that $p \in \mathbb{R}^2$ is an equilibrium for every λ. There are several non-equivalent definitions of bifurcation at p when $\{p\}$ is an attractor for φ_0. We adopt the following one, which conveys the idea that a new continuum, evolving from p, is created in the bifurcation.

Definition 3. Let $\varphi_\lambda : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, with $\lambda \in I$, be a continuous family of flows. Suppose that p is a fixed point for every φ_λ and $\{p\}$ is an attractor for φ_0. Suppose also that $(M^\lambda)_{\lambda \in I}$, with $M^0 = \{p\}$, is a continuation of $\{p\}$. If there is a $\lambda_0 \in (0,1]$ and a Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ of M^λ into two continua, where one of them is $\{p\}$ for every λ with $0 < \lambda \le \lambda_0$, we say that a bifurcation takes place in p.

Concerning the former definition we remark that the order is essential in the Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ and that we admit the two possibilities $M_a^\lambda = \{p\}$ for every λ with $0 < \lambda \le \lambda_0$ or $M_b^\lambda = \{p\}$ for every λ with $0 < \lambda \le \lambda_0$. Since $\{p\}$ is an attractor for φ_0 we can select λ_0 so small that M^λ is an attractor of trivial shape for φ_λ with $0 < \lambda \le \lambda_0$ (see [39] for properties of continuations of attractors). Since M_a^λ is an attractor for the restricted flow $\varphi_\lambda | M^\lambda$, then M_a^λ is also an attractor for the flow φ_λ. The most notorious particular case is when $M_b^\lambda = \{p\}$ is a repeller for φ_λ with $0 < \lambda \le \lambda_0$ and M_a^λ is a periodic orbit. In this case we say that a *Hopf bifurcation* takes place at p.

The bifurcation may be embedded in a more complex process of continuation of an isolated invariant continuum. Suppose we have a continuum $K = K_0$ which is invariant and isolated for φ_0, endowed with a Morse decomposition $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ with $M_1 = \{p\}$ and suppose that K continues to a family of continua $(K_\lambda)_{\lambda \in I}$. Then $\mathcal{M}$ also continues to Morse decompositions $\mathcal{M}^\lambda = \{M_1^\lambda, M_2^\lambda, \ldots, M_k^\lambda\}$ of the K_λ and we suppose that simultaneously a bifurcation takes place at p according to the previous definition, i.e. that M_1^λ has itself a Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ as in Definition 3. Then $\hat{\mathcal{M}}^\lambda = \{M_a^\lambda, M_b^\lambda, M_2^\lambda, \ldots, M_k^\lambda\}$ is also a Morse decomposition of K_λ which embodies information about the bifurcation and about the continuation. We call $\hat{\mathcal{M}}^\lambda$ the Morse decomposition associated to the bifurcation. We write the Morse equation of $\hat{\mathcal{M}}^\lambda$ in the usual form $P^\lambda(t) = R^\lambda(t) + (1+t)Q^\lambda(t)$, where $Q^\lambda(t)$ is a polynomial whose coefficients are non-negative integers.

Theorem 18. *Let K be an isolated invariant continuum of a flow φ in $\mathbb{R}^2$ and let $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ be a Morse decomposition of K with $M_1 = \{p\}$. Suppose that a Hopf bifurcation takes place at p for a continuation φ_λ of φ and denote by $\hat{\mathcal{M}}^\lambda = \{M_a^\lambda, M_b^\lambda, M_2^\lambda, \ldots, M_k^\lambda\}$ the associated Morse decomposition. Then $P^\lambda - P = t^2 + t$, where P corresponds to the Morse equation of $\mathcal{M}$.*

Proof. The main difference of $\hat{\mathcal{M}}^\lambda$ with the initial Morse decomposition $\mathcal{M}$ is that the point p becomes repelling and an attracting periodic orbit M_a^λ evolves from p. The repelling point is responsible for the term t^2 and the attracting orbit adds the term t to the Morse equations. The contribution of the rest of the Morse sets remains the same, since they are continuations of the Morse sets of the initial decomposition. □

We shall see now that the relation $P^\lambda - P = t^2 + t$ captures some of the topology involved in the Hopf bifurcation, although not the whole of the dynamics: if we have a bifurcation (not necessarily Hopf) whose Morse equation satisfies this particular relation then we shall show that a new attractor with the shape of S^1 (although not necessarily a periodic orbit) is created in the bifurcation. The following result enumerates all the possible types of bifurcations. We see that the rest of the bifurcations have no effect on the Morse equations.

Theorem 19. *Let $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ be a Morse decomposition of K with $M_1 = \{p\}$. Suppose that a bifurcation (not necessarily Hopf) takes place at p for a continuation φ_λ of φ and denote by $\hat{\mathcal{M}}^\lambda = \{M_a^\lambda, M_b^\lambda, M_2^\lambda, \ldots, M_k^\lambda\}$ the associated Morse decomposition. Then there are the following possibilities: (1) $M_a^\lambda = \{p\}$ is an attractor and M_b^λ is a non-saddle set with the shape of S^1, (2) $M_a^\lambda = \{p\}$ is an attractor and M_b^λ a saddle-set with trivial shape, (3) M_a^λ is an attractor of trivial shape and $M_b^\lambda = \{p\}$ is a saddle-set, (4) M_a^λ is an attractor with the shape of S^1 and $M_b^\lambda = \{p\}$ is a repeller. In case (4) we have the relation $P^\lambda - P = t^2 + t$ for the Morse equations and in cases (1), (2), (3) the Morse equations remain unaltered.*

Proof. The Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ of M_1^λ consists of two sets, one of them, for instance M_b^λ, is equal to $\{p\}$ and the other, M_a^λ, is a plane continuum. This plane continuum cannot separate the plane into more than two components since, being M_1^λ of trivial shape, all the bounded components of $\mathbb{R}^2 - M_a^\lambda$ must be contained in M_1^λ and, thus, each of them must contain a Morse set of the decomposition of M_1^λ other than M_a^λ, and there is only one. As a consequence we have the following possibilities: (1) $M_a^\lambda = \{p\}$ and M_b^λ a continuum with the shape of S^1, (2) $M_a^\lambda = \{p\}$ and M_b^λ a continuum with trivial shape, (3) M_a^λ a continuum of trivial shape and $M_b^\lambda = \{p\}$, (4) M_a^λ a continuum with the shape of S^1 and $M_b^\lambda = \{p\}$. We discuss first the case (4). As we remarked before, since M_1^λ is an attractor and M_a^λ is an attractor of the restriction of the flow $\varphi_\lambda|M_1^\lambda$ then M_a^λ is, in fact, an attractor of φ_λ. The bounded component of $\mathbb{R}^2 - M_a^\lambda$ must be contained in M_1^λ and p must lie there. As a consequence, the bounded component of $\mathbb{R}^2 - M_a^\lambda$ is the basin of repulsion of $\{p\}$, which means that $\{p\}$ is a repeller for φ_λ (and not only for the restriction $\varphi_\lambda|M_1^\lambda$). If we calculate now the Morse equations of the associated Morse decomposition we see that the repeller $\{p\}$ contributes with the term t^2 and the evolving attractor M_a^λ contributes with a new t. The rest of the Morse sets have the same contribution to the Morse equations as in P since they are continuations of those of the decomposition $\mathcal{M}$. Hence $P^\lambda - P = t^2 + t$. The rest of the cases are similarly discussed. Case (1) is very similar to case (4) and we leave it to the reader. Cases (2) and (3) have in common the fact that M_1^λ has a Morse decomposition $\{M_a^\lambda, M_b^\lambda\}$ into two sets of trivial shape. The Conley index of M_a^λ is the index of an attractor of trivial shape and the Conley index of M_b^λ can be easily calculated from the long exact sequence of the Morse decomposition of M_1^λ, from which it results a trivial Conley index. A consequence of this is that M_b^λ is a saddle-set and the Morse equation P^λ is not changed after the bifurcation. □

For a discussion of generalized Poincaré–Andronov–Hopf bifurcations we refer the reader to the paper [41].

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